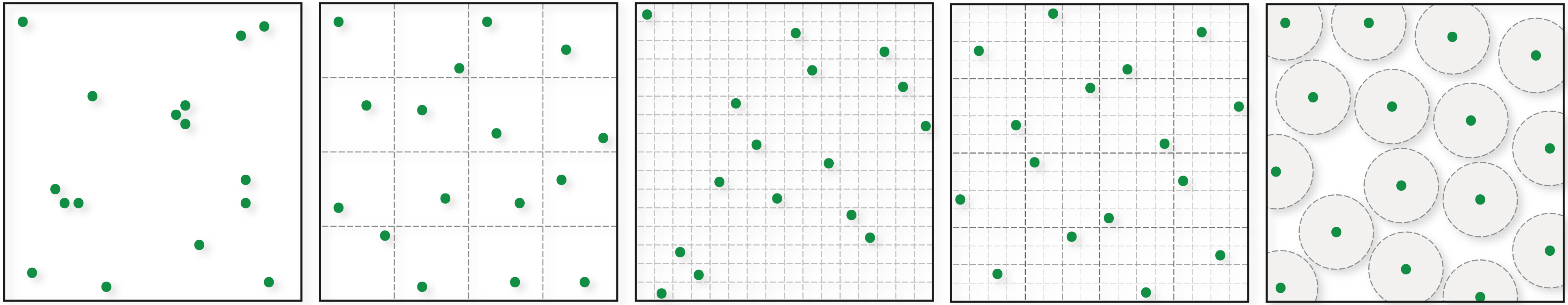
ADVANCED SAMPLING



Gurprit Singh

Last week

Volumetric Processes:

Absorption

Scattering

Transmittance

Phase Functions

Volumetric Rendering Equation

Volumetric Path Tracing

Woodcock Tracking

Volumetric Rendering Equation

$$L(\mathbf{x}, \vec{\omega}) = T_r(\mathbf{x}, \mathbf{x}_z) L(\mathbf{x}_z, \vec{\omega}) + \int_0^z T_r(\mathbf{x}, \mathbf{x}_t) \sigma_a(\mathbf{x}_t) L_e(\mathbf{x}_t, \vec{\omega}) dt + \int_0^z T_r(\mathbf{x}, \mathbf{x}_t) \sigma_s(\mathbf{x}_t) \int_{S^2} f_p(\mathbf{x}_t, \vec{\omega}', \vec{\omega}) L_i(\mathbf{x}_t, \vec{\omega}') d\omega' dt$$

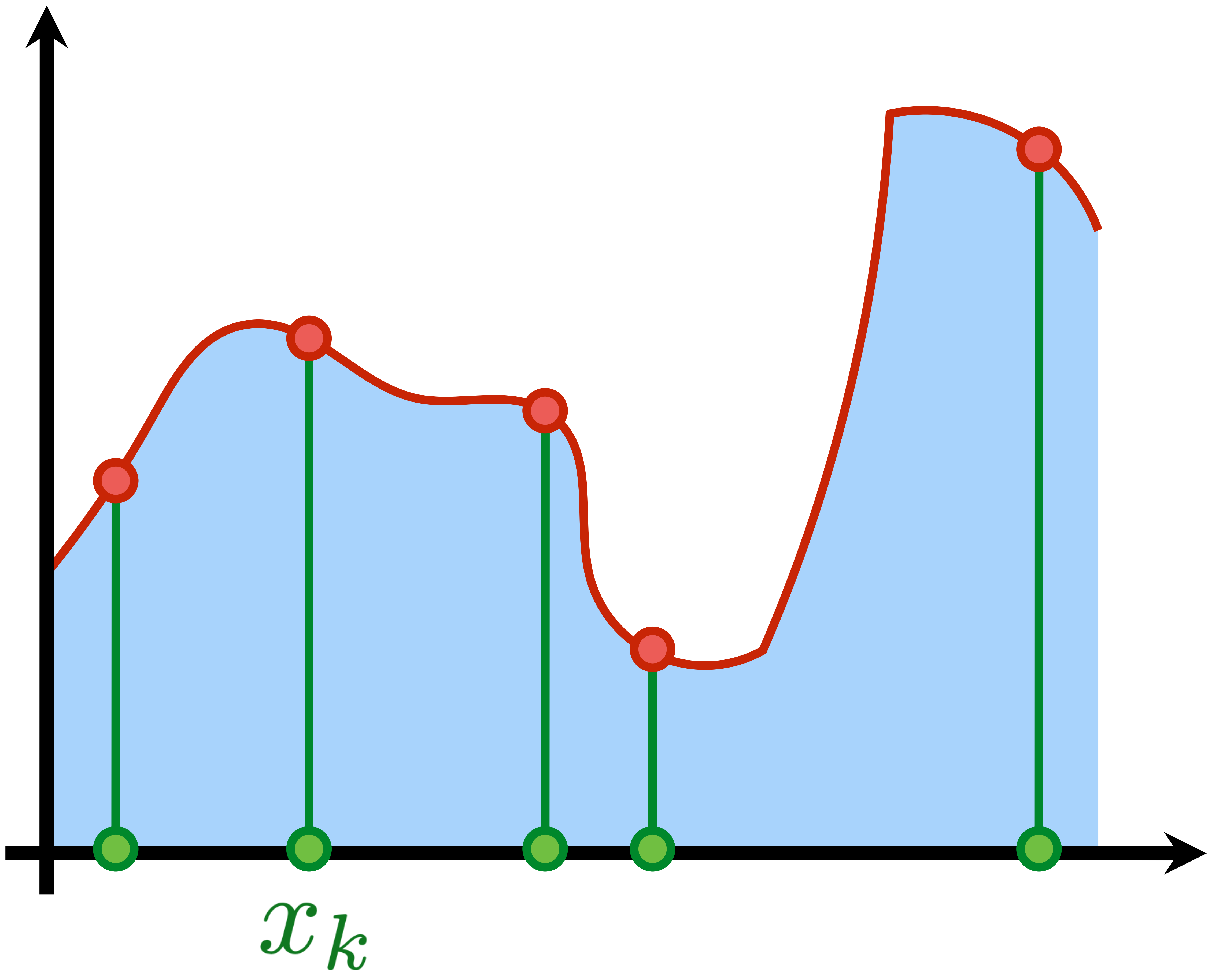
Recall: Monte Carlo Integration

$$I = \int_D f(x)\, \mathrm{d}x$$

$$\approx \int_D f(x)\, \mathbf{S}(x)\, \mathrm{d}x$$

$$\mathbf{S}(x) = \frac{1}{N} \sum_{k=1}^{N} \delta(x - x_k)$$

How to generate the locations x_k?



Independent Random Sampling

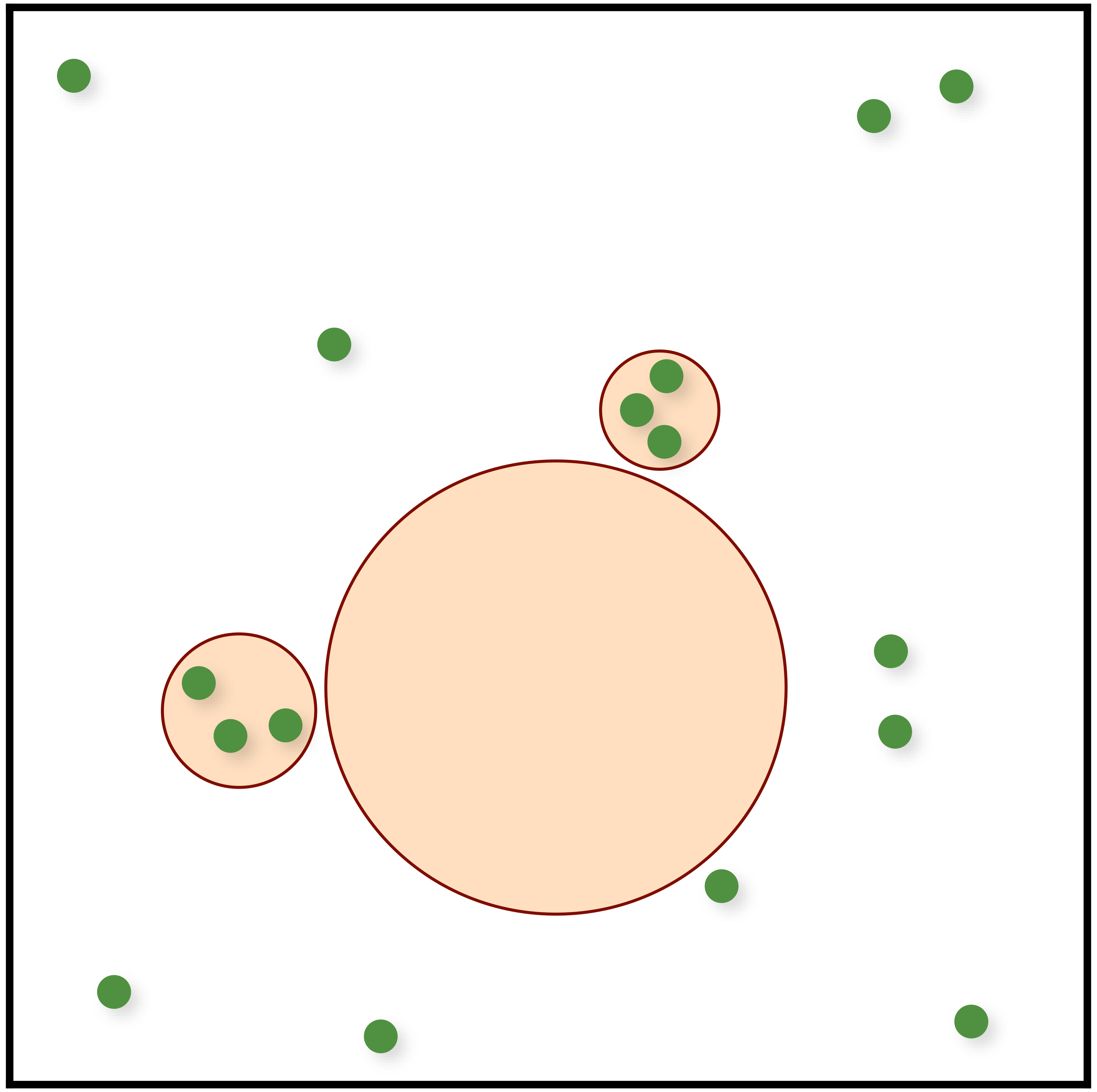
```
for (int k = 0; k < num; k++)
{
    samples(k).x = randf();
    samples(k).y = randf();
}
```

✓ Trivially extends to higher dimensions

✓ Trivially progressive and memory-less

✗ Big gaps

✗ Clumping



Recall: Fourier Theory

Input Image

Power Spectrum

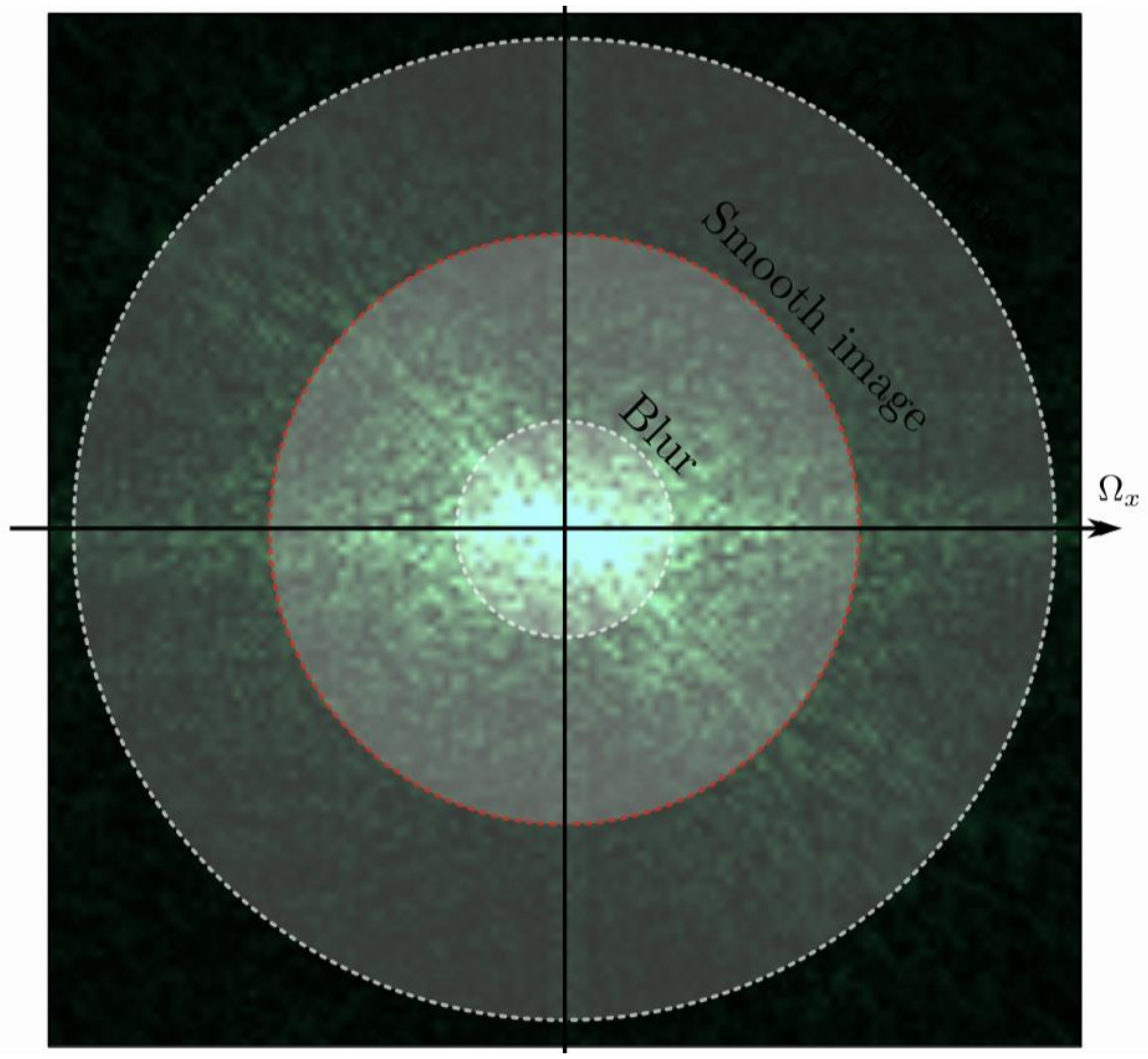


Image courtesy: Laurent Belcour

Recall: Fourier theory

Fourier transform: $\hat{f}(\vec{\omega}) = \int_D f(\vec{x})\, \mathrm{e}^{-2\,\pi\,\imath\,(\vec{\omega}\cdot\vec{x})}\, \mathrm{d}\vec{x}$

$$\hat{f}(\vec{\omega}) = \int_D f(\vec{x})\, \mathrm{e}^{-2\,\pi\,\imath\,(\vec{\omega}\vec{x})}\, \mathrm{d}x$$

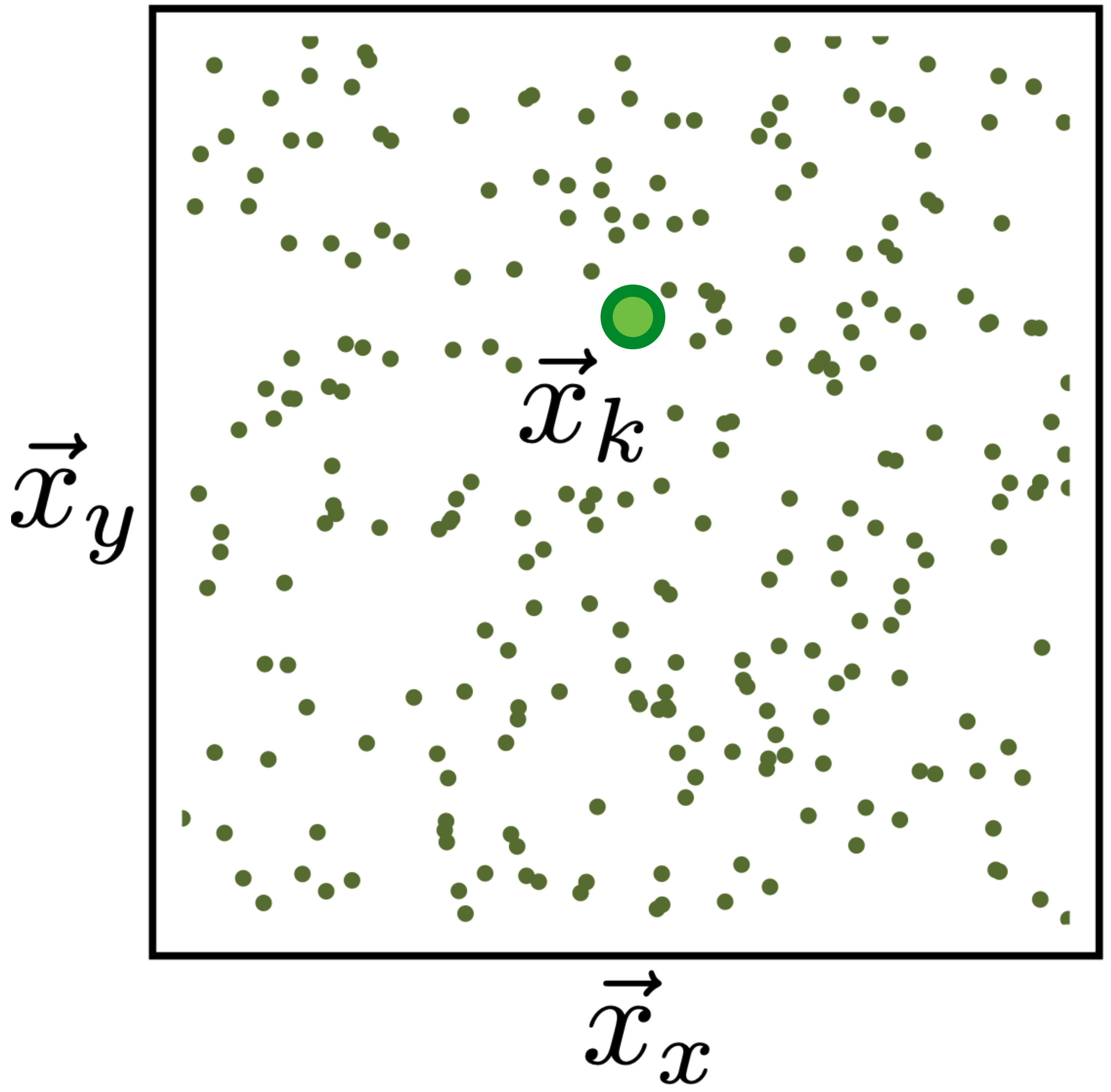
Sampling function: $\hat{\mathbf{S}}(\vec{\omega}) = \int_D \mathbf{S}(\vec{x})\, \mathrm{e}^{-2\,\pi\,\imath\,(\vec{\omega}\cdot\vec{x})}\, \mathrm{d}\vec{x}$

$$\hat{\mathbf{S}}(\vec{\omega}) = \int_D \frac{1}{N} \sum_{k=1}^{N} \mathrm{e}^{-2\,\pi\,\imath\,(\vec{\omega}\cdot\vec{x})}\, \delta(|\vec{x} - \vec{x}_k|)\, \mathrm{d}\vec{x}$$

$$= \frac{1}{N} \sum_{k=1}^{N} \mathrm{e}^{-2\,\pi\,\imath\,(\vec{\omega}\cdot\vec{x}_k)}$$

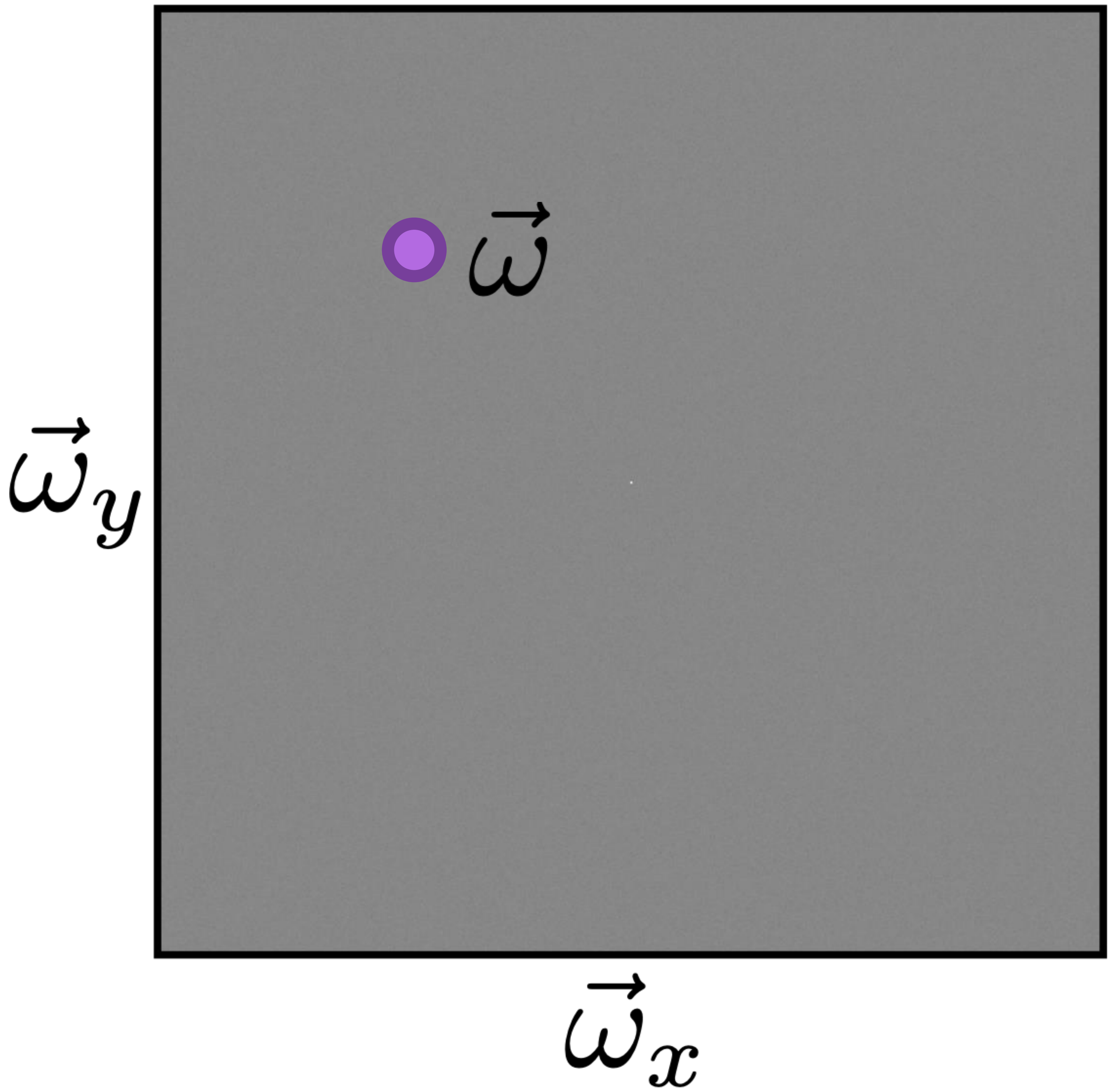
Independent Random Sampling

Samples



$$\frac{1}{N}\sum_{k=1}^{N}\delta(|\vec{x}-\vec{x}_k|)$$

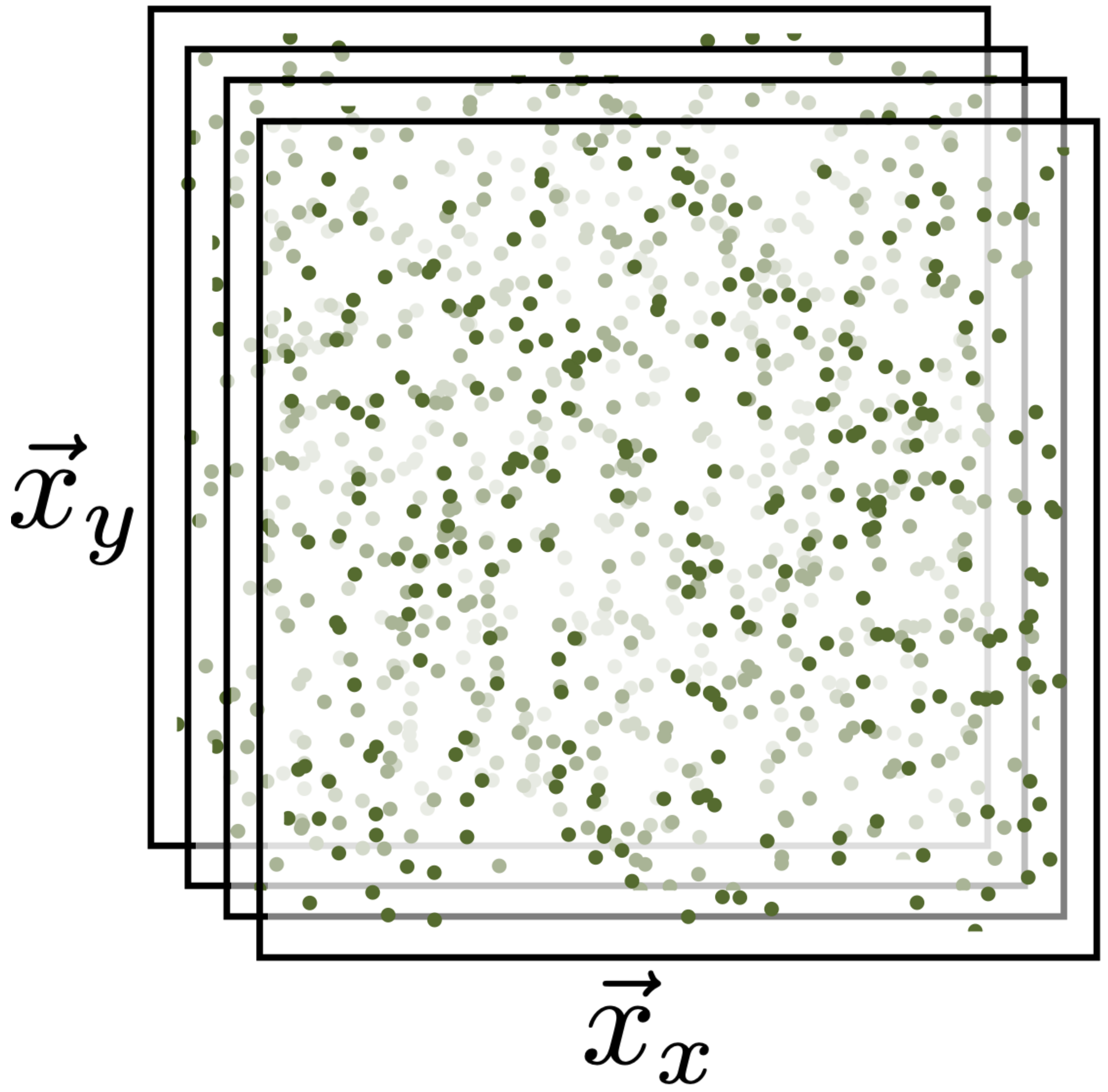
Power spectrum



$$\left|\frac{1}{N}\sum_{k=1}^{N}e^{-2\pi\imath(\vec{\omega}\cdot\vec{x}_k)}\right|^2$$

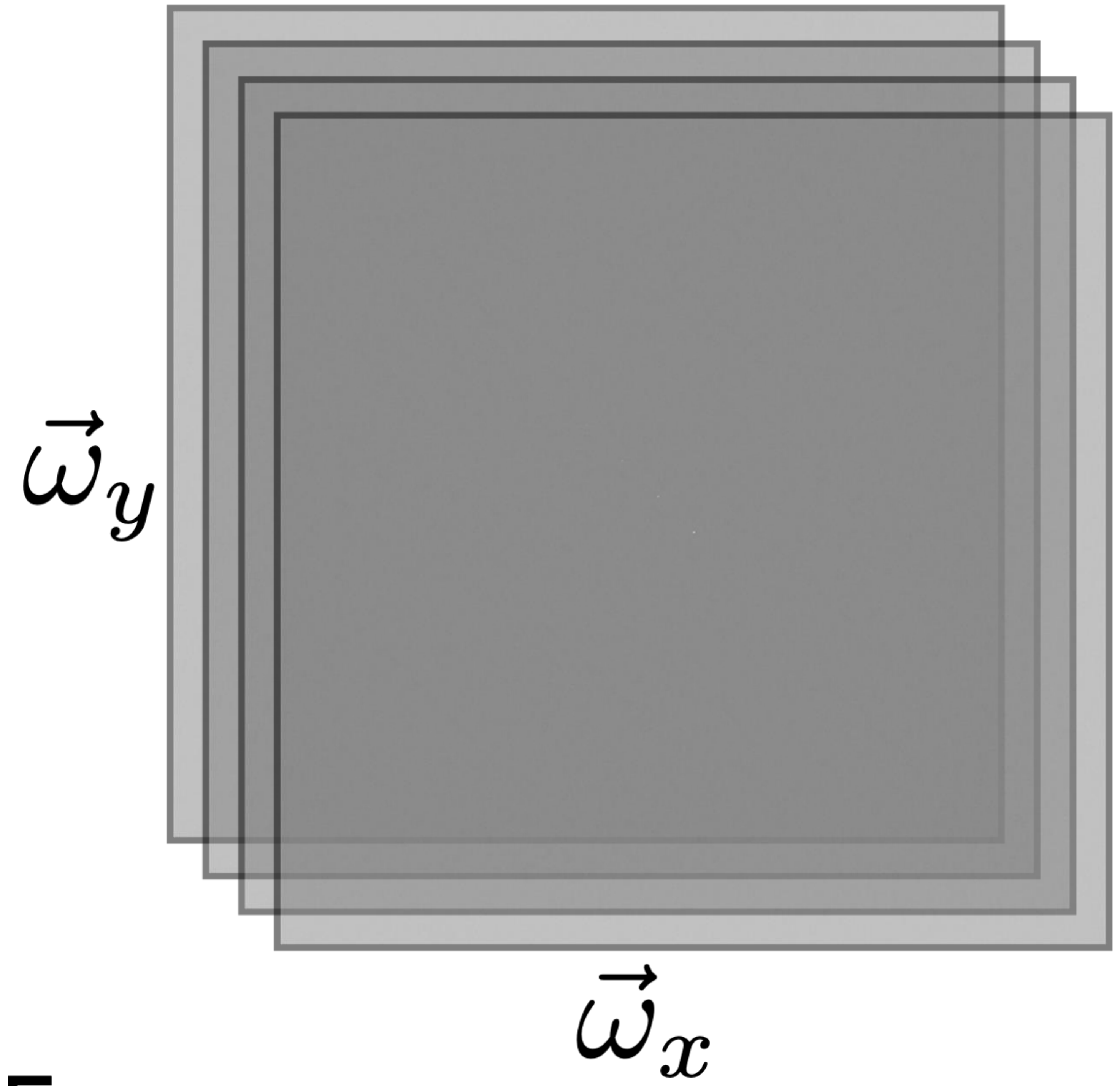
Independent Random Sampling

Many sample set realizations



$$\frac{1}{N}\sum_{k=1}^{N}\delta(|\vec{x}-\vec{x}_k|)$$

Expected power spectrum



$$\mathrm{E}\left[\left|\frac{1}{N}\sum_{k=1}^{N}\mathrm{e}^{-2\pi\imath(\vec{\omega}\cdot\vec{x}_k)}\right|^2\right]$$

Independent Random Sampling

$$\frac{1}{N}\sum_{k=1}^{N}\delta(|\vec{x}-\vec{x}_k|) \quad \mathrm{E}\left[\left|\frac{1}{N}\sum_{k=1}^{N}\mathrm{e}^{-2\pi\imath(\vec{\omega}\cdot\vec{x}_k)}\right|^2\right]$$

Source code: Power spectrum

```
procedure powerSpectrum(samples, spectrumWidth, spectrumHeight)
  int N = samples.size()
  for u: 0 → spectrumWidth{
    for v: 0 → spectrumHeight{
    double real = 0, imag = 0;

    }
  }
  return power;
}
```

Source code: Power spectrum

```
procedure powerSpectrum(samples, spectrumWidth, spectrumHeight)
  int N = samples.size()
  for u: 0 → spectrumWidth{
    for v: 0 → spectrumHeight{
    double real = 0, imag = 0;

    //compute the real and imaginary fourier coefficients
    for(int k=0;k<N;k++){



    }
  }
  return power;
}
```

Source code: Power spectrum

```
procedure powerSpectrum(samples, spectrumWidth, spectrumHeight)
  int N = samples.size()
  for u: 0 → spectrumWidth{
    for v: 0 → spectrumHeight{
    double real = 0, imag = 0;

    //compute the real and imaginary fourier coefficients
    for(int k=0;k<N;k++){
      real += cos(2 * π * (u * samples[k].x + v * samples[k].y));
      imag += sin(2 * π * (u * samples[k].x + v * samples[k].y));
    }



    }
  }
  return power;
}
```

Source code: Power spectrum

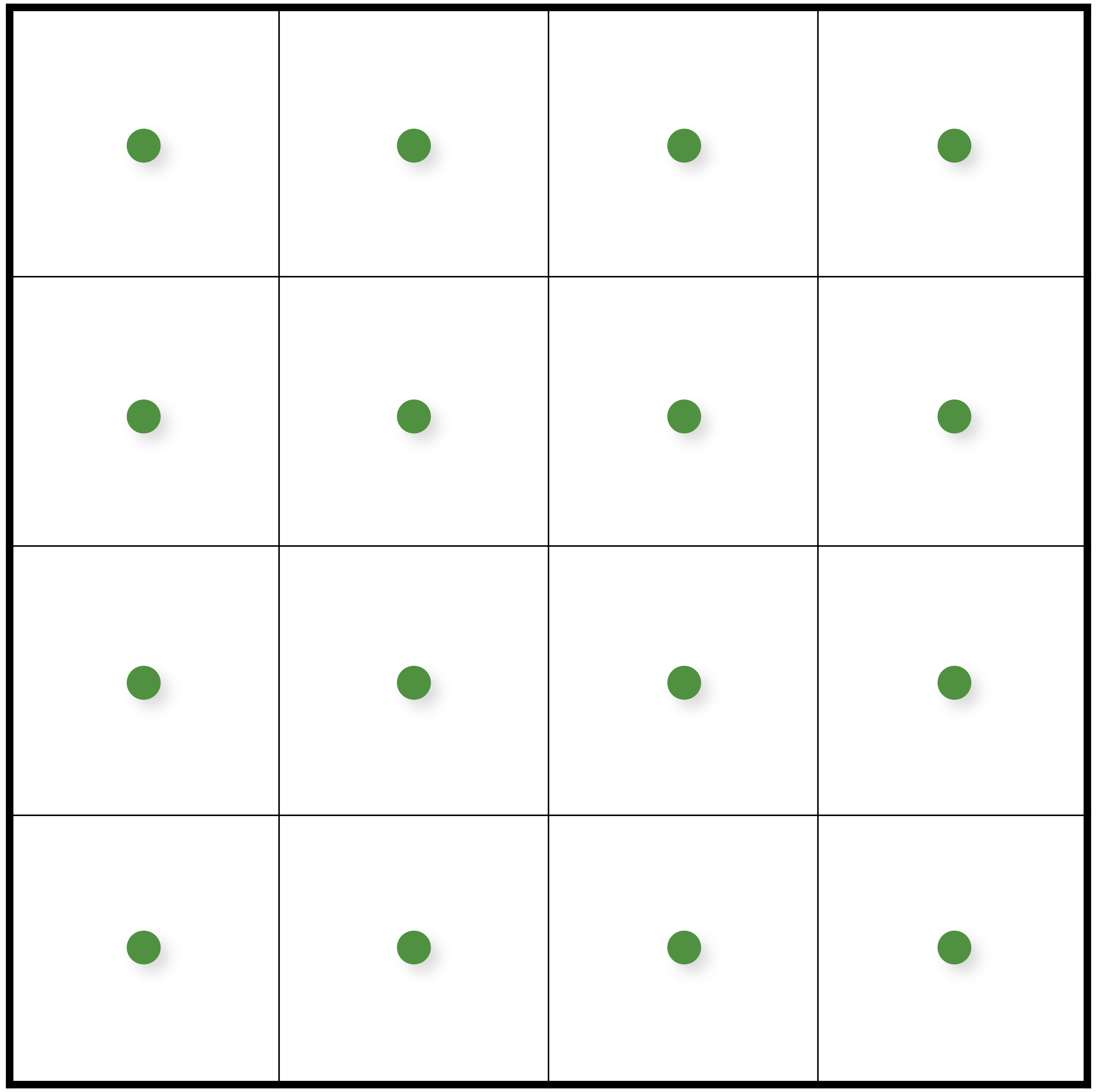
```
procedure powerSpectrum(samples, spectrumWidth, spectrumHeight)
  int N = samples.size()
  for u: 0 → spectrumWidth{
    for v: 0 → spectrumHeight{
    double real = 0, imag = 0;

    //compute the real and imaginary fourier coefficients
    for(int k=0;k<N;k++){
      real += cos(2 * π * (u * samples[k].x + v * samples[k].y));
      imag += sin(2 * π * (u * samples[k].x + v * samples[k].y));
    }

    //power spectrum is the magnitude square value of the coefficients
    power[u * spectrumWidth + v] = (real*real + imag * imag) / N;
    }
  }
  return power;
}
```

Regular Sampling

```
for (uint i = 0; i < numX; i++)
    for (uint j = 0; j < numY; j++)
    {
        samples(i,j).x = (i + 0.5)/numX;
        samples(i,j).y = (j + 0.5)/numY;
    }
```



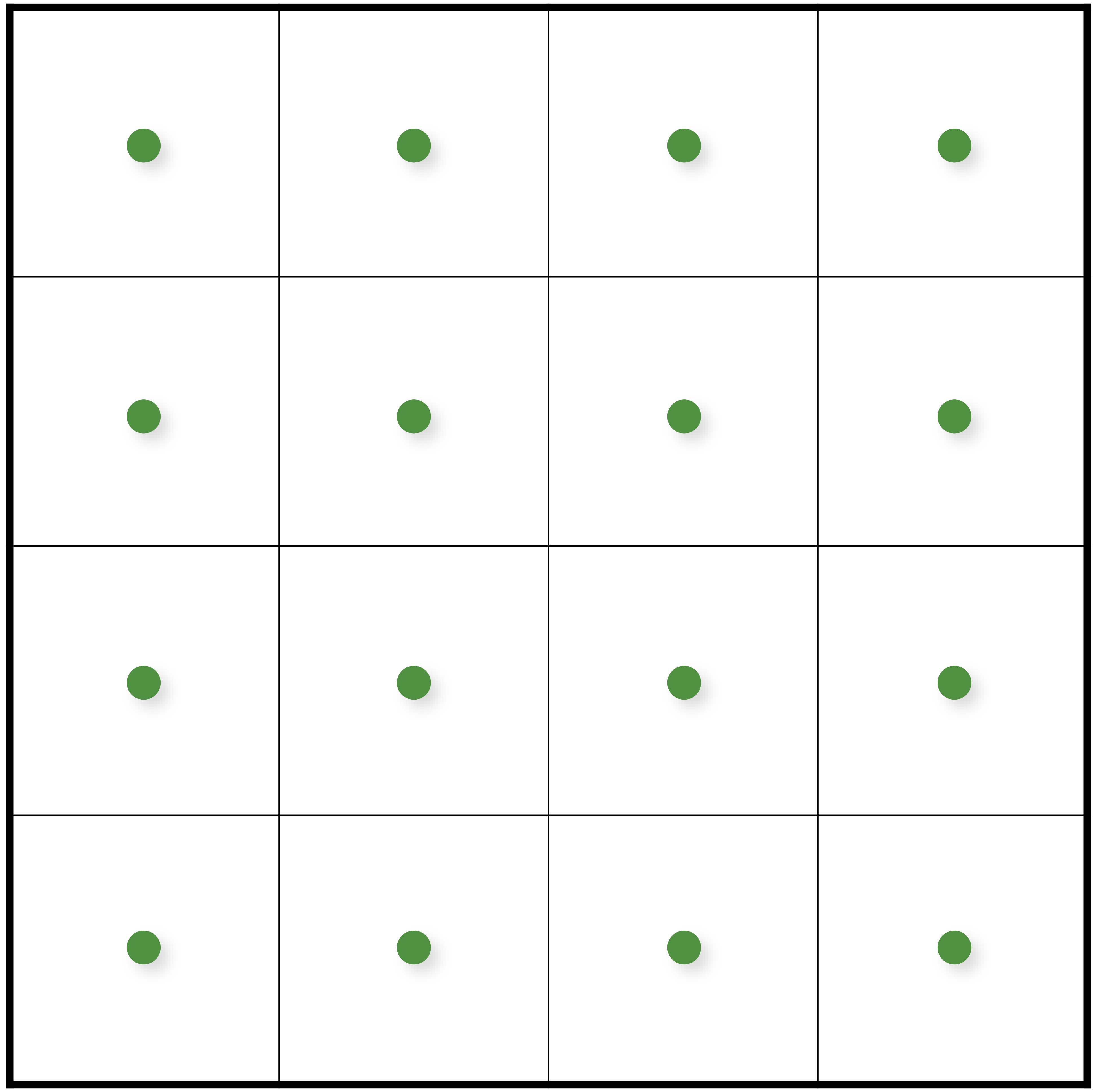
✓ Extends to higher dimensions, but...

✗ Curse of dimensionality

✗ Aliasing

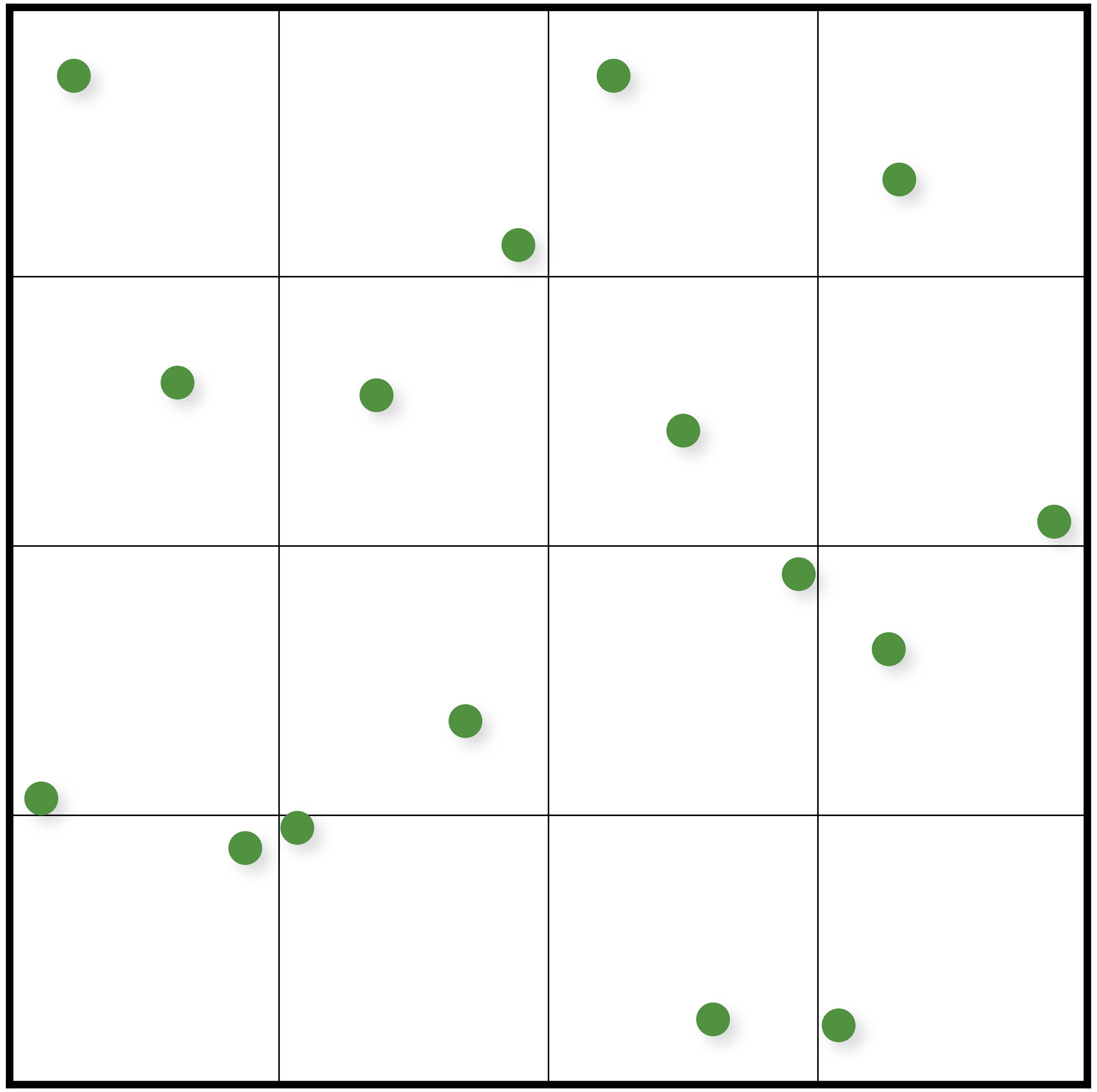
Regular Sampling

```
for (uint i = 0; i < numX; i++)
    for (uint j = 0; j < numY; j++)
    {
        samples(i,j).x = (i + 0.5)/numX;
        samples(i,j).y = (j + 0.5)/numY;
    }
```



Jittered/Stratified Sampling

```
for (uint i = 0; i < numX; i++)
    for (uint j = 0; j < numY; j++)
    {
        samples(i,j).x = (i + randf())/numX;
        samples(i,j).y = (j + randf())/numY;
    }
```



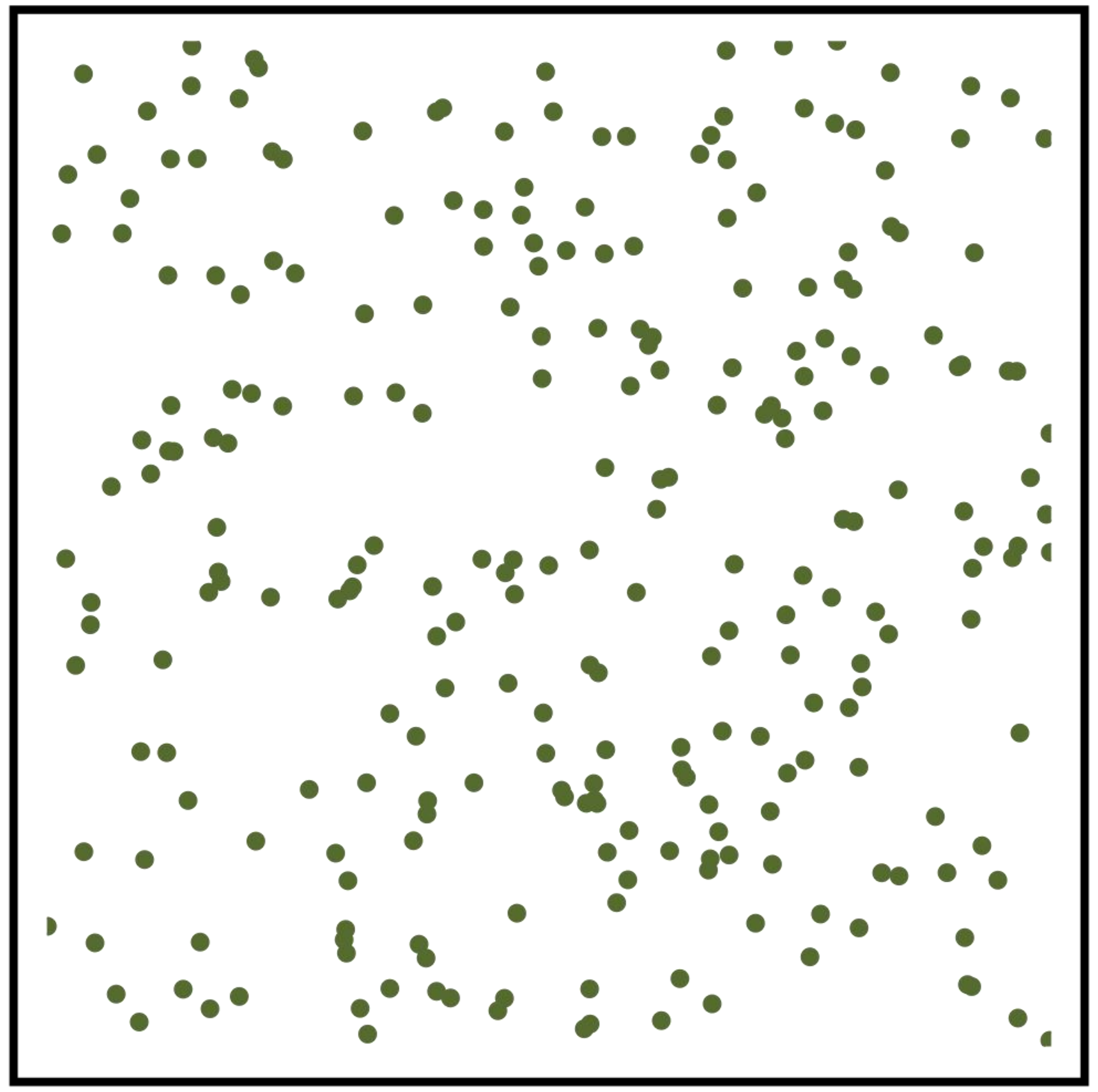
- ✓ Provably cannot increase variance
- ✓ Extends to higher dimensions, but…
- ✗ Curse of dimensionality
- ✗ Not progressive

Jittered Sampling

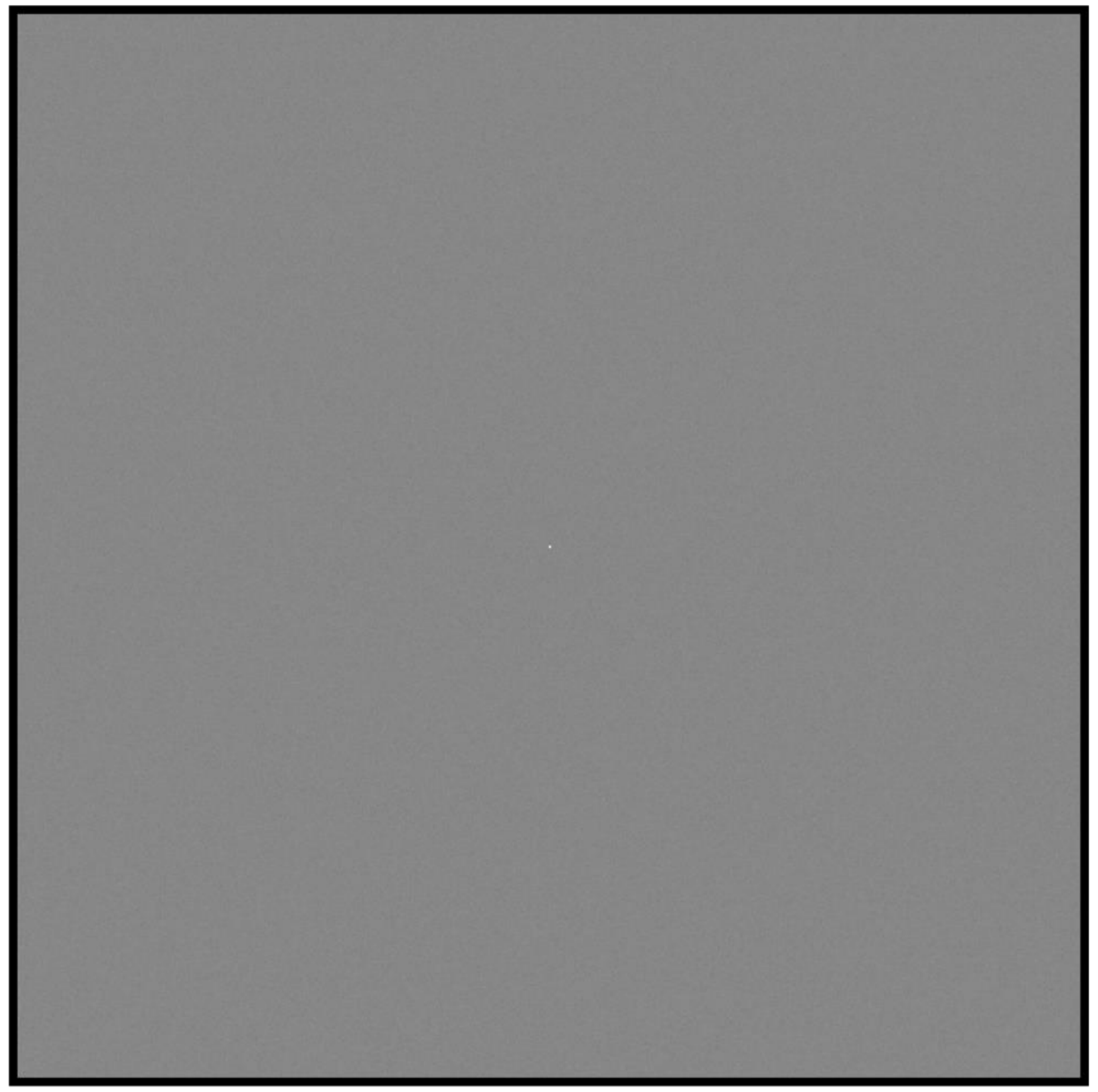
Samples | Expected power spectrum | Radial mean

Independent Random Sampling

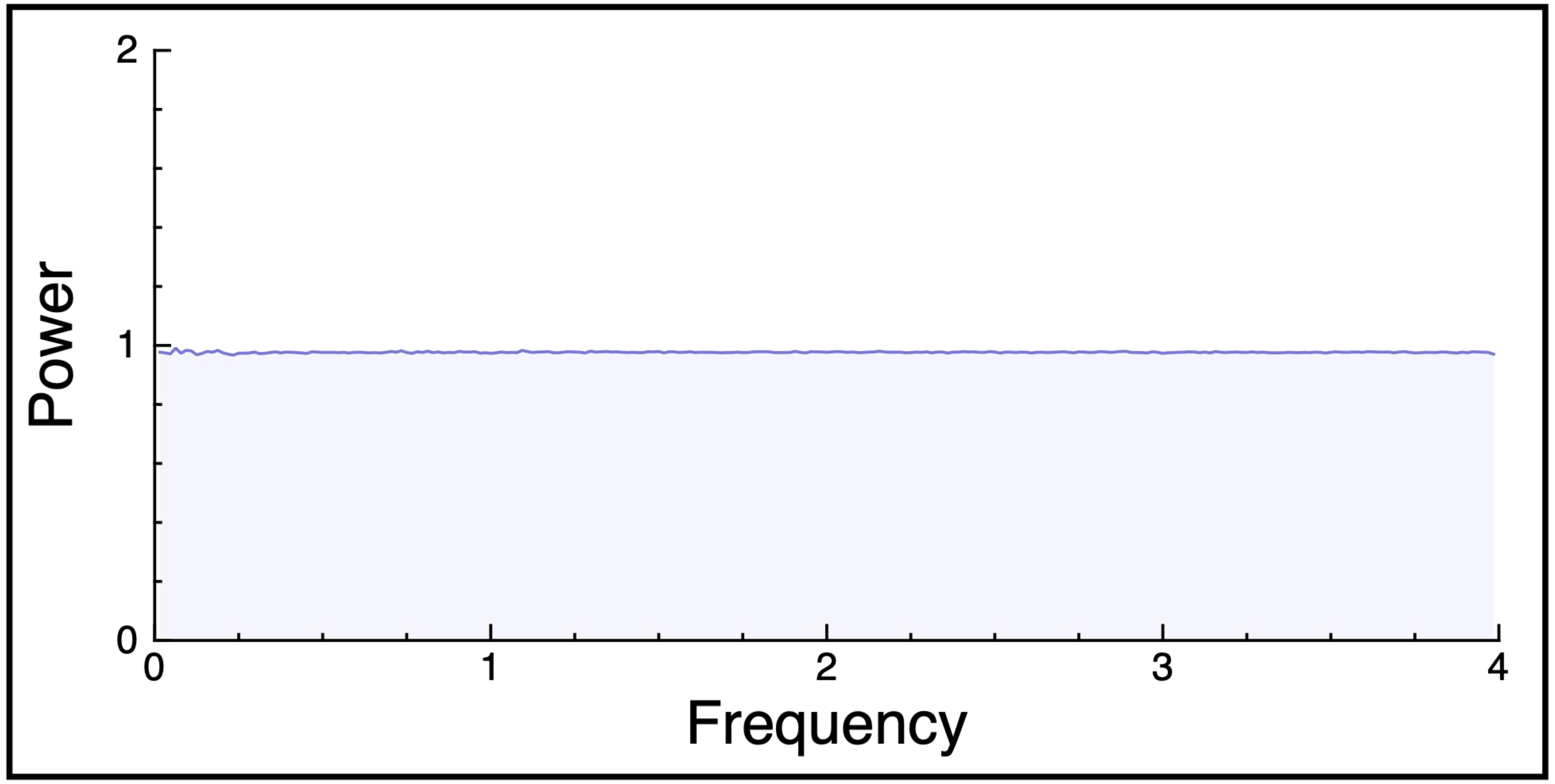
Samples



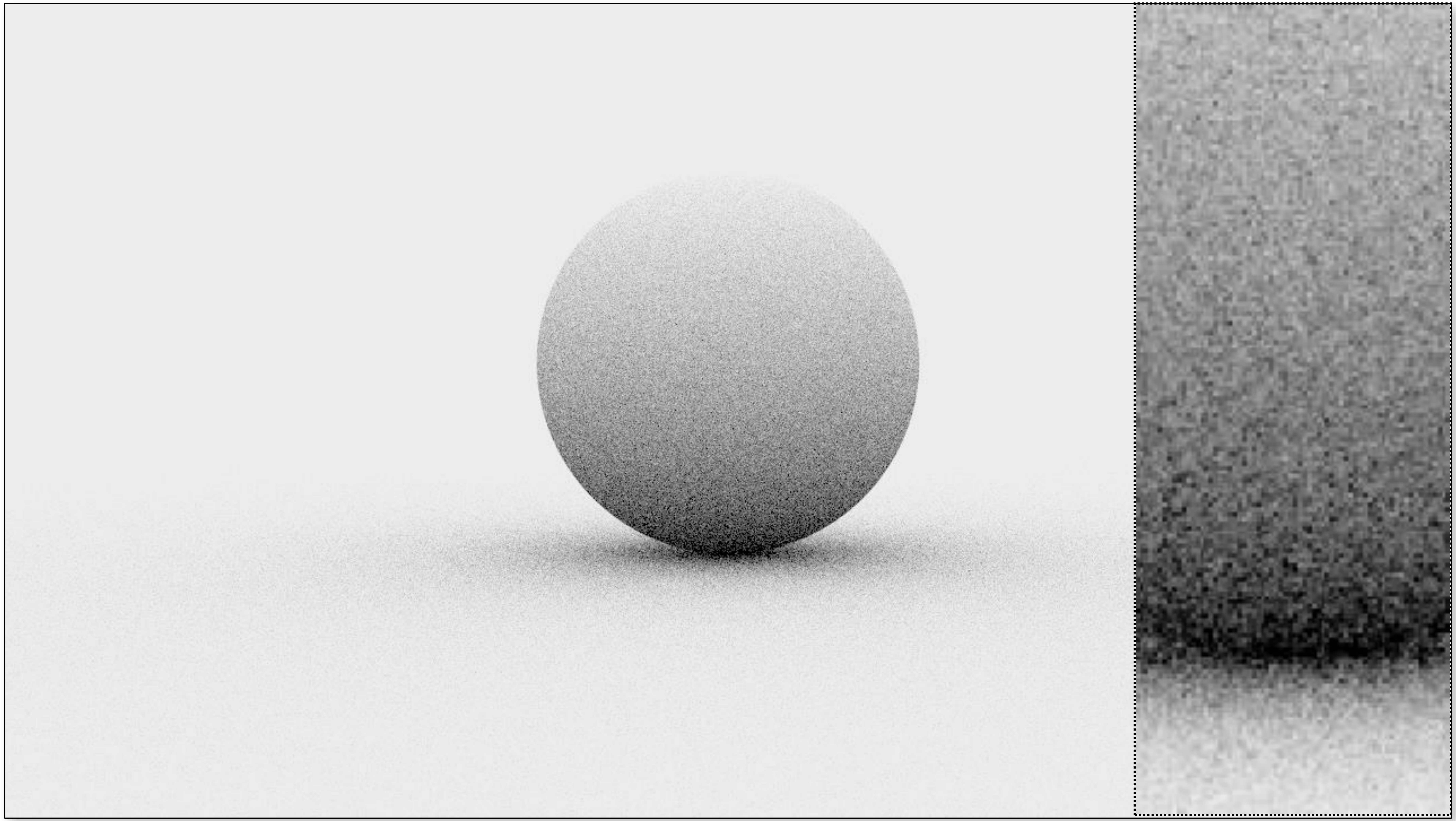
Expected power spectrum



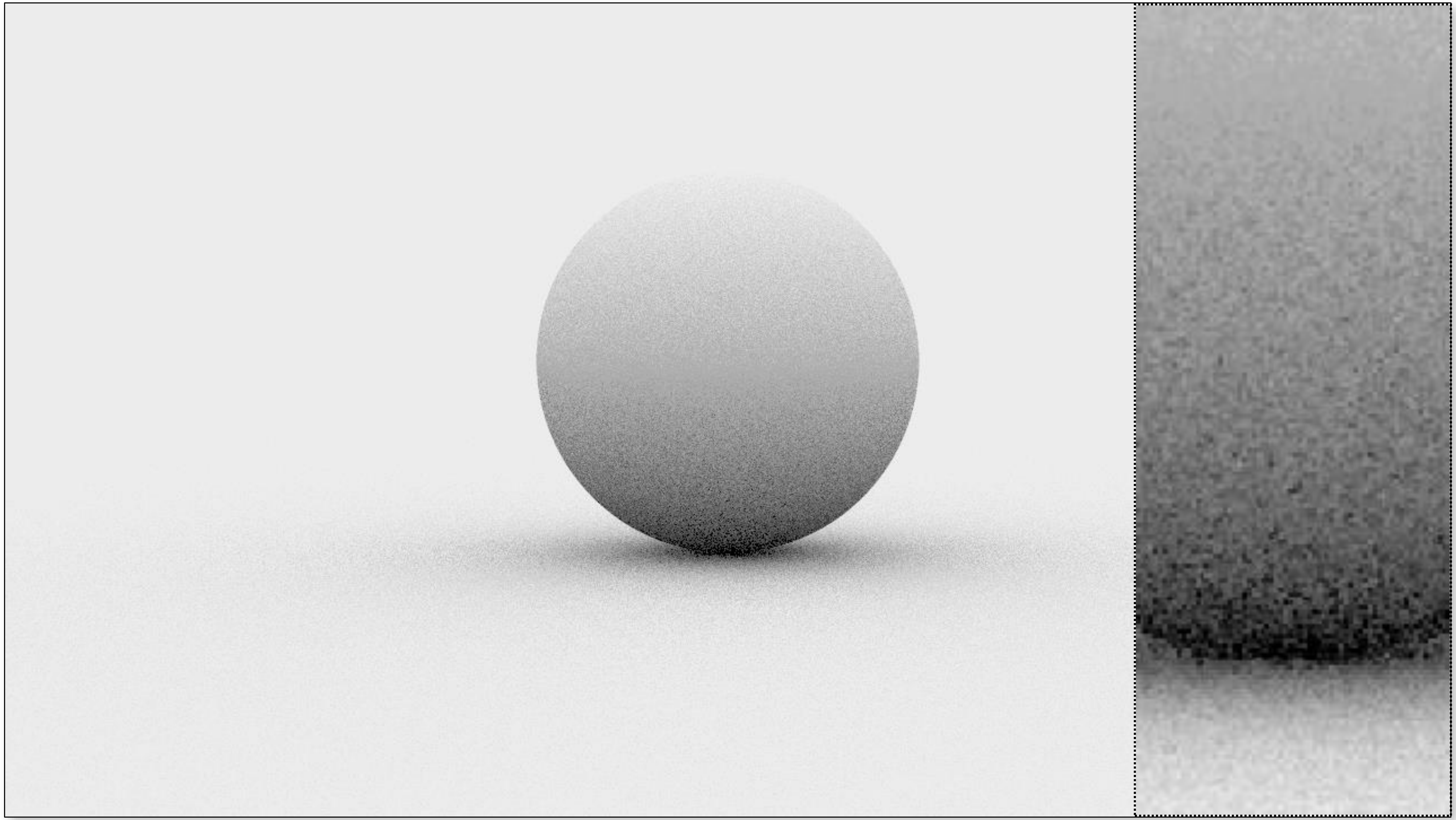
Radial mean



Monte Carlo (16 random samples)



Monte Carlo (16 jittered samples)



Stratifying in Higher Dimensions

Stratification requires O(N^d) samples

- e.g. pixel (2D) + lens (2D) + time (1D) = 5D
  - splitting 2 times in 5D = 2^5 = 32 samples
  - splitting 3 times in 5D = 3^5 = 243 samples!

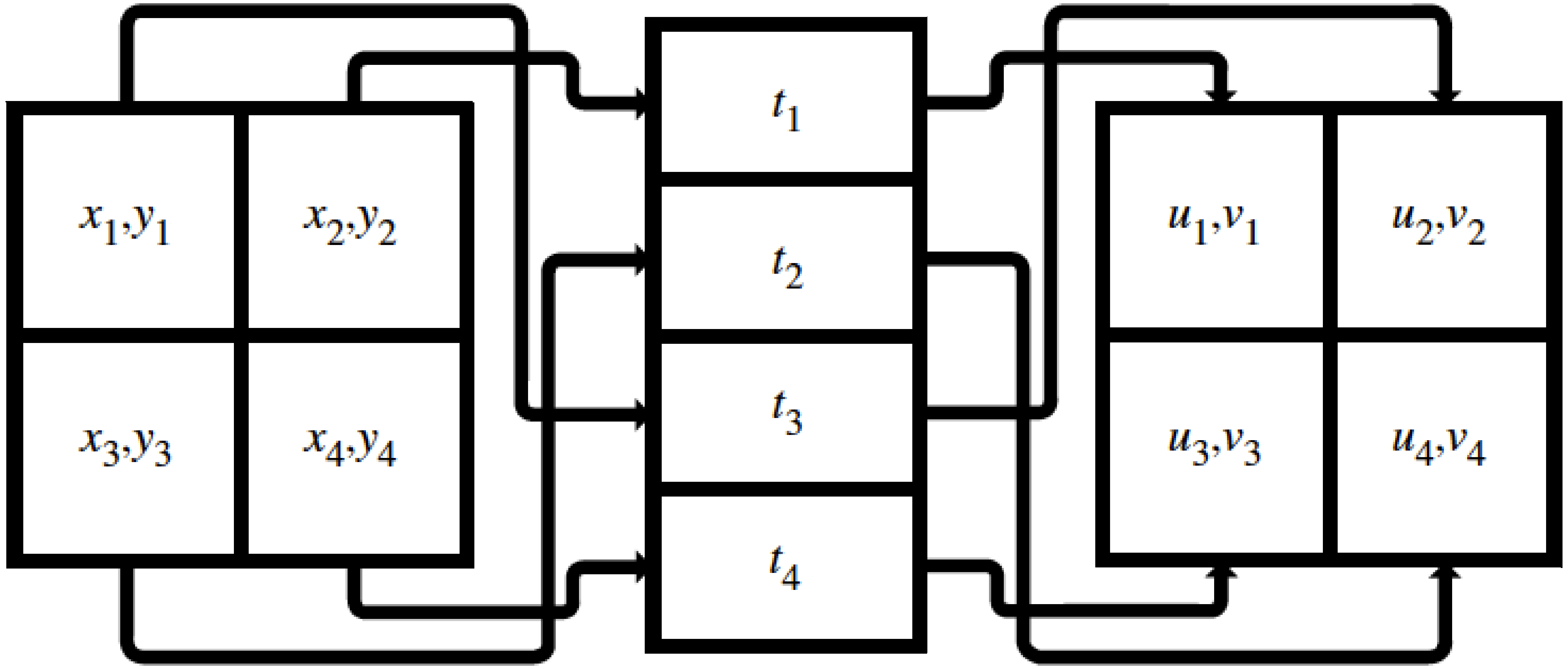
Inconvenient for large d

- cannot select sample count with fine granularity

Uncorrelated Jitter [Cook et al. 84]

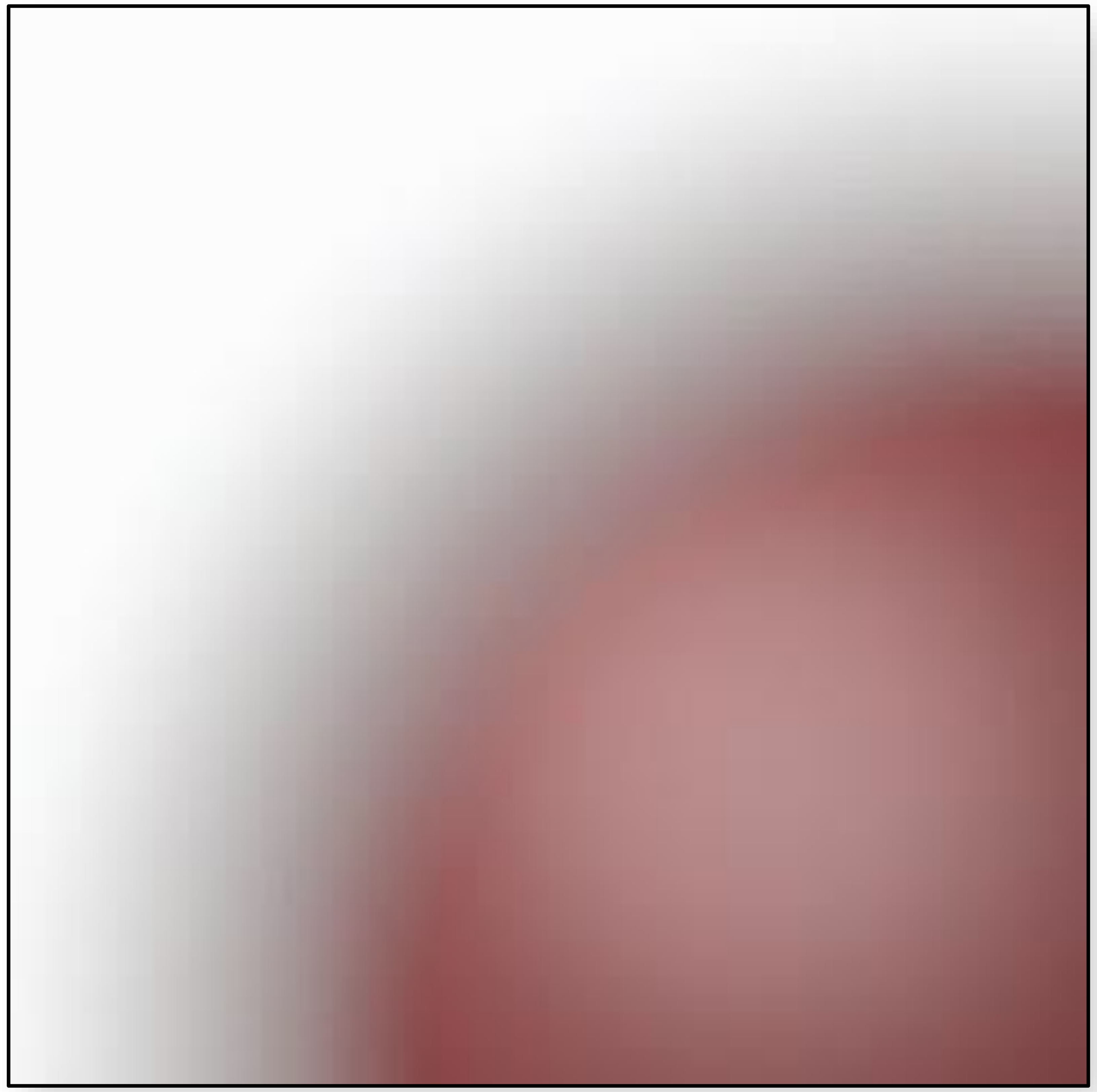
Compute stratified samples in sub-dimensions

- 2D jittered (x,y) for pixel
- 2D jittered (u,v) for lens
- 1D jittered (t) for time
- combine dimensions
  in random order

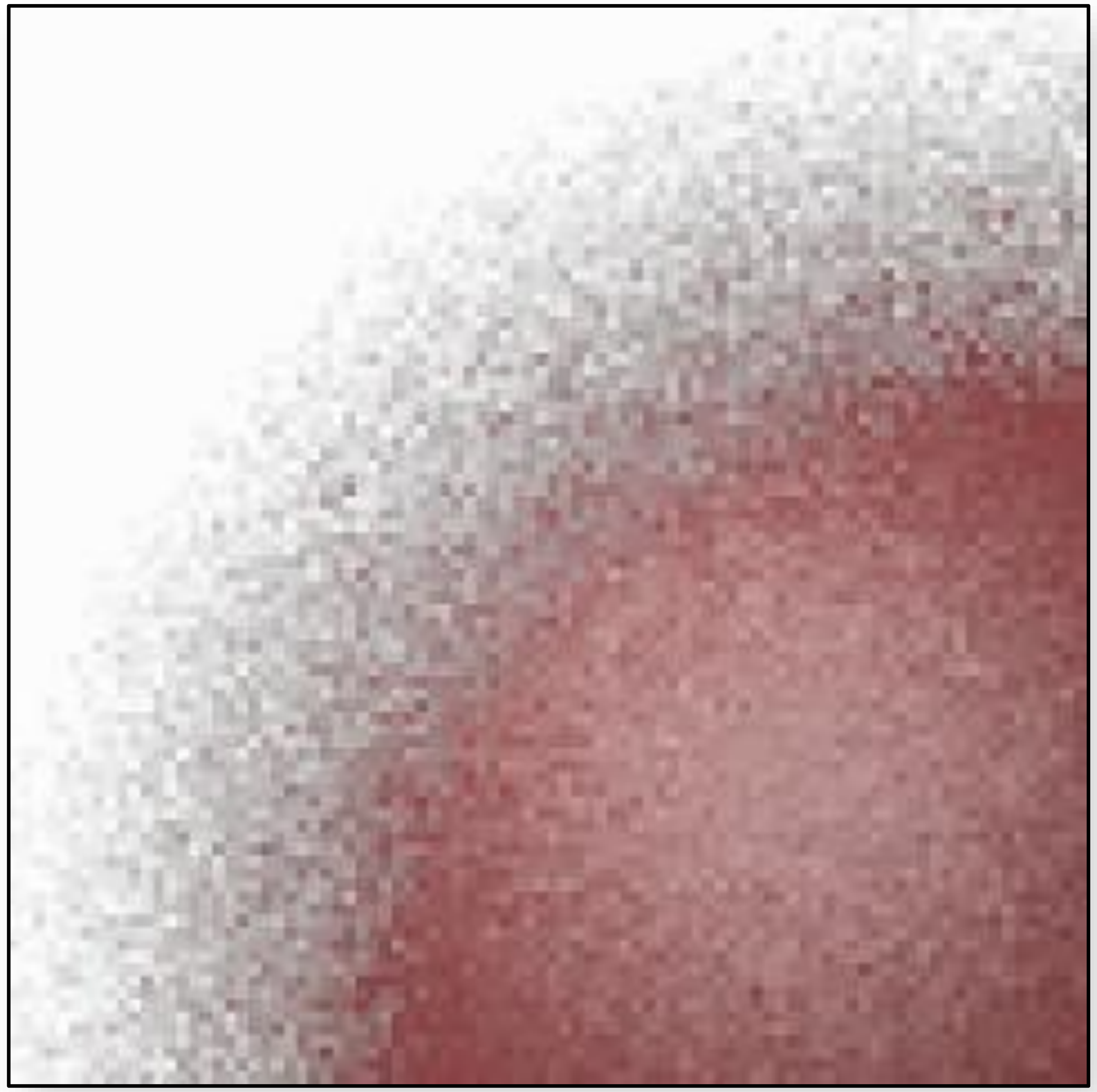


Depth of Field (4D)

Reference



Random Sampling



Uncorrelated Jitter

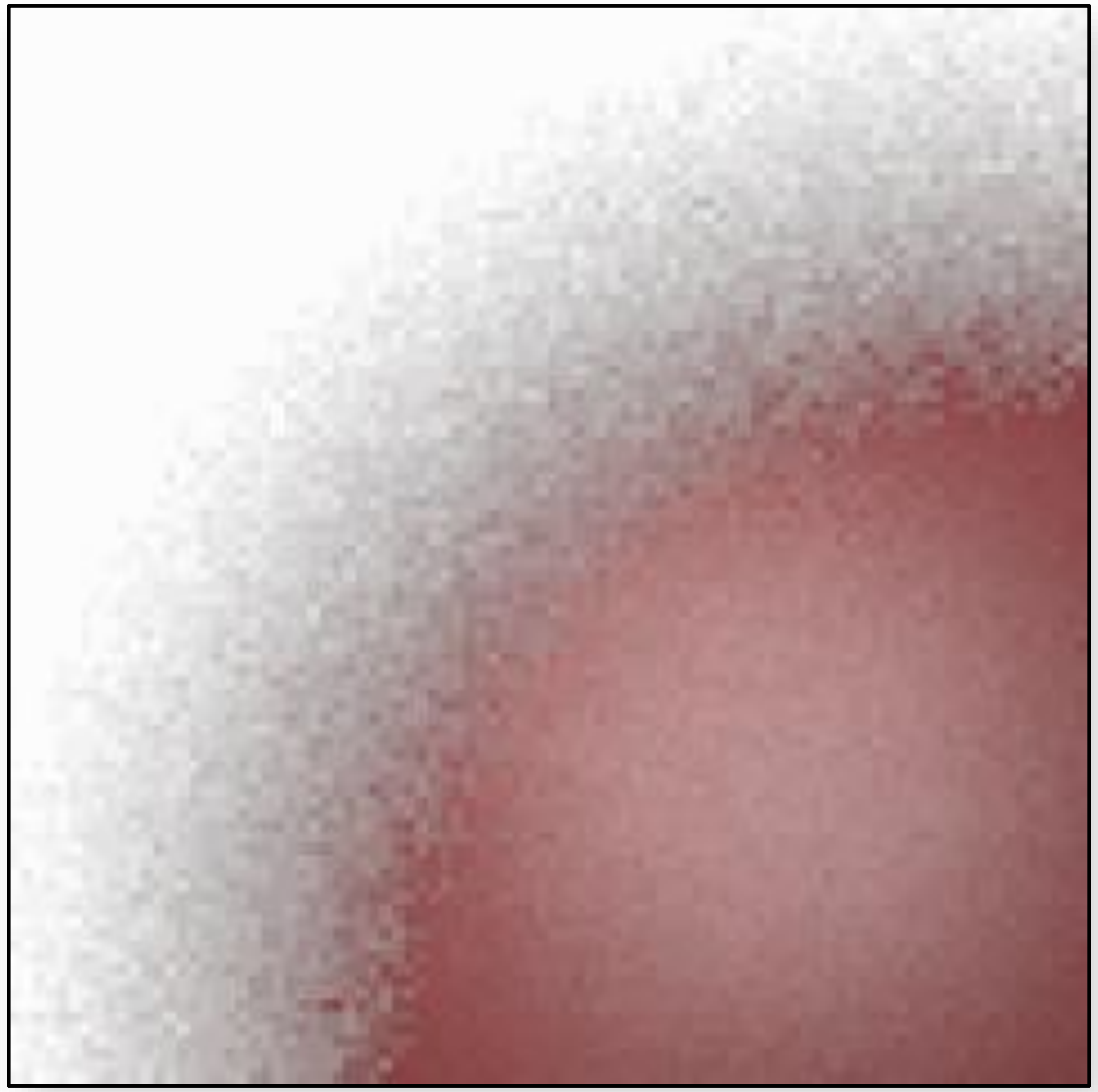


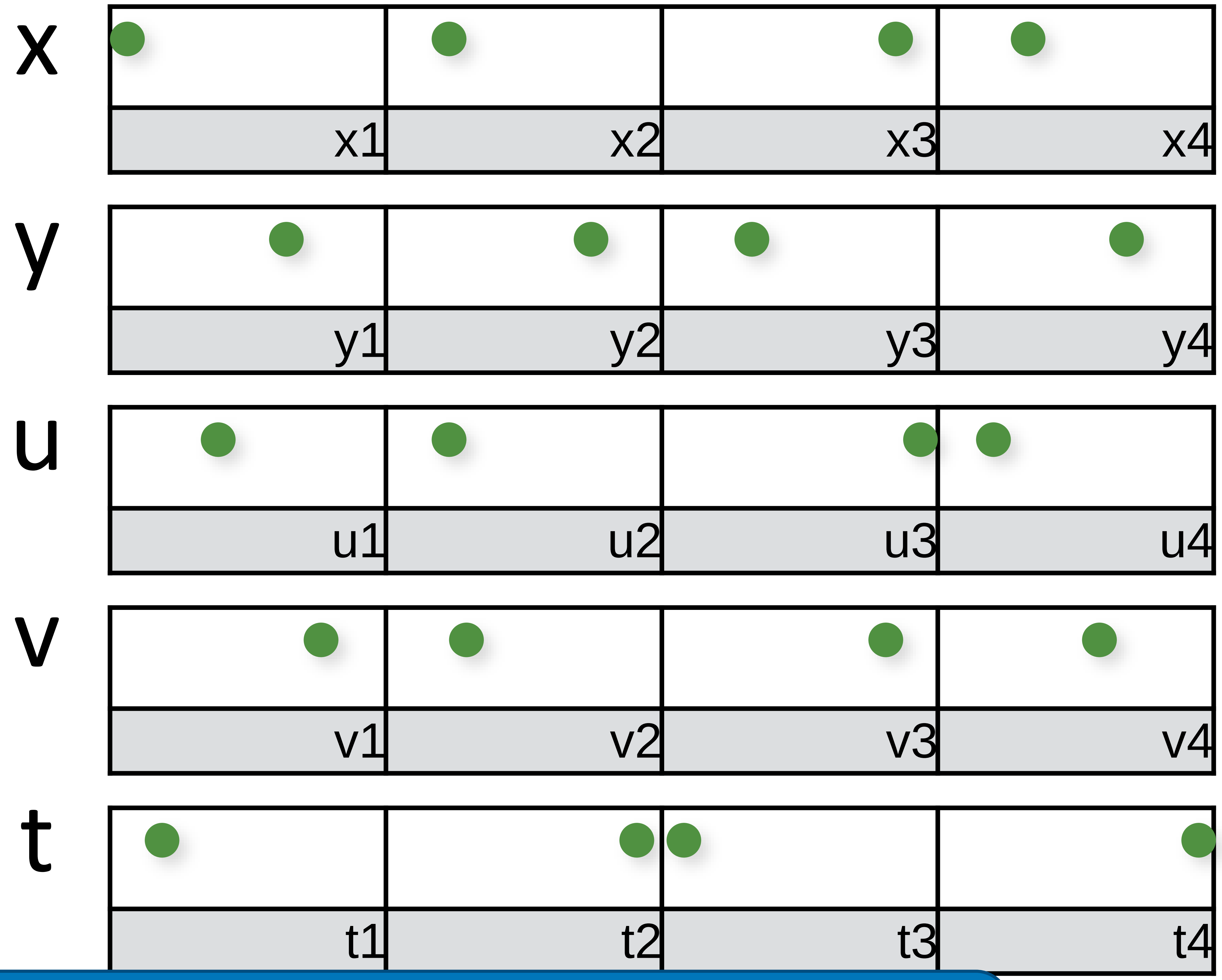
Image source: PBRTe2 [Pharr & Humphreys 2010]

Uncorrelated Jitter ➔ Latin Hypercube

Stratify samples in each dimension separately

- for 5D: 5 separate 1D jittered point sets

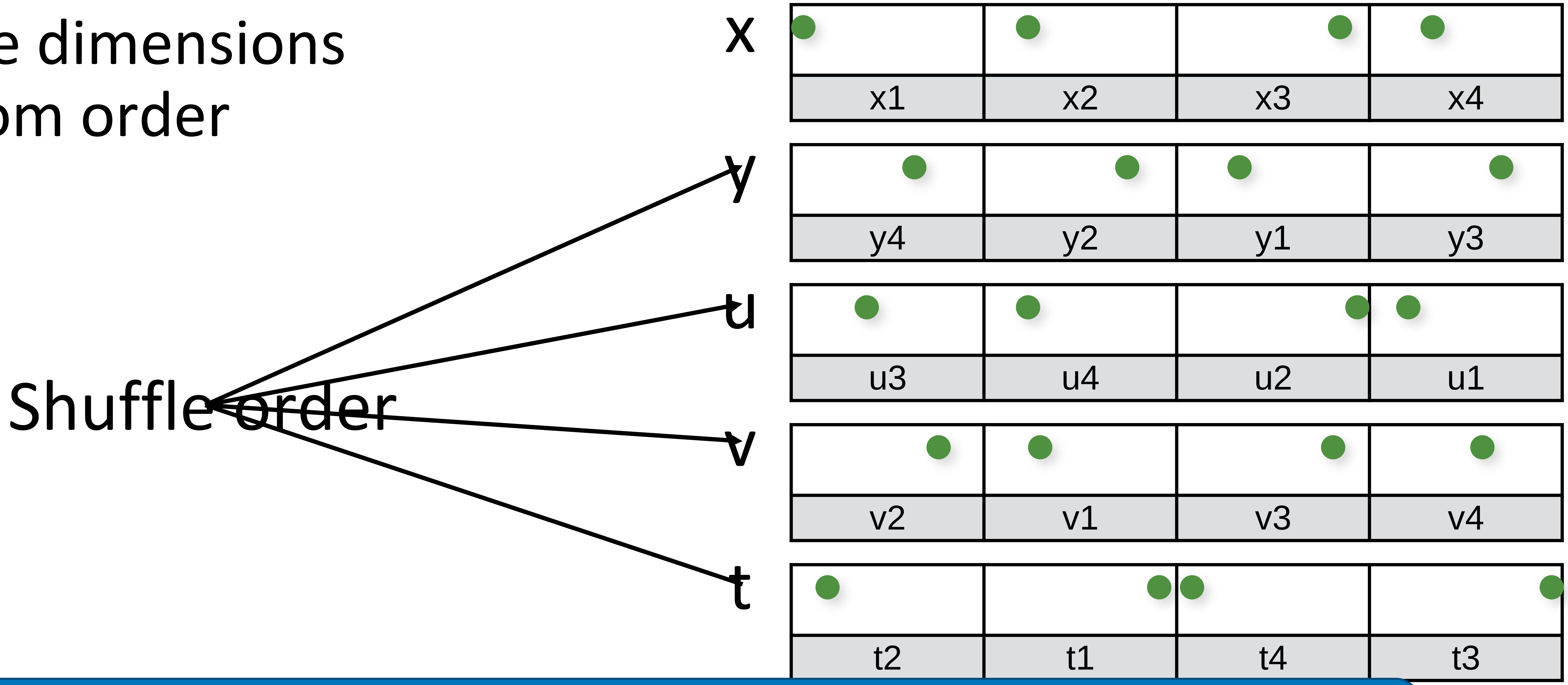
- combine dimensions
  in random order



Uncorrelated Jitter ➔ Latin Hypercube

Stratify samples in each dimension separately

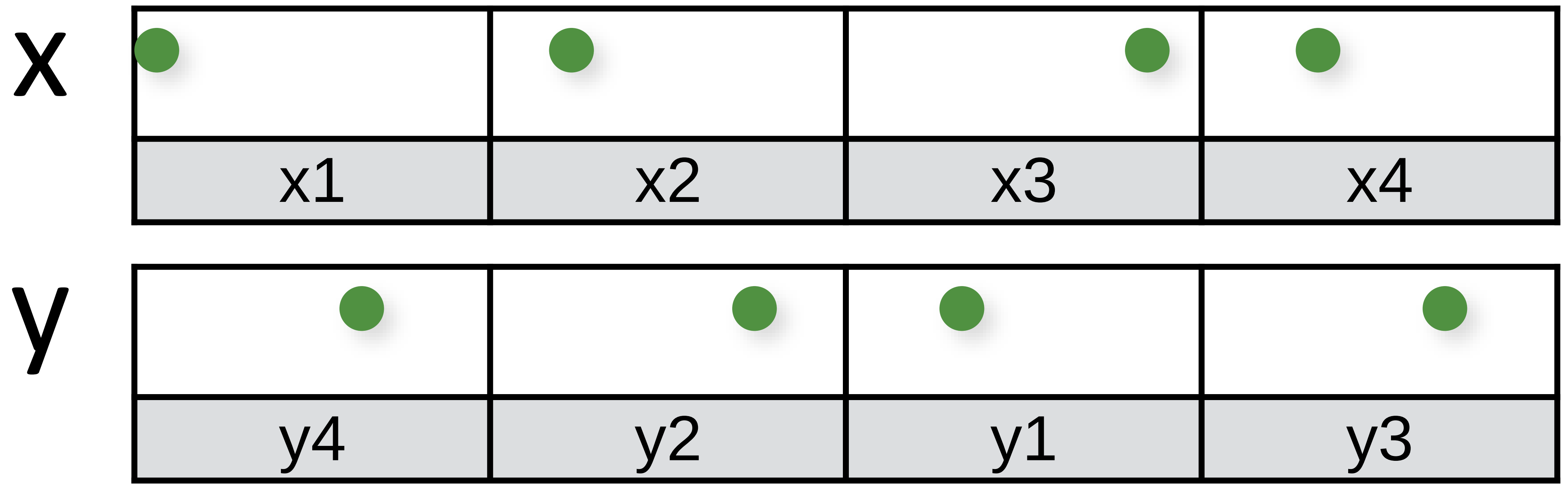
- for 5D: 5 separate 1D jittered point sets
- combine dimensions in random order



N-Rooks = 2D Latin Hypercube [Shirley 91]

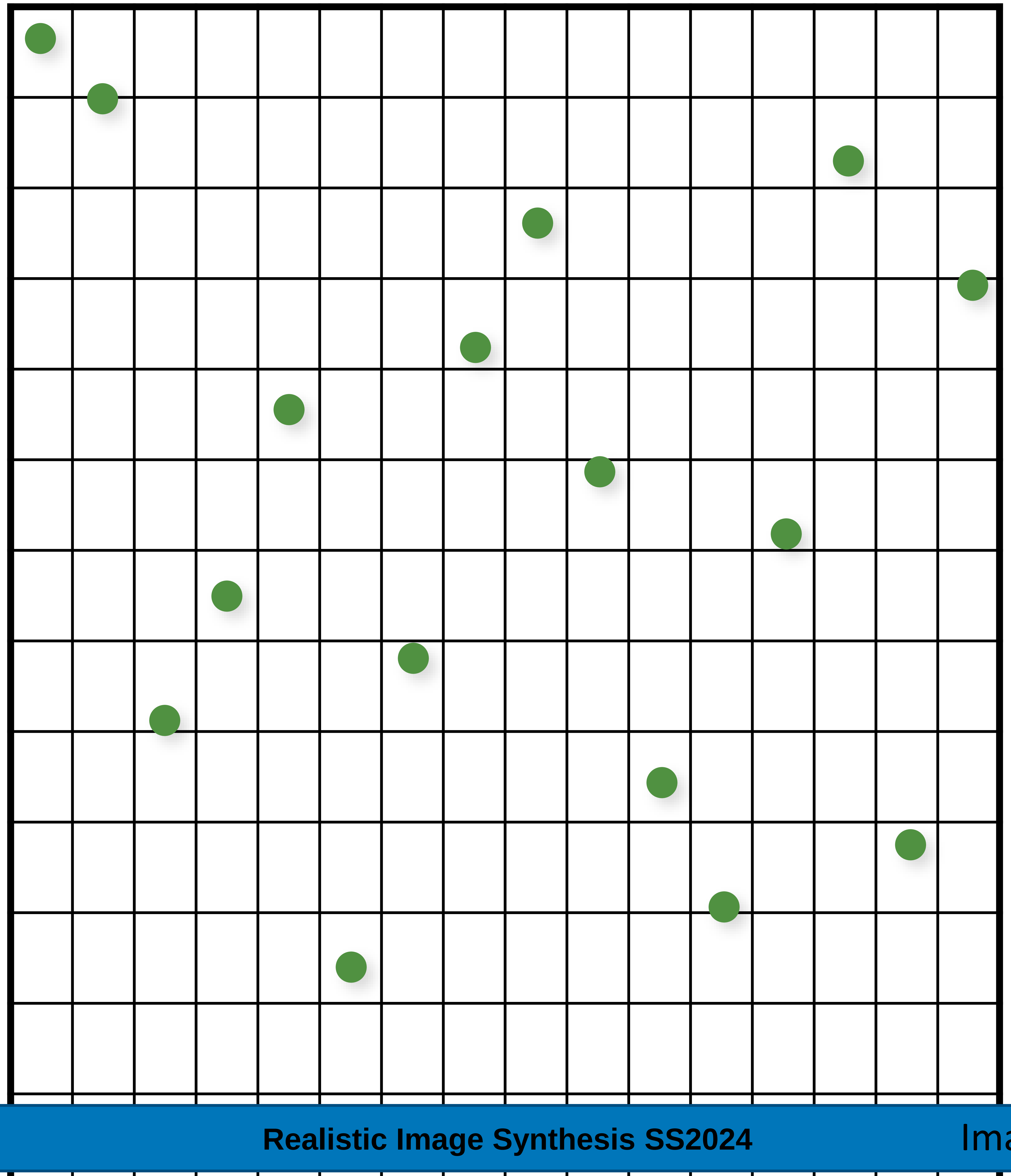
Stratify samples in each dimension separately

- for **2D**: **2** separate 1D jittered point sets
- combine dimensions
  in random order



Latin Hypercube (N-Rooks) Sampling

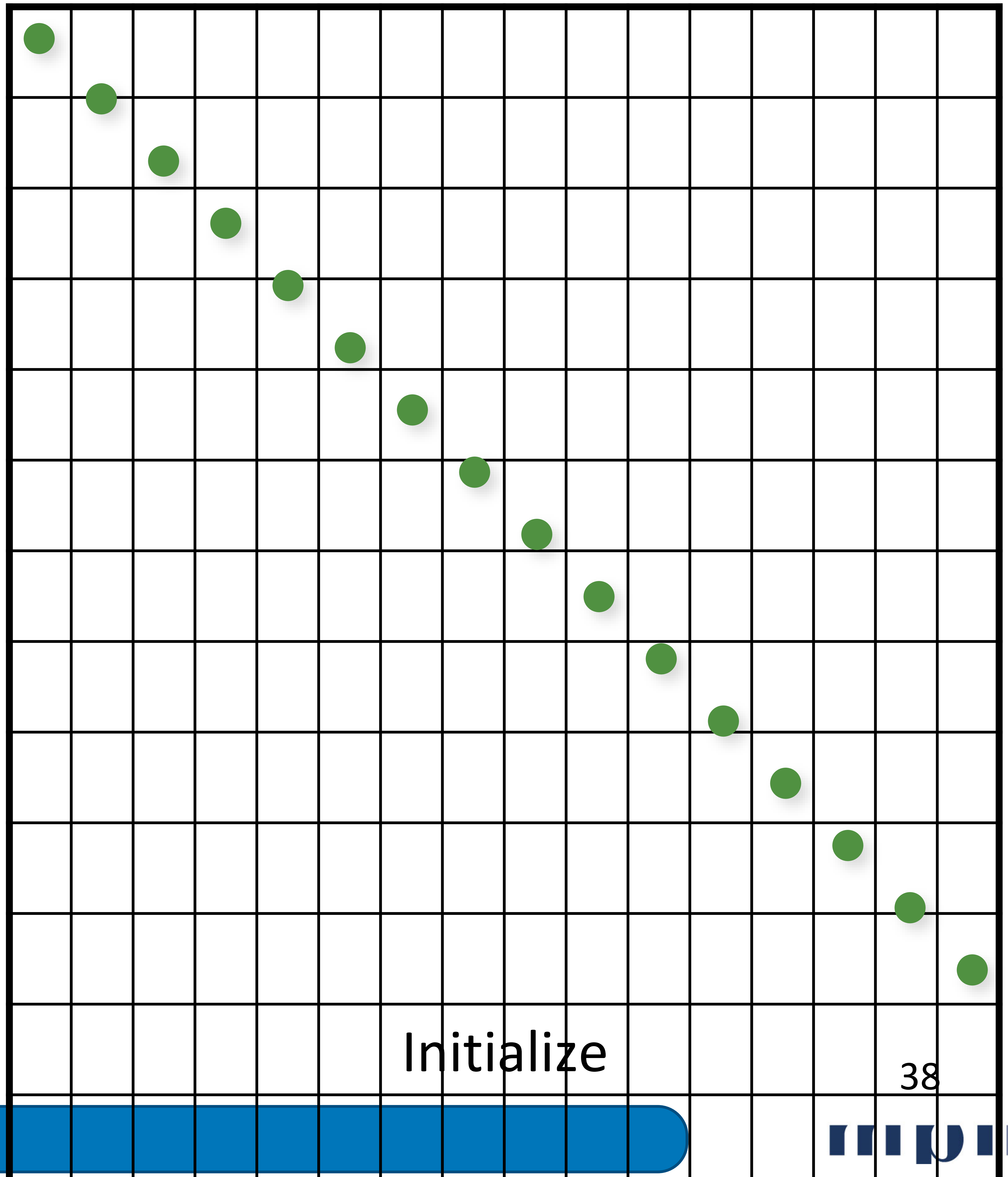
[Shirley 91]



Latin Hypercube (N-Rooks) Sampling

```
// initialize the diagonal
for (uint d = 0; d < numDimensions; d++)
    for (uint i = 0; i < numS; i++)
        samples(d,i) = (i + randf())/numS;
```

```
// shuffle each dimension independently
for (uint d = 0; d < numDimensions; d++)
    shuffle(samples(d,:));
```



Latin Hypercube (N-Rooks) Sampling

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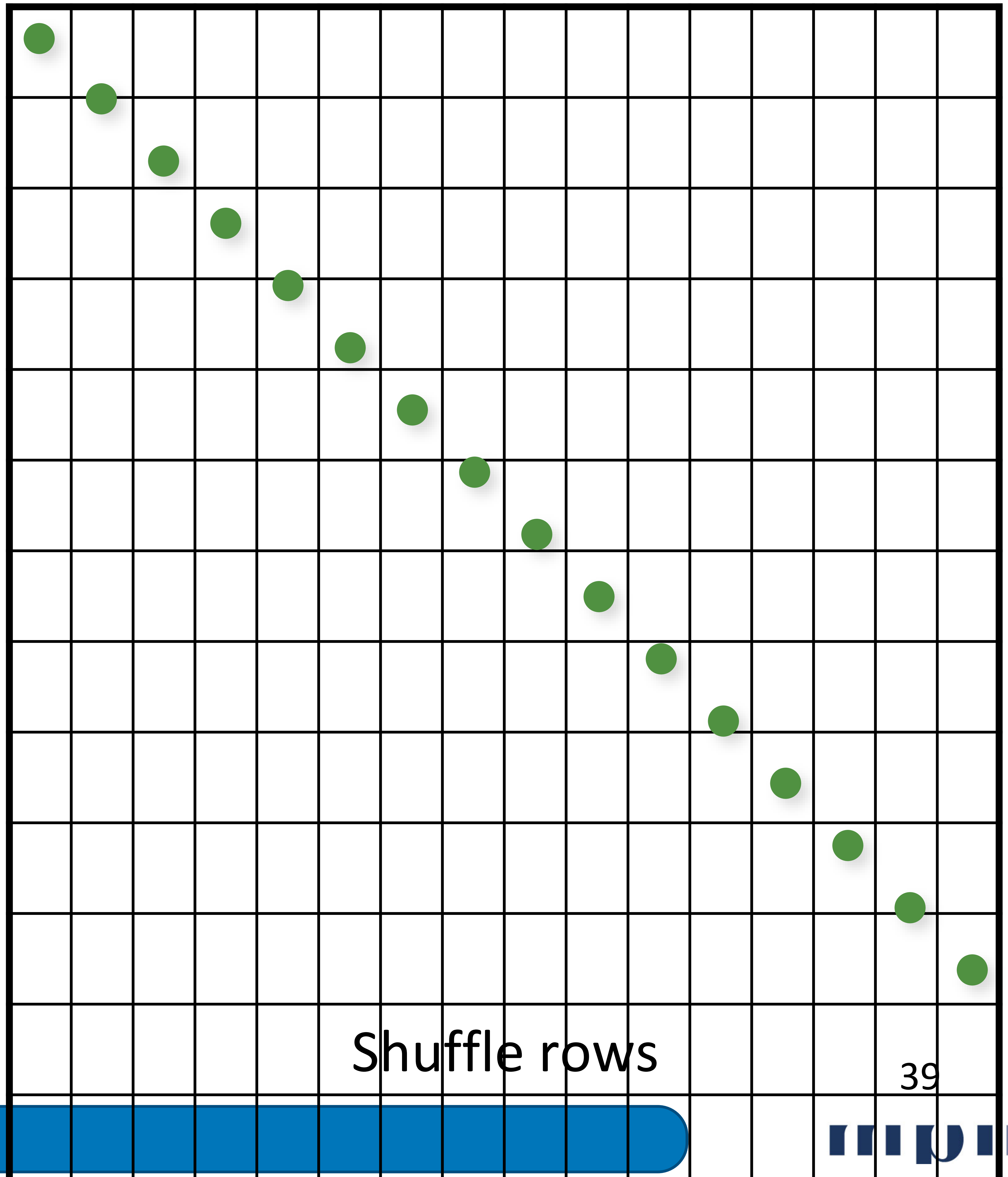
        samples(d,i) = (i + randf())/numS;

// shuffle each dimension independently
for (uint d = 0; d < numDimensions; d++)
    shuffle(samples(d,:));
```

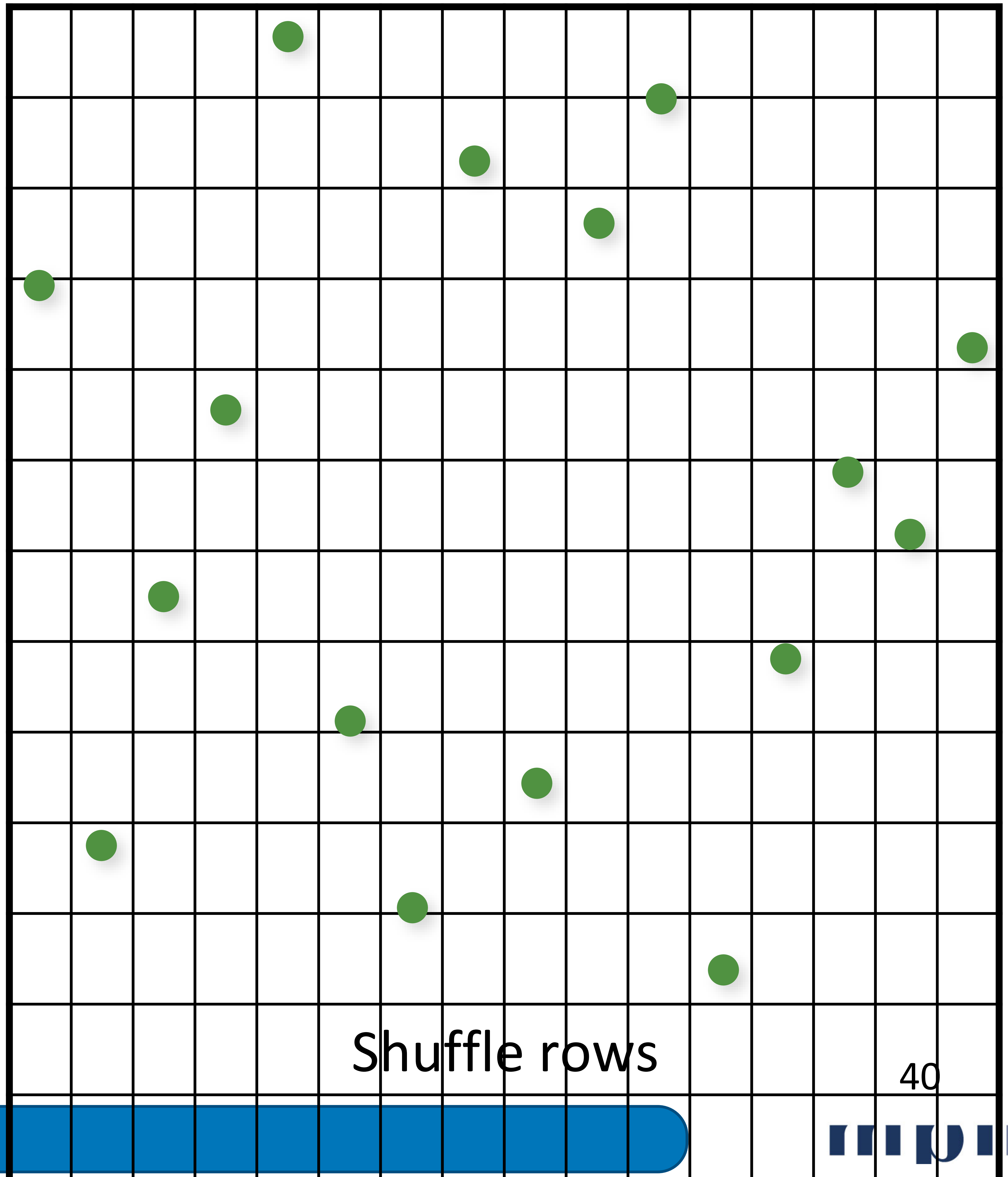


Shuffle rows

Latin Hypercube (N-Rooks) Sampling

```
// initialize the diagonal
for (uint d = 0; d < numDimensions; d++)
    for (uint i = 0; i < numS; i++)
        samples(d,i) = (i + randf())/numS;

// shuffle each dimension independently
for (uint d = 0; d < numDimensions; d++)
    shuffle(samples(d,:));
```



Latin Hypercube (N-Rooks) Sampling

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// initialize the diagonal
for (uint d = 0; d < numDimensions; d++)
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        samples(d,i) = (i + randf())/numS;

// shuffle each dimension independently
for (uint d = 0; d < numDimensions; d++)
    shuffle(samples(d,:));
```
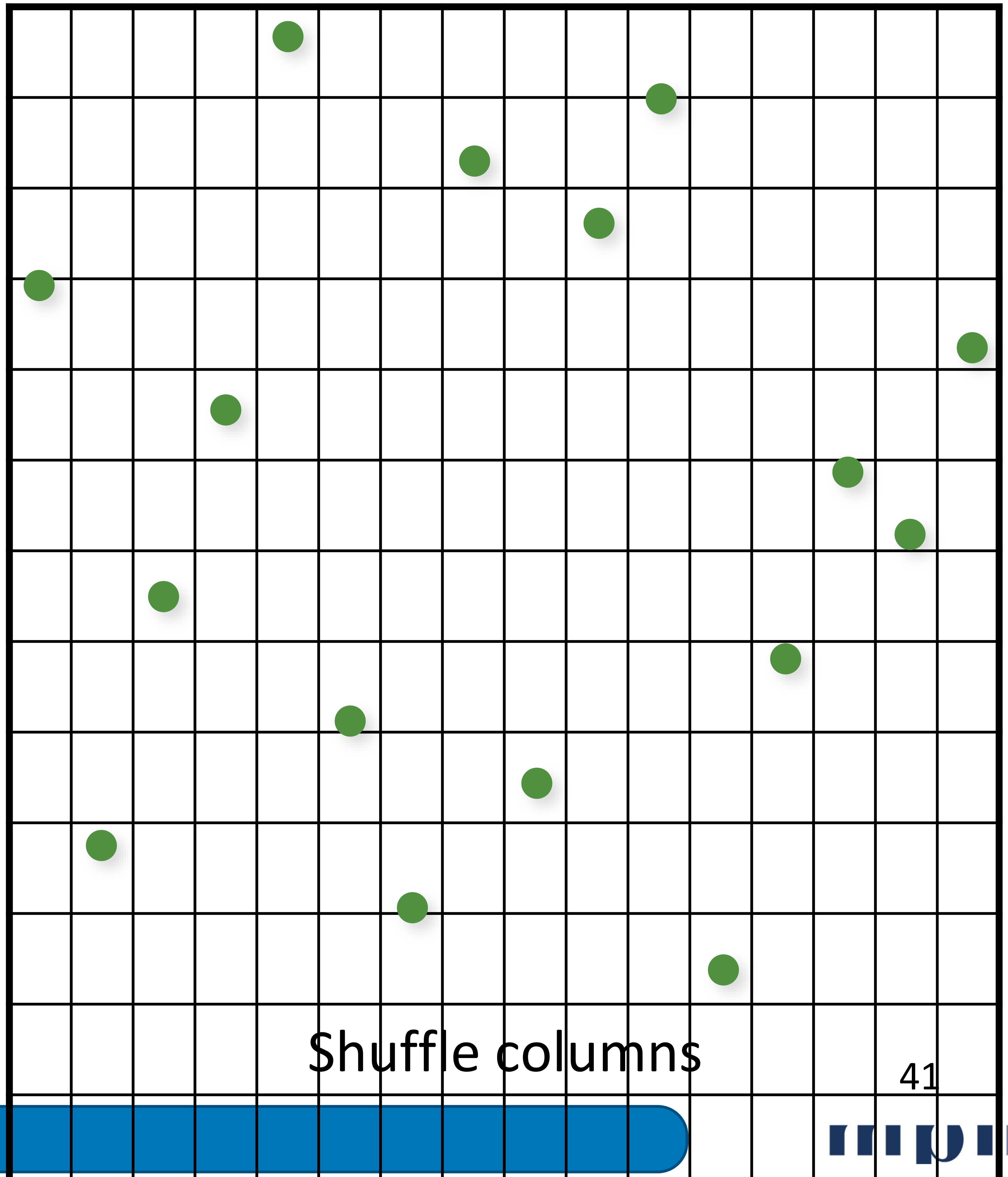
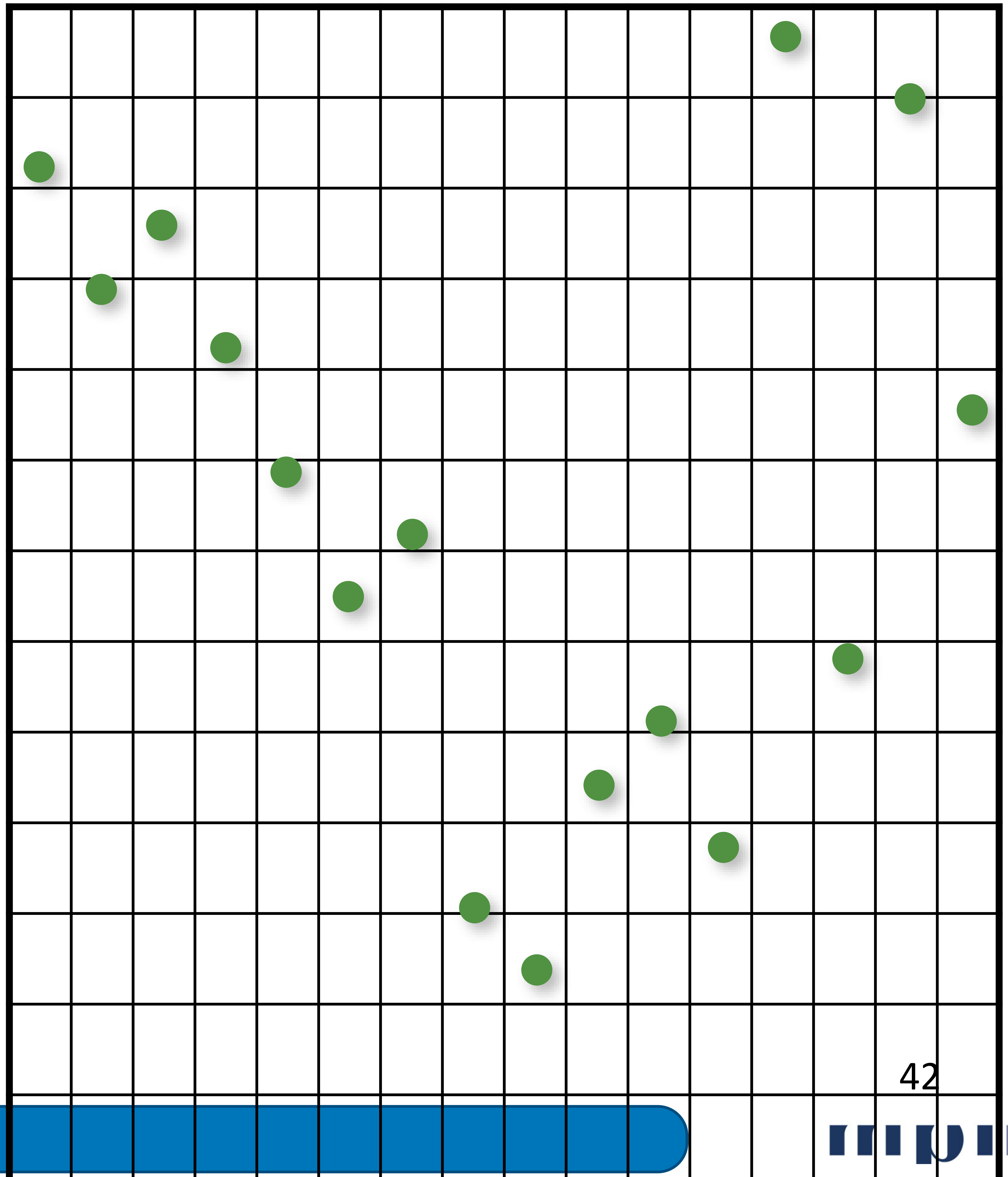
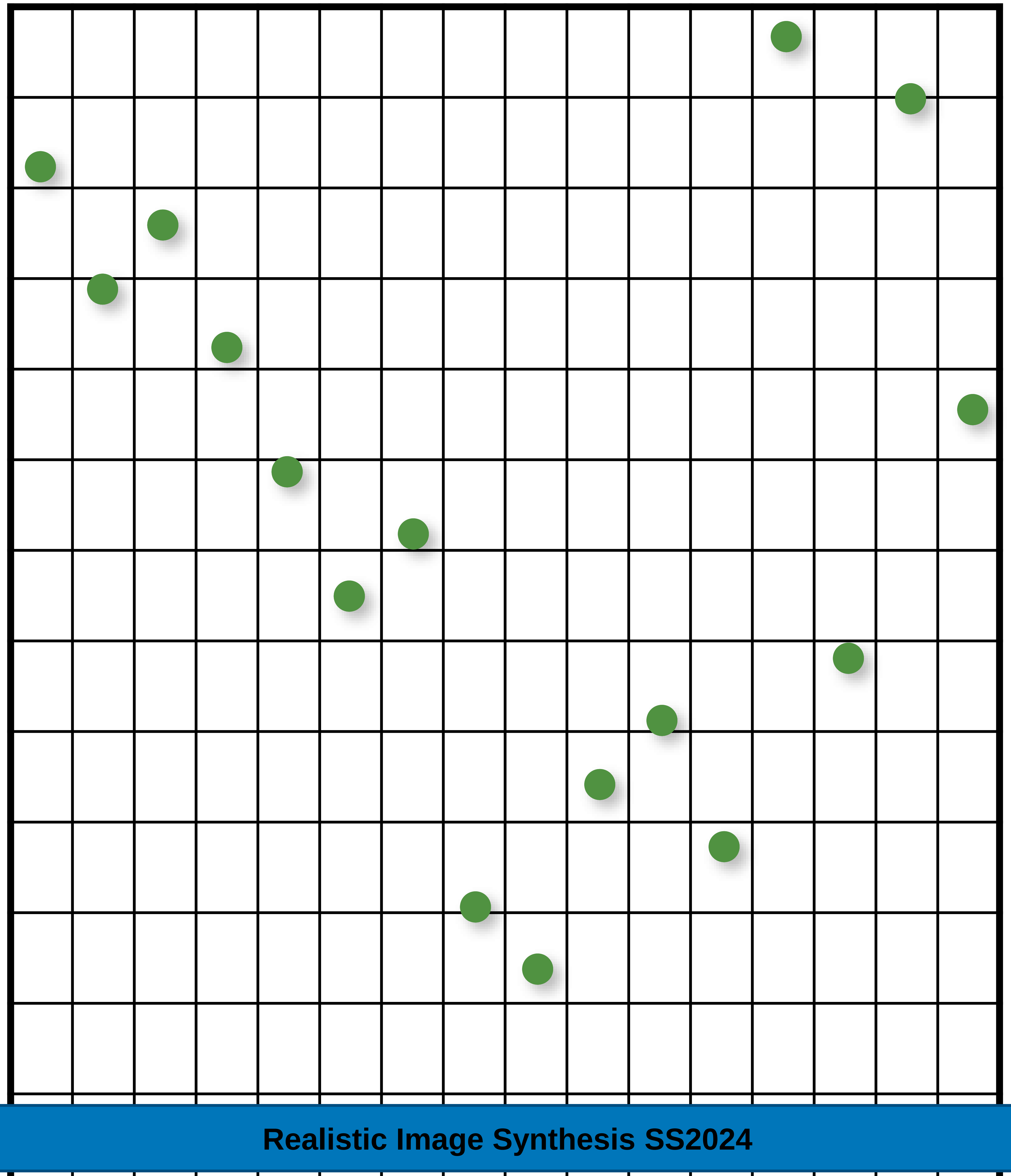



Latin Hypercube (N-Rooks) Sampling

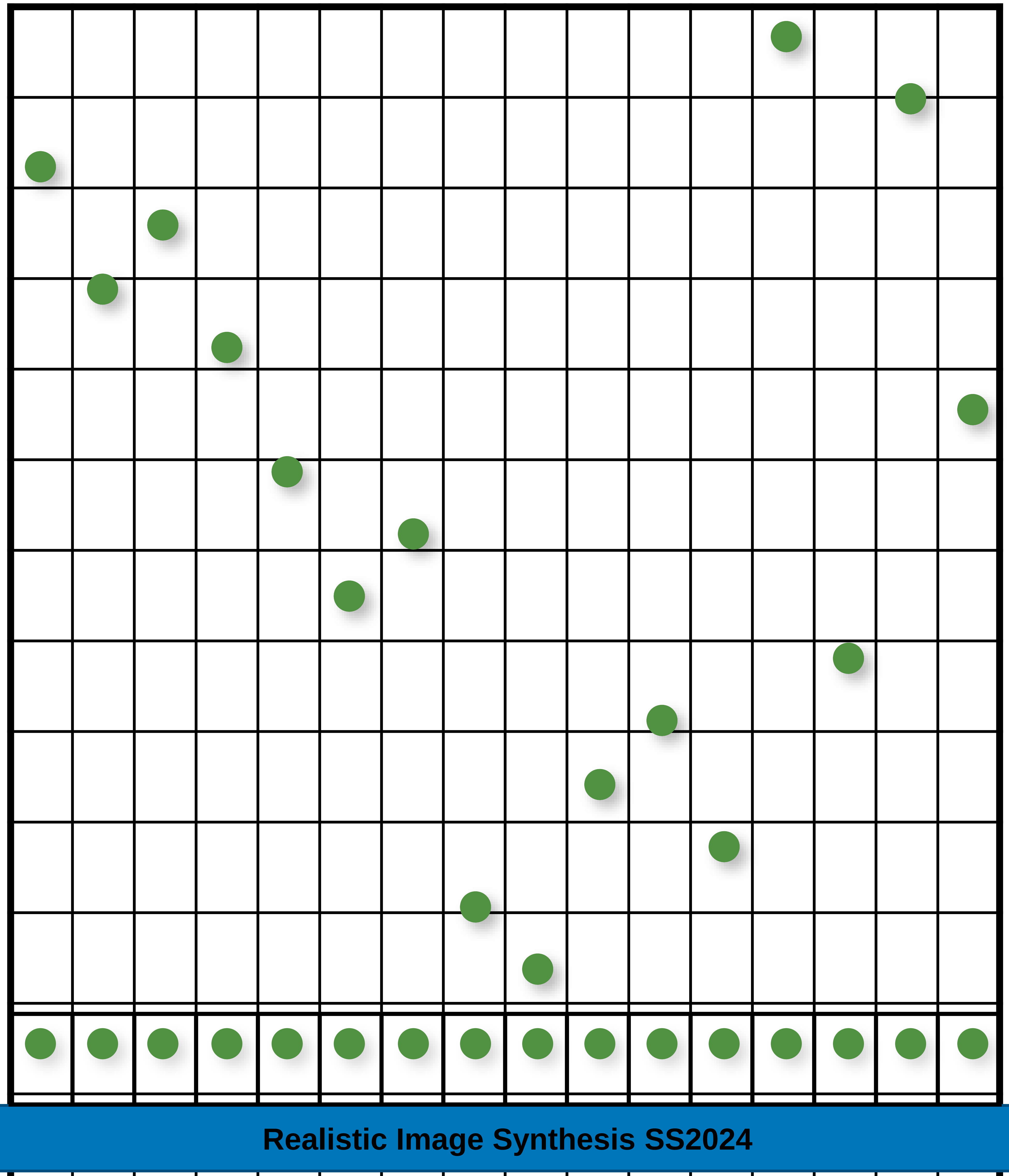
```
// initialize the diagonal
for (uint d = 0; d < numDimensions; d++)
    for (uint i = 0; i < numS; i++)
        samples(d,i) = (i + randf())/numS;

// shuffle each dimension independently
for (uint d = 0; d < numDimensions; d++)
    shuffle(samples(d,:));
```



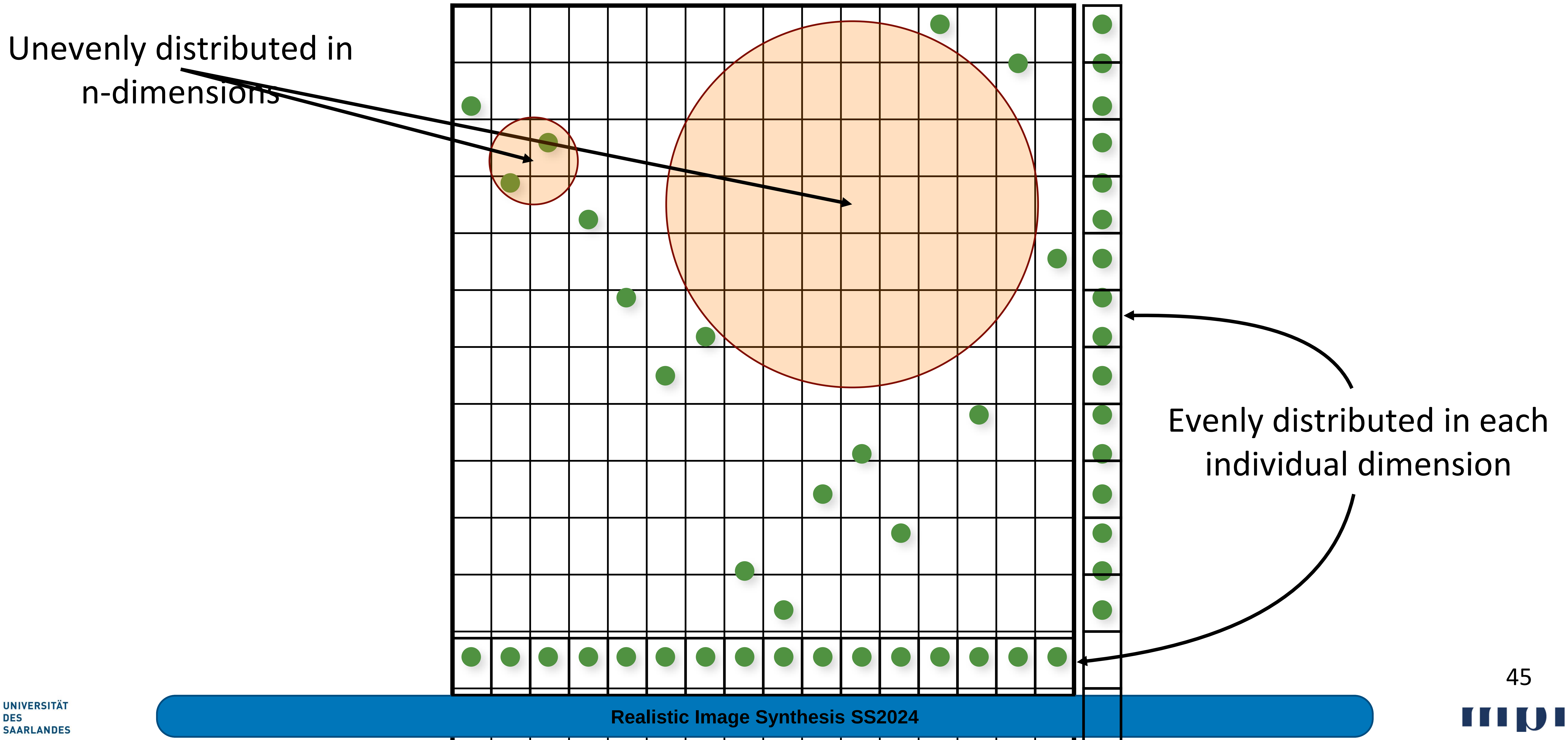
Latin Hypercube (N-Rooks) Sampling



Latin Hypercube (N-Rooks) Sampling

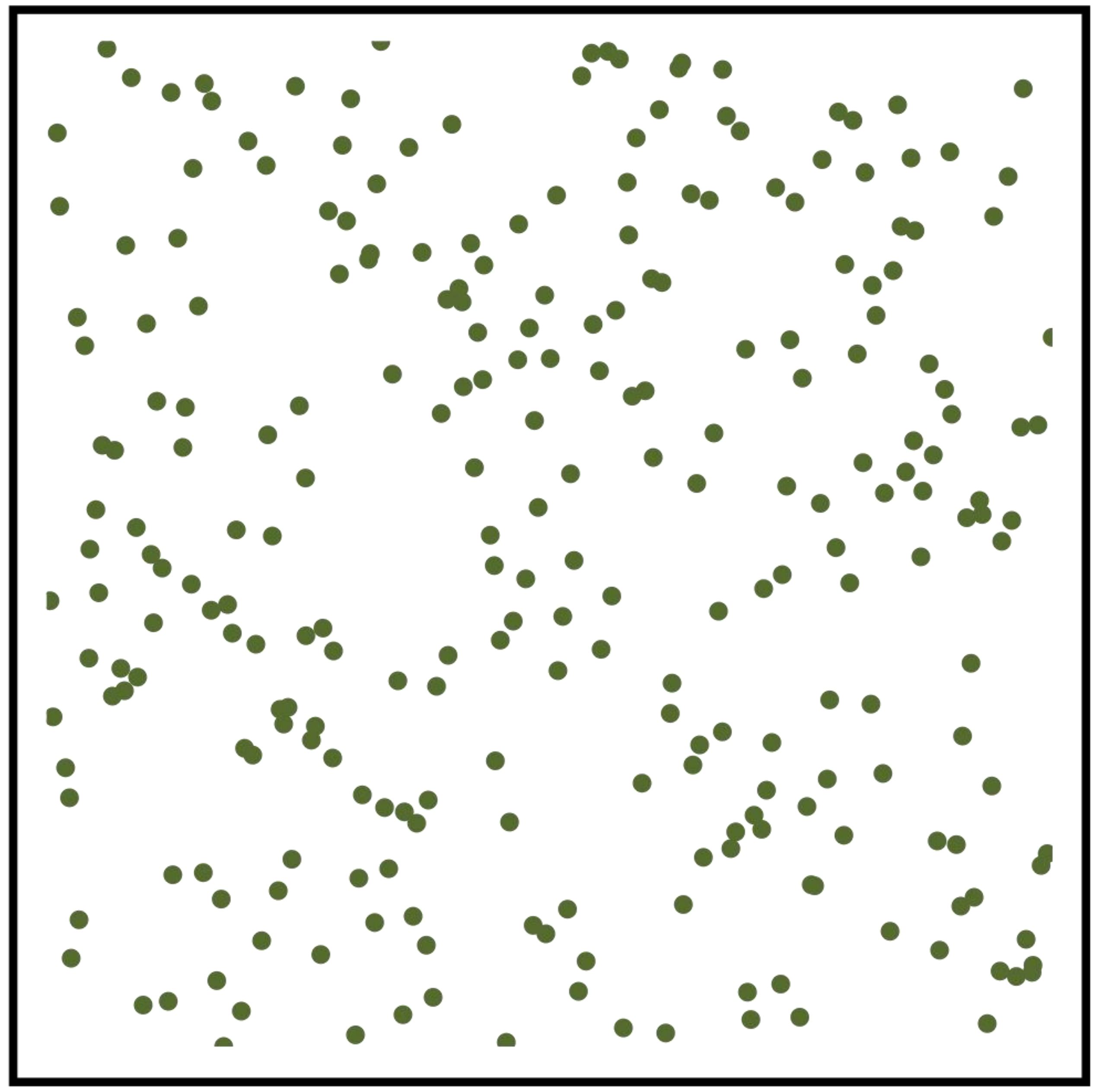


Latin Hypercube (N-Rooks) Sampling

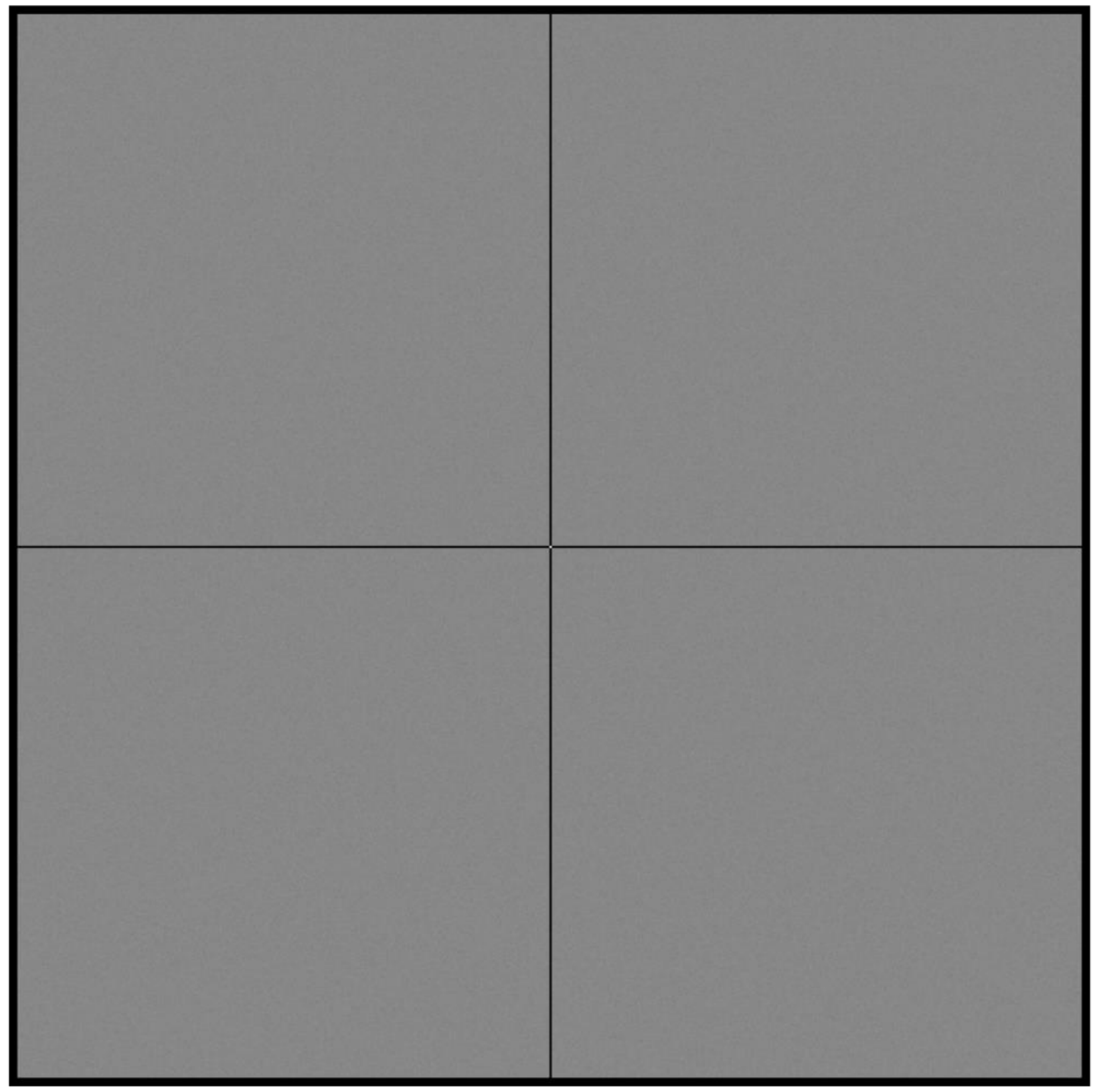


N-Rooks Sampling

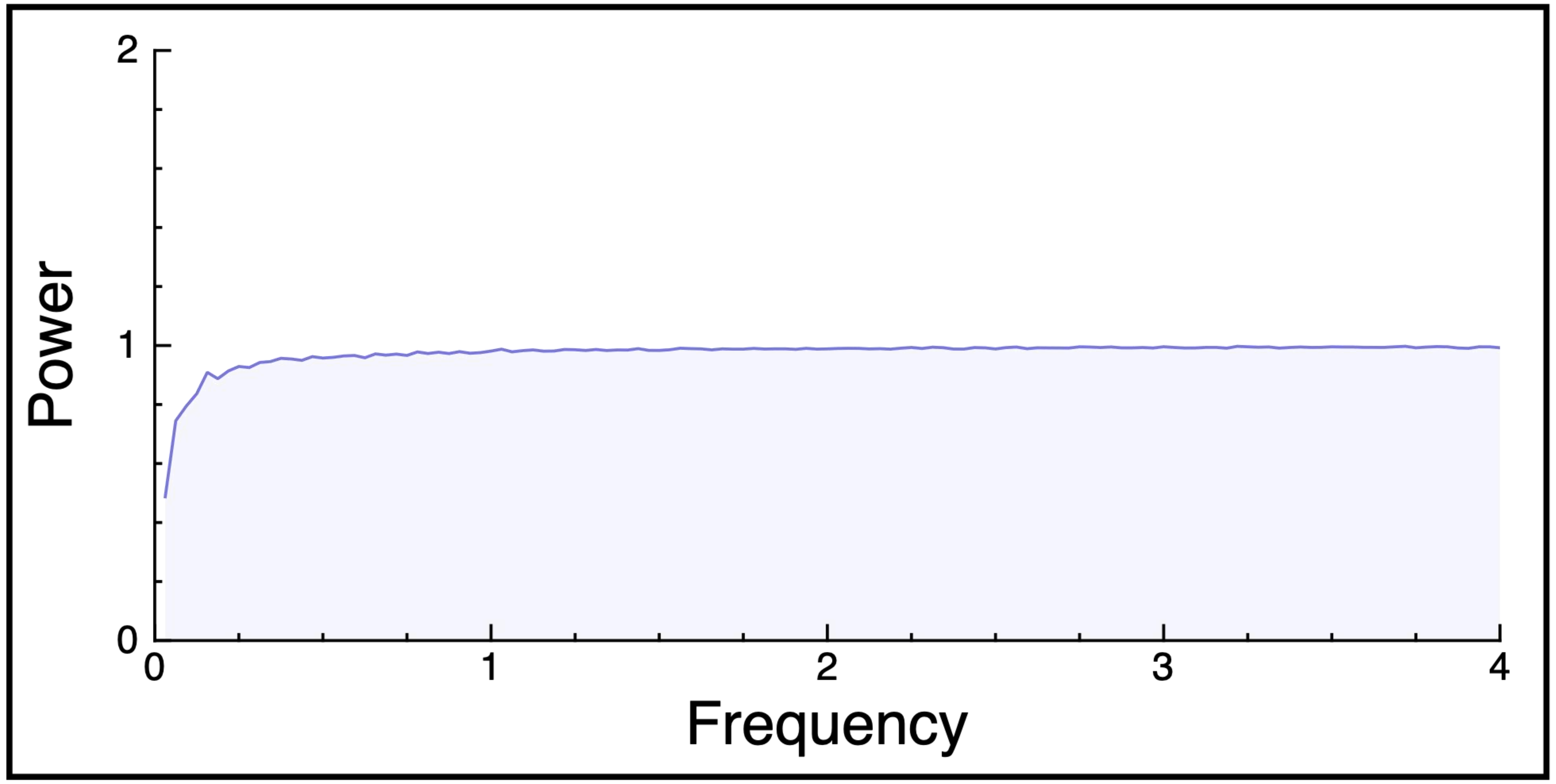
Samples



Expected power spectrum



Radial mean

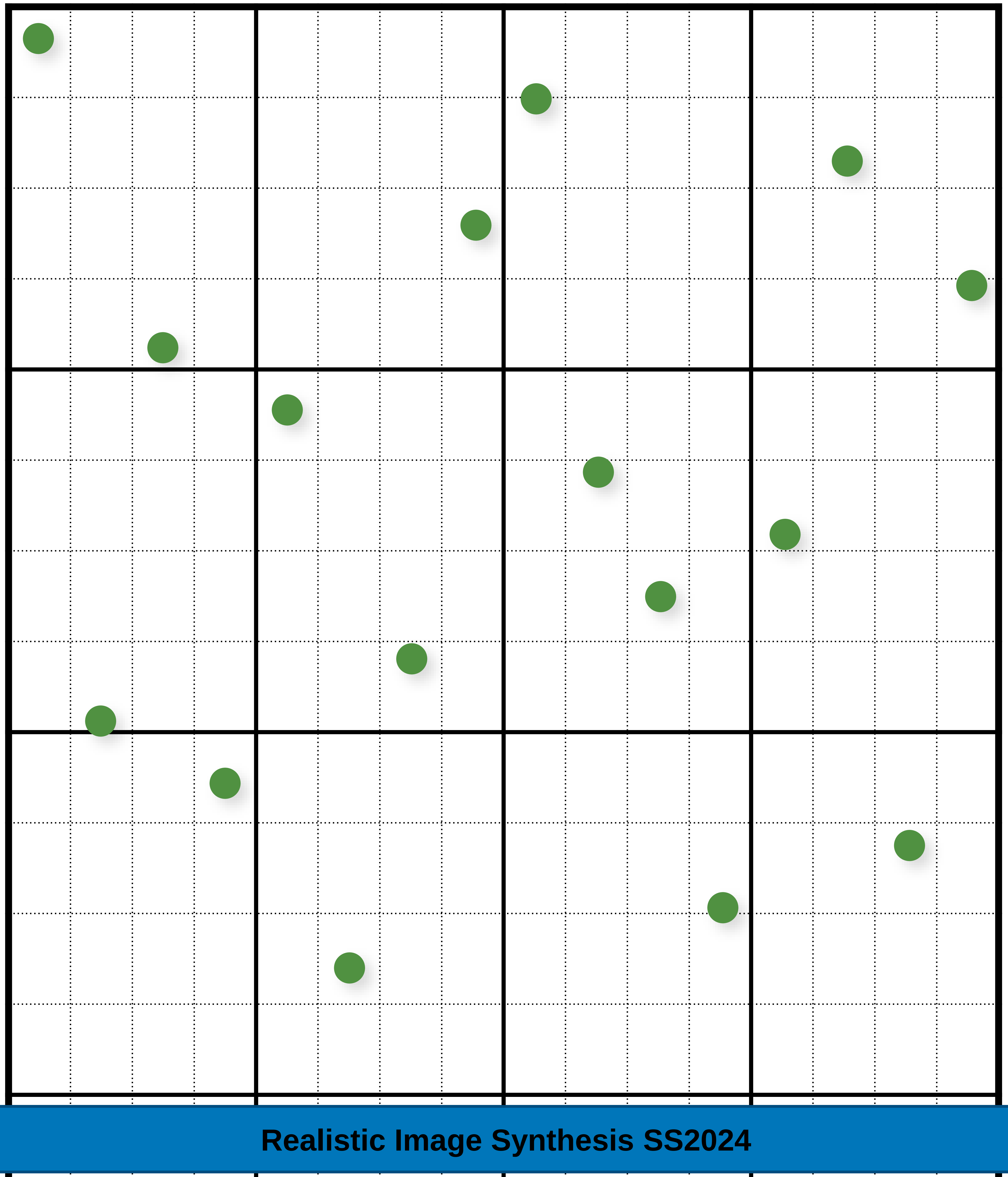


Multi-Jittered Sampling

Kenneth Chiu, Peter Shirley, and Changyaw Wang. “Multi-jittered sampling.” In *Graphics Gems IV*, pp. 370–374. Academic Press, May 1994.

- combine N-Rooks and Jittered stratification constraints

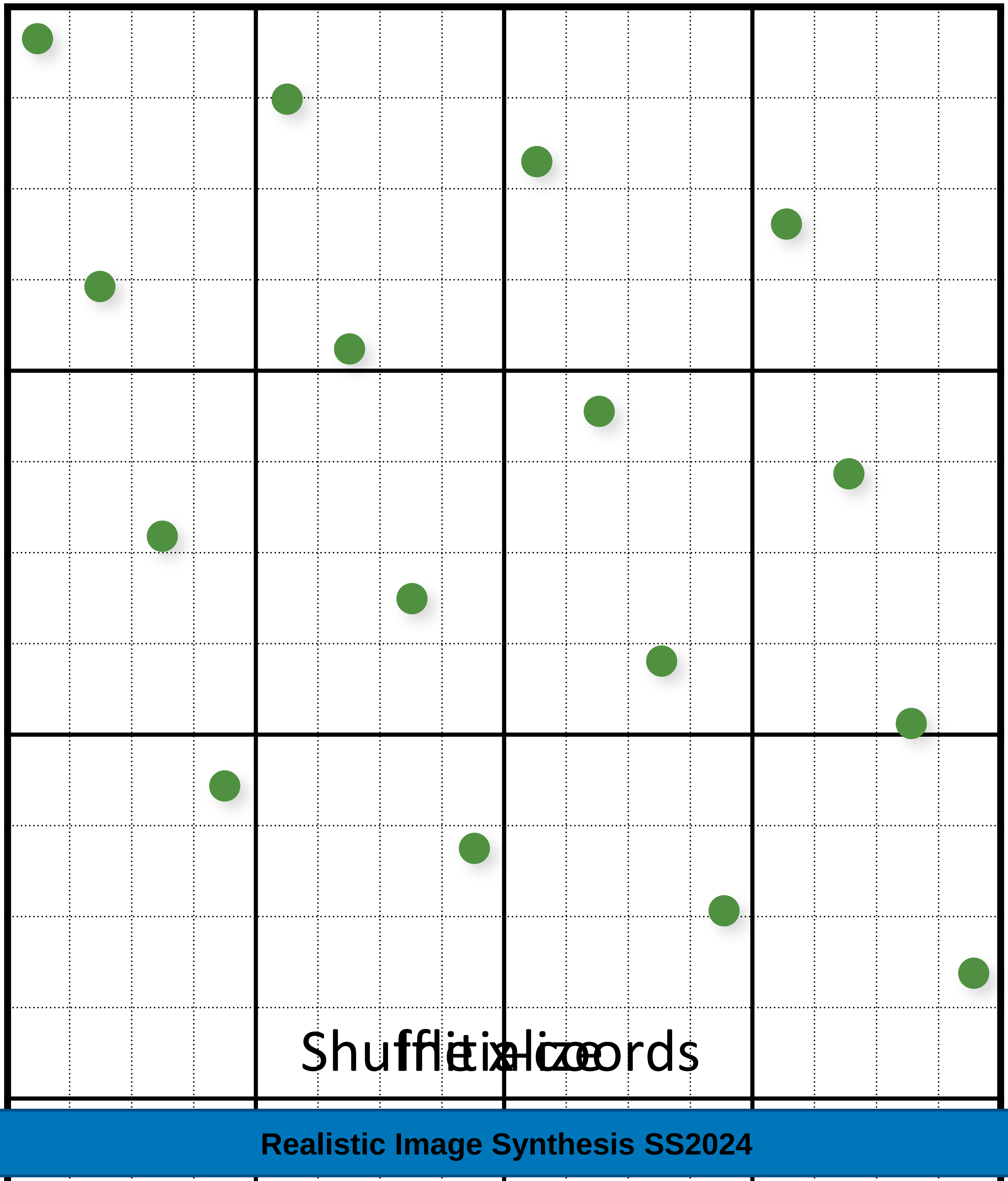
Multi-Jittered Sampling



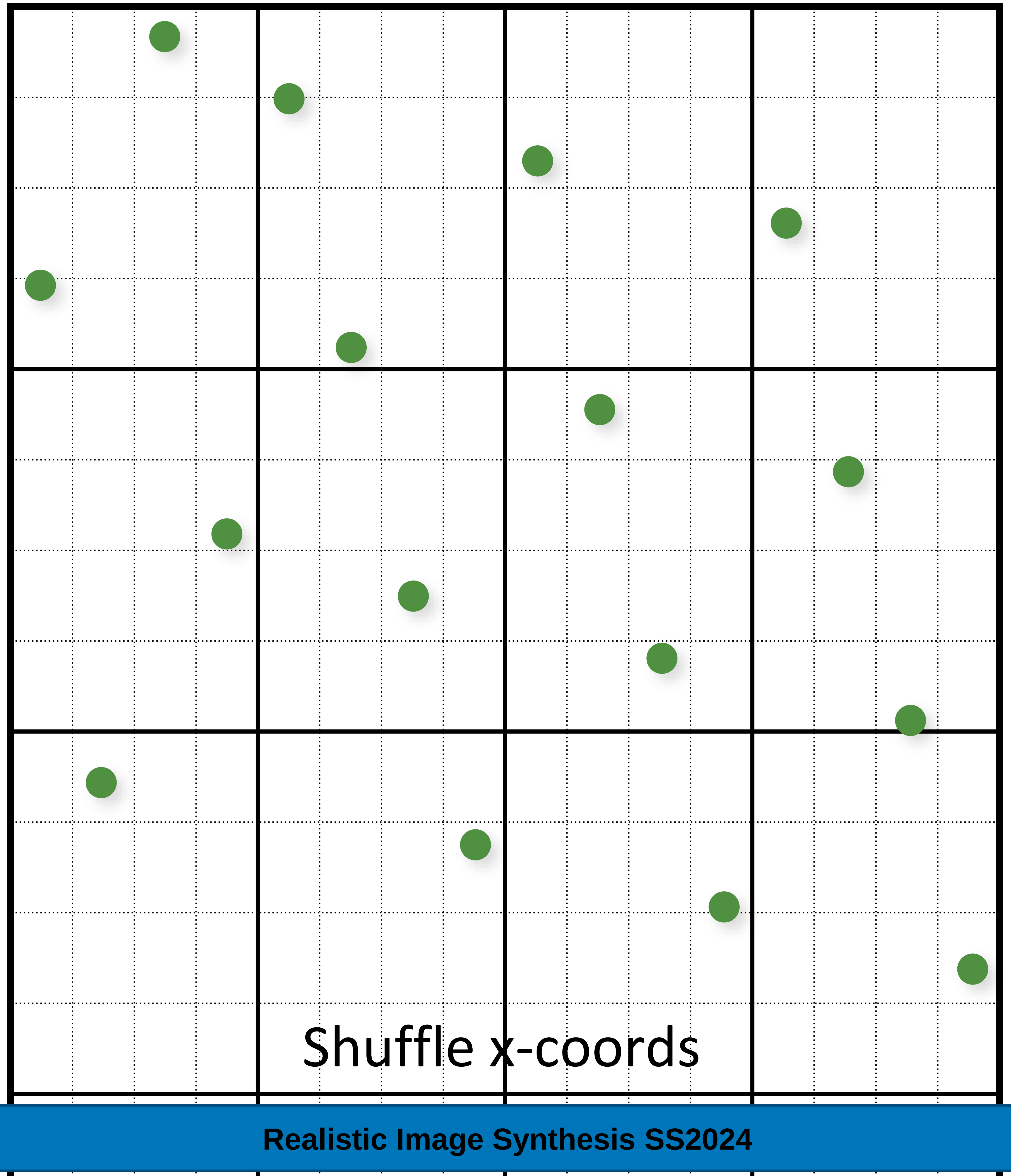
Multi-Jittered Sampling

```
// initialize
float cellSize = 1.0 / (resX*resY);
for (uint i = 0; i < resX; i++)
   for (uint j = 0; j < resY; j++)
   {
      samples(i,j).x = i/resX + (j+randf()) / (resX*resY);
      samples(i,j).y = j/resY + (i+randf()) / (resX*resY);
   }

// shuffle x coordinates within each column of cells
for (uint i = 0; i < resX; i++)
   for (uint j = resY-1; j >= 1; j--)
      swap(samples(i, j).x, samples(i, randi(0, j)).x);

// shuffle y coordinates within each row of cells
for (unsigned j = 0; j < resY; j++)
   for (unsigned i = resX-1; i >= 1; i--)
      swap(samples(i, j).y, samples(randi(0, i), j).y);
```

Multi-Jittered Sampling

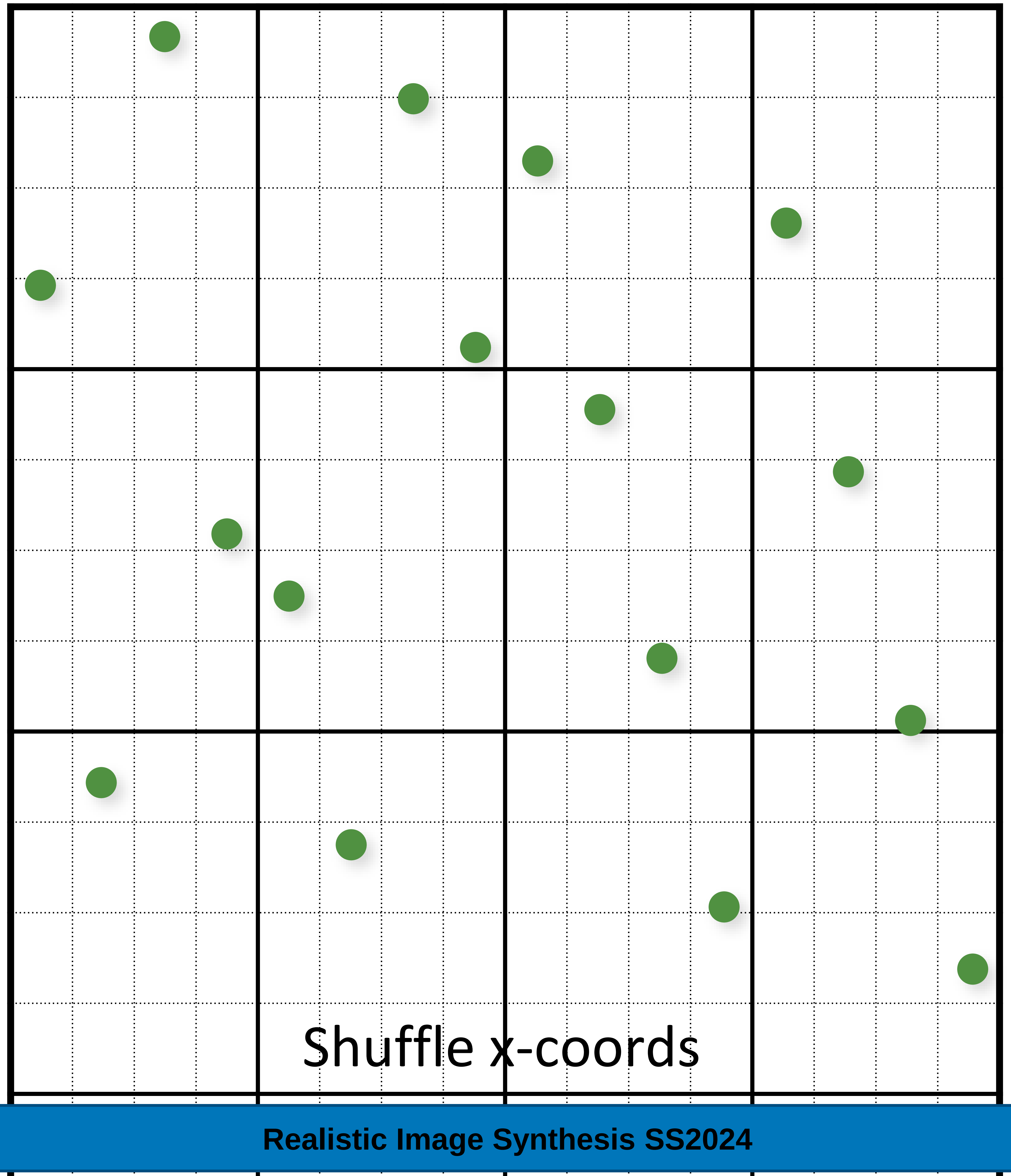


Multi-Jittered Sampling

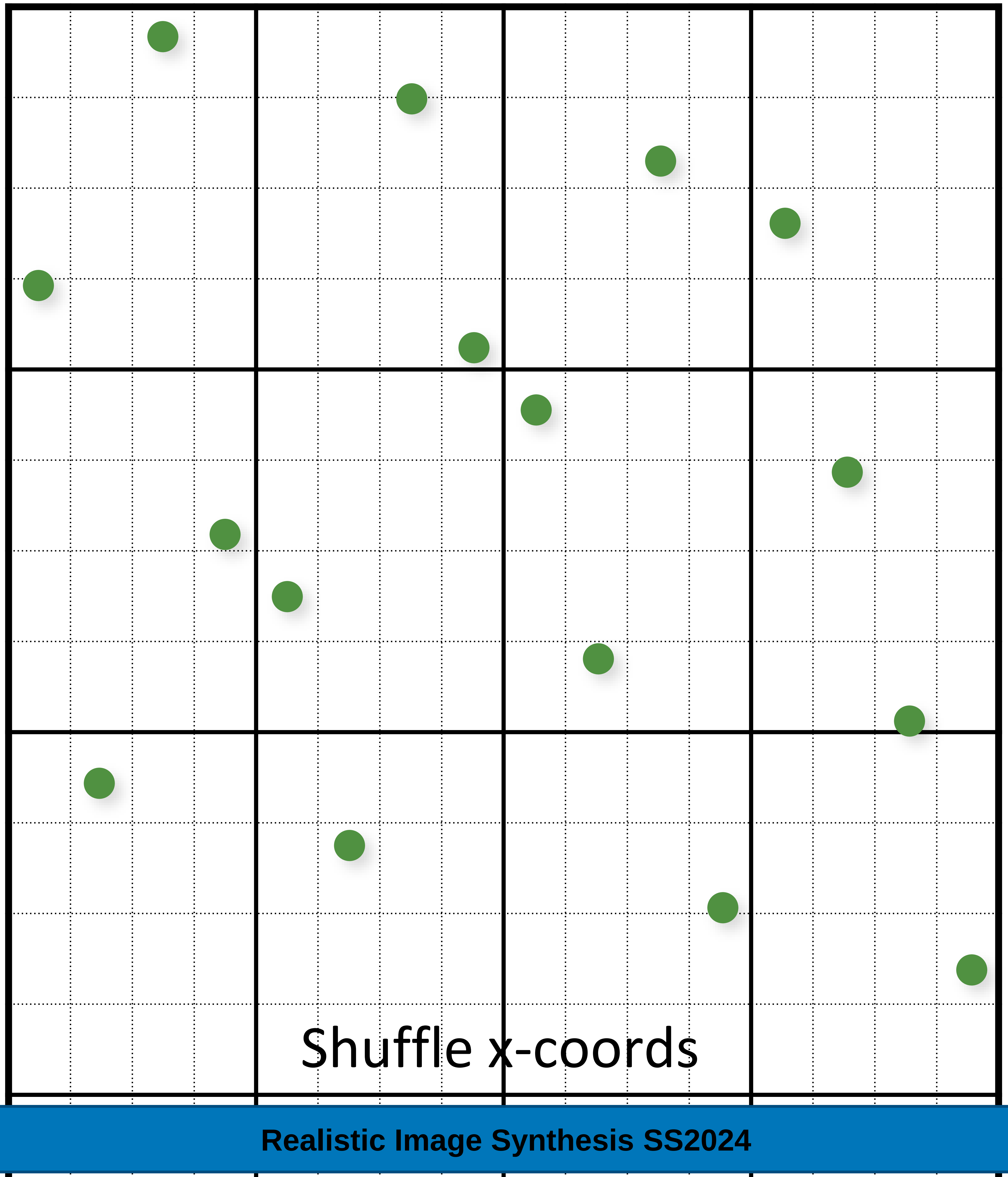


Shuffle x-coords

Multi-Jittered Sampling

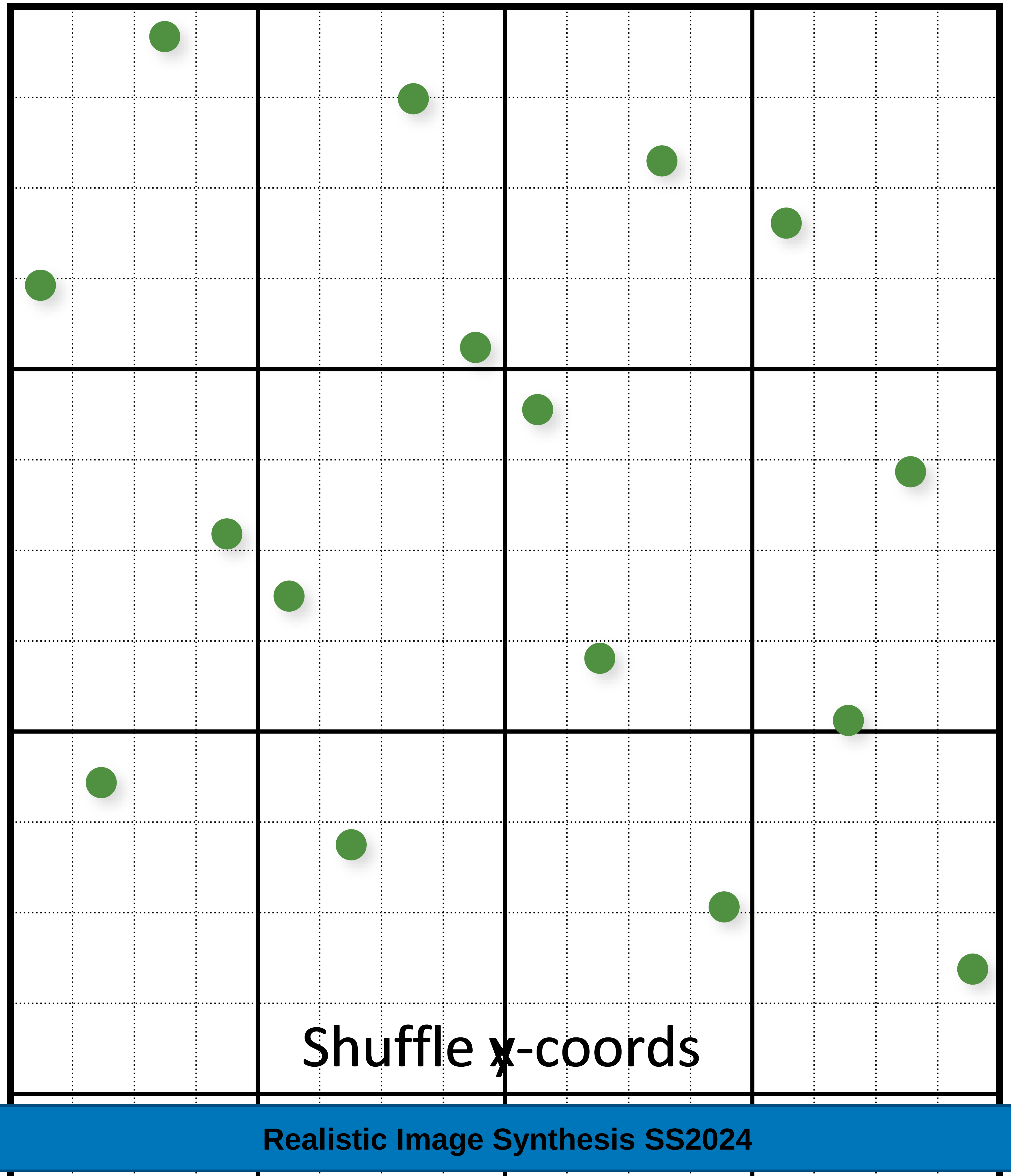


Multi-Jittered Sampling



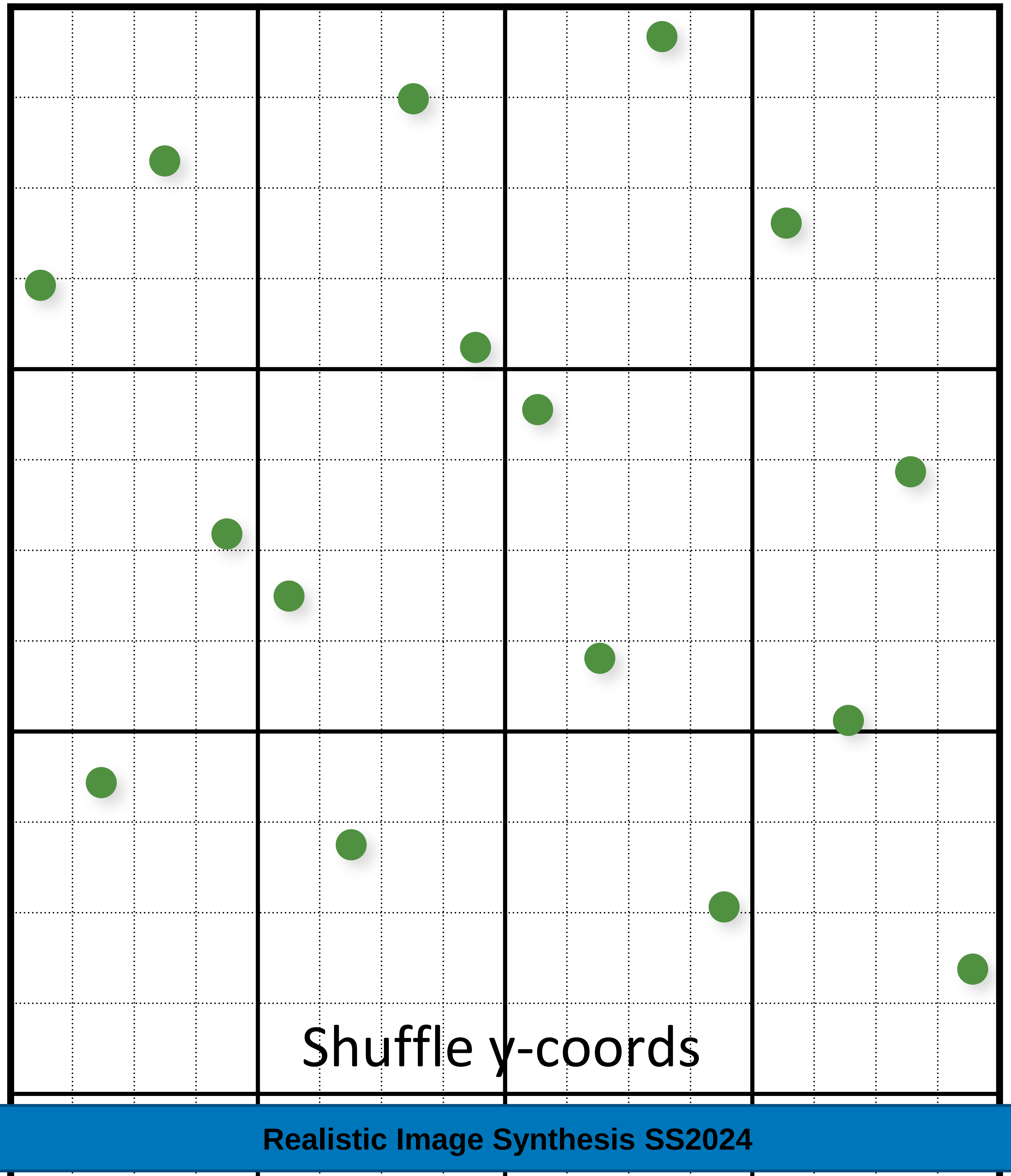
Shuffle x-coords

Multi-Jittered Sampling

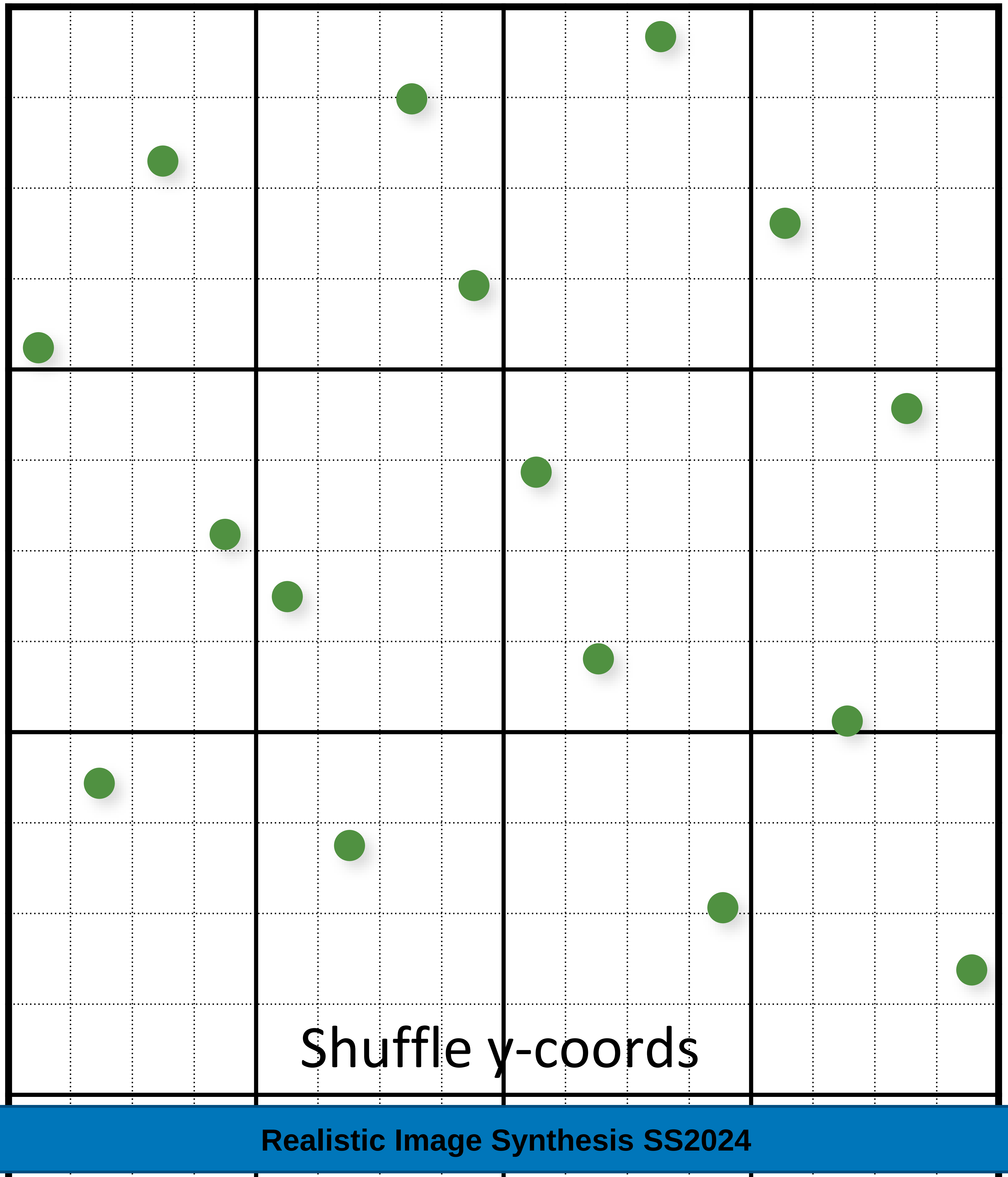


Shuffle x-coords

Multi-Jittered Sampling

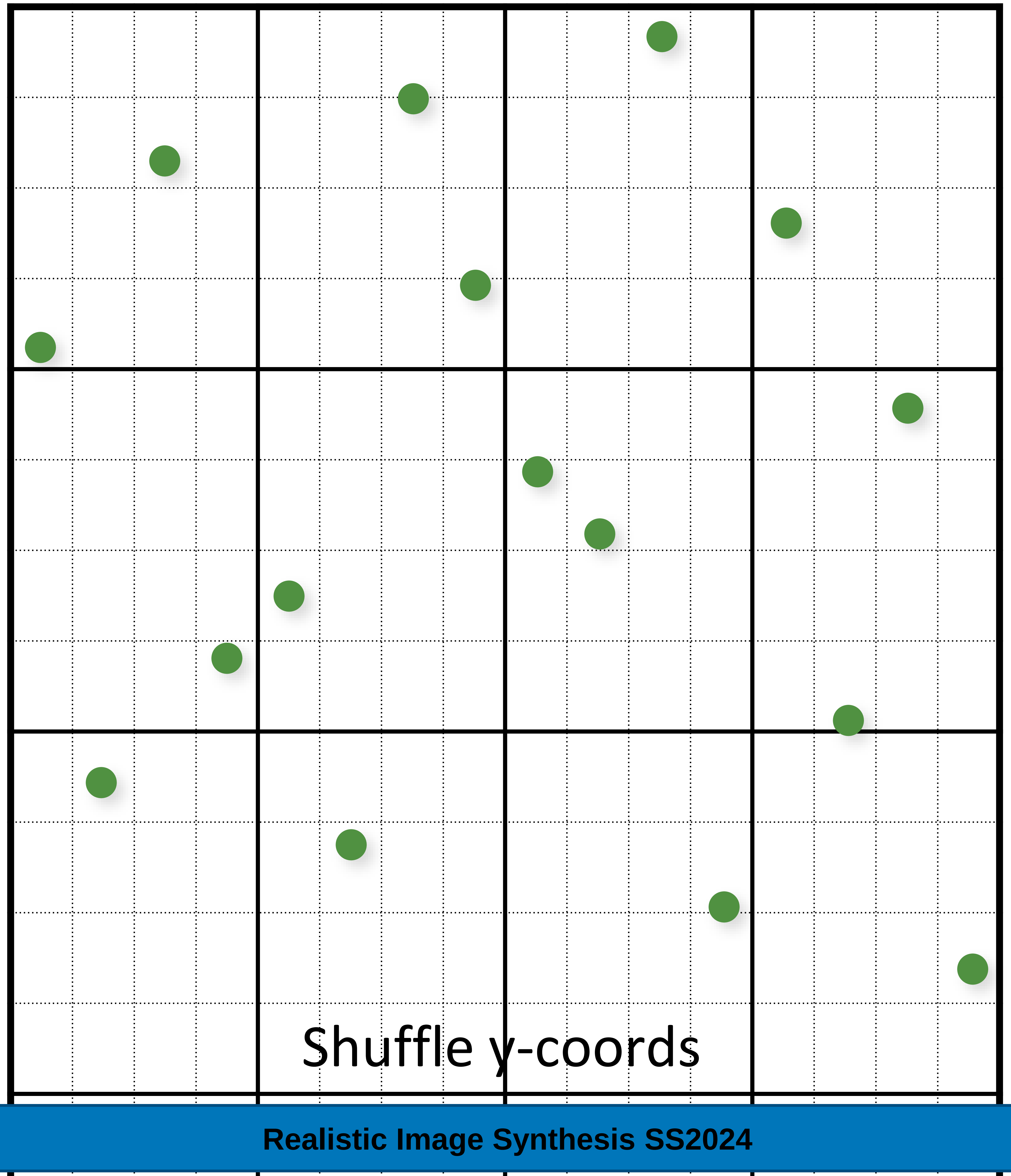


Multi-Jittered Sampling



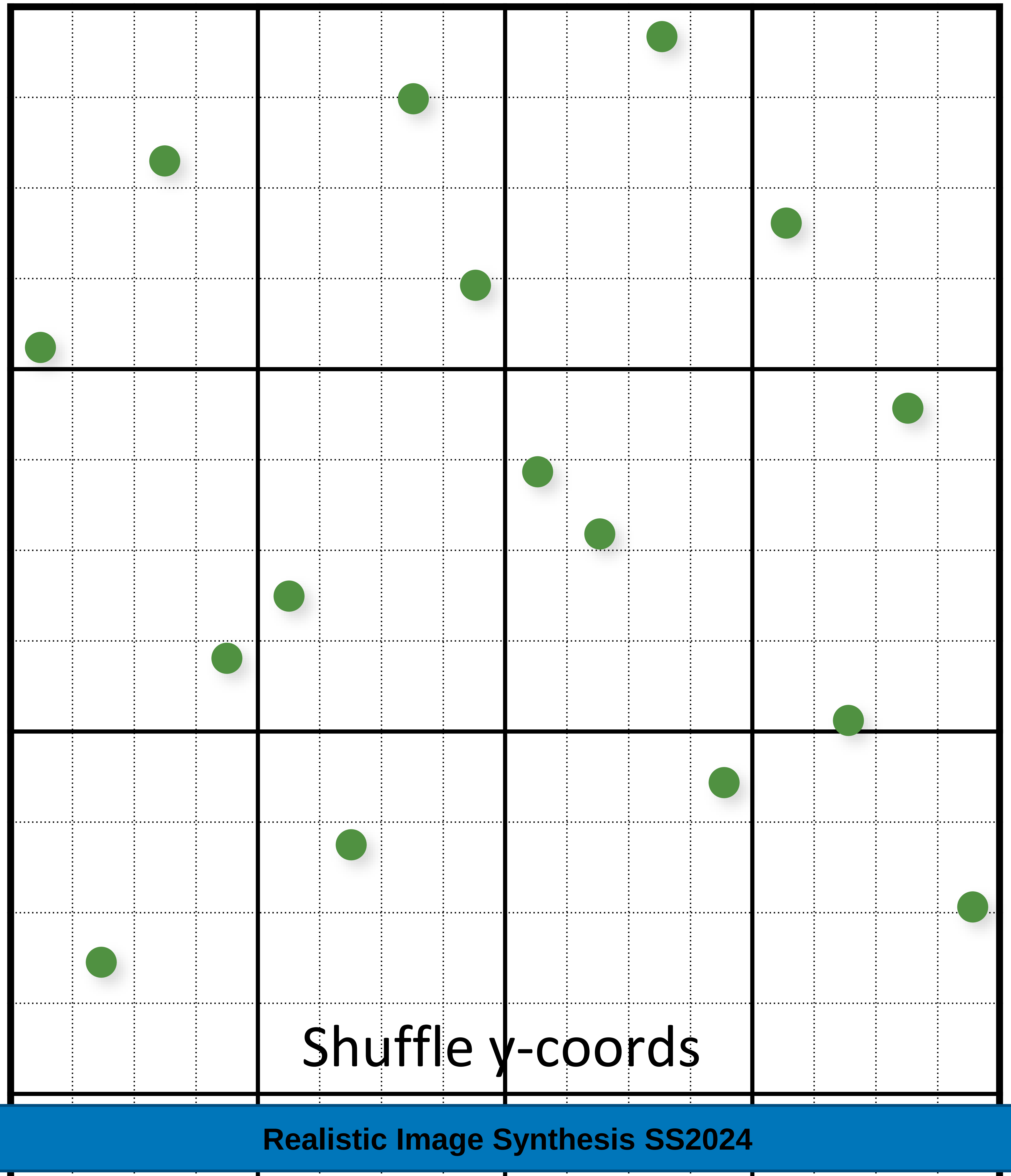
Shuffle y-coords

Multi-Jittered Sampling

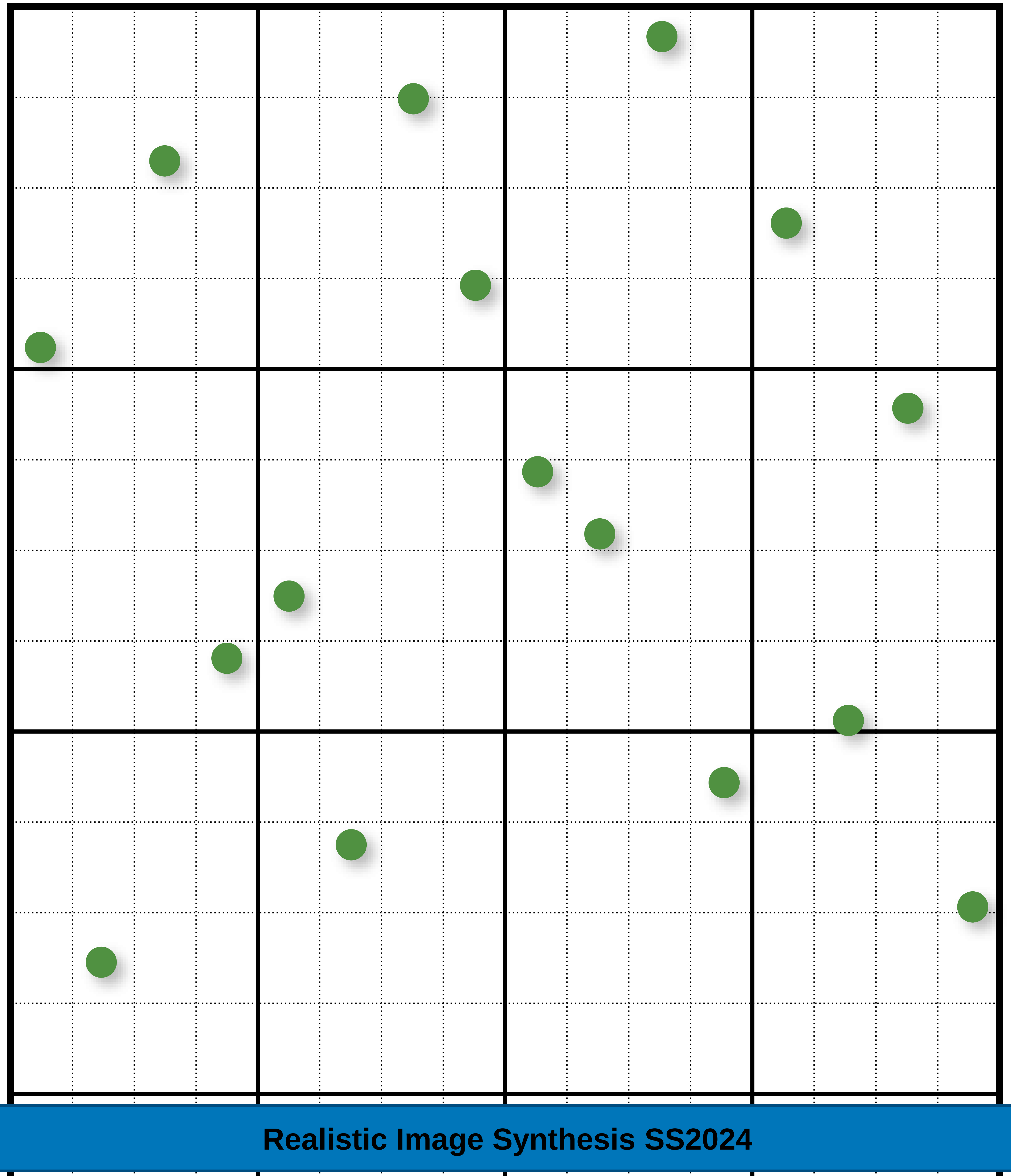


Shuffle y-coords

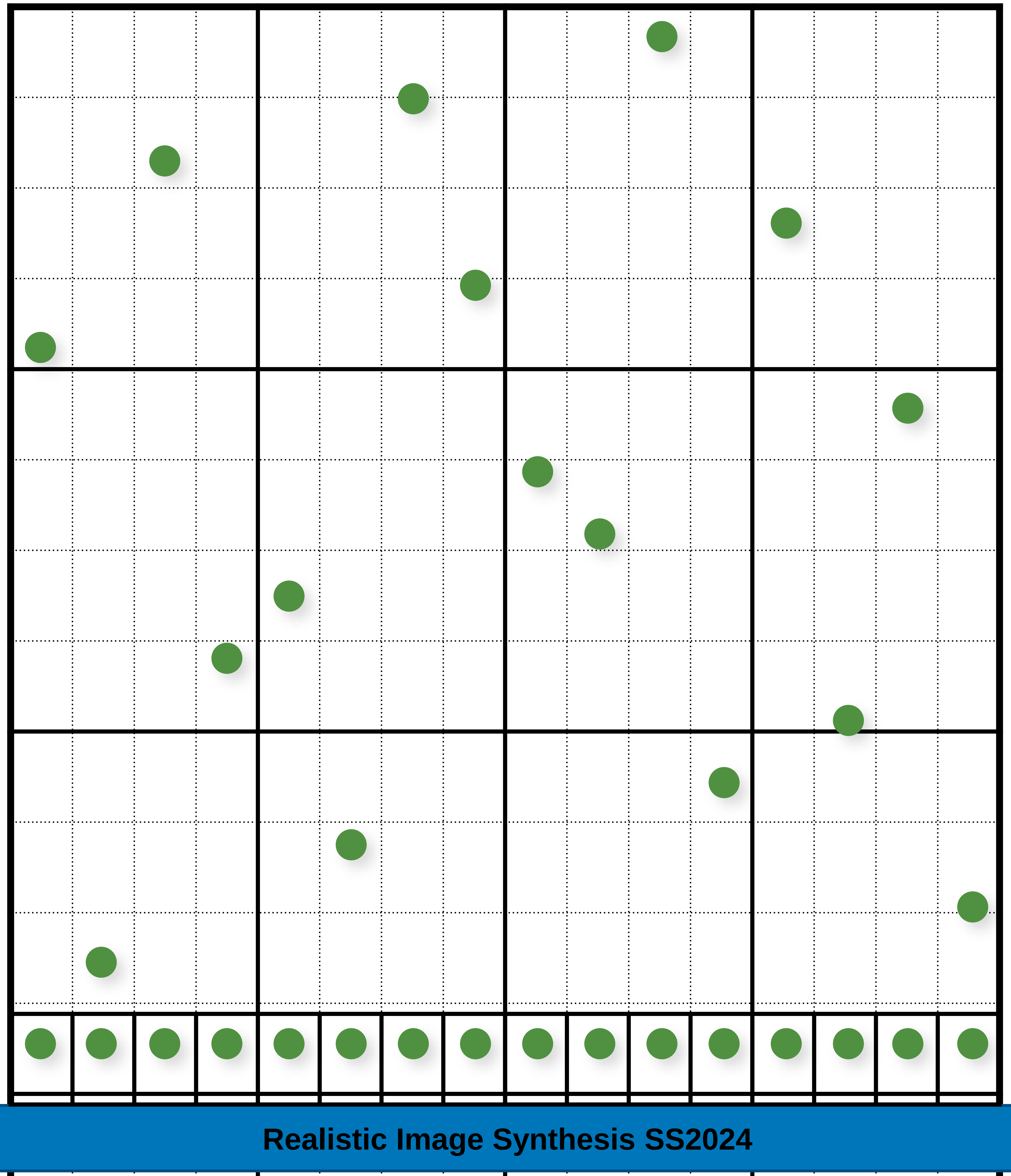
Multi-Jittered Sampling



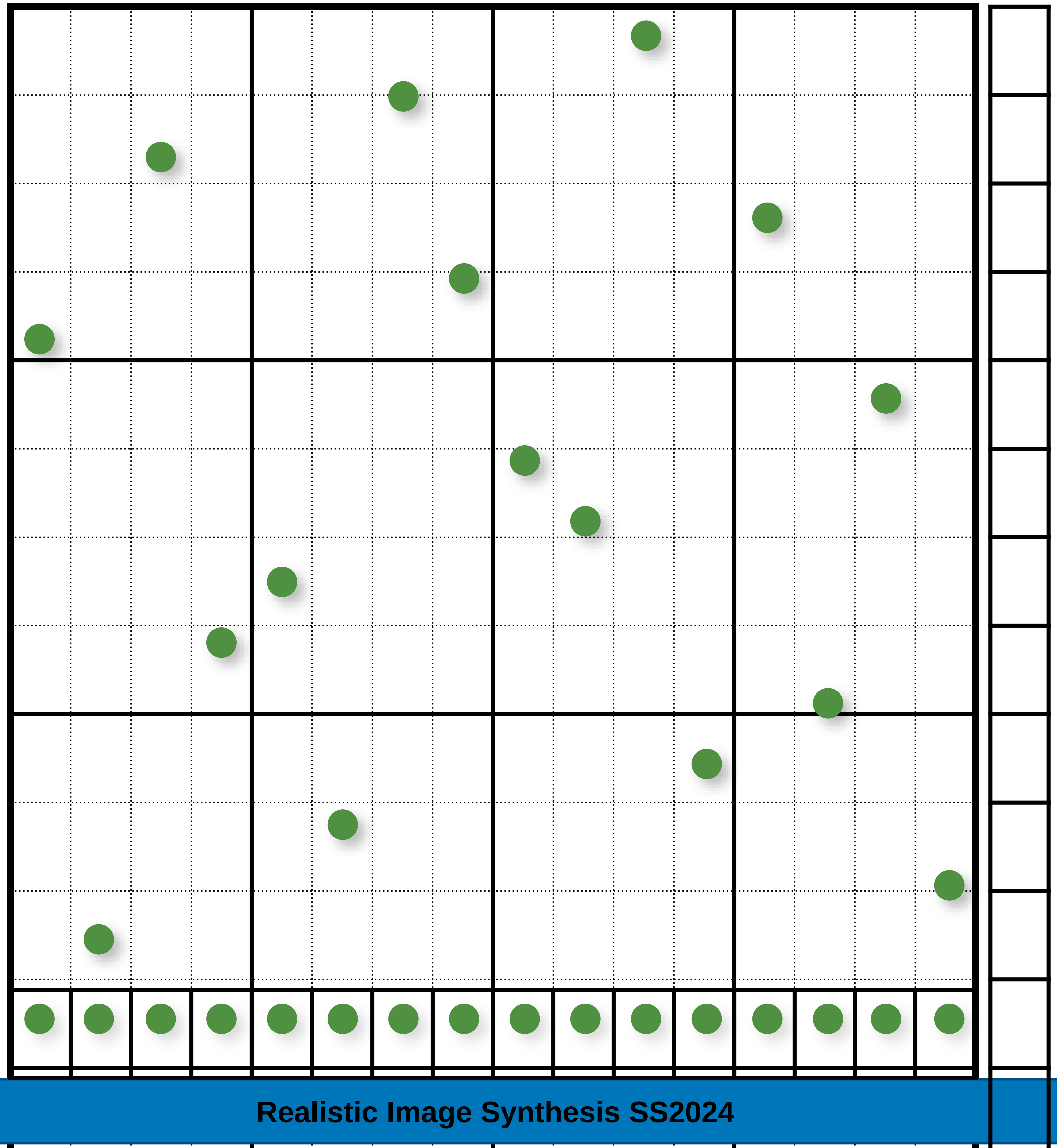
Multi-Jittered Sampling (Projections)



Multi-Jittered Sampling (Projections)

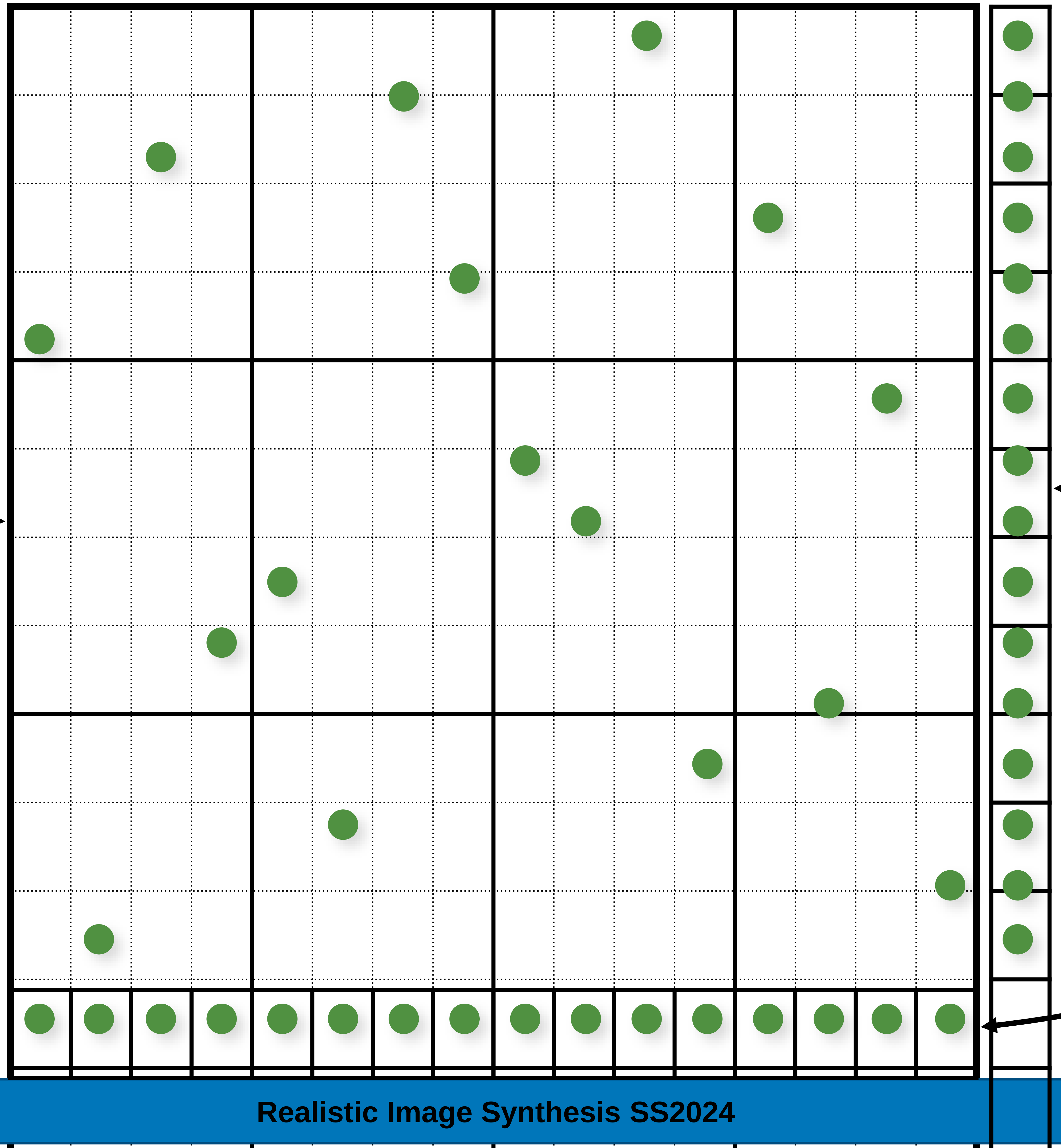


Multi-Jittered Sampling (Projections)



Multi-Jittered Sampling (Projections)

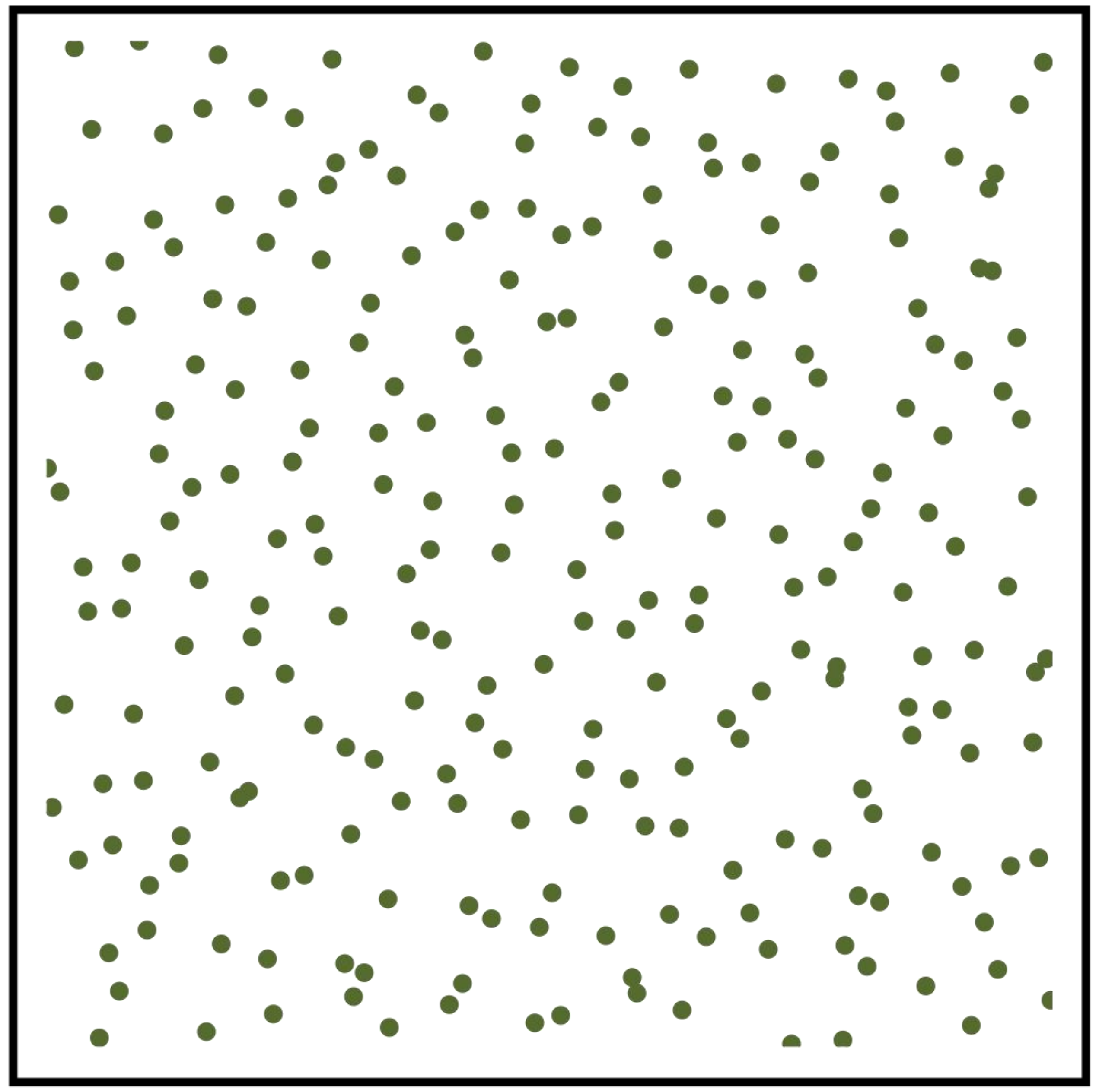
Evenly distributed in 2D!



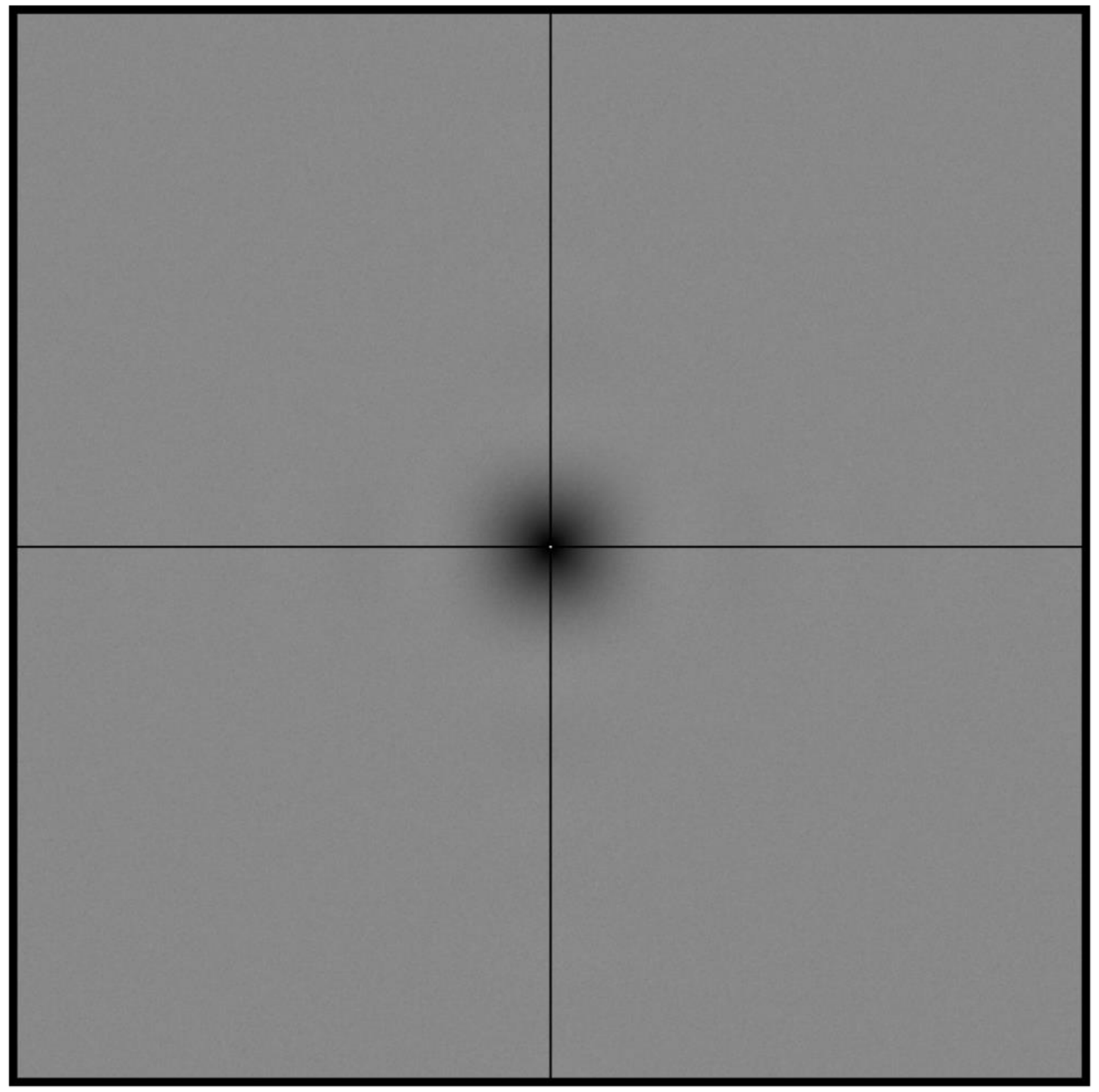
Evenly distributed in each individual dimension

Multi-Jittered Sampling

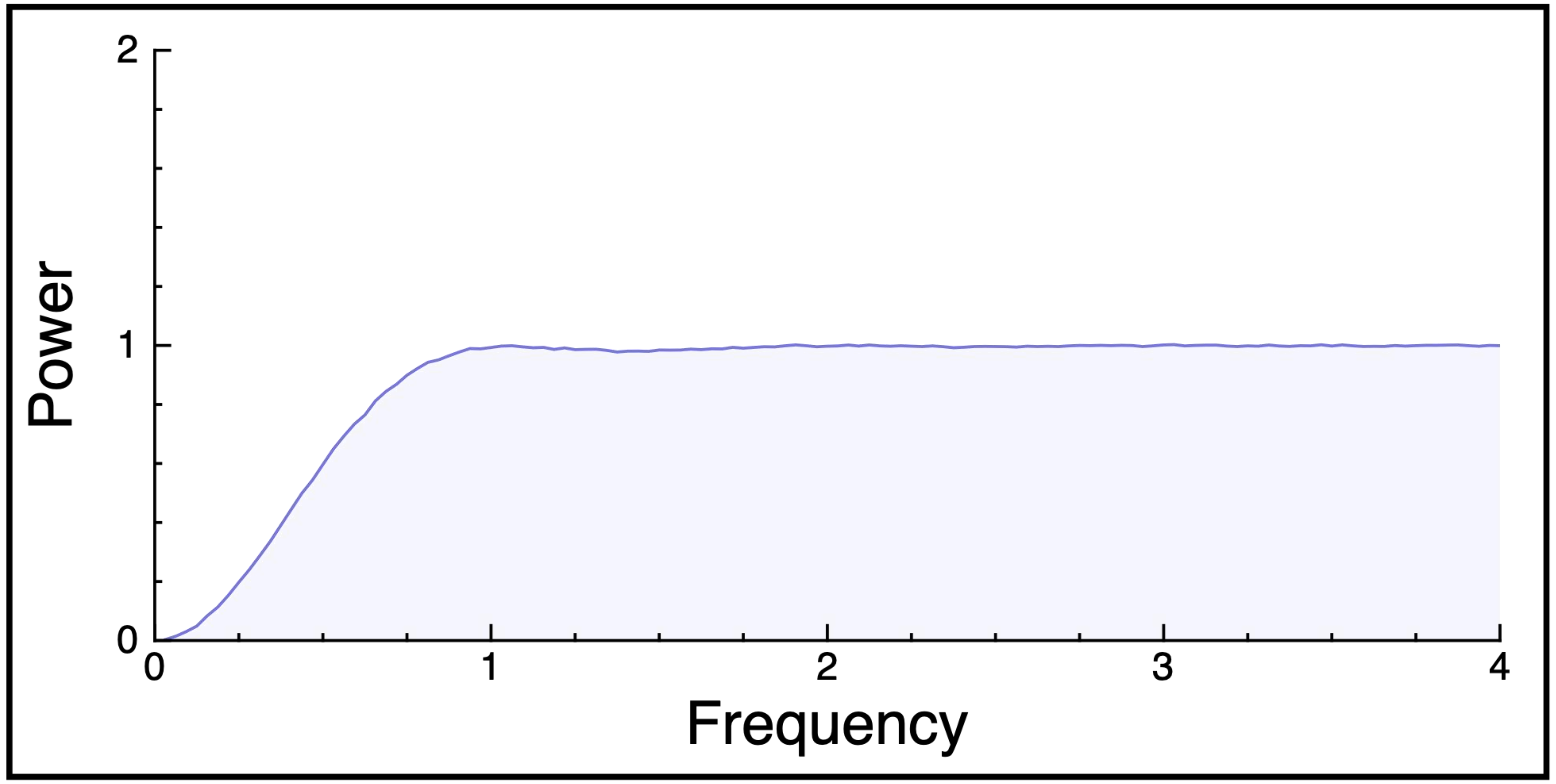
Samples



Expected power spectrum

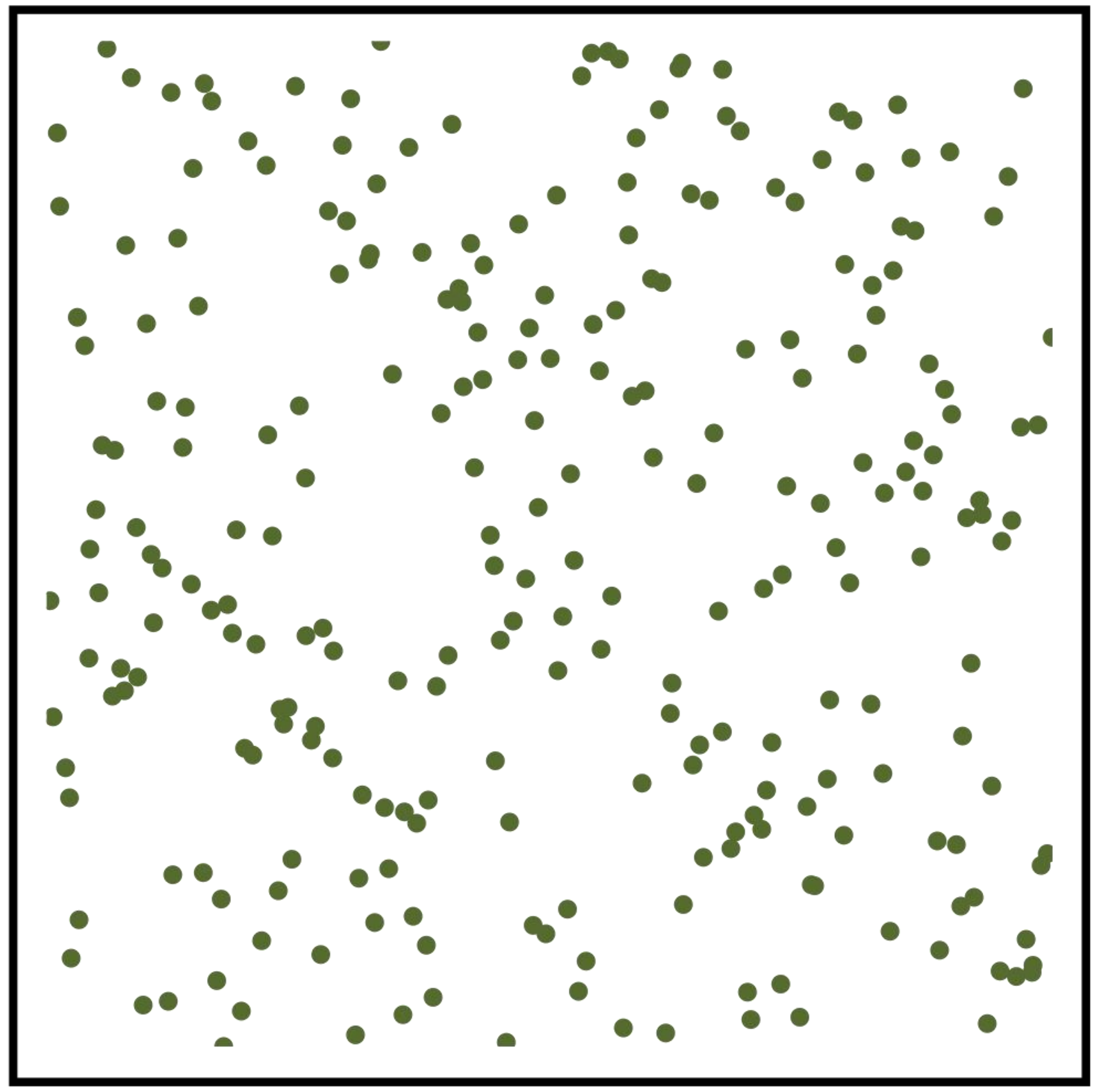


Radial mean

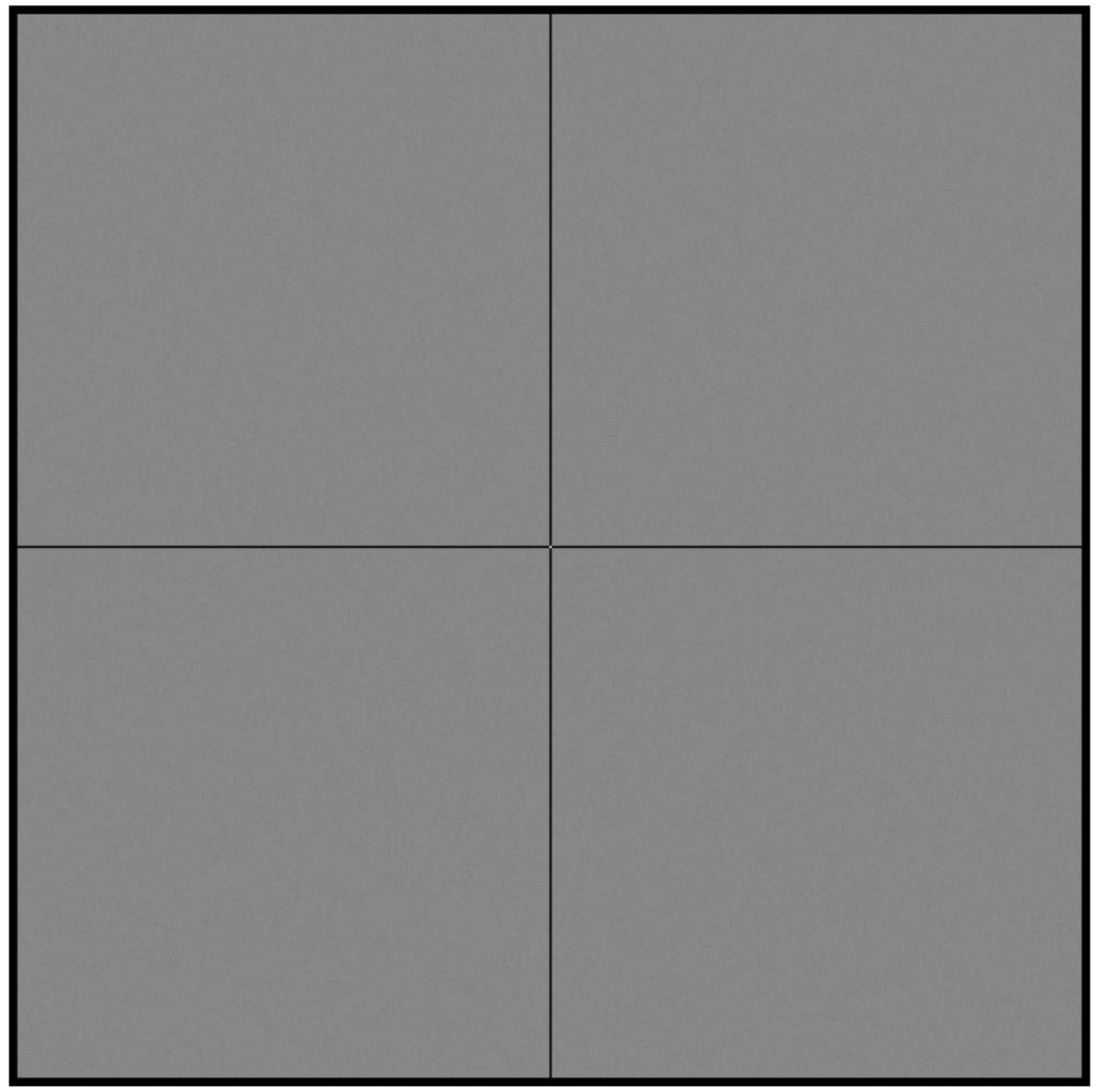


N-Rooks Sampling

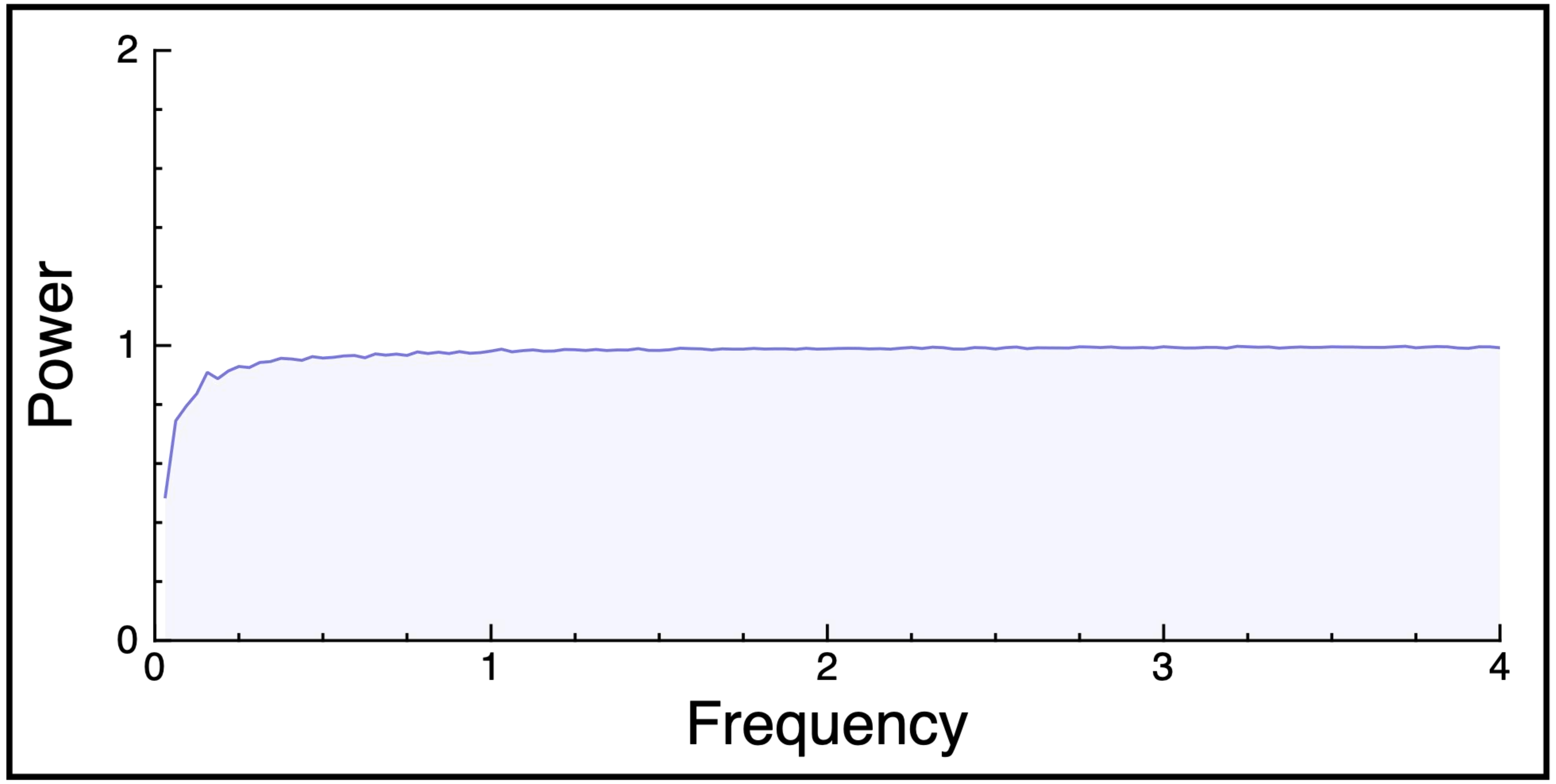
Samples



Expected power spectrum

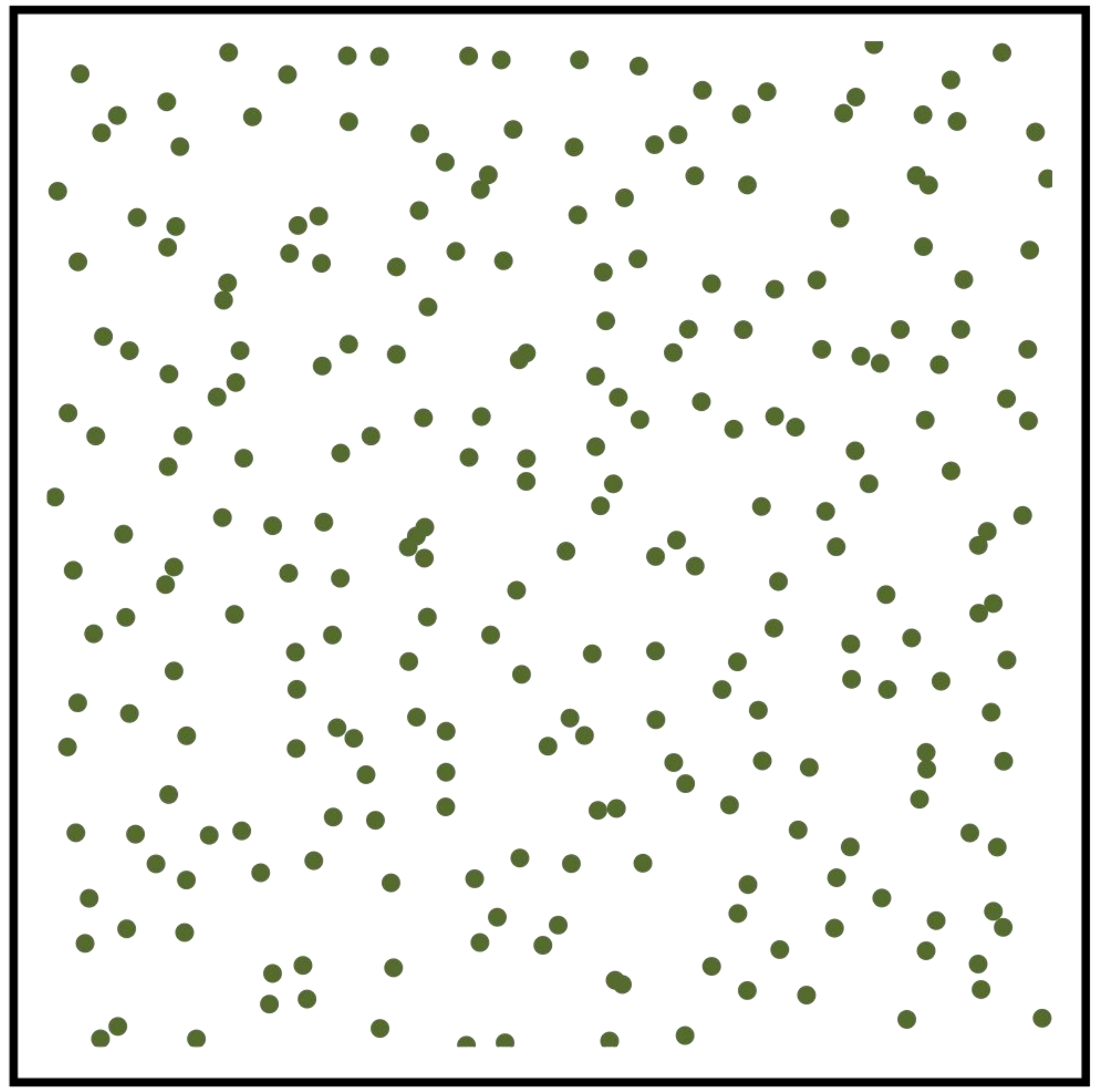


Radial mean

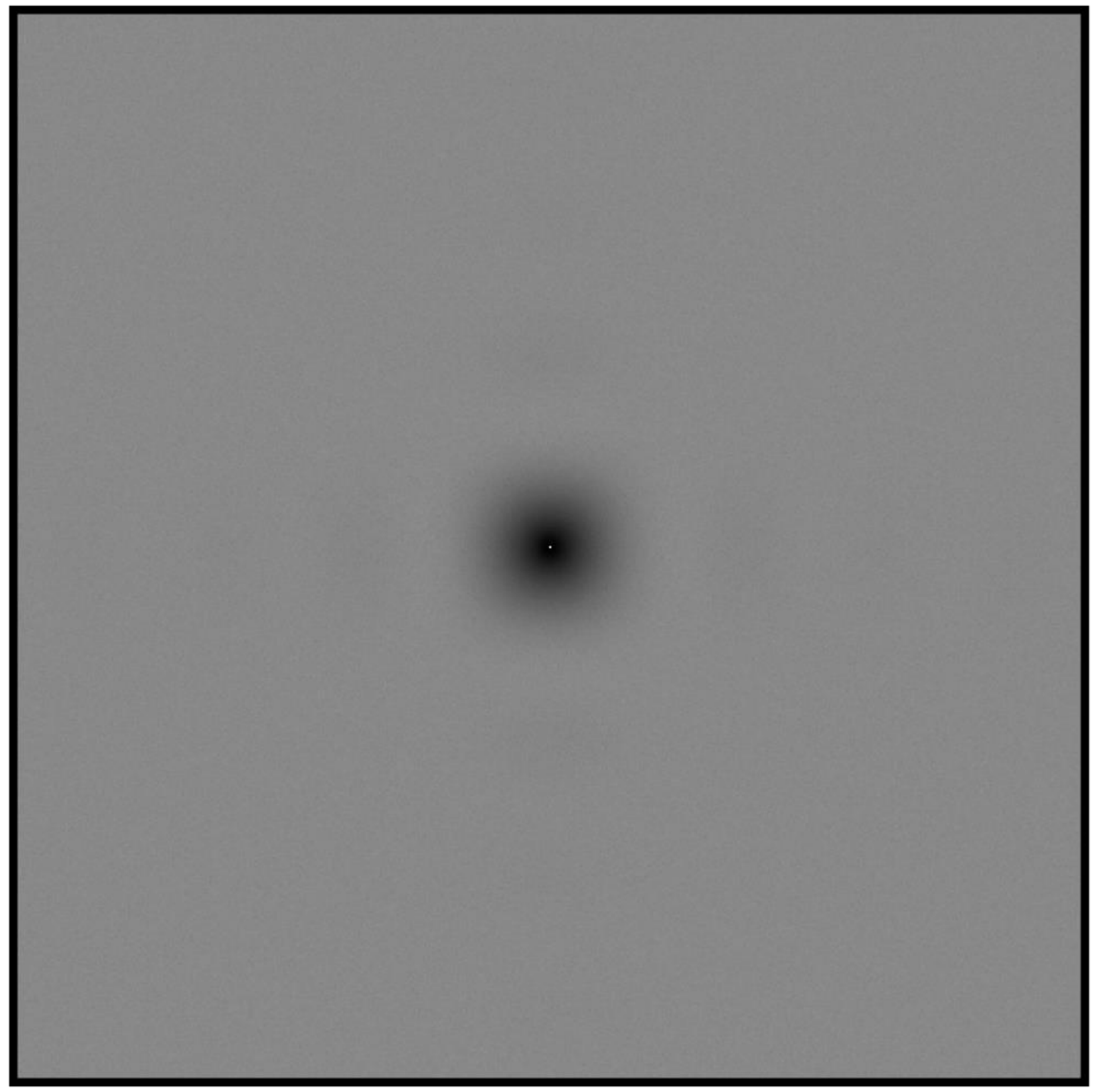


Jittered Sampling

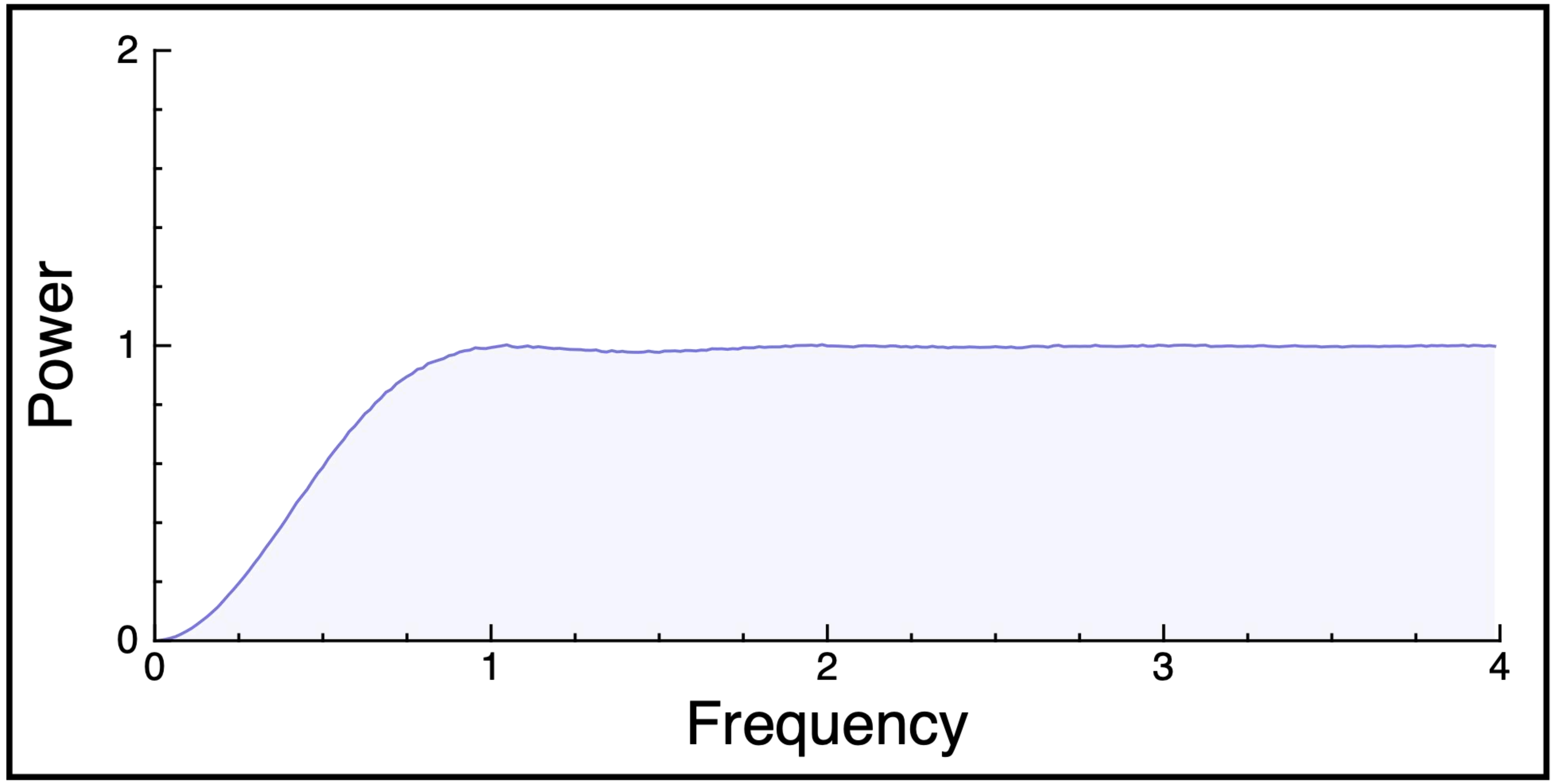
Samples



Expected power spectrum



Radial mean



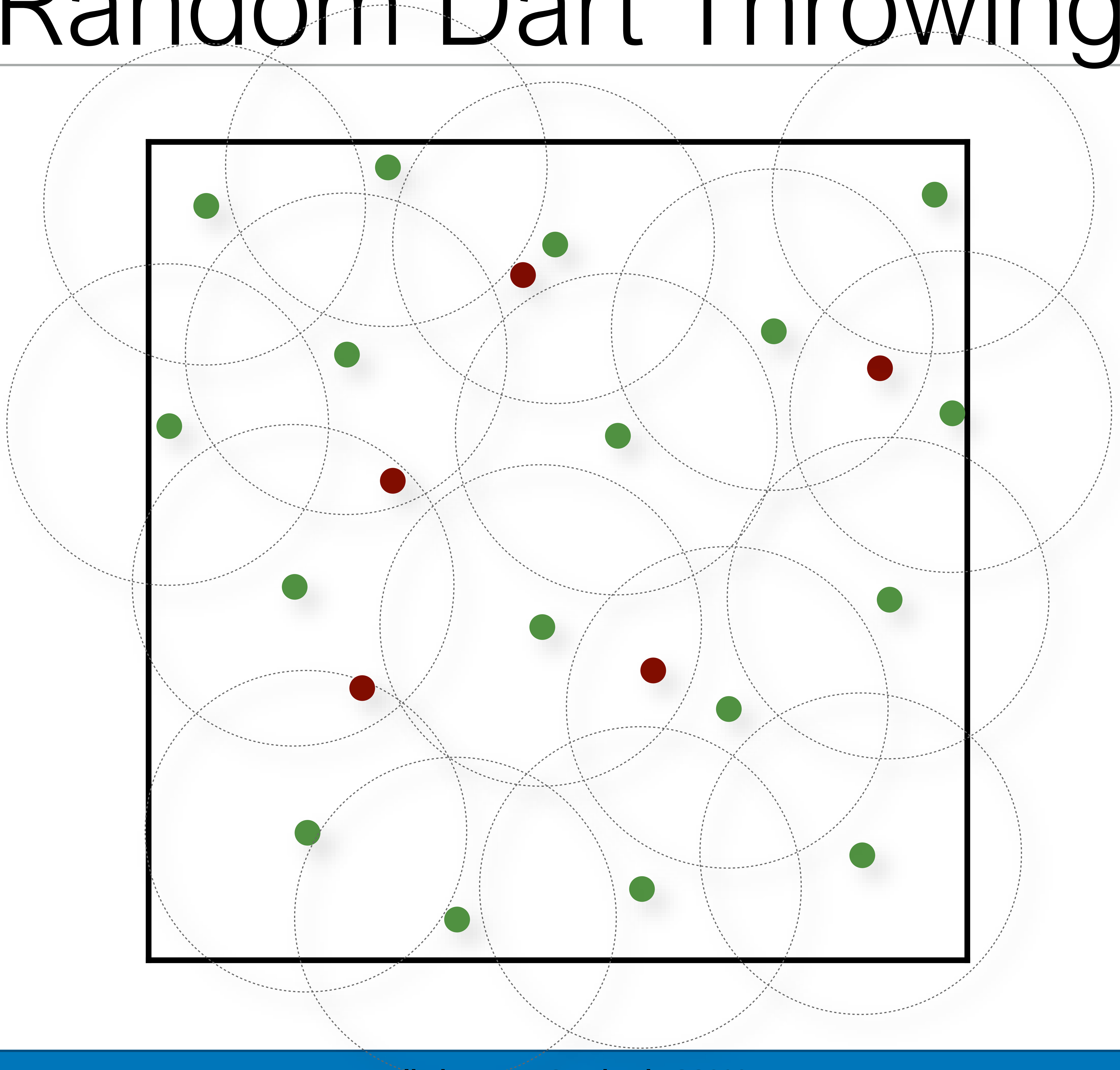
Poisson-Disk/Blue-Noise Sampling

Enforce a minimum distance between points

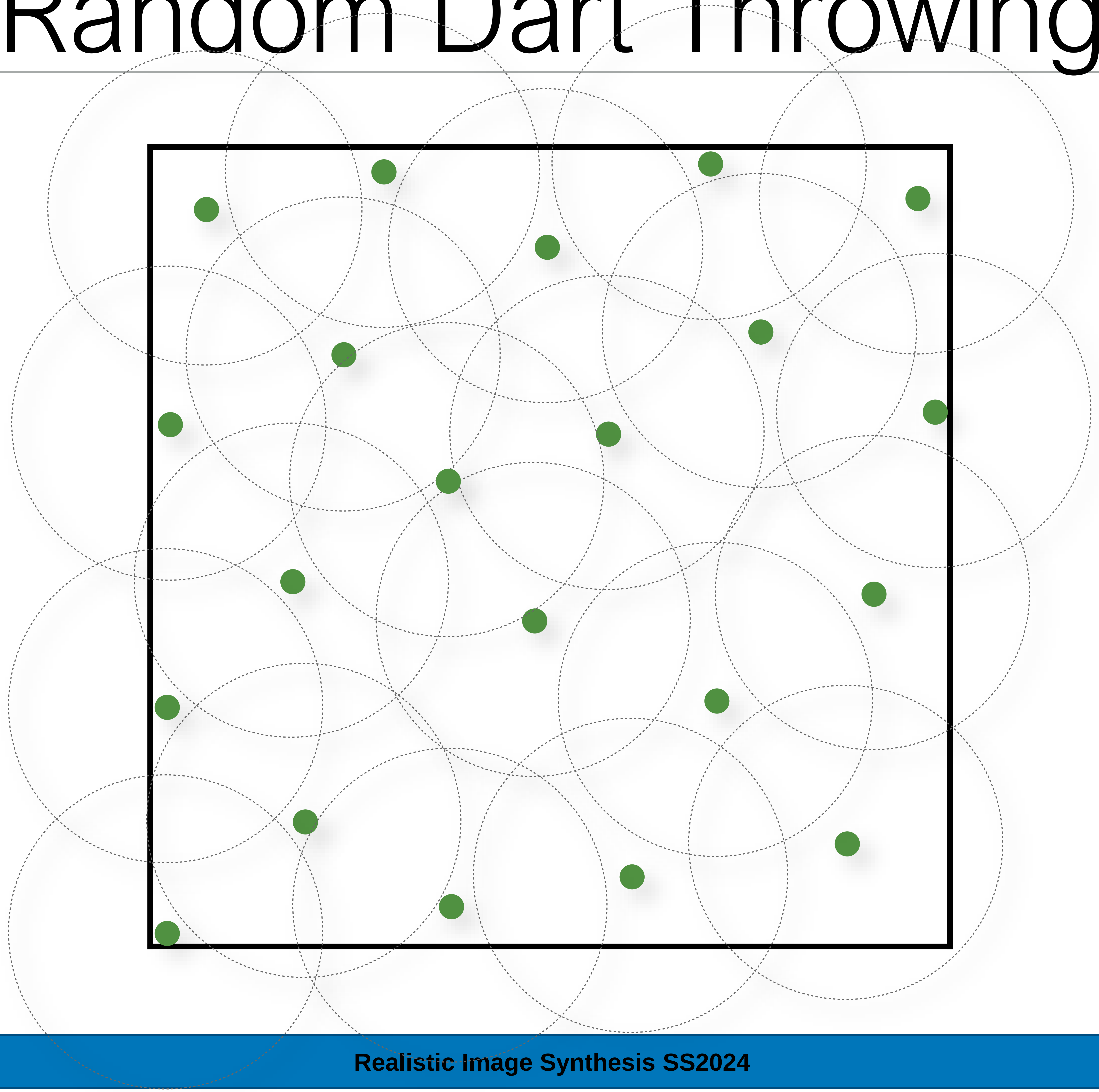
Poisson-Disk Sampling:

- Mark A. Z. Dippé and Erling Henry Wold. “Antialiasing through stochastic sampling.” *ACM SIGGRAPH,* 1985.
- Robert L. Cook. “Stochastic sampling in computer graphics.” *ACM Transactions on Graphics,* 1986.
- Ares Lagae and Philip Dutré. “A comparison of methods for generating Poisson disk distributions.” *Computer Graphics Forum*, 2008.

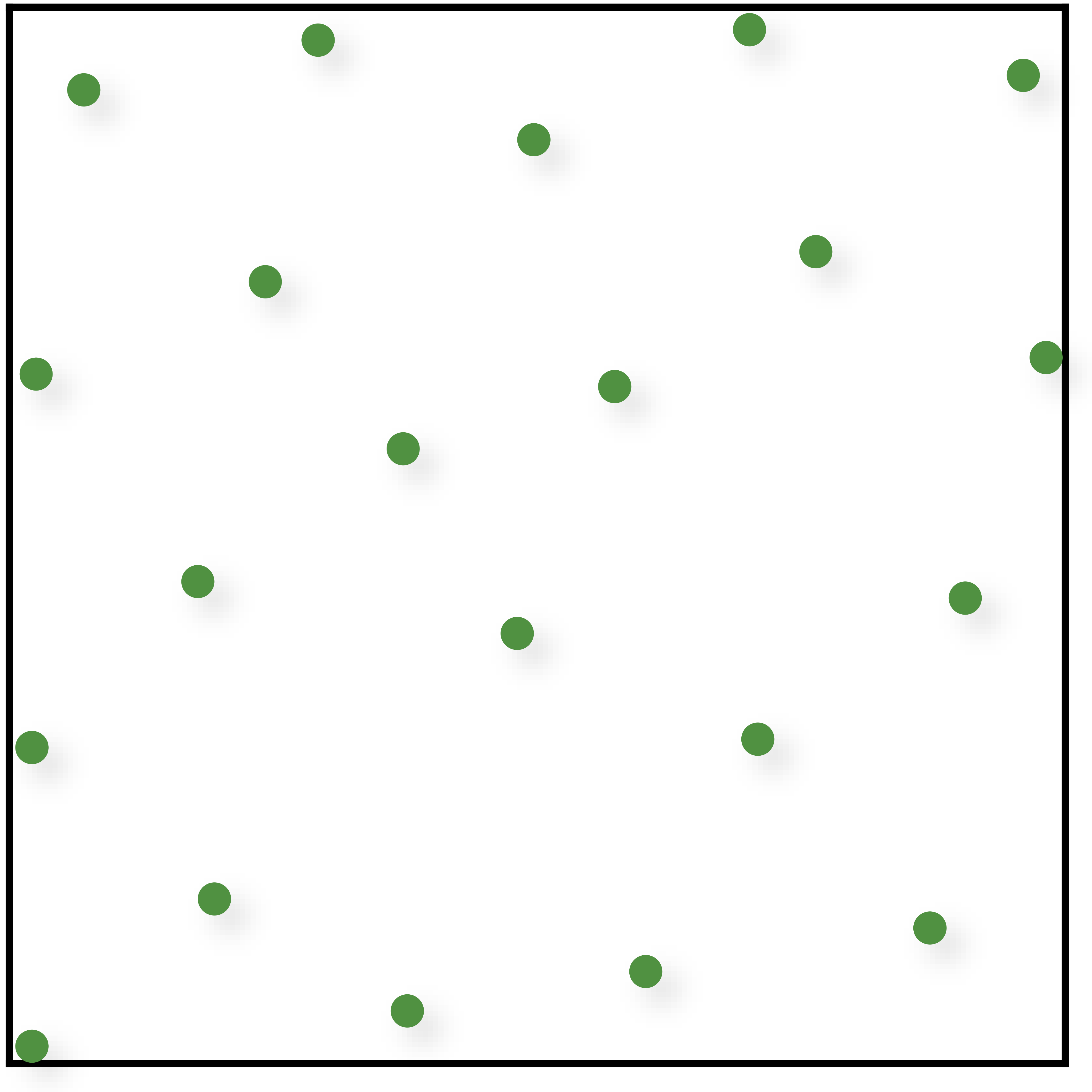
Random Dart Throwing



Random Dart Throwing

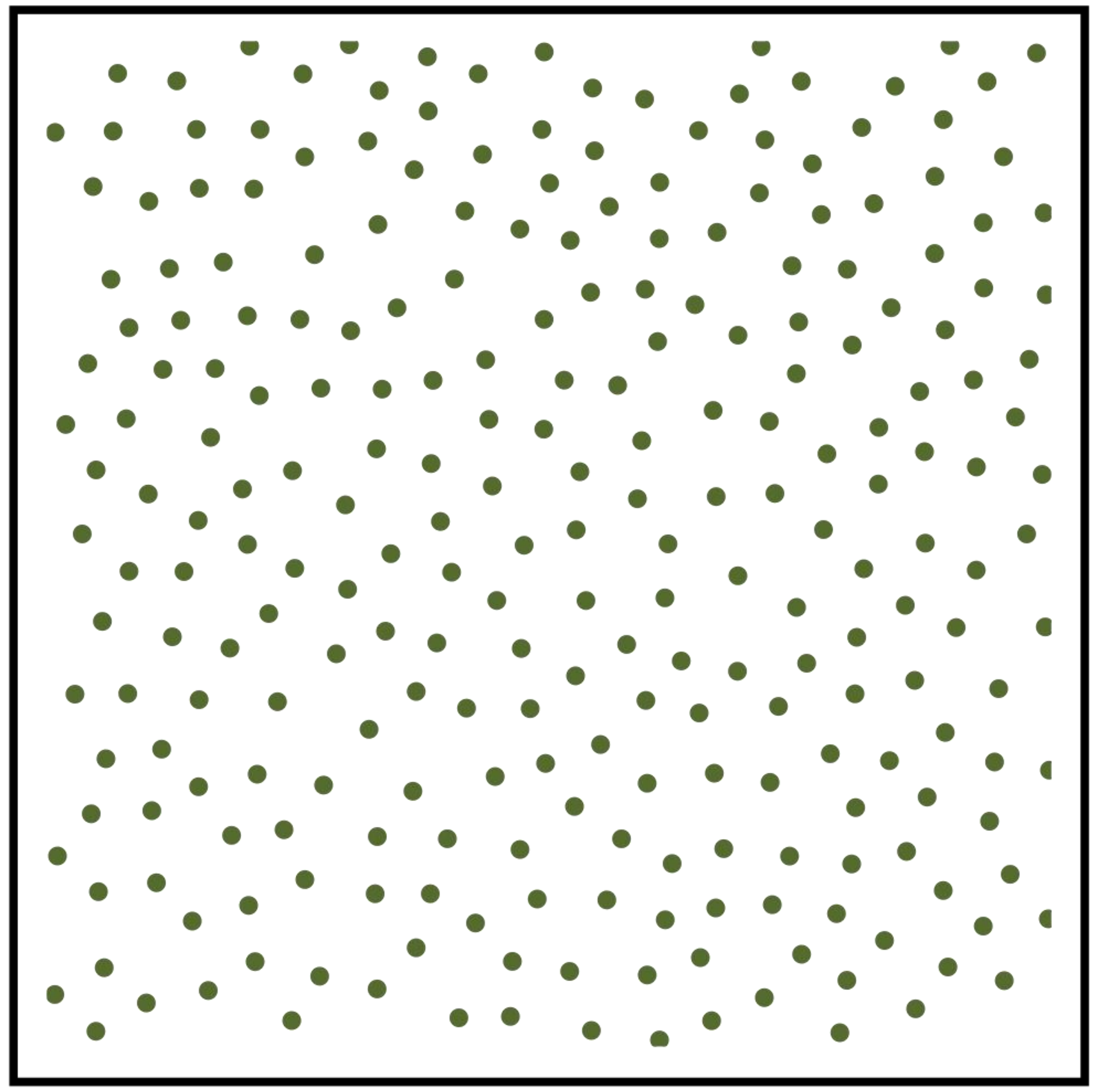


Random Dart Throwing

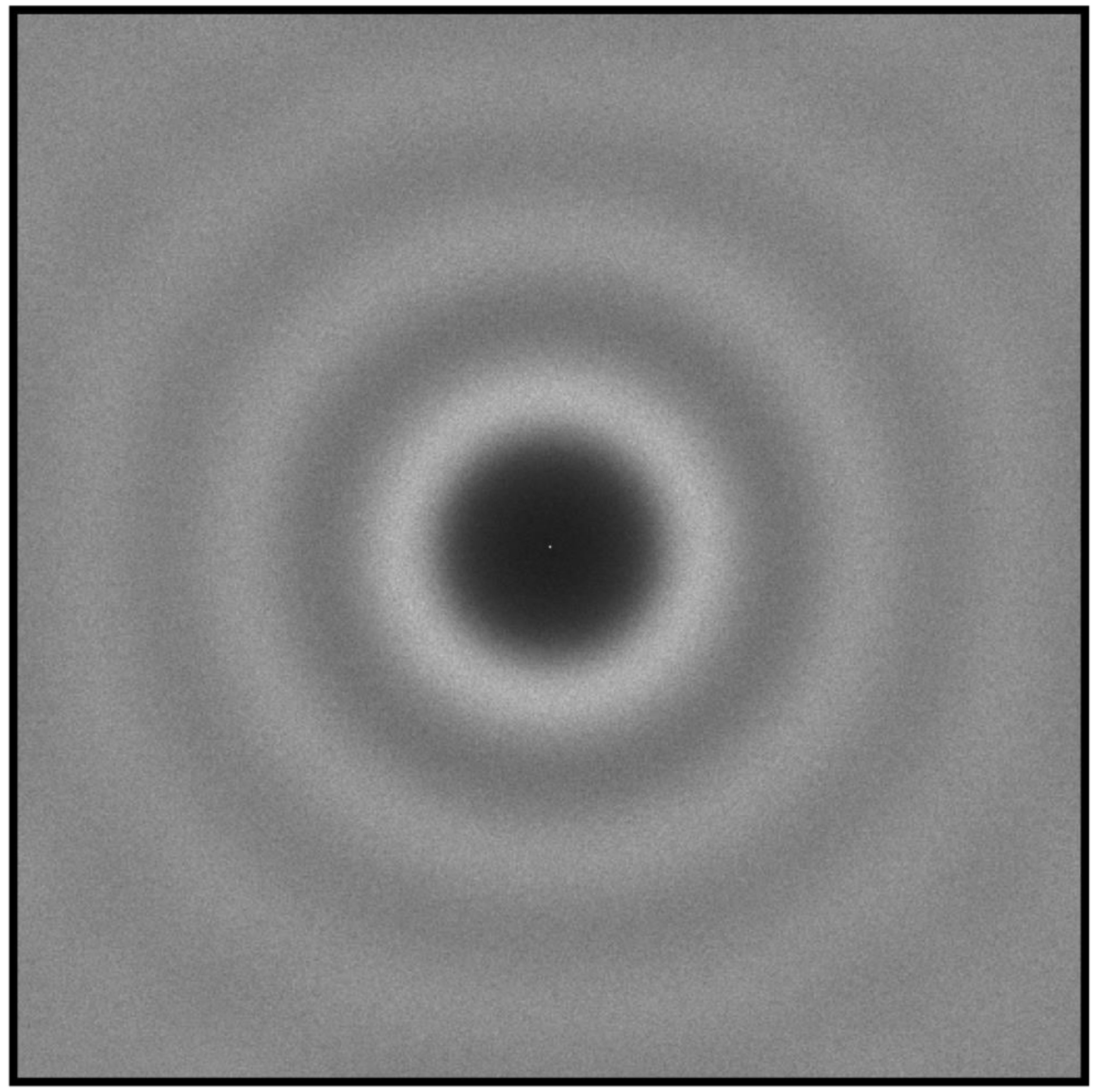


Poisson Disk Sampling

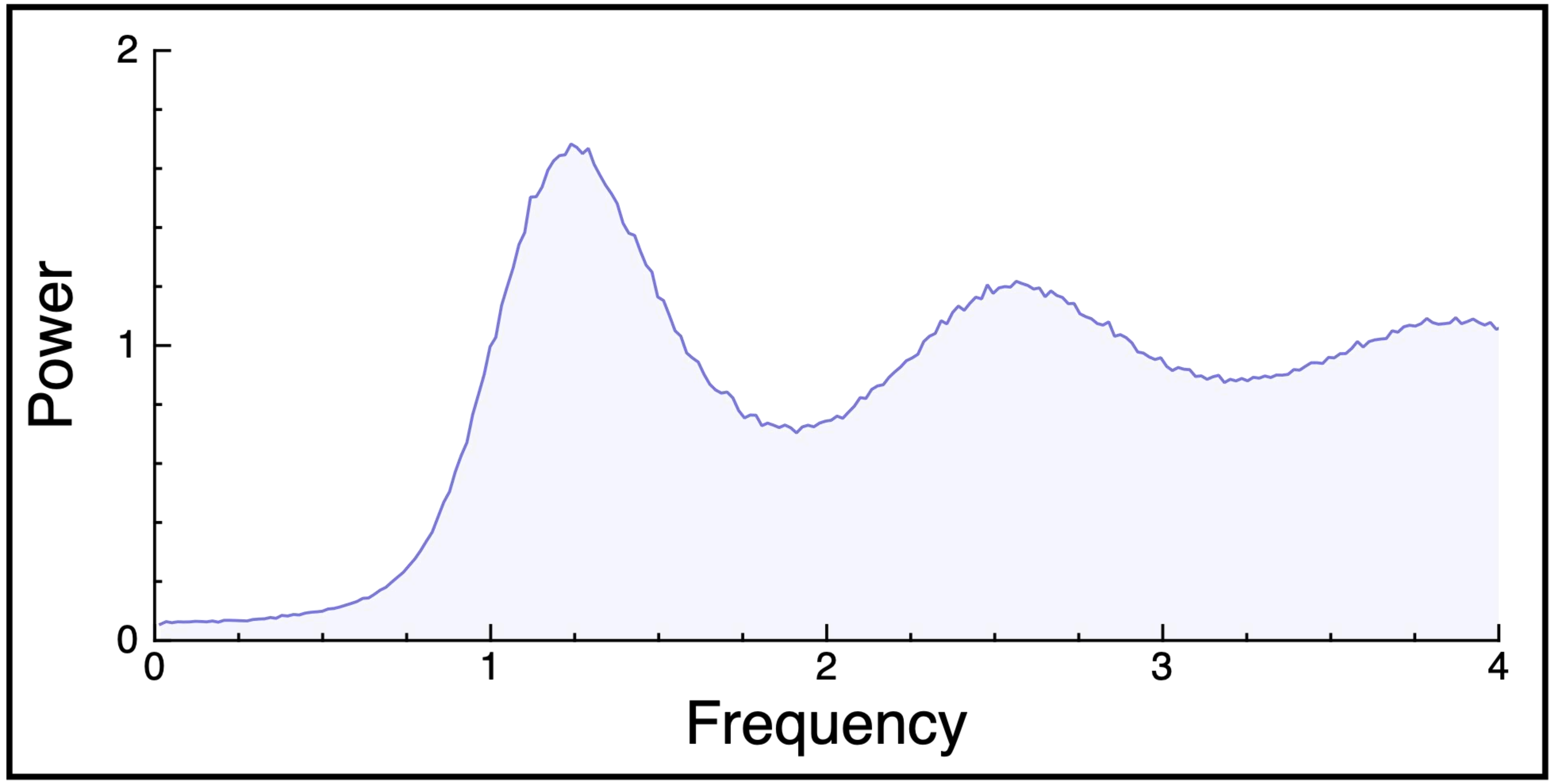
Samples



Expected power spectrum



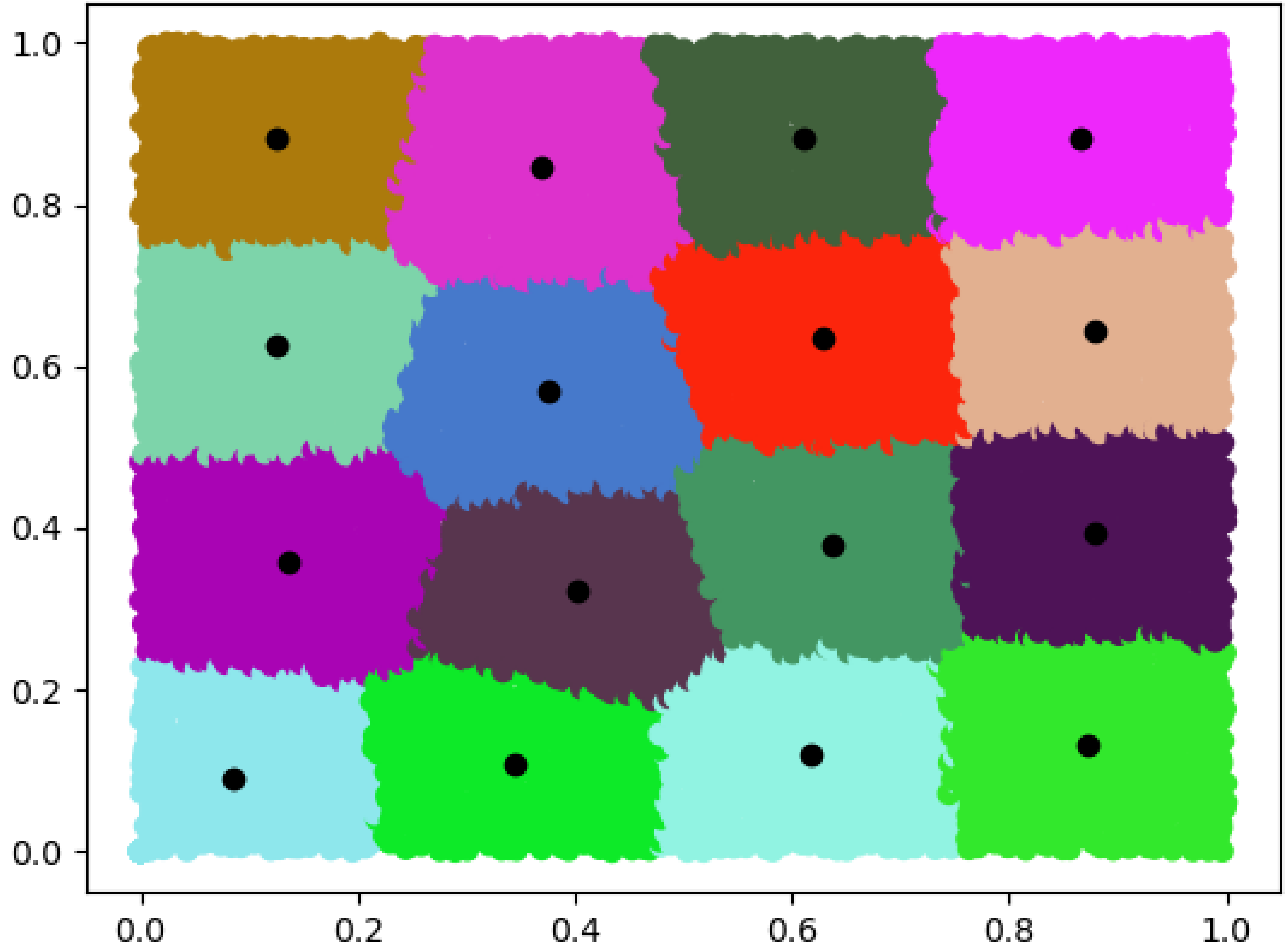
Radial mean



Blue-Noise Sampling (Relaxation-based)

1. Initialize sample positions (e.g. random)
2. Use an iterative relaxation to move samples away from each other.

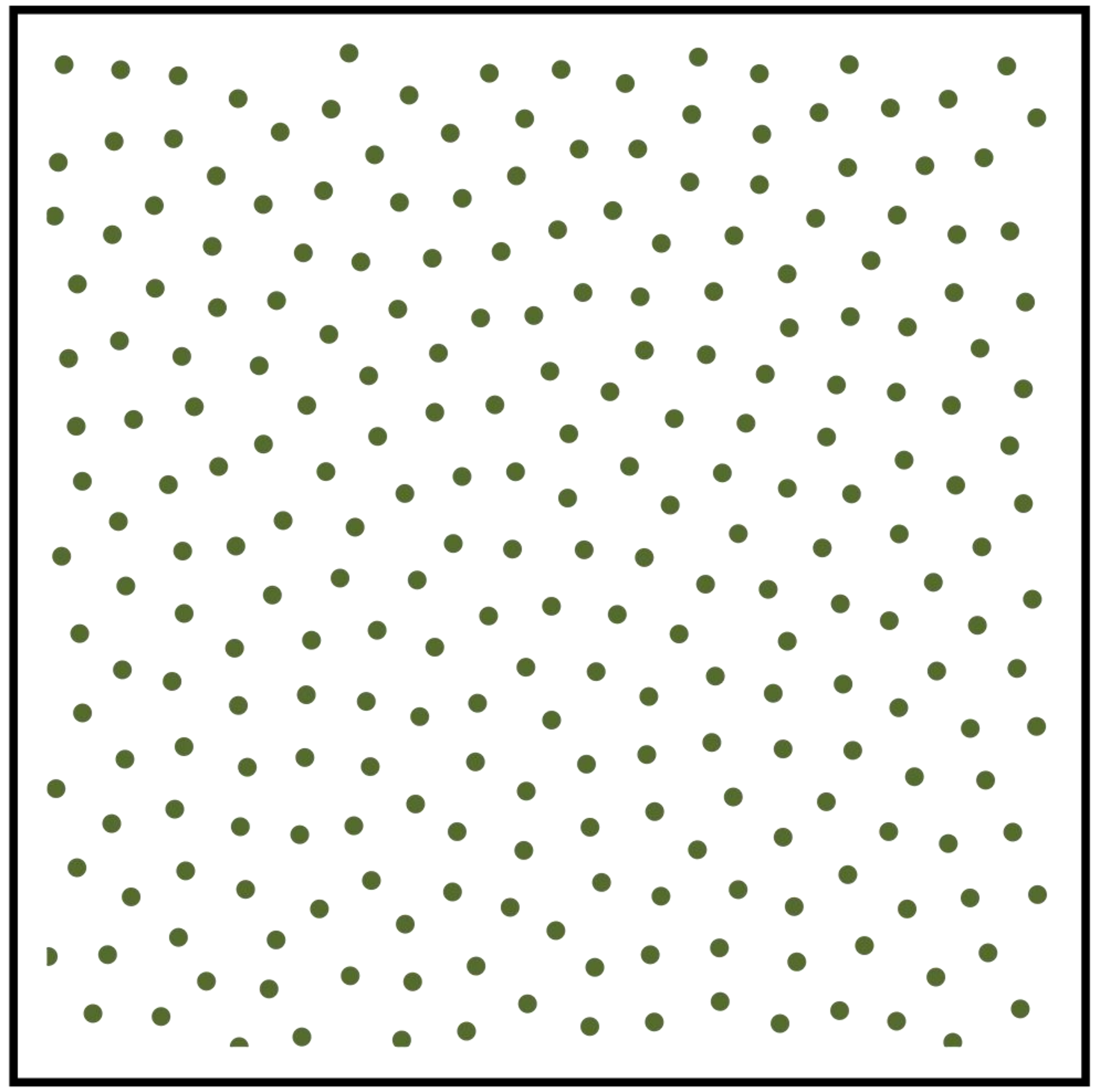
Lloyd-Relaxation Method



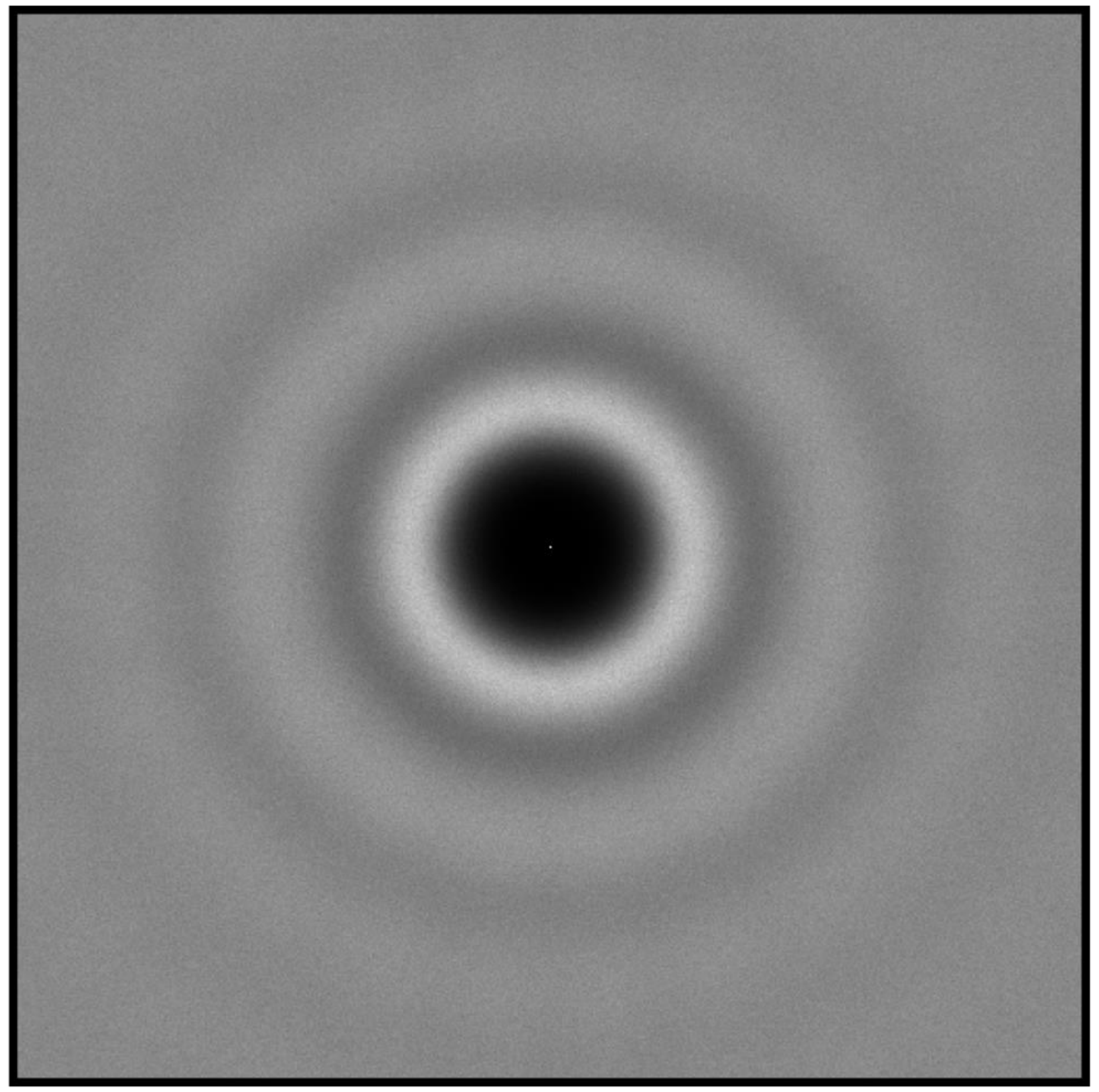
CCVT Sampling [Balzer et al. 2009]

CCVT Sampling [Balzer et al. 2009]

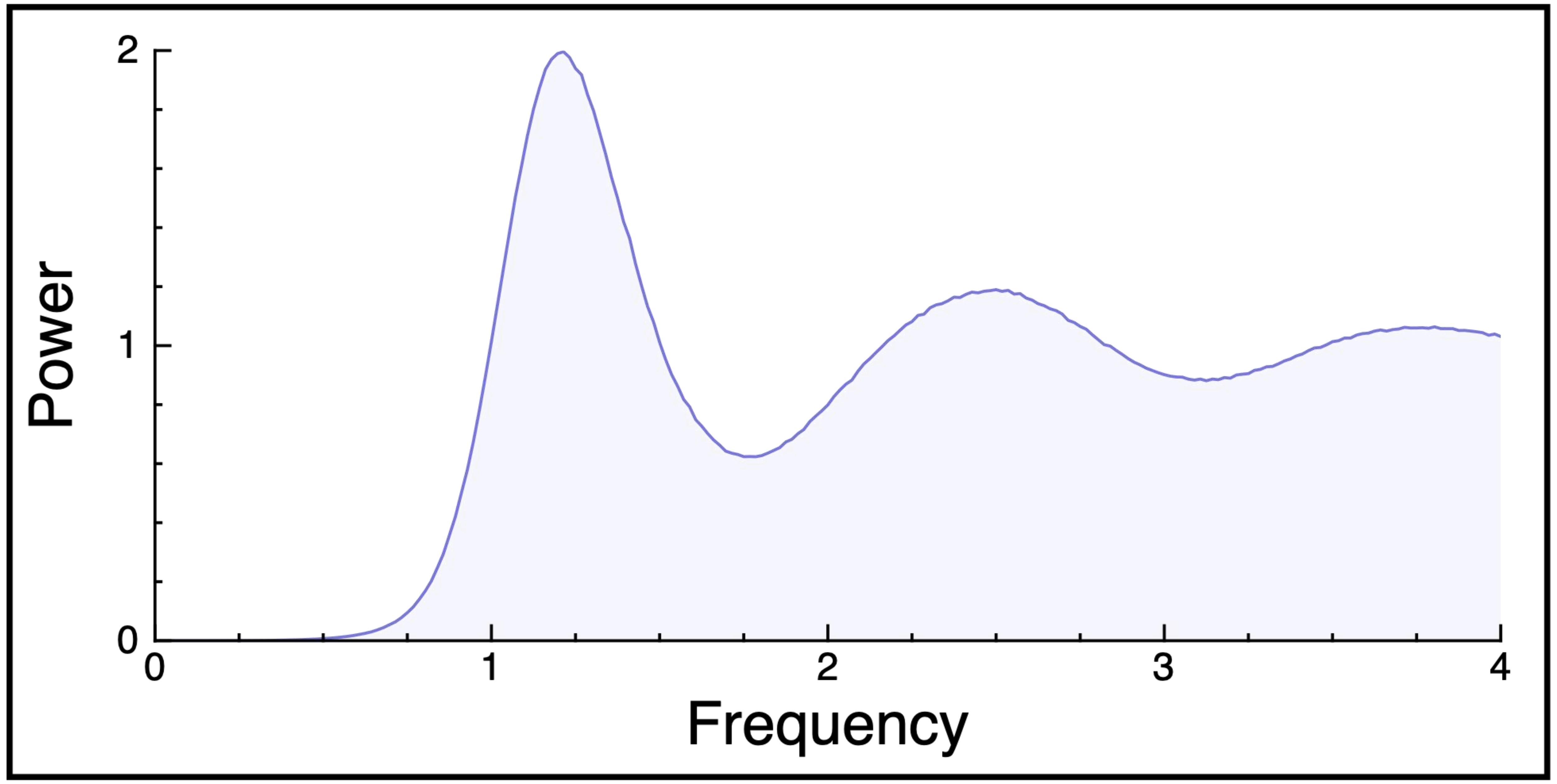
Samples



Expected power spectrum

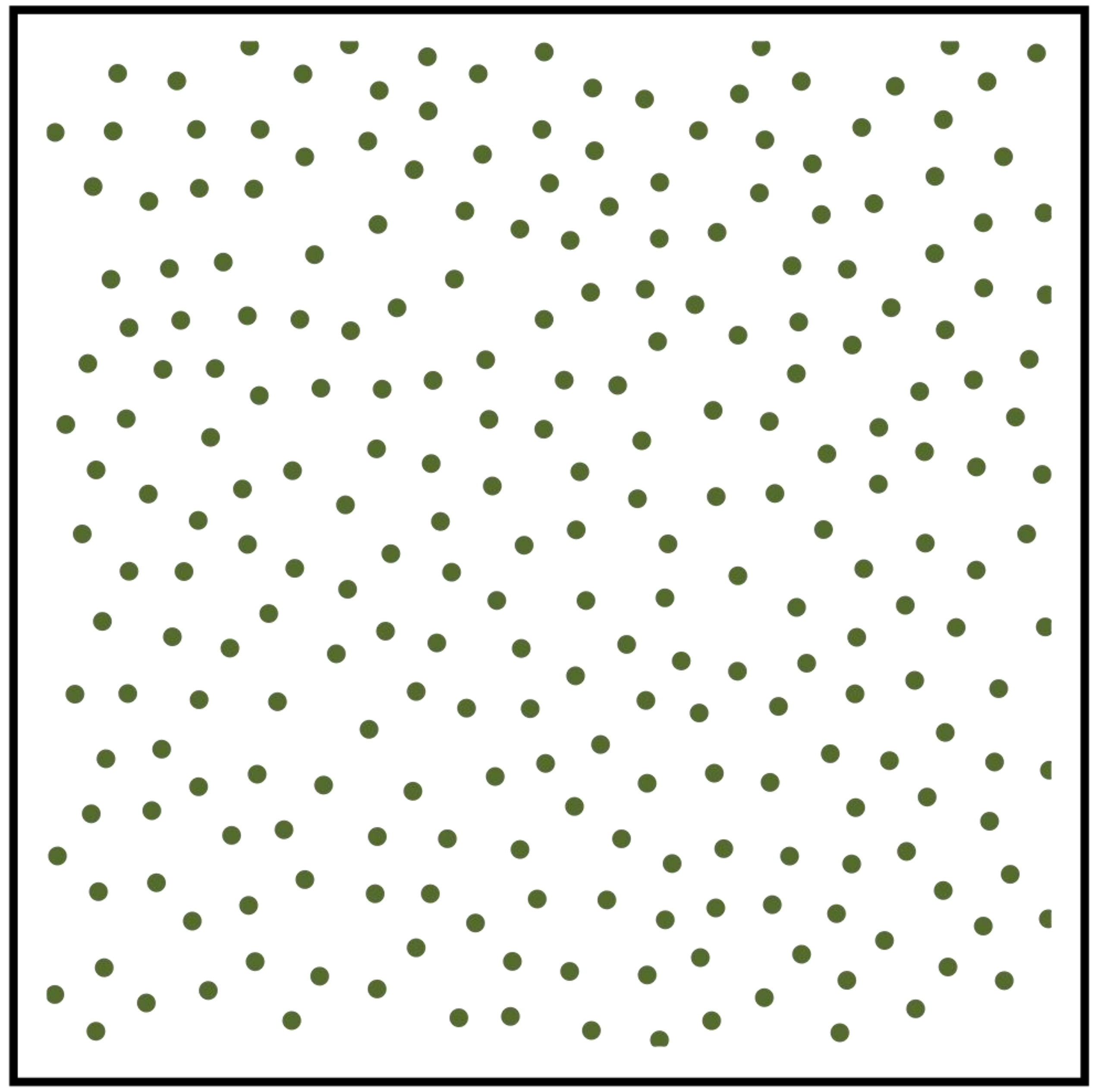


Radial mean

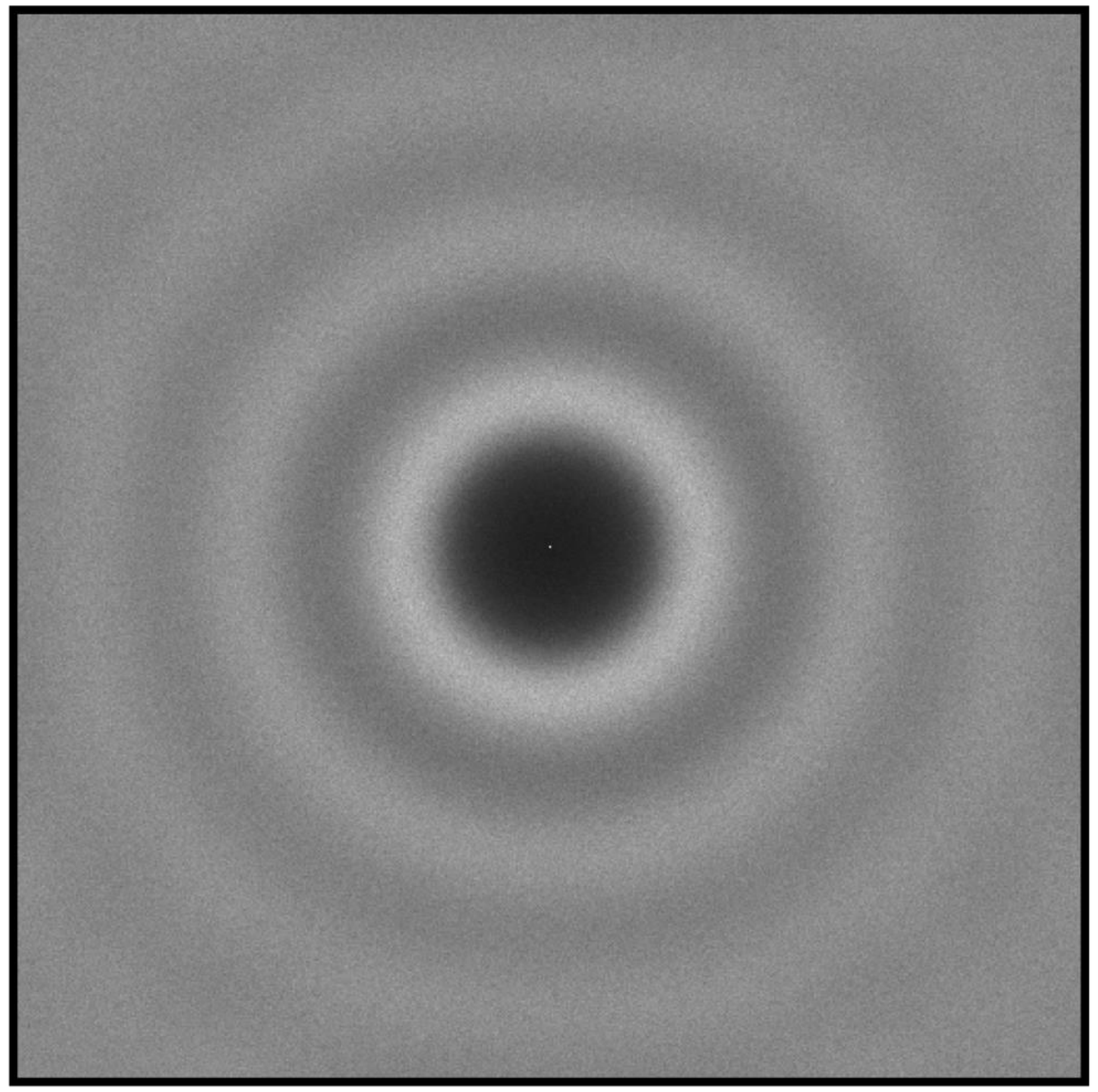


Poisson Disk Sampling

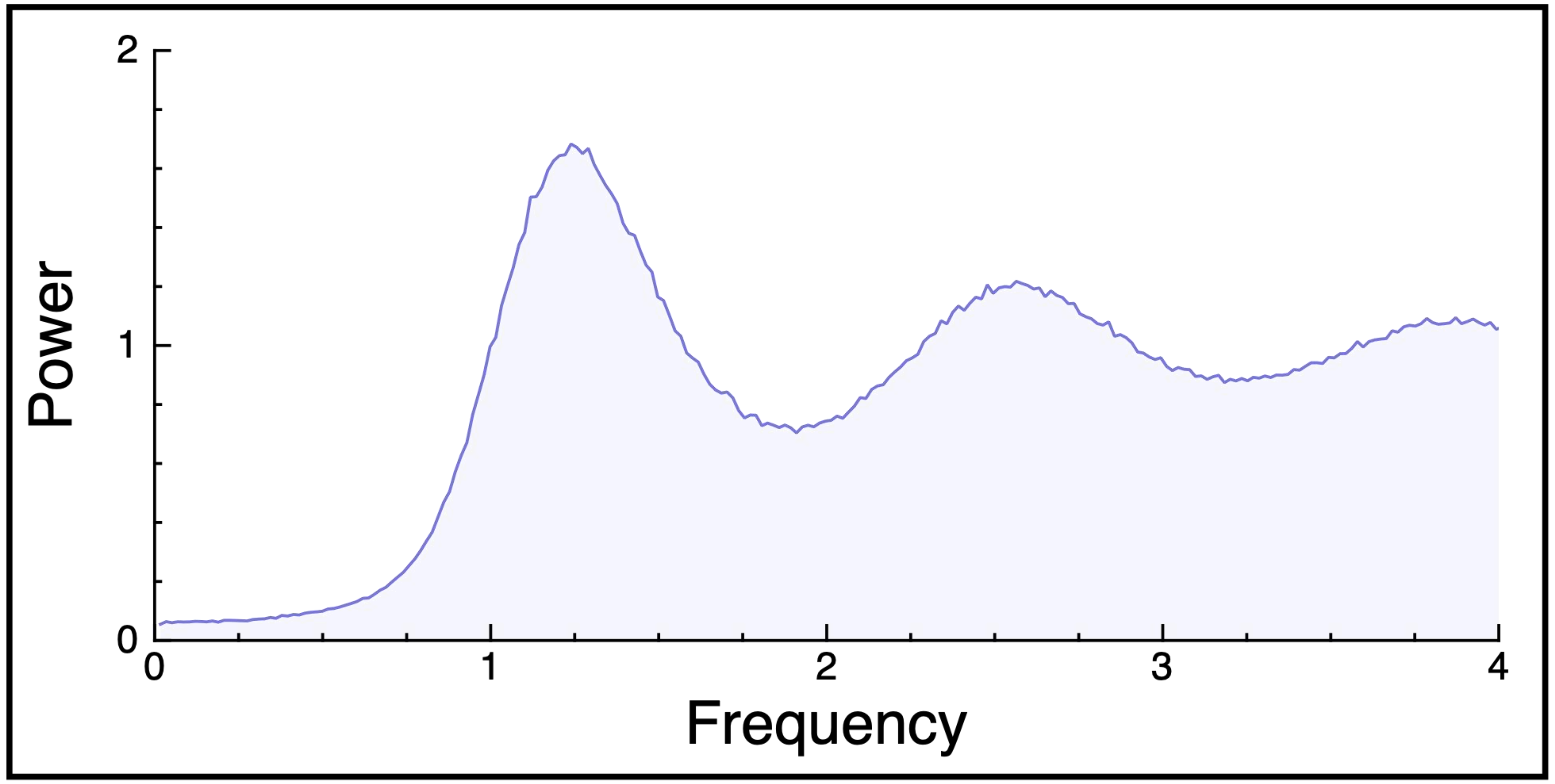
Samples



Expected power spectrum



Radial mean



Low-Discrepancy Sampling

Deterministic sets of points specially crafted to be evenly distributed (have low discrepancy).

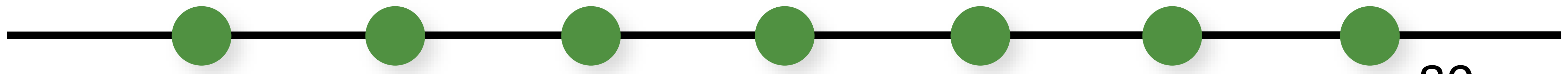
Entire field of study called Quasi-Monte Carlo (QMC)

The Van der Corput Sequence

Radical Inverse Φ_b in base 2

Subsequent points “fall into biggest holes”

k	Base 2	Φ_b
1	1	.1 = 1/2
2	10	.01 = 1/4
3	11	.11 = 3/4
4	100	.001 = 1/8
5	101	.101 = 5/8
6	110	.011 = 3/8
7	111	.111 = 7/8
…		



Halton and Hammersley Points

Halton: Radical inverse with different base for each dimension:

$$\vec{x}_k = (\Phi_2(k), \Phi_3(k), \Phi_5(k), \ \dots\ , \Phi_{p_n}(k))$$

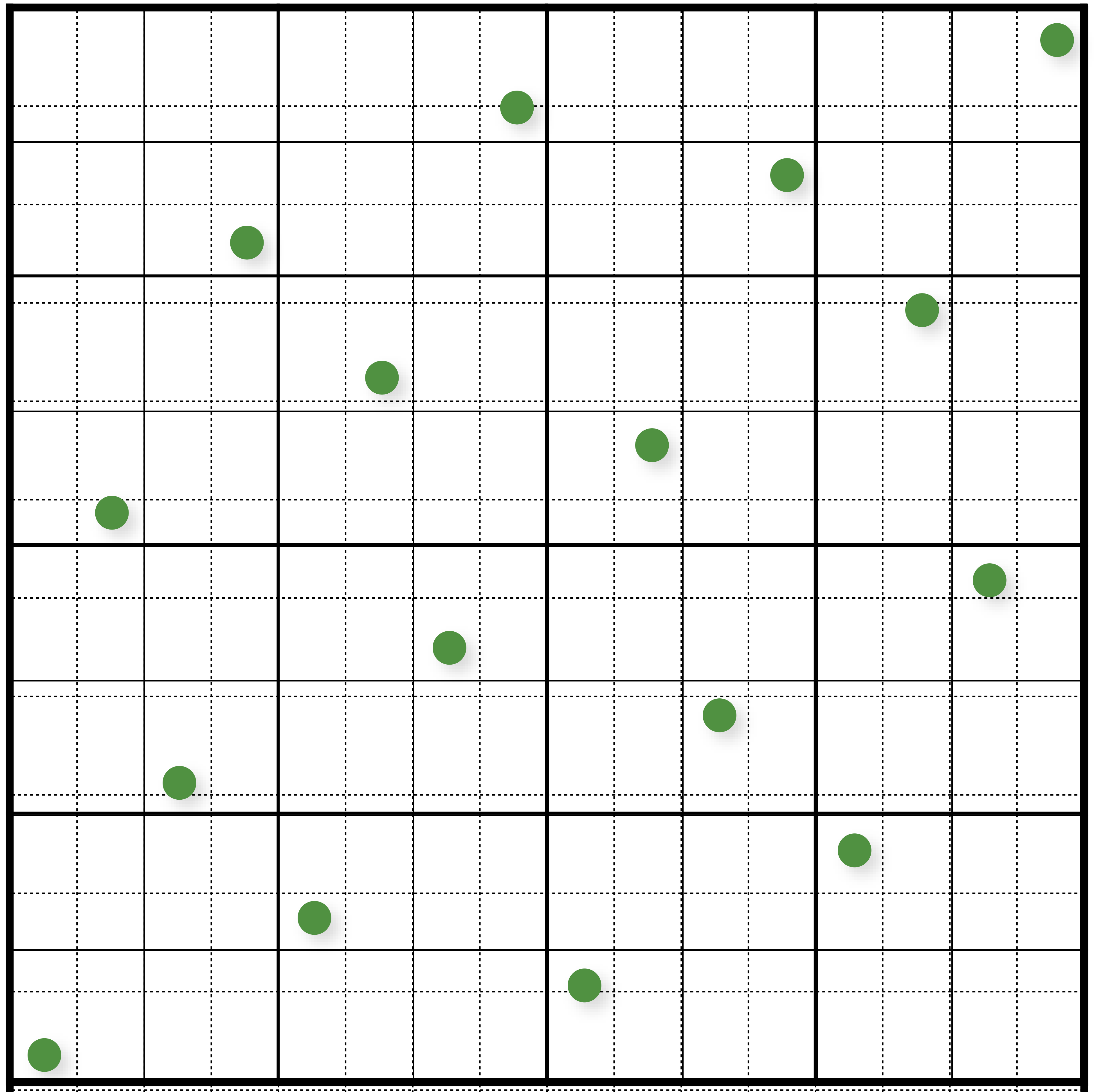
- The bases should all be relatively prime.
- Incremental/progressive generation of samples

Hammersley: Same as Halton, but first dimension is k/N:

$$\vec{x}_k = ({}^{k}/_{N}, \Phi_2(k), \Phi_3(k), \Phi_5(k), \ \dots\ , \Phi_{p_n}(k))$$

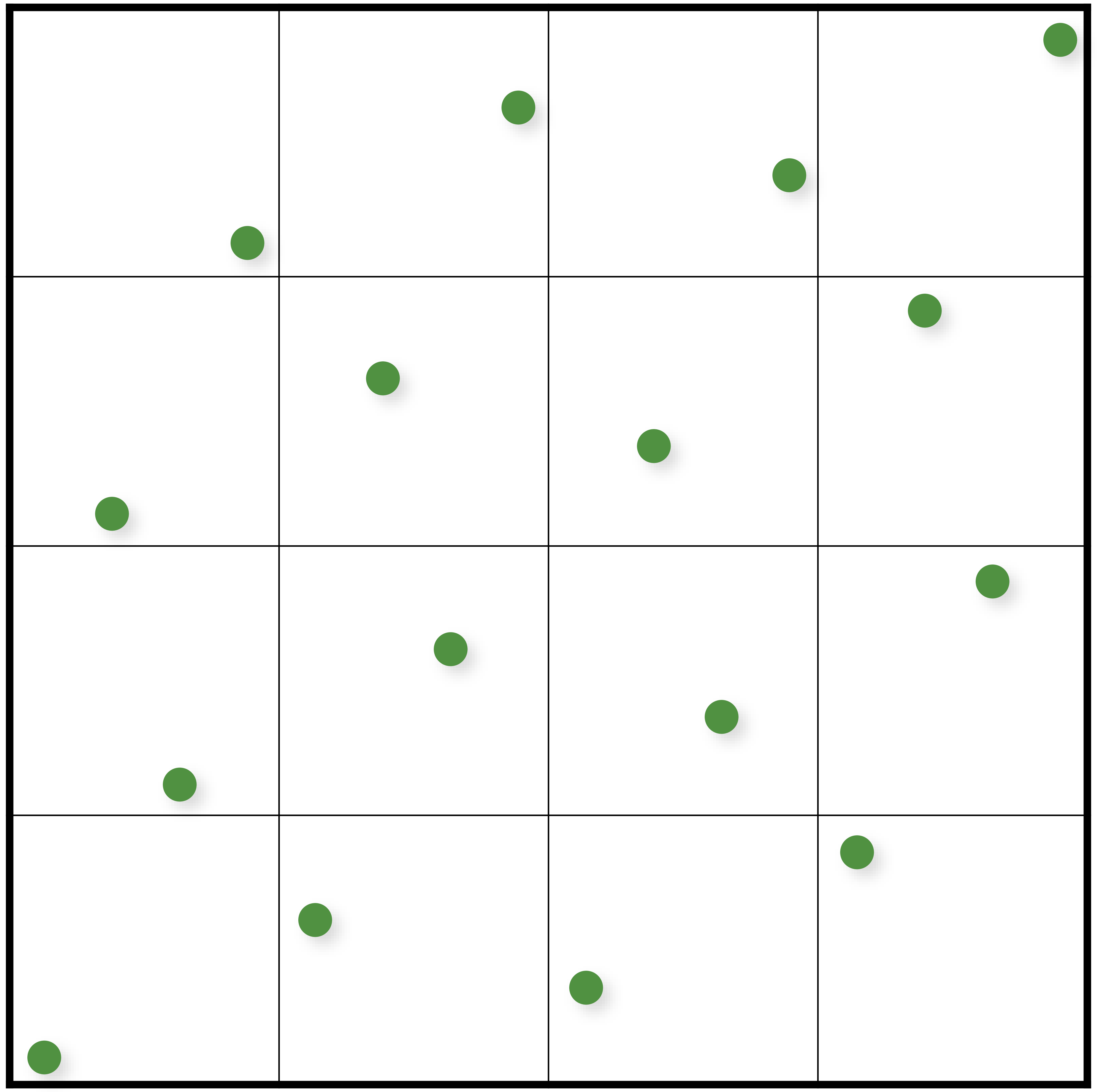
- Not incremental, need to know sample count, N, in advance

The Hammersley Sequence



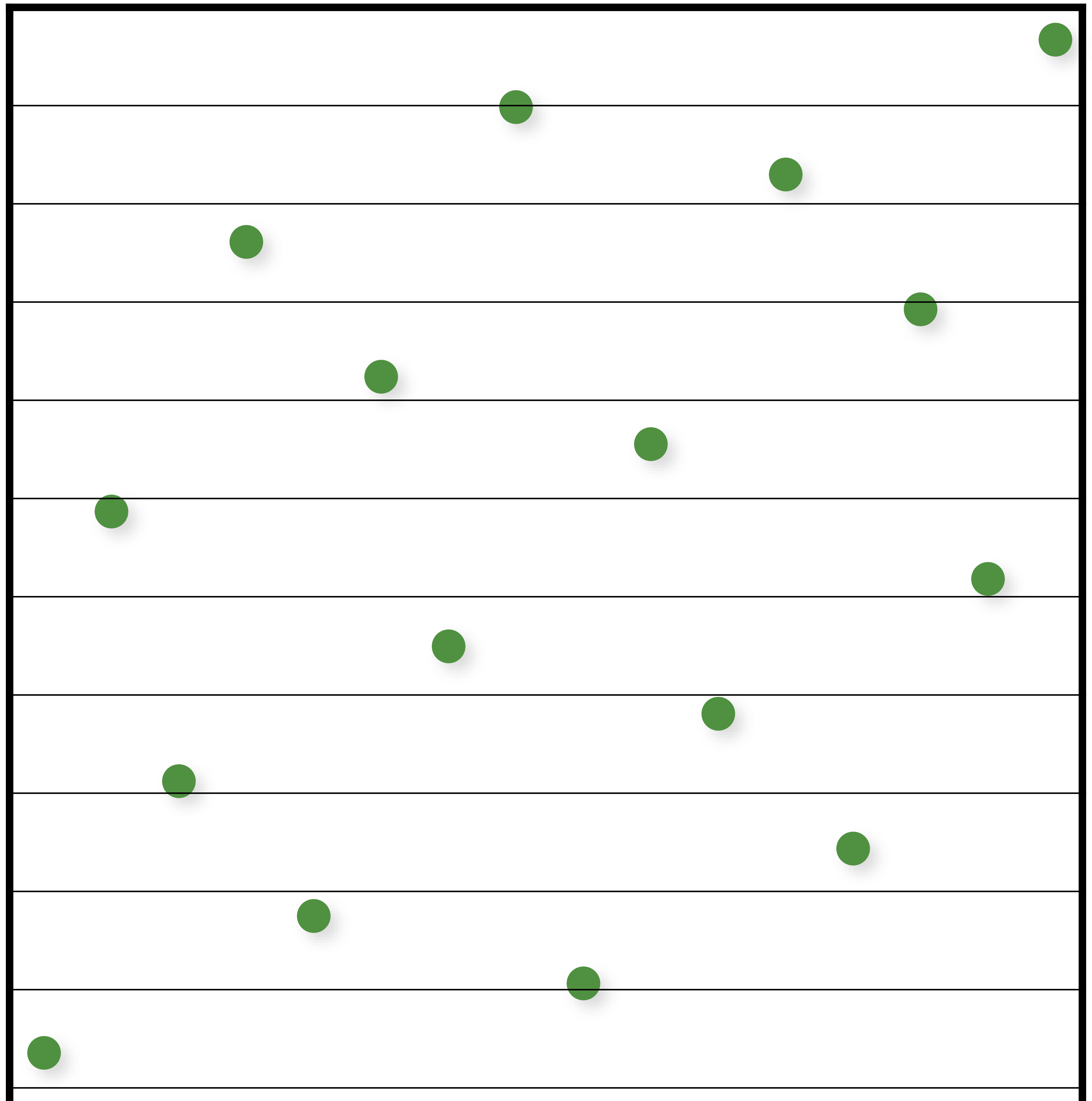
1 sample in each “elementary interval”

The Hammersley Sequence



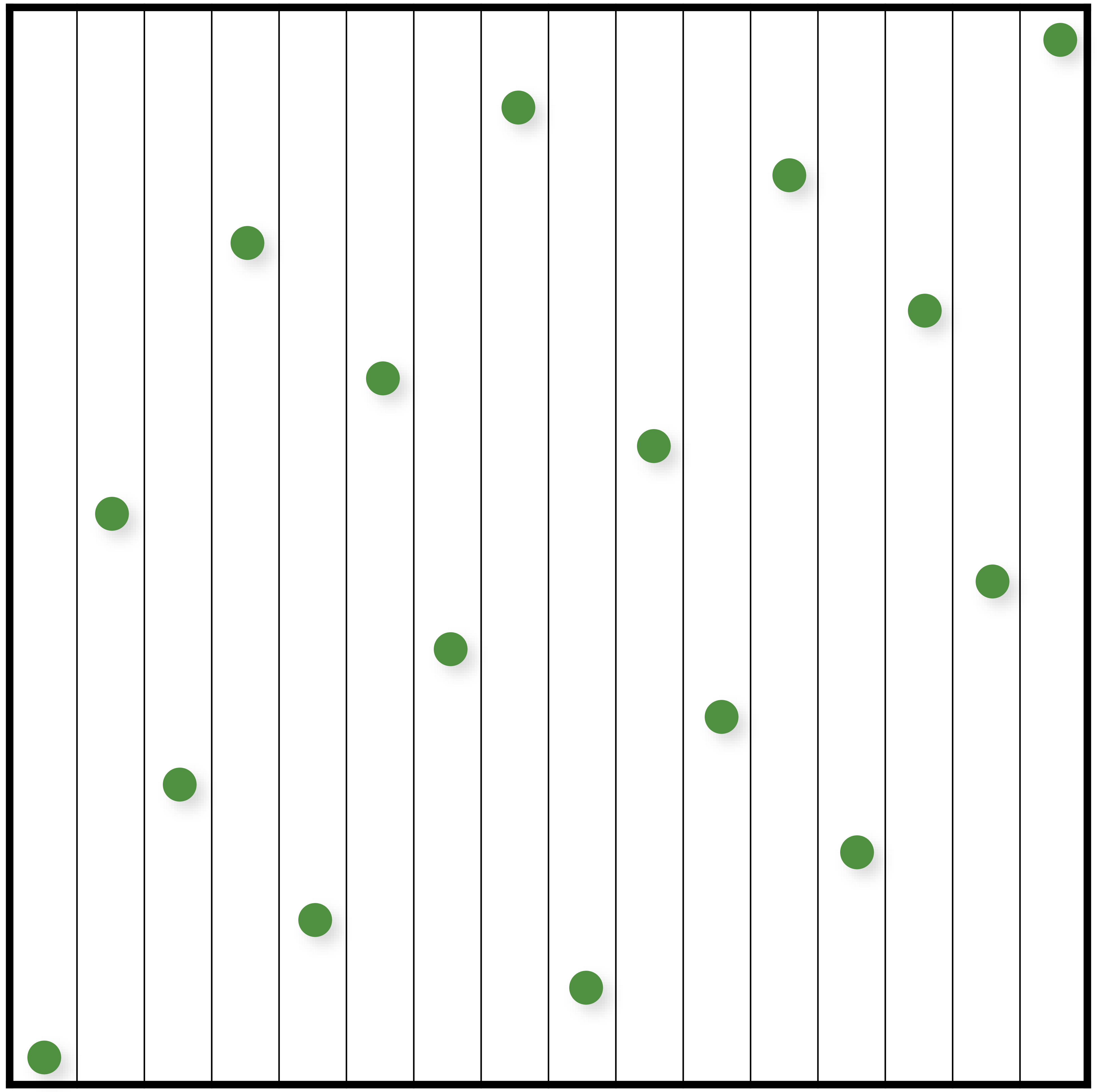
1 sample in each “elementary interval”

The Hammersley Sequence



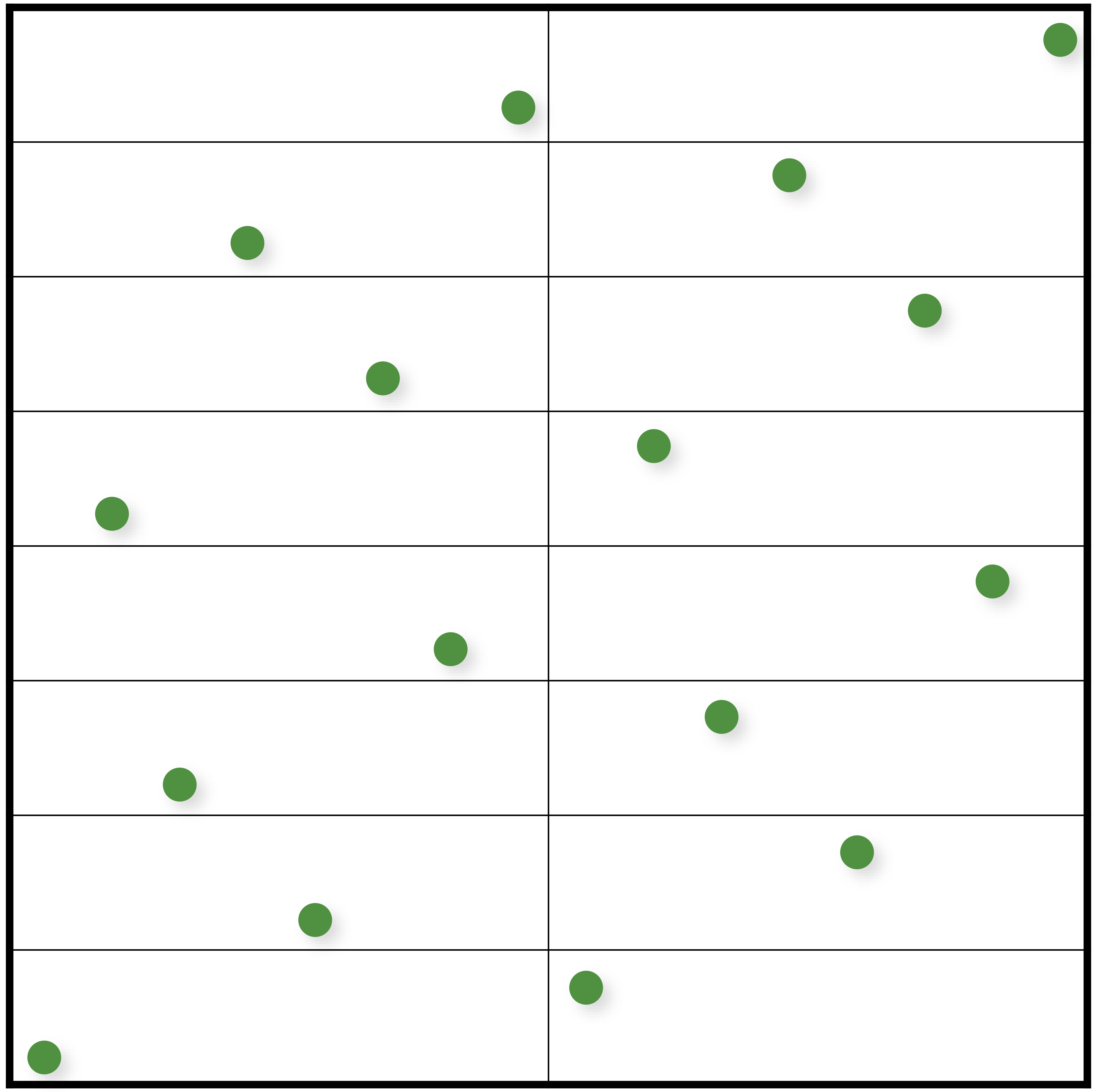
1 sample in each “elementary interval”

The Hammersley Sequence



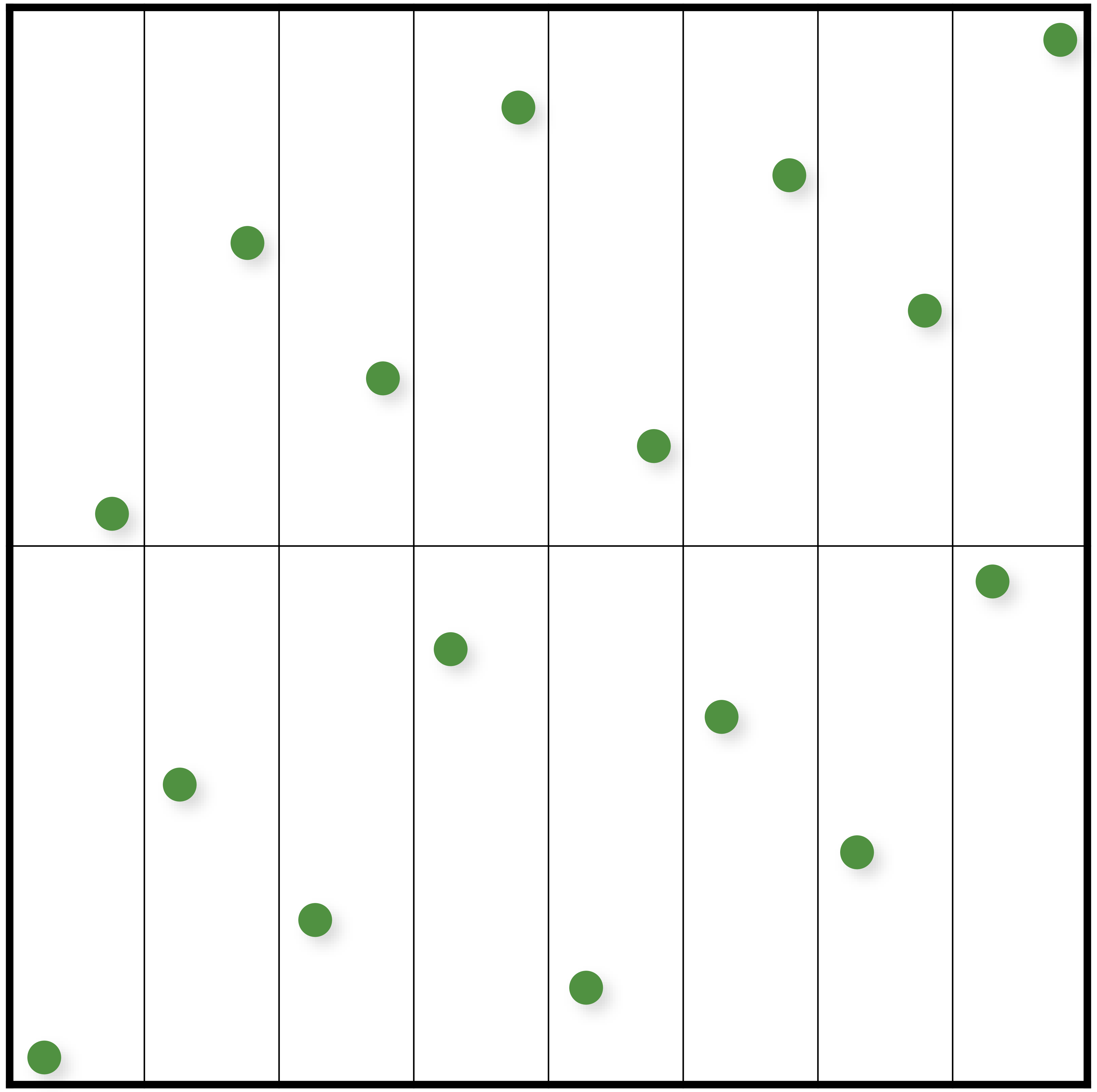
1 sample in each “elementary interval”

The Hammersley Sequence



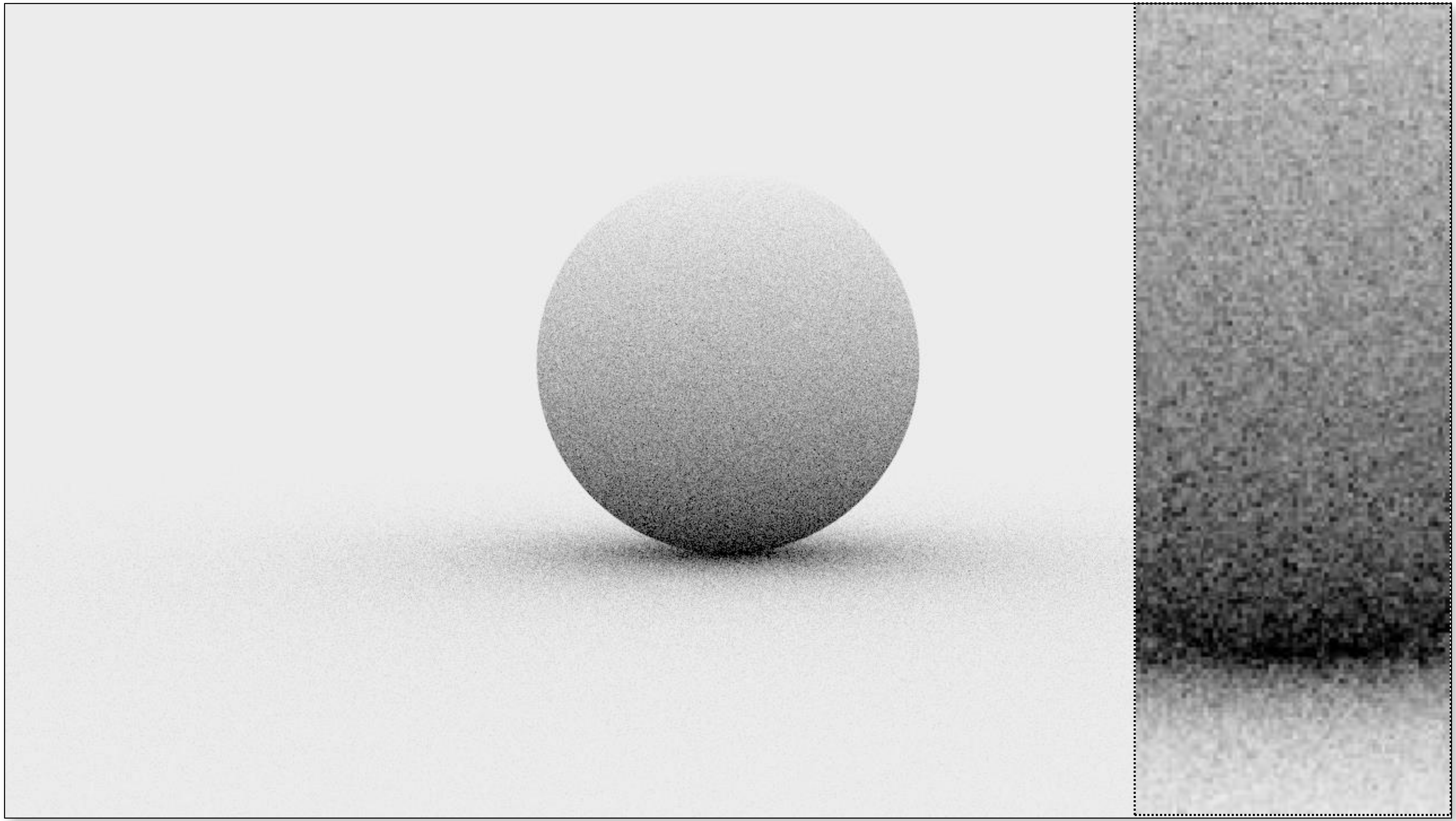
1 sample in each “elementary interval”

The Hammersley Sequence

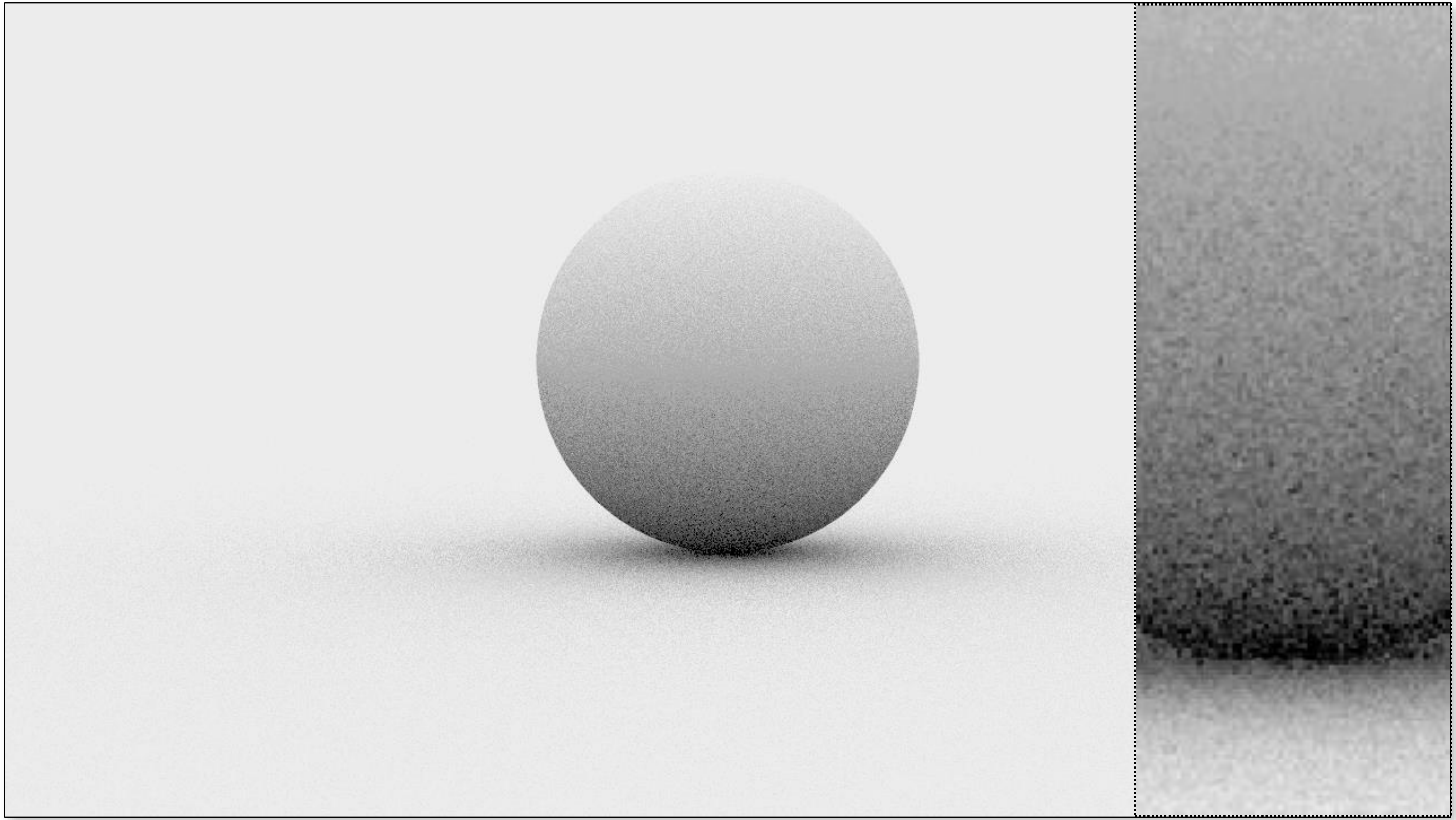


1 sample in each “elementary interval”

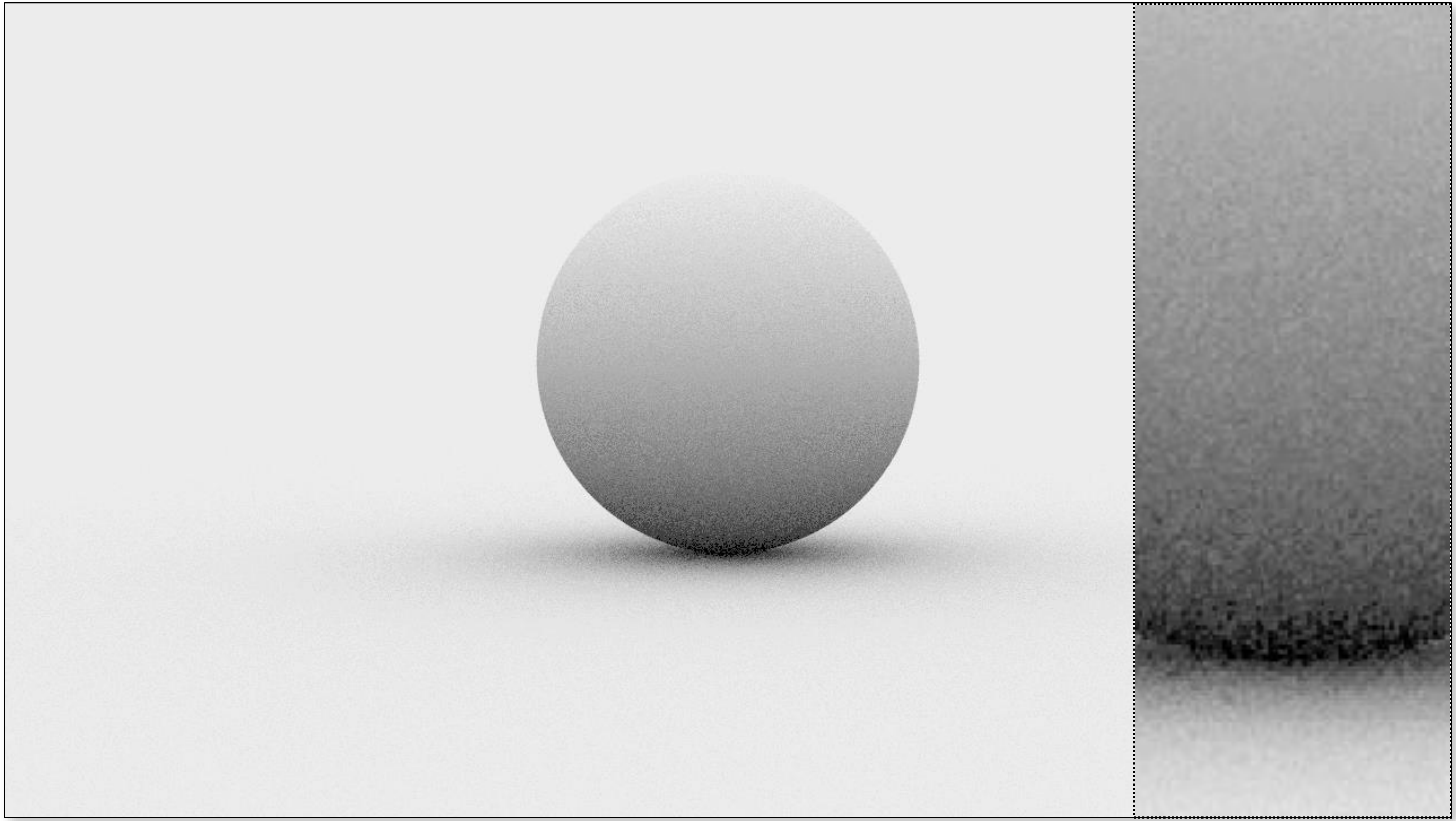
Monte Carlo (16 random samples)



Monte Carlo (16 jittered samples)



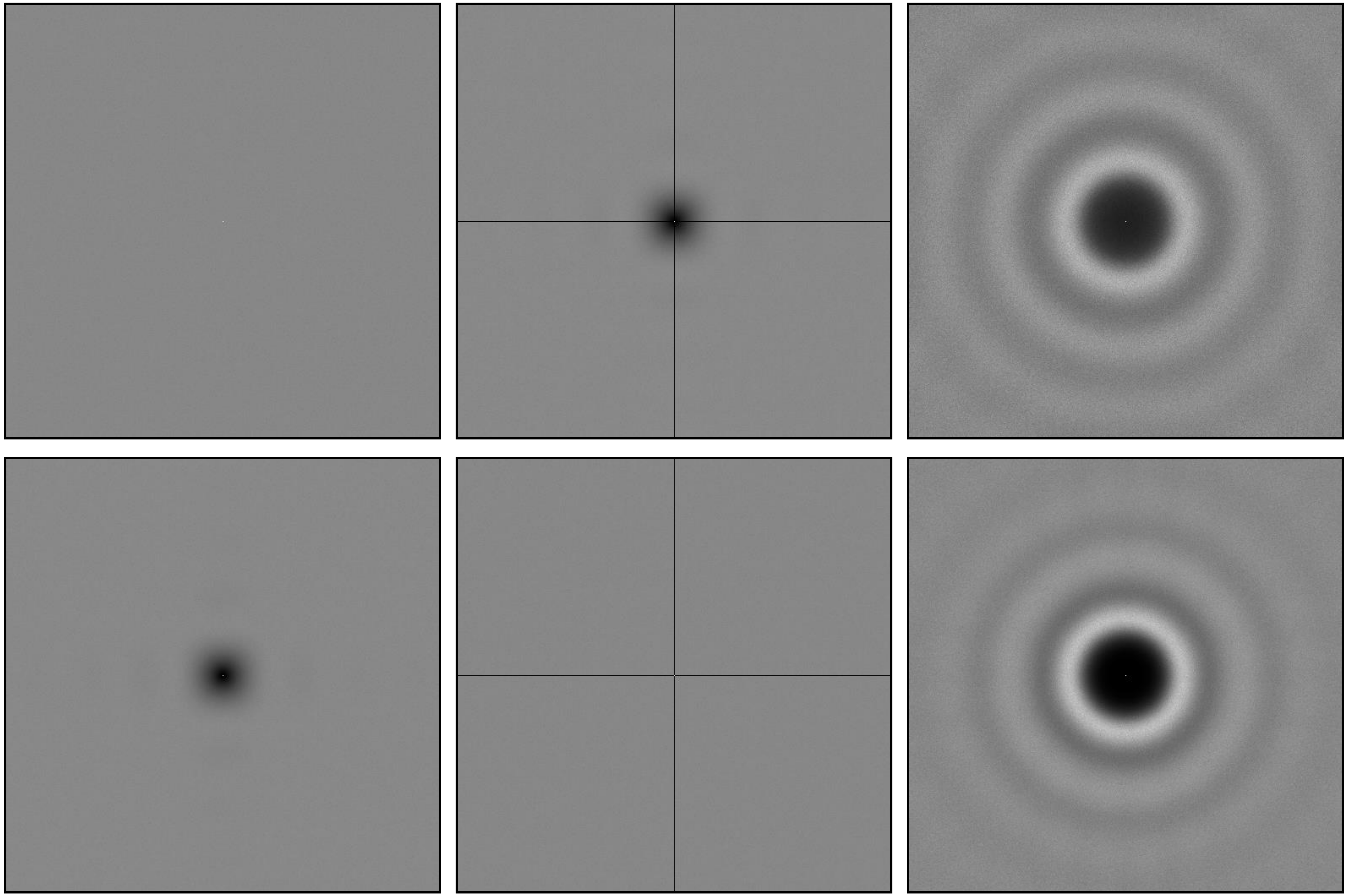
Scrambled Low-Discrepancy Sampling



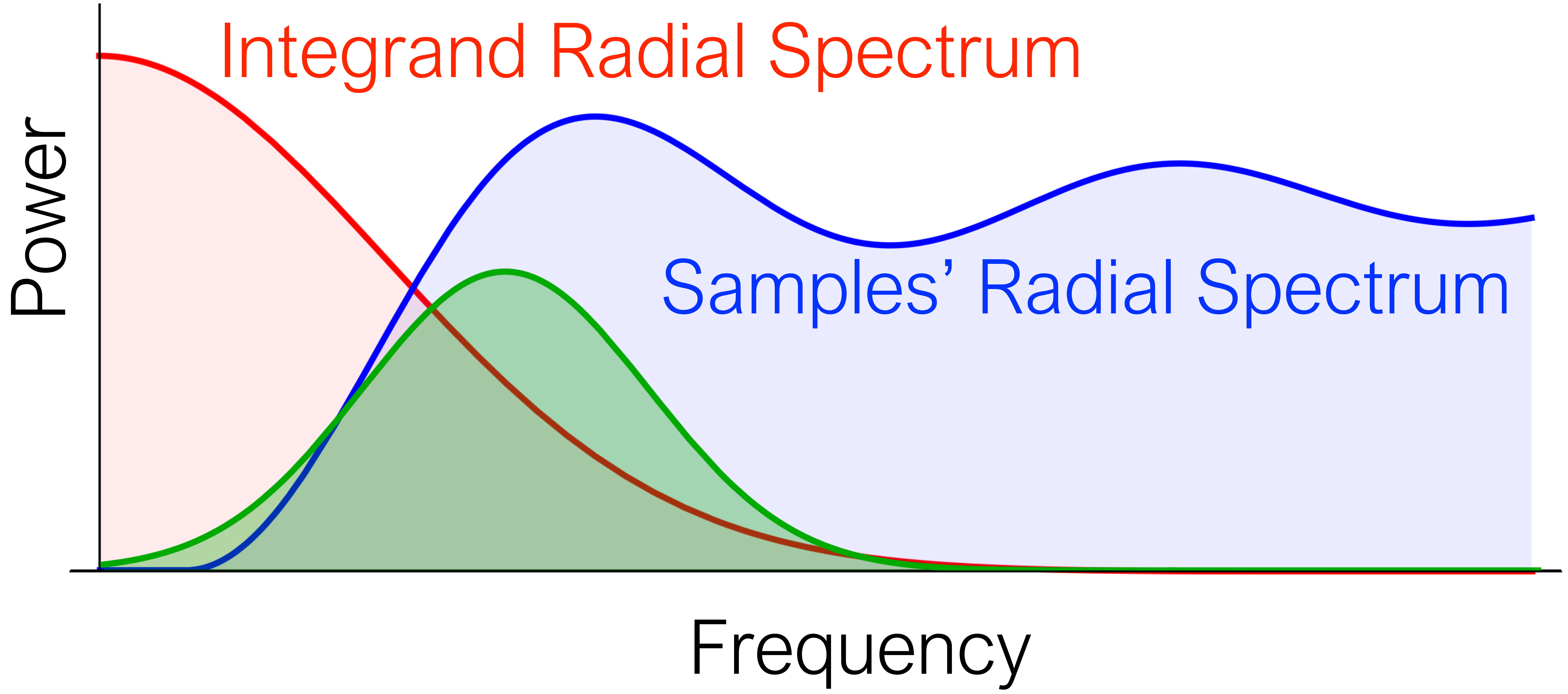
More info on QMC in Rendering

S. Premoze, A. Keller, and M. Raab.
Advanced (Quasi-) Monte Carlo Methods for Image Synthesis. In SIGGRAPH 2012 courses.

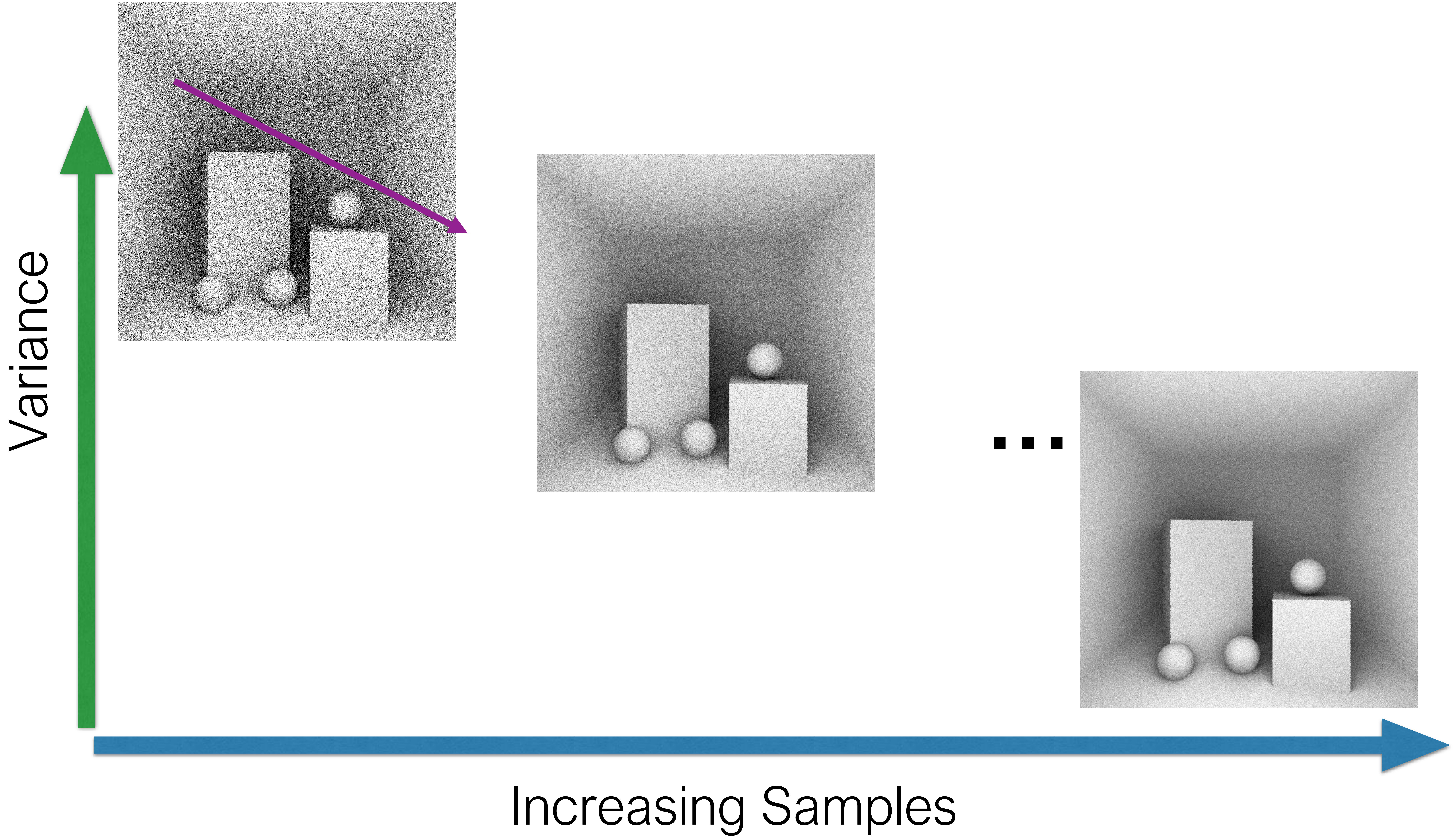
How can we predict error from these?



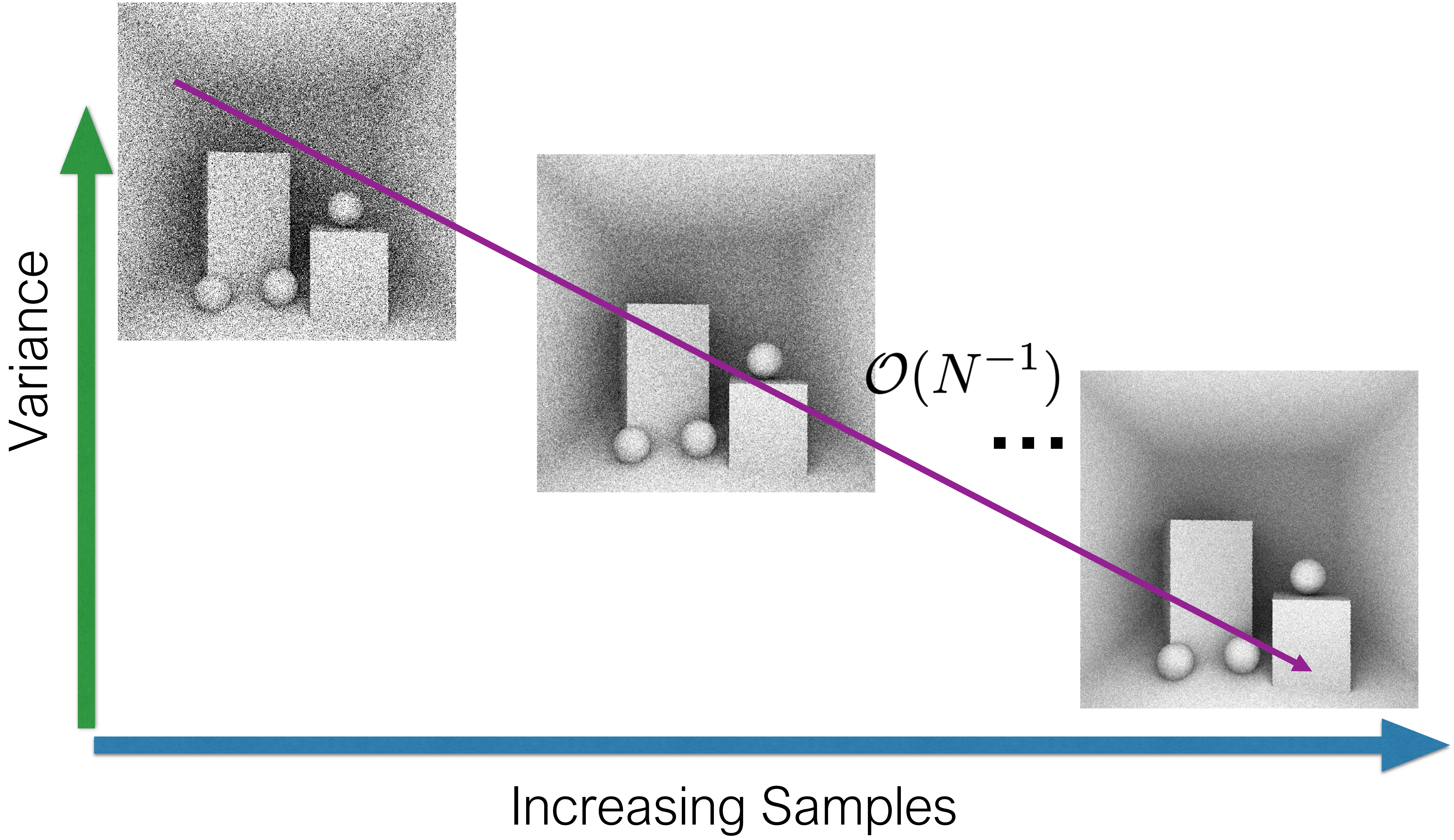
Part 2: Formal Treatment of MSE, Bias and Variance



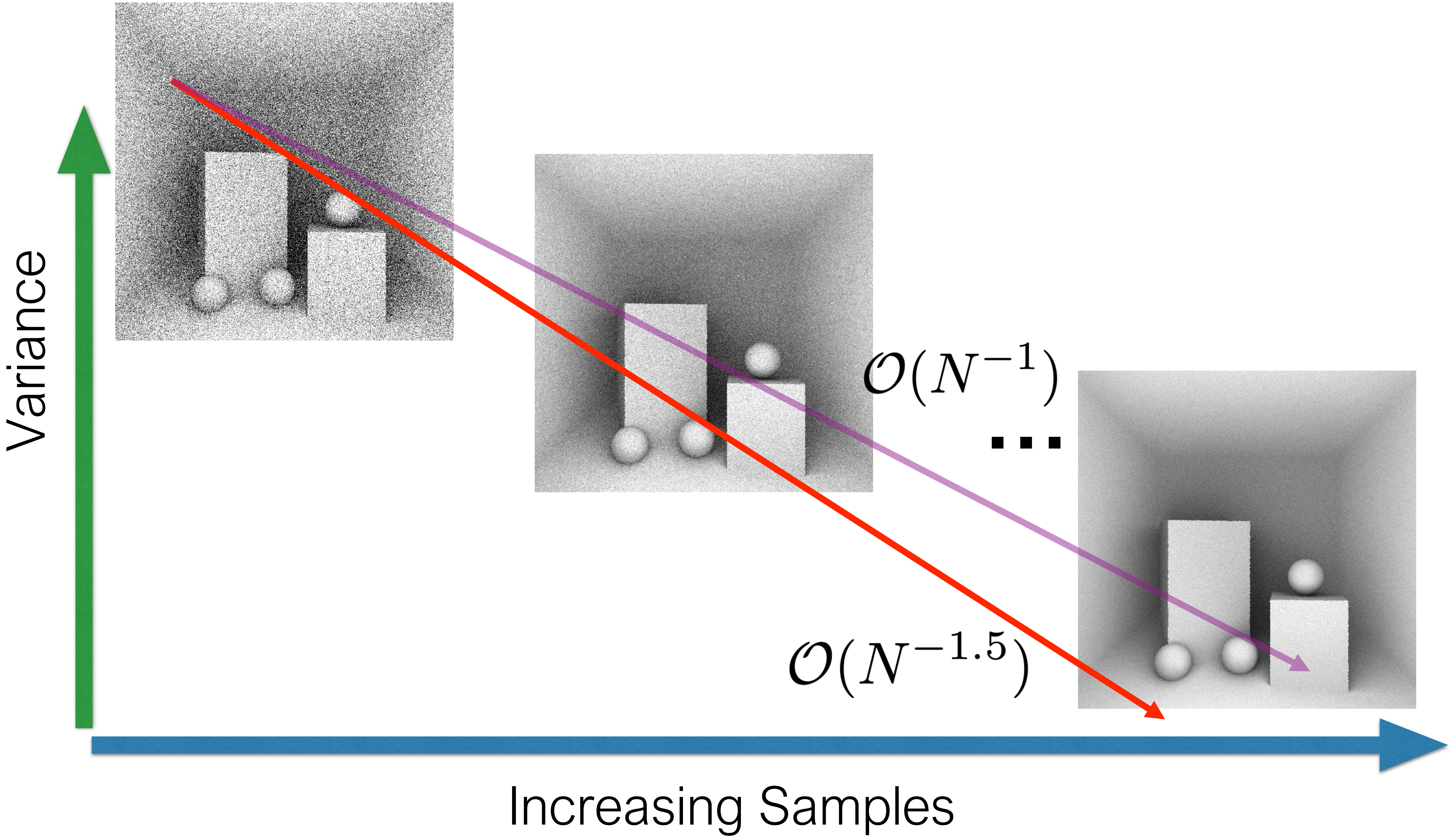
Convergence rate for Random Samples



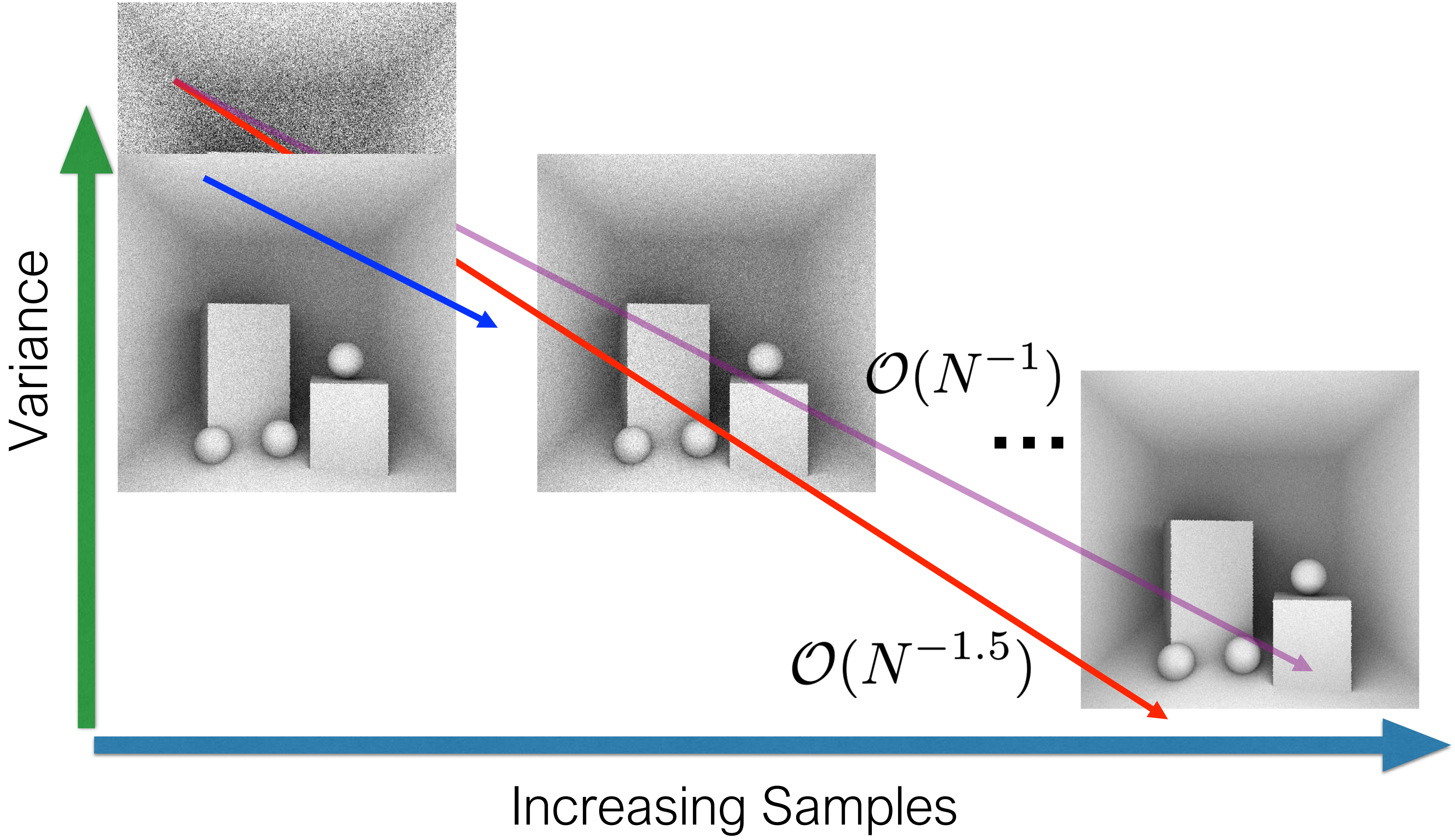
Convergence rate for Random Samples



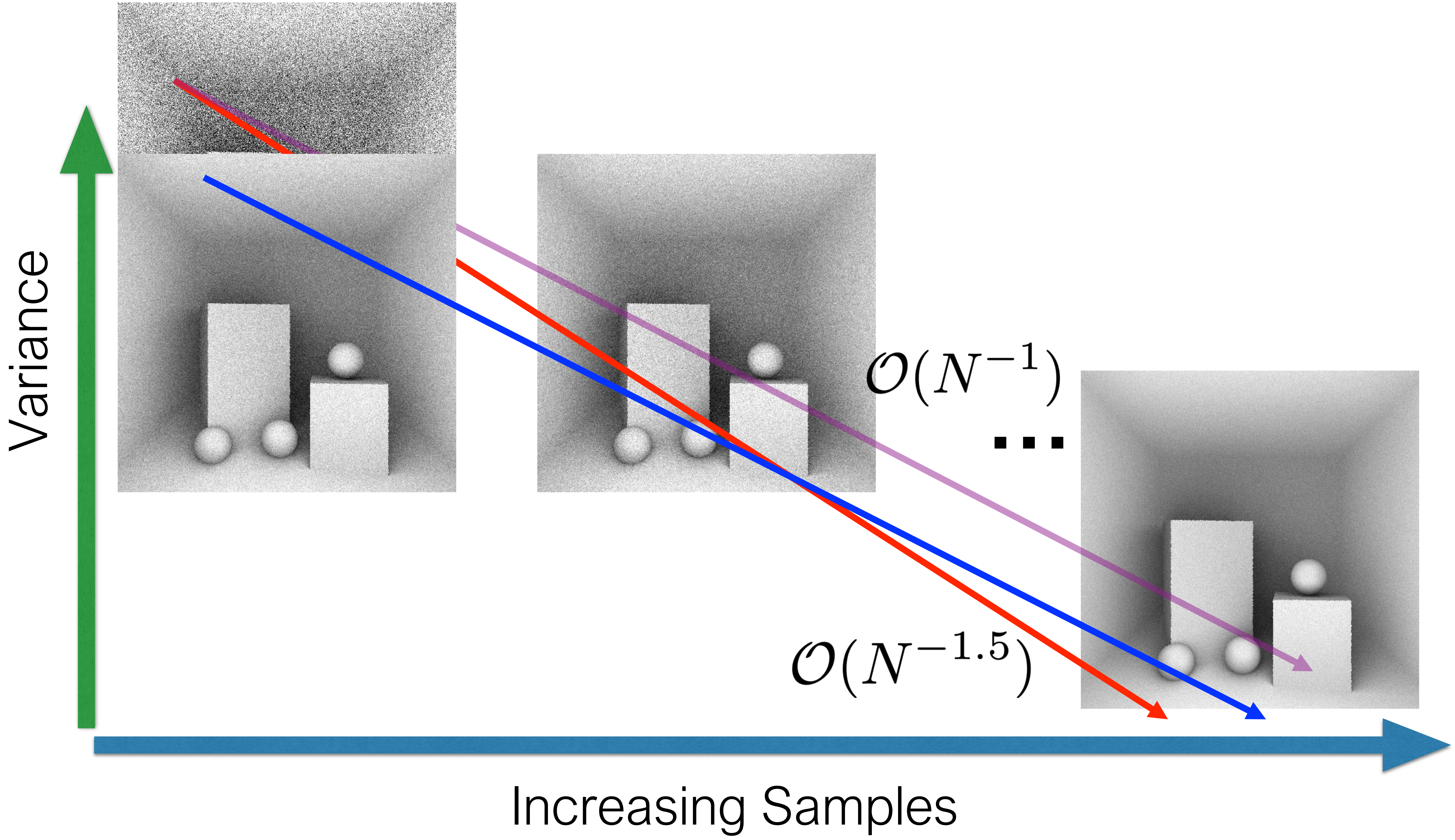
Convergence rate for Jittered Samples



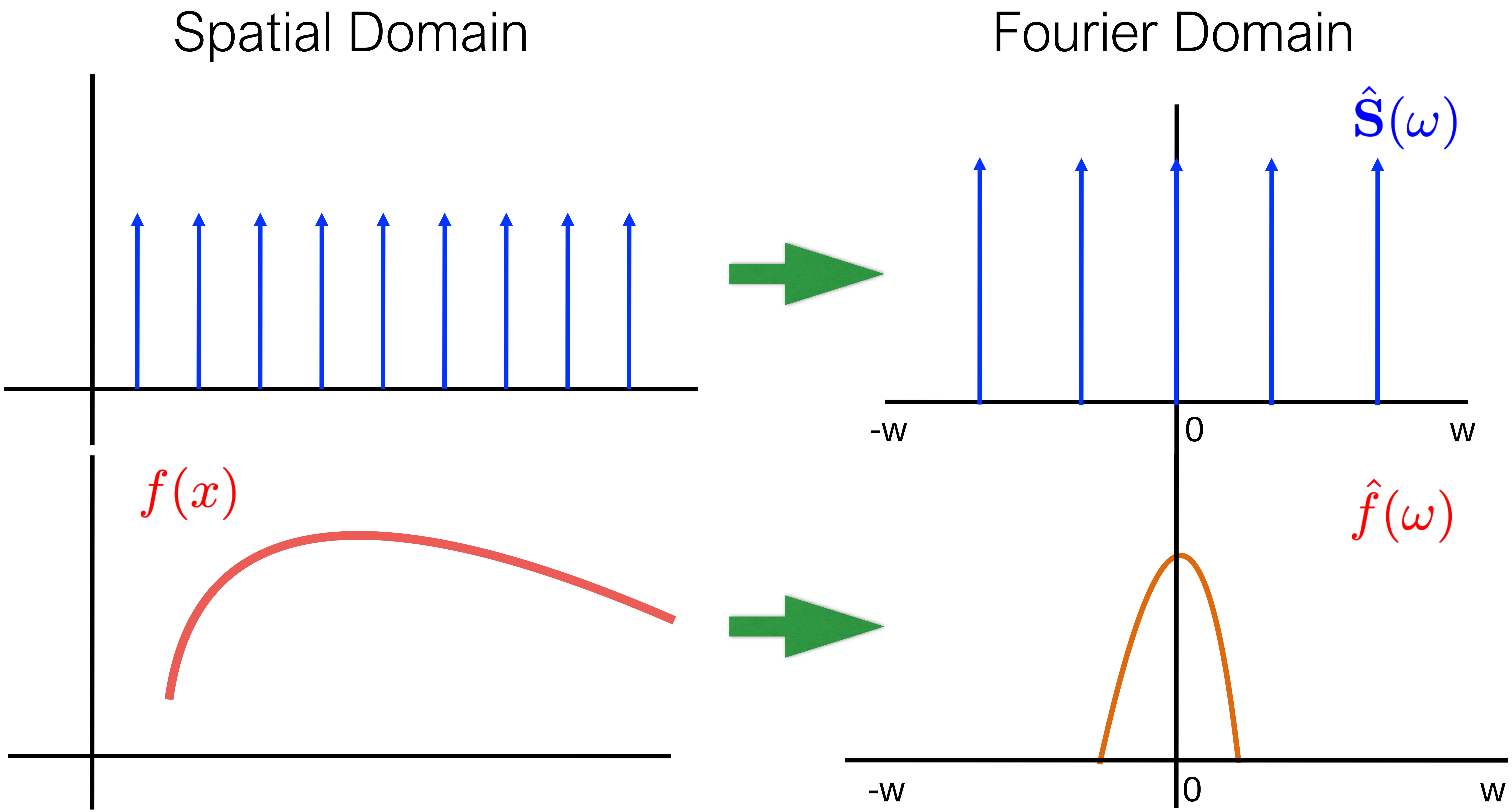
Convergence rate
Jittered vs Poisson Disk



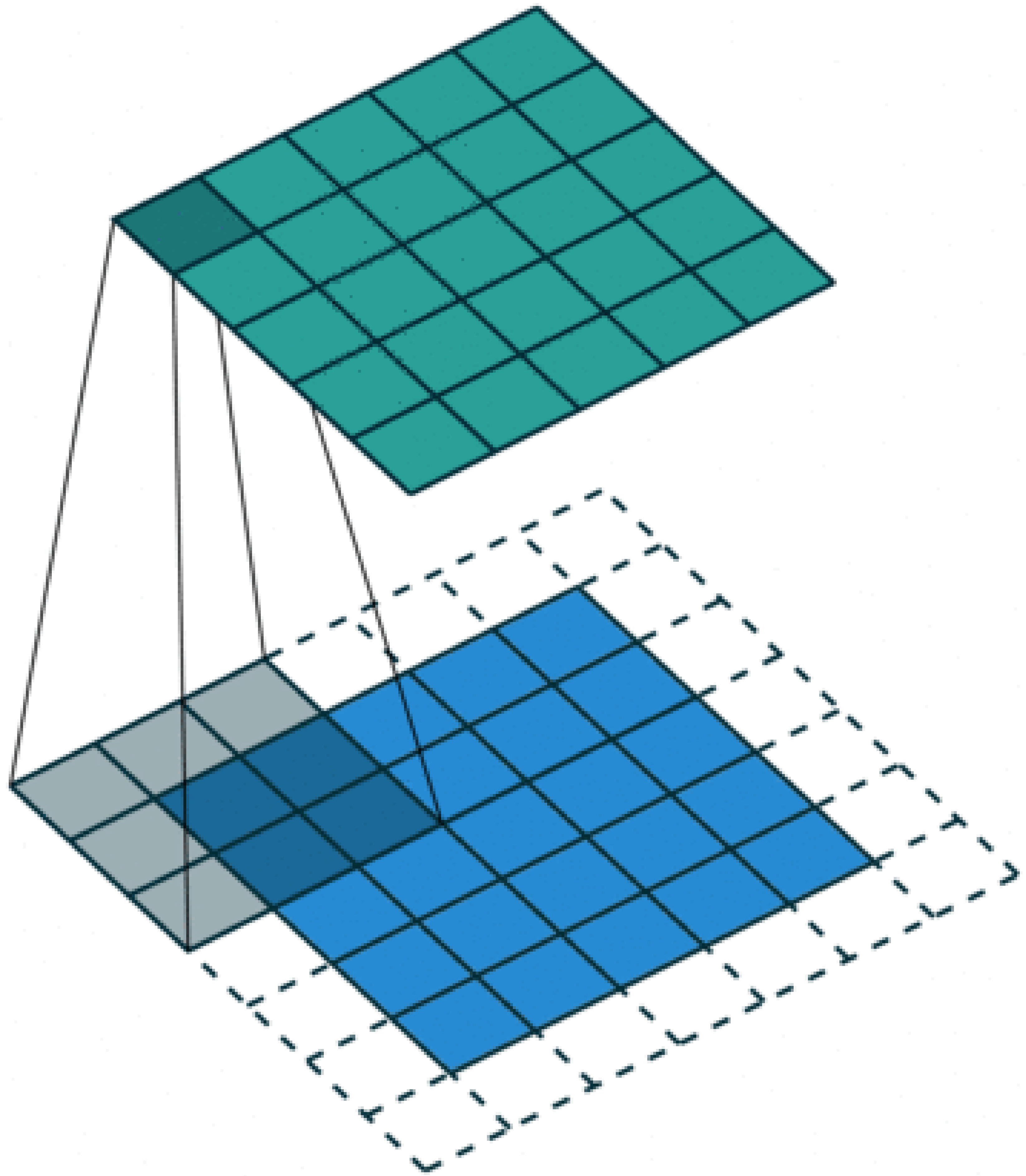
Convergence rate
Jittered vs Poisson Disk



Samples and function in Fourier Domain

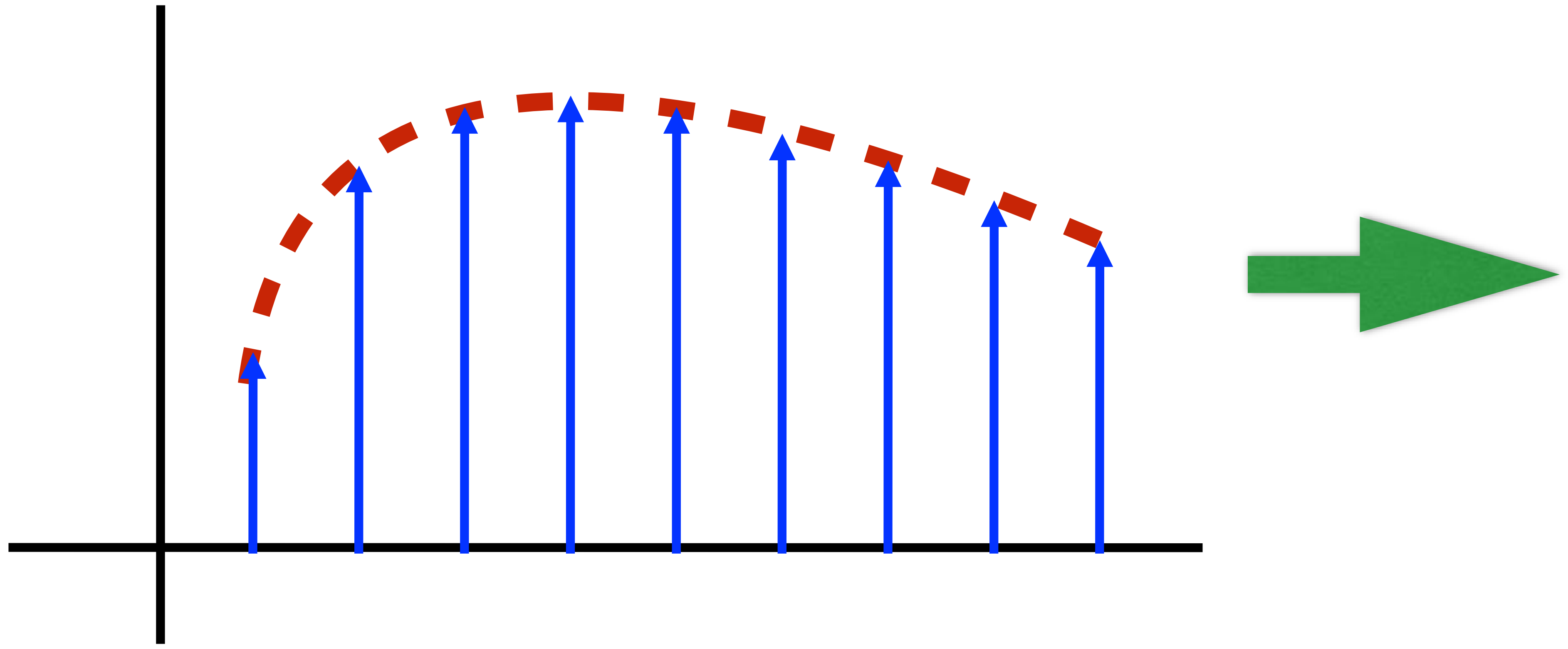


Convolution



Source: vdumoulin-github

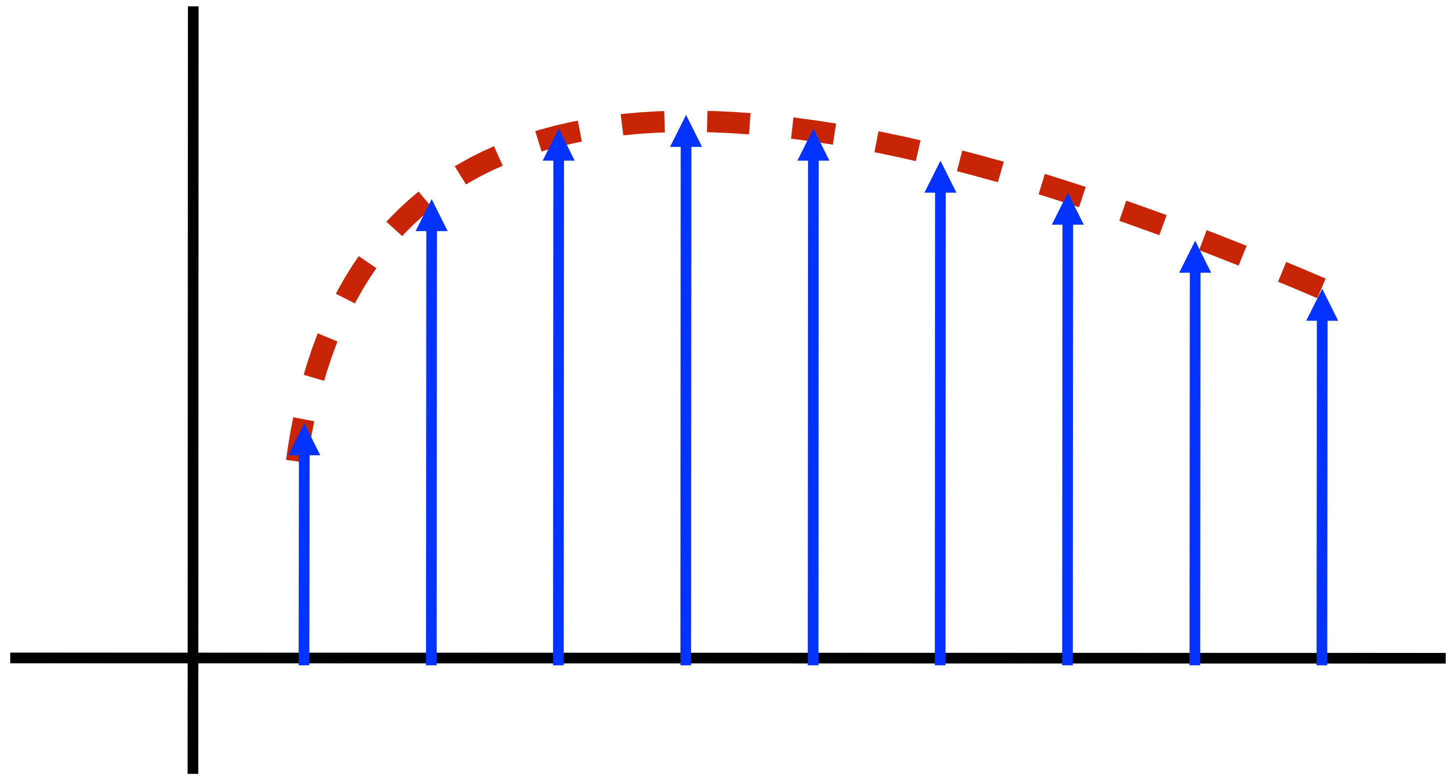
Sampling in Primal Domain is Convolution in Fourier Domain



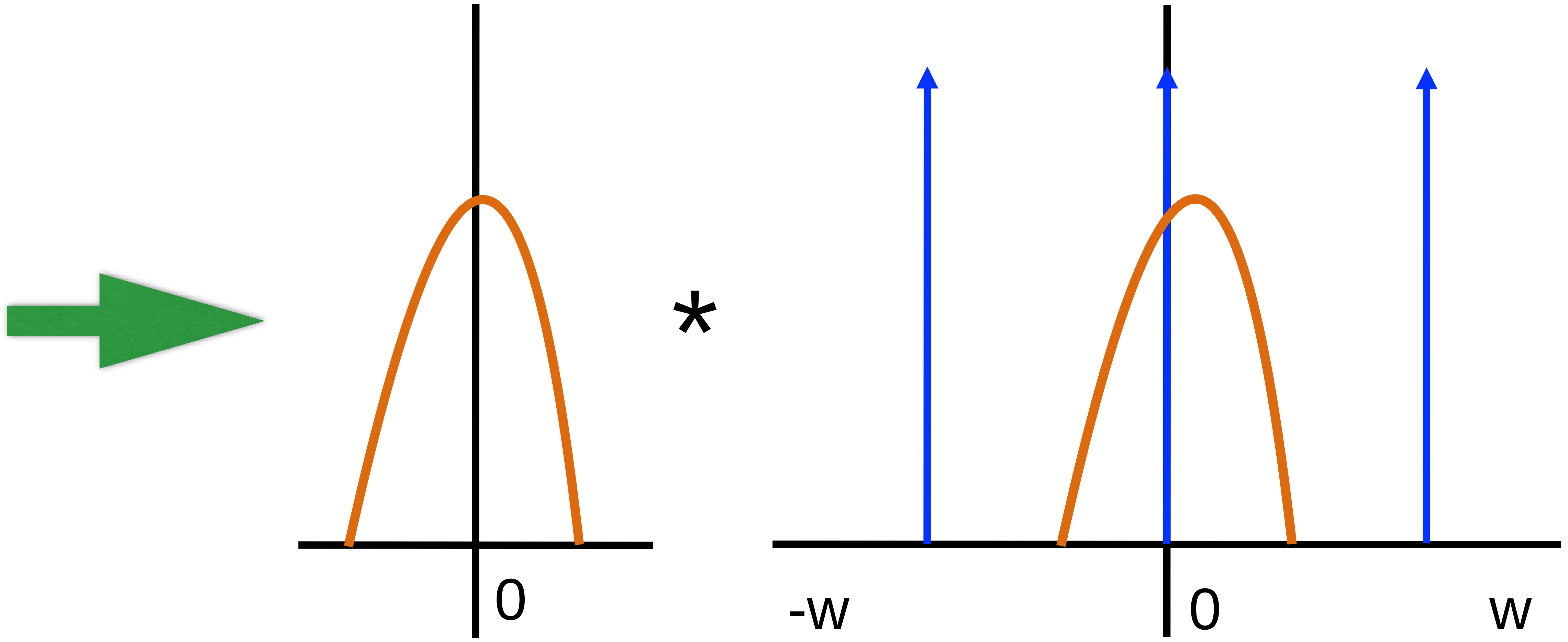
$f(x)\,\mathbf{S}(x)$

Fredo Durand [2011]

Sampling in Primal Domain is Convolution in Fourier Domain



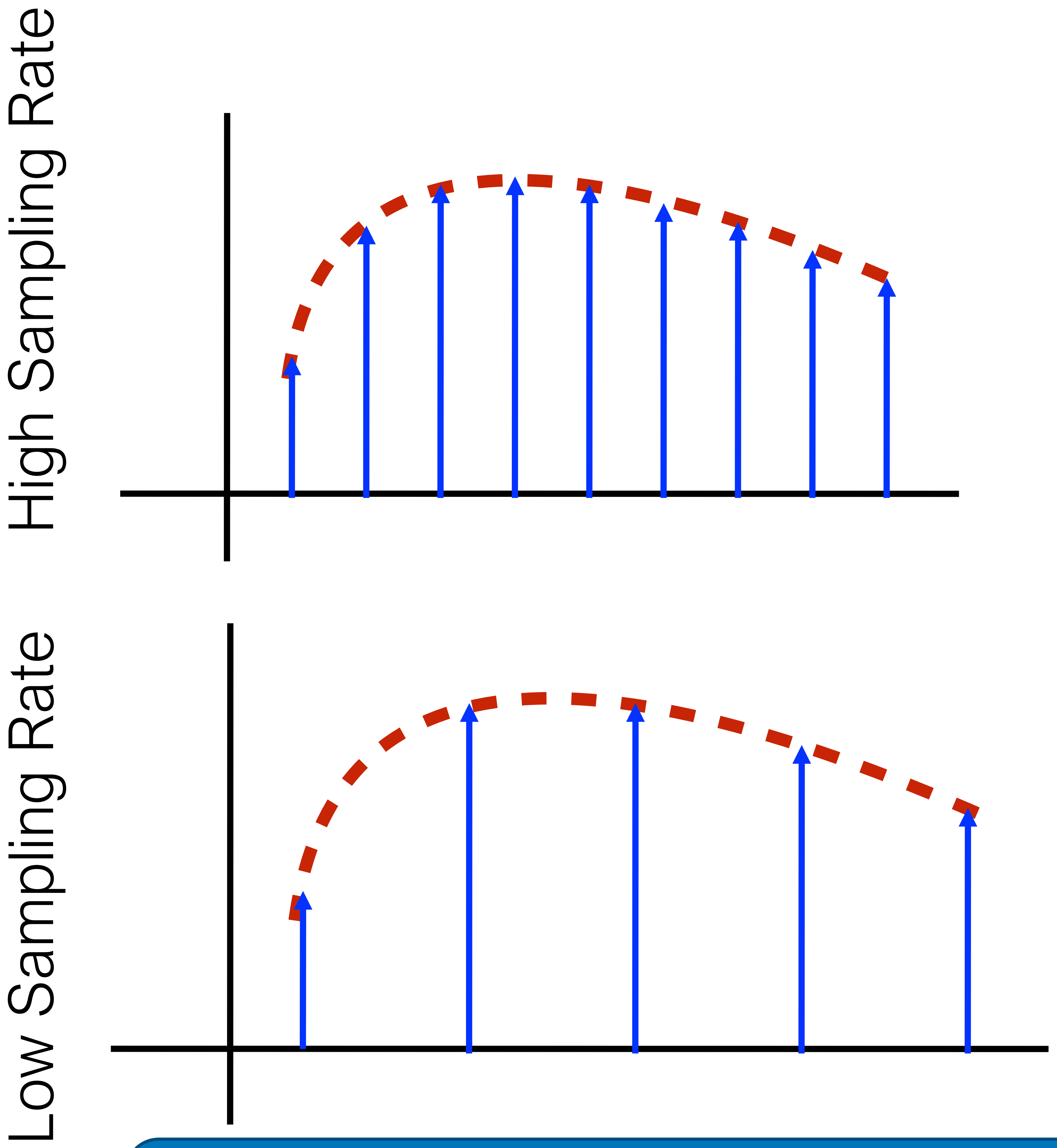
$f(x)\,\mathbf{S}(x)$

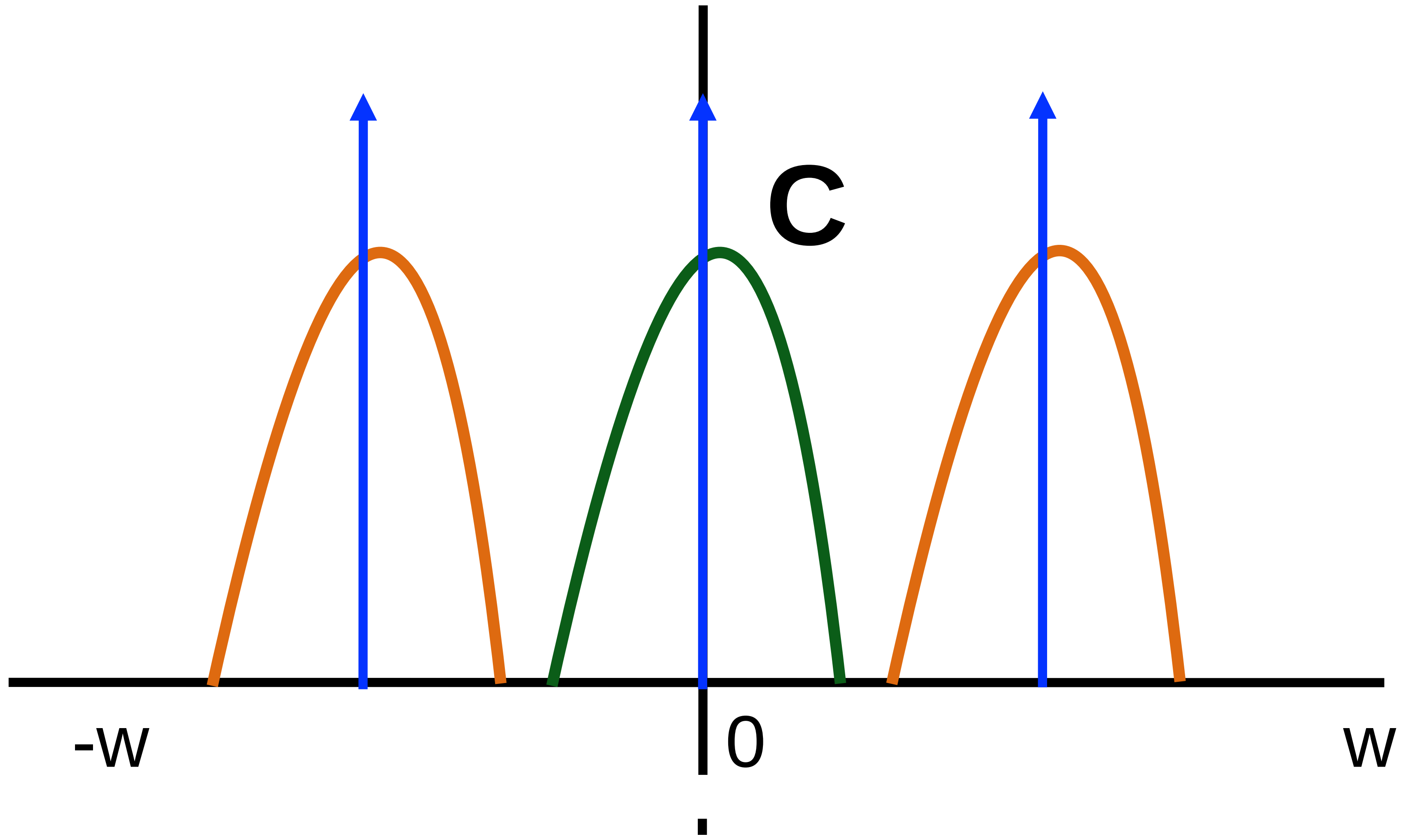


$\hat{f}(\omega) \otimes \hat{\mathbf{S}}(\omega)$

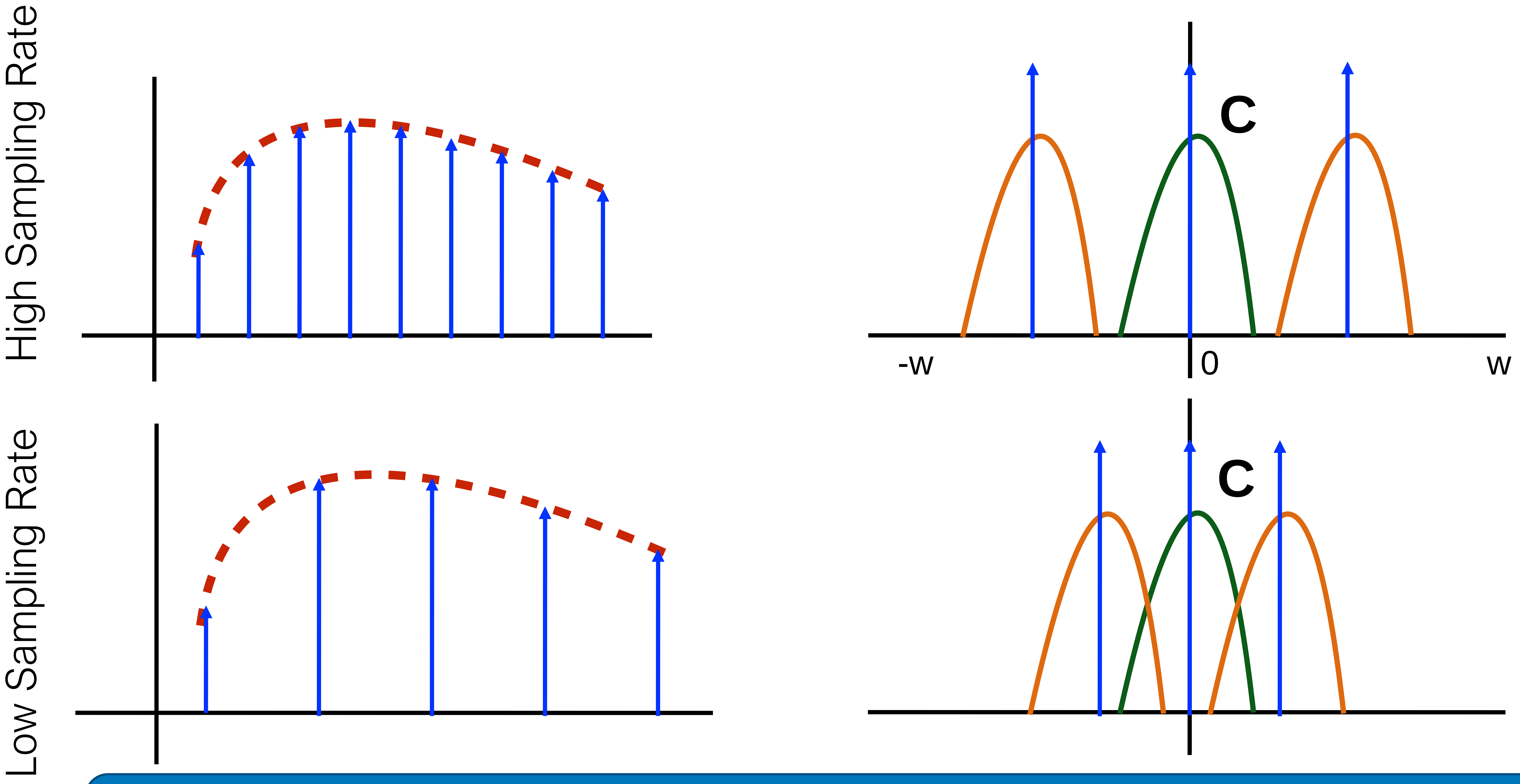
Fredo Durand [2011]

Aliasing in Reconstruction

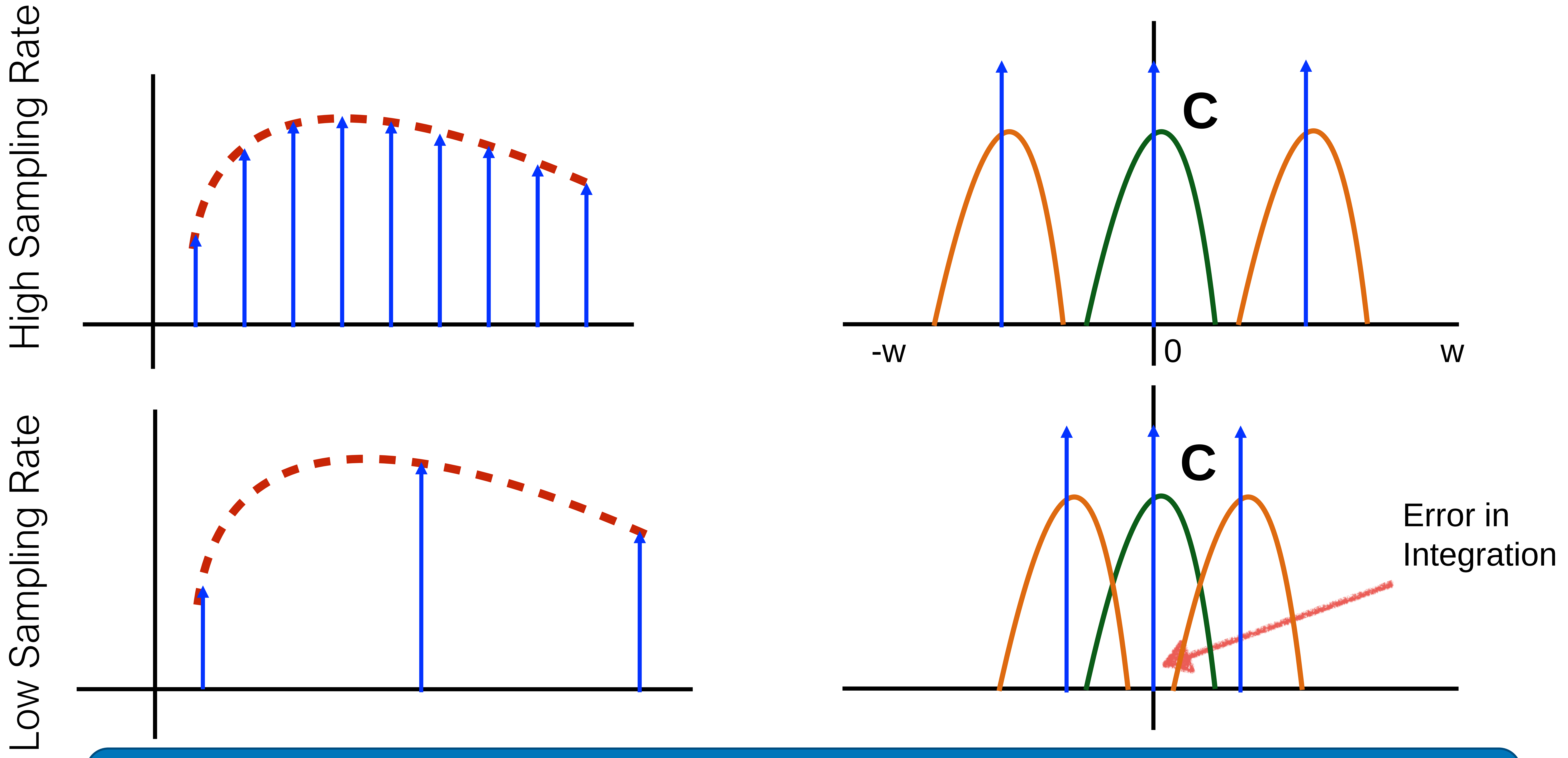




Aliasing in Reconstruction



Error in Monte Carlo Integration



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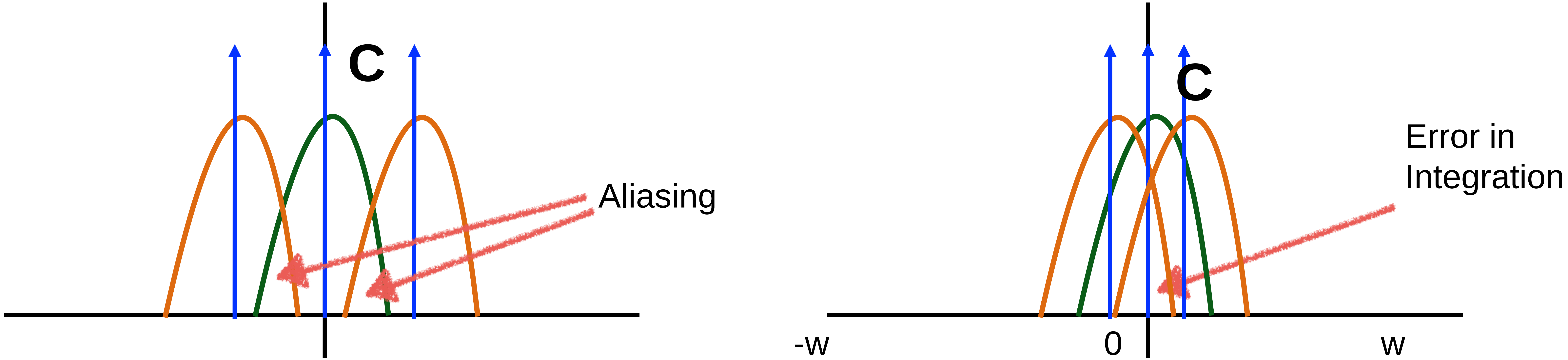
Aliasing (Reconstruction) vs. Error (Integration)

Fredo Durand [2011]
Belcour et al. [2013]



Integration in the Fourier Domain

Integration is the DC term in the Fourier Domain

Spatial Domain:

$$I = \int_D f(x)dx$$

Fourier Domain:

$$\hat{f}(0)$$

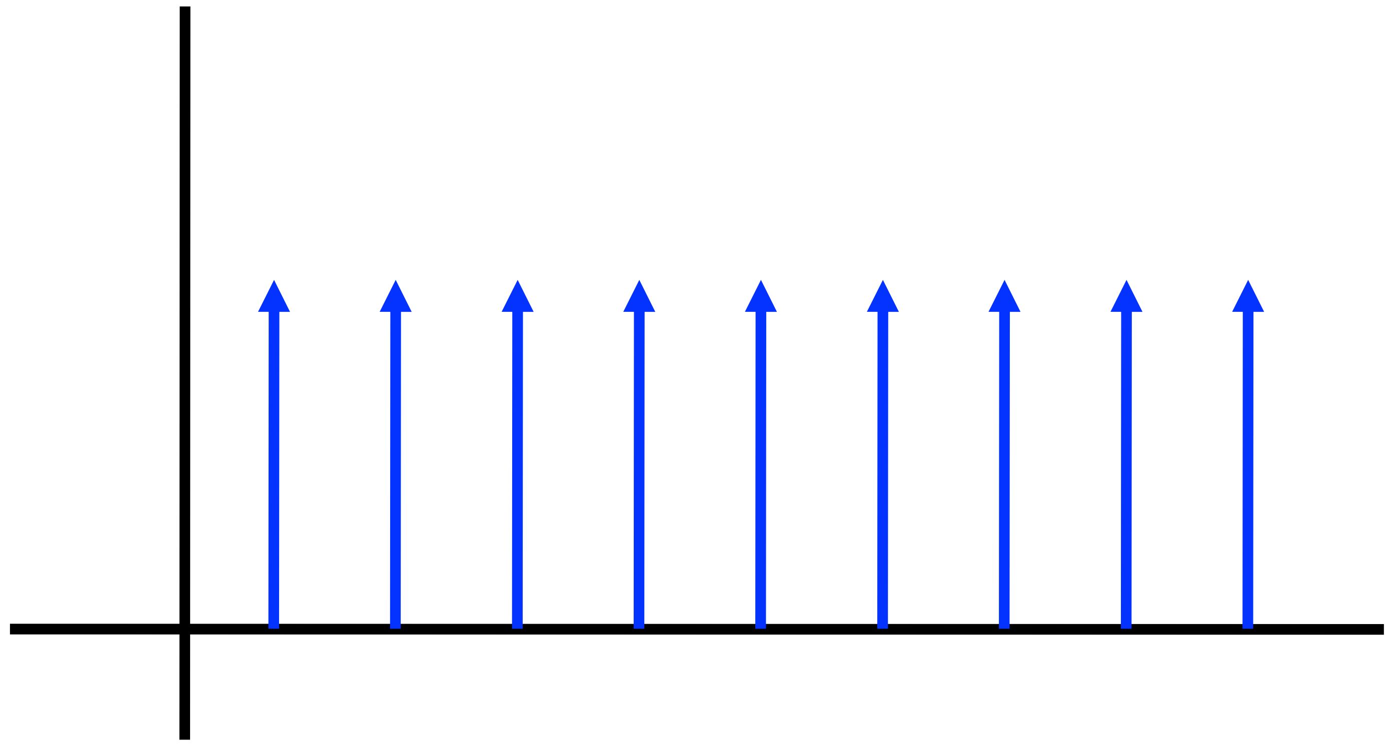
Monte Carlo Estimator in Spatial Domain

$$\tilde{\mu}_N = \int_D f(x)\mathbf{S}(x)dx$$

Monte Carlo Estimator in Spatial Domain

$$\tilde{\mu}_N = \int_D f(x)\mathbf{S}(x)dx$$

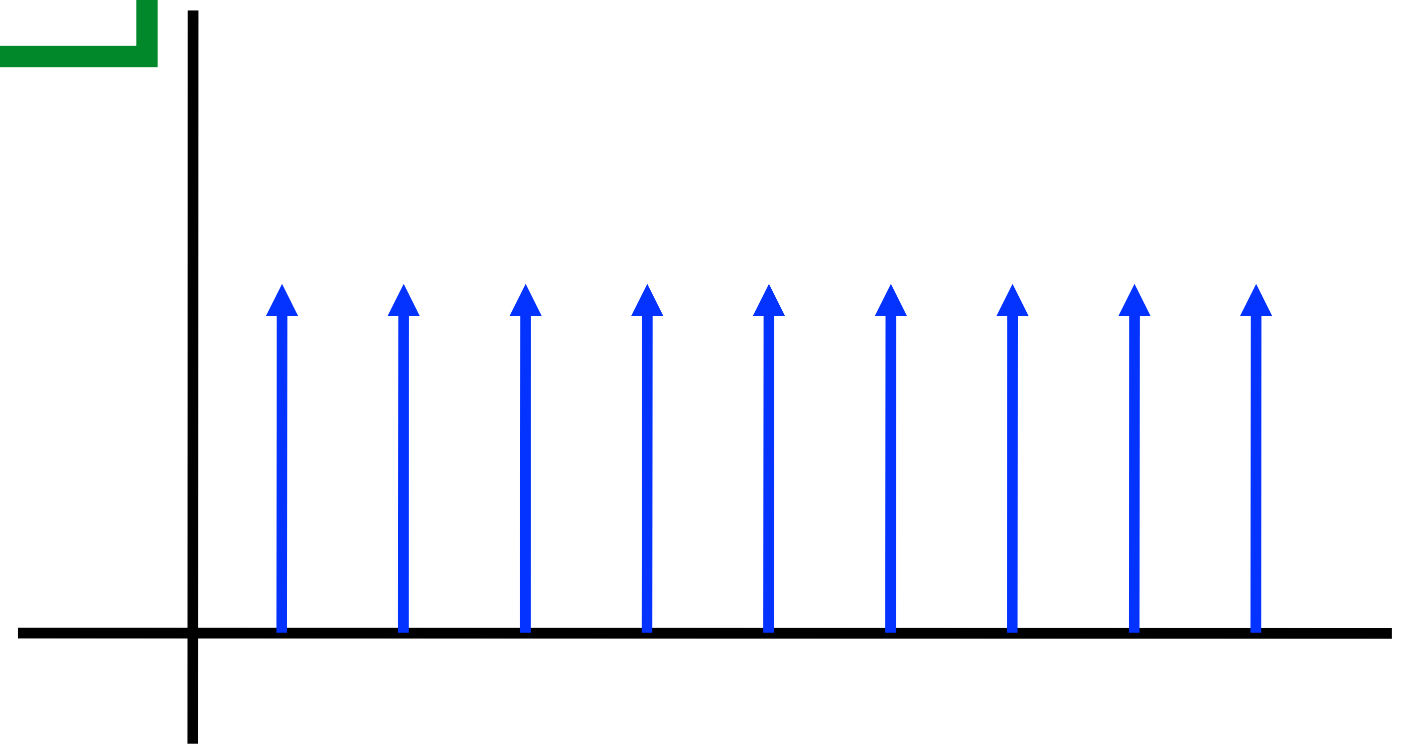
$$\mathbf{S}(x) = \frac{1}{N}\sum_{k=1}^{N}\delta(x - x_k)$$



Monte Carlo Estimator in Fourier Domain

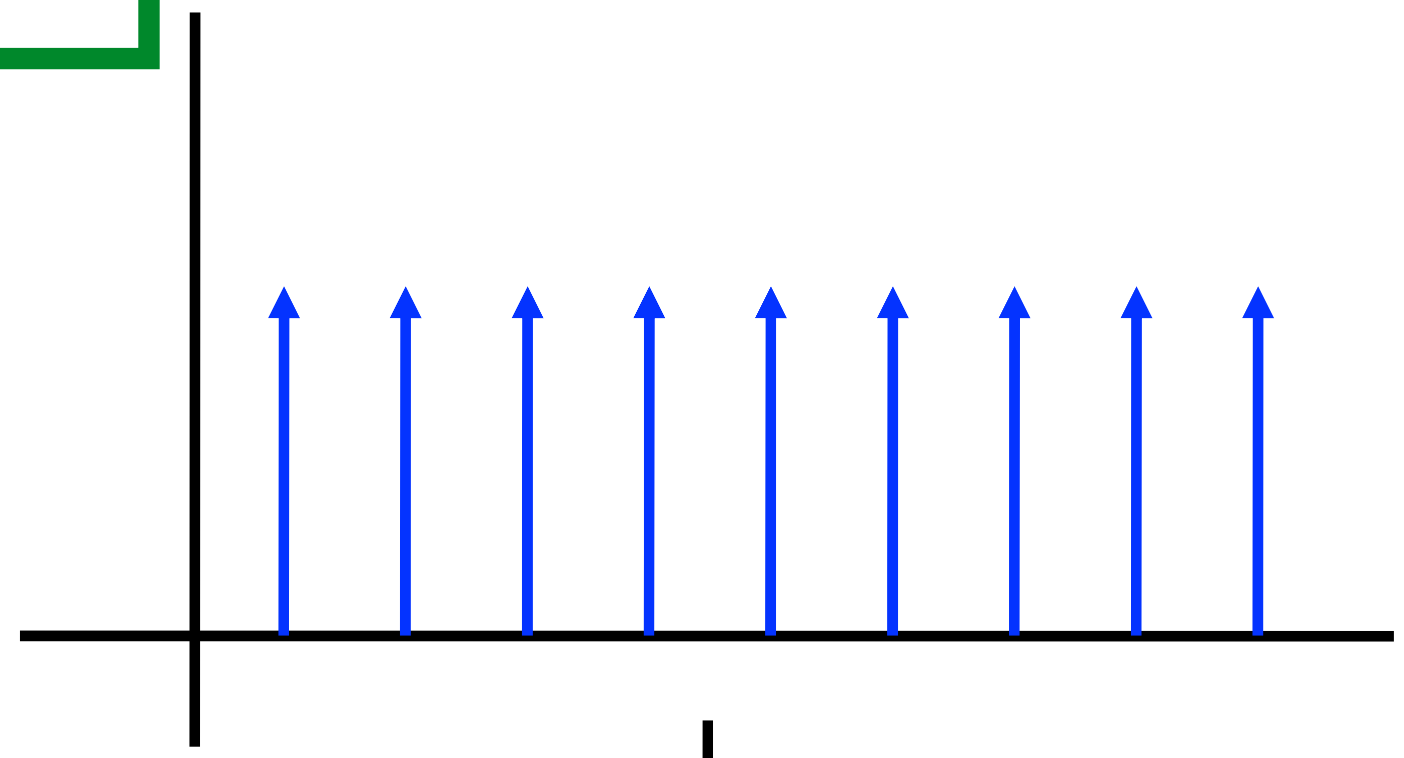
$$\tilde{\mu}_N = \int_D f(x)\mathbf{S}(x)dx = \int_\Omega \hat{f}^*(\omega)\hat{\mathbf{S}}(\omega)d\omega$$

$$\mathbf{S}(x) = \frac{1}{N}\sum_{k=1}^{N}\delta(x - x_k)$$



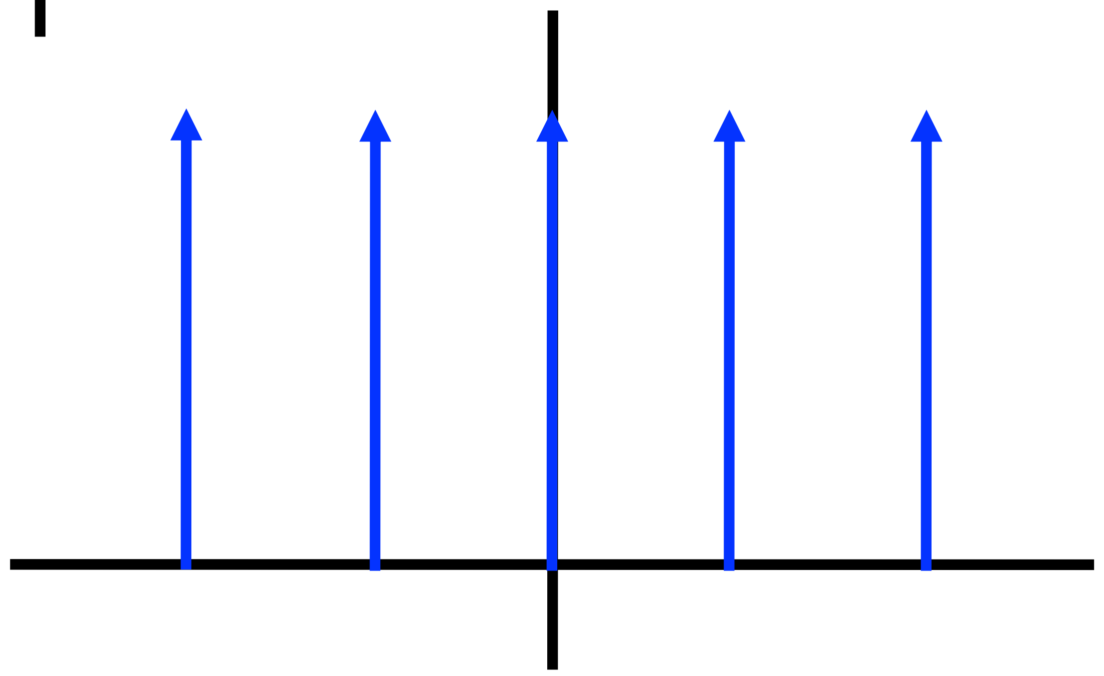
Monte Carlo Estimator in Fourier Domain

$$\tilde{\mu}_N = \int_D f(x)\mathbf{S}(x)dx = \int_\Omega \hat{f}^*(\omega)\hat{\mathbf{S}}(\omega)d\omega$$



$$\mathbf{S}(x) = \frac{1}{N}\sum_{k=1}^{N}\delta(x - x_k)$$

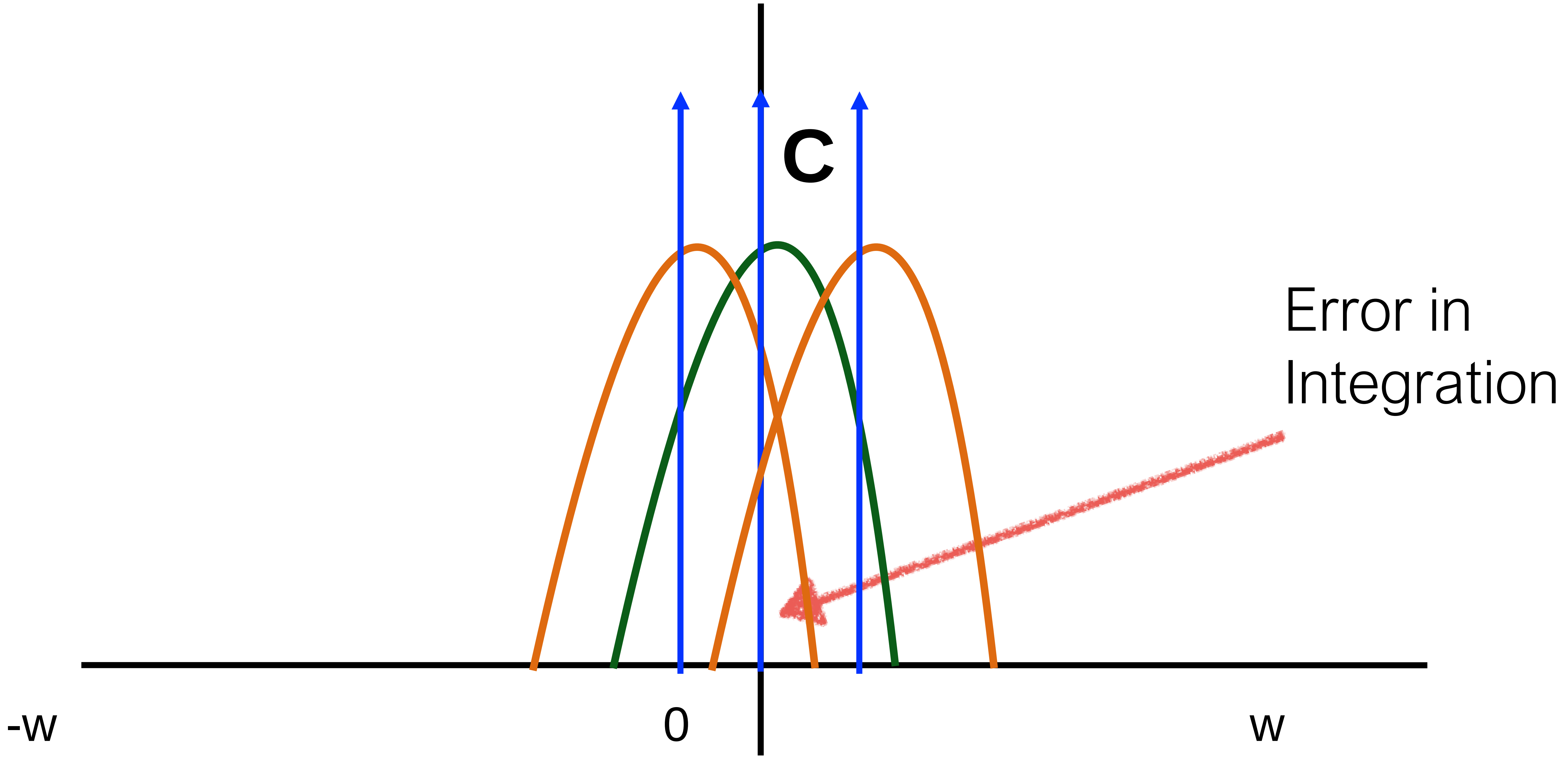
$$\hat{\mathbf{S}}(\omega) = \frac{1}{N}\sum_{k=1}^{N}e^{-i2\pi\omega x_k}$$



How to Formulate Error in Fourier Domain ?

$$I = \hat{f}(0)$$

$$\tilde{\mu}_N = \int_\Omega \hat{f}^*(\omega)\hat{\mathbf{S}}(\omega)d\omega$$



Fredo Durand [2011]

Error in Spatial Domain

$$I = \hat{f}(0)$$

$$\tilde{\mu}_N = \int_\Omega \hat{f}^*(\omega)\hat{\mathbf{S}}(\omega)d\omega$$

True Integral

$$I - \tilde{\mu}_N = \int_D f(x)dx - \int_D f(x)\mathbf{S}(x)dx$$

Monte Carlo Estimator

Error in Spatial Domain

$$I = \hat{f}(0)$$

$$\tilde{\mu}_N = \int_\Omega \hat{f}^*(\omega)\hat{\mathbf{S}}(\omega)d\omega$$

$$I - \tilde{\mu}_N = \int_D f(x)dx - \int_D f(x)\mathbf{S}(x)dx$$

Error in Fourier Domain

$$I = \hat{f}(0)$$

$$\tilde{\mu}_N = \int_\Omega \hat{f}^*(\omega)\hat{\mathbf{S}}(\omega)d\omega$$

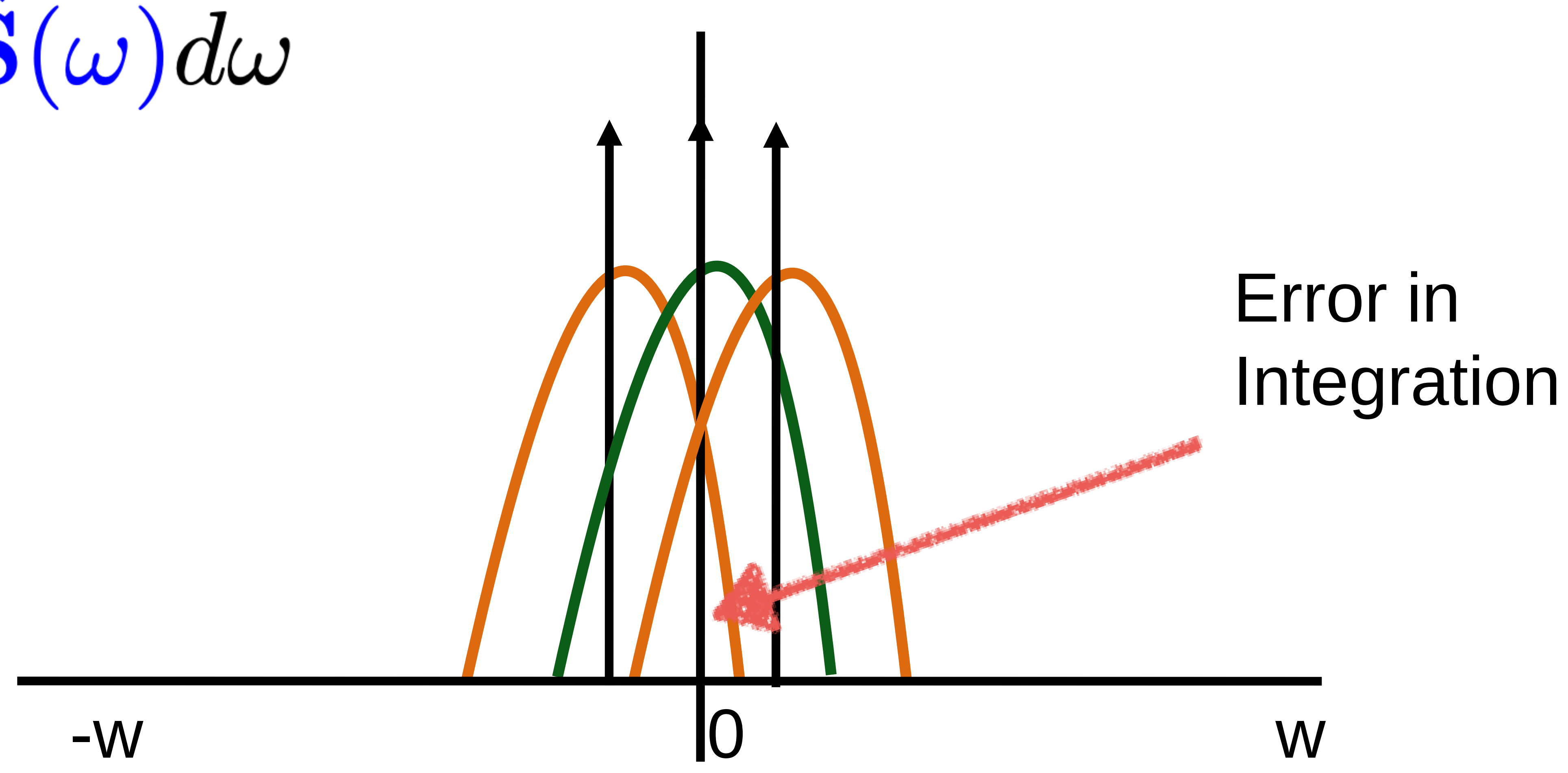
$$I - \tilde{\mu}_N = \int_D f(x)dx - \int_D f(x)\mathbf{S}(x)dx$$

$$I - \tilde{\mu}_N = \hat{f}(0) - \int_\Omega \hat{f}^*(\omega)\hat{\mathbf{S}}(\omega)d\omega$$

Fredo Durand [2011]

Error in Fourier Domain

$$I - \tilde{\mu}_N = \hat{f}(0) - \int_{\Omega} \hat{f}^*(\omega)\hat{\mathbf{S}}(\omega)d\omega$$



Fredo Durand [2011]

$$\text{Error} = \text{Bias}^2 + \text{Variance}$$

Properties of Error

- Bias:
- Variance:

Subr and Kautz [2013]

Bias in the Monte Carlo Estimator

Bias in Fourier Domain

Error: $$I - \tilde{\mu}_N = \hat{f}(0) - \int_{\Omega} \hat{f}^*(\omega)\hat{\mathbf{S}}(\omega)d\omega$$

Bias in Fourier Domain

Error: $$I - \tilde{\mu}_N = \hat{f}(0) - \int_{\Omega} \hat{f}^*(\omega)\hat{\mathbf{S}}(\omega)d\omega$$

Bias in Fourier Domain

$$\langle I - \tilde{\mu}_N \rangle = \hat{f}(0) - \left\langle \int_{\Omega} \hat{f}^*(\omega) \hat{\mathbf{S}}(\omega) d\omega \right\rangle$$

Bias in Fourier Domain

$$\langle I - \tilde{\mu}_N \rangle = \hat{f}(0) - \left\langle \int_\Omega \hat{f}^*(\omega)\hat{\mathbf{S}}(\omega) d\omega \right\rangle$$

$$\langle I - \tilde{\mu}_N \rangle = \hat{f}(0) - \int_\Omega \hat{f}^*(\omega) \left\langle \hat{\mathbf{S}}(\omega) \right\rangle d\omega$$

Subr and Kautz [2013]

Bias in Fourier Domain

$$\langle I - \tilde{\mu}_N \rangle = \hat{f}(0) - \boxed{\int_\Omega \hat{f}^*(\omega) \, \langle \hat{\mathbf{S}}(\omega) \rangle \, d\omega}$$

To obtain an unbiased estimator:

Subr and Kautz [2013]

$$\langle \hat{\mathbf{S}}(\omega) \rangle = 0$$

for frequencies other than zero

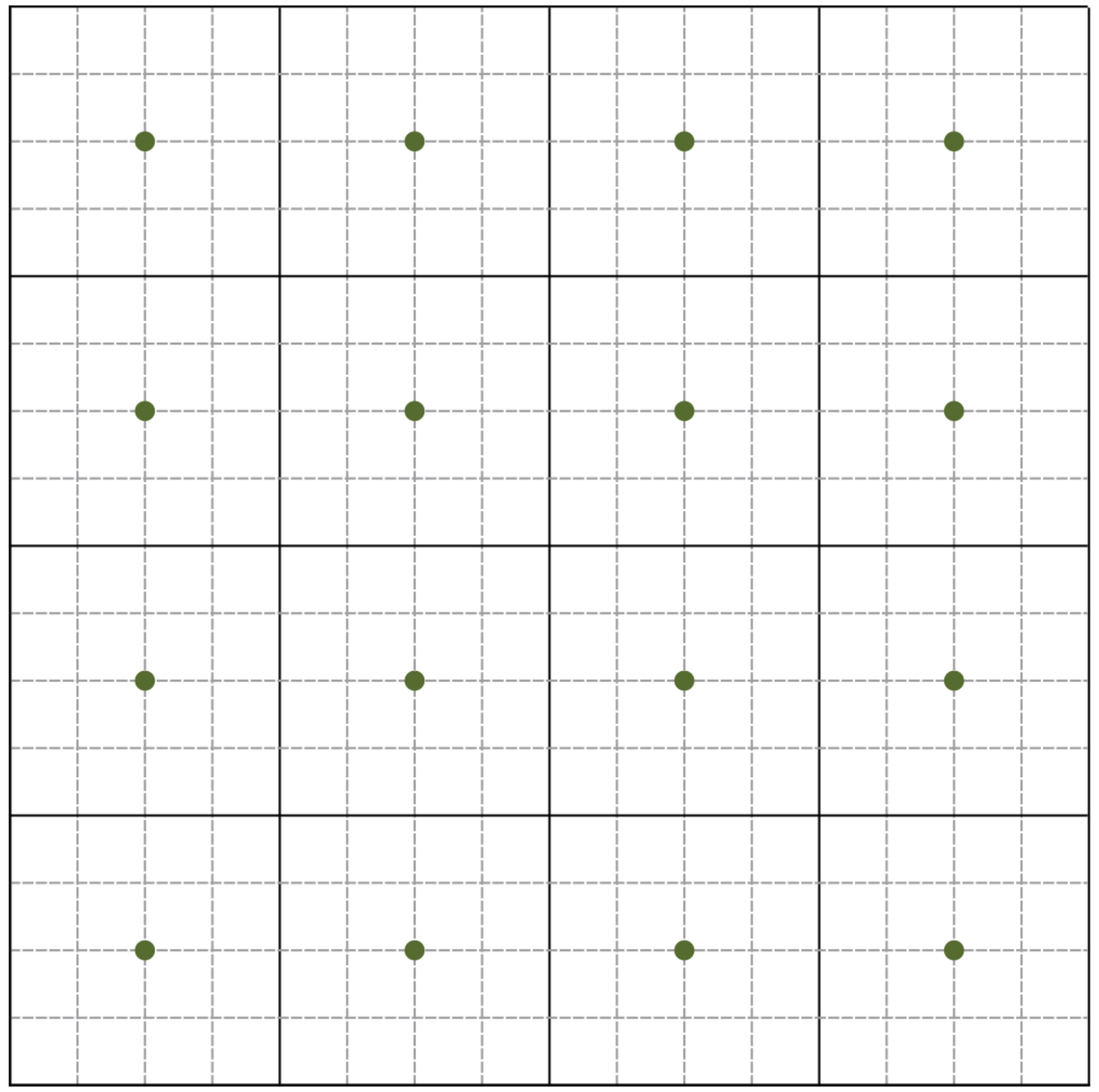
How to obtain $\langle \hat{\mathbf{S}}(\omega) \rangle = 0$?

Complex form in Amplitude and Phase

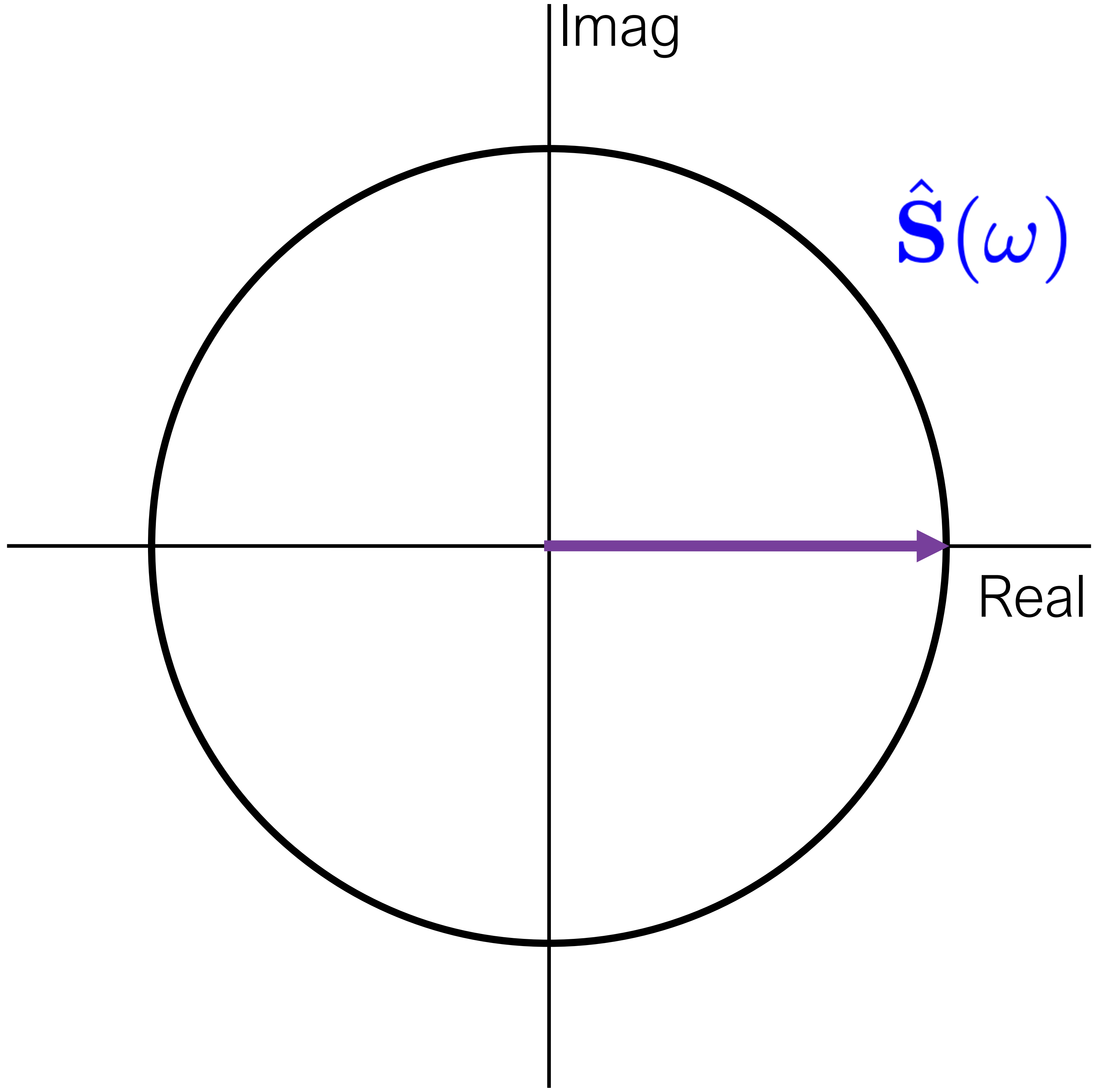
Amplitude | Phase

$$\langle \hat{\mathbf{S}}(\omega) \rangle = \boxed{|\langle \hat{\mathbf{S}}(\omega) \rangle|} \, e^{-\boxed{\Phi(\langle \hat{\mathbf{S}}(\omega) \rangle)}}$$

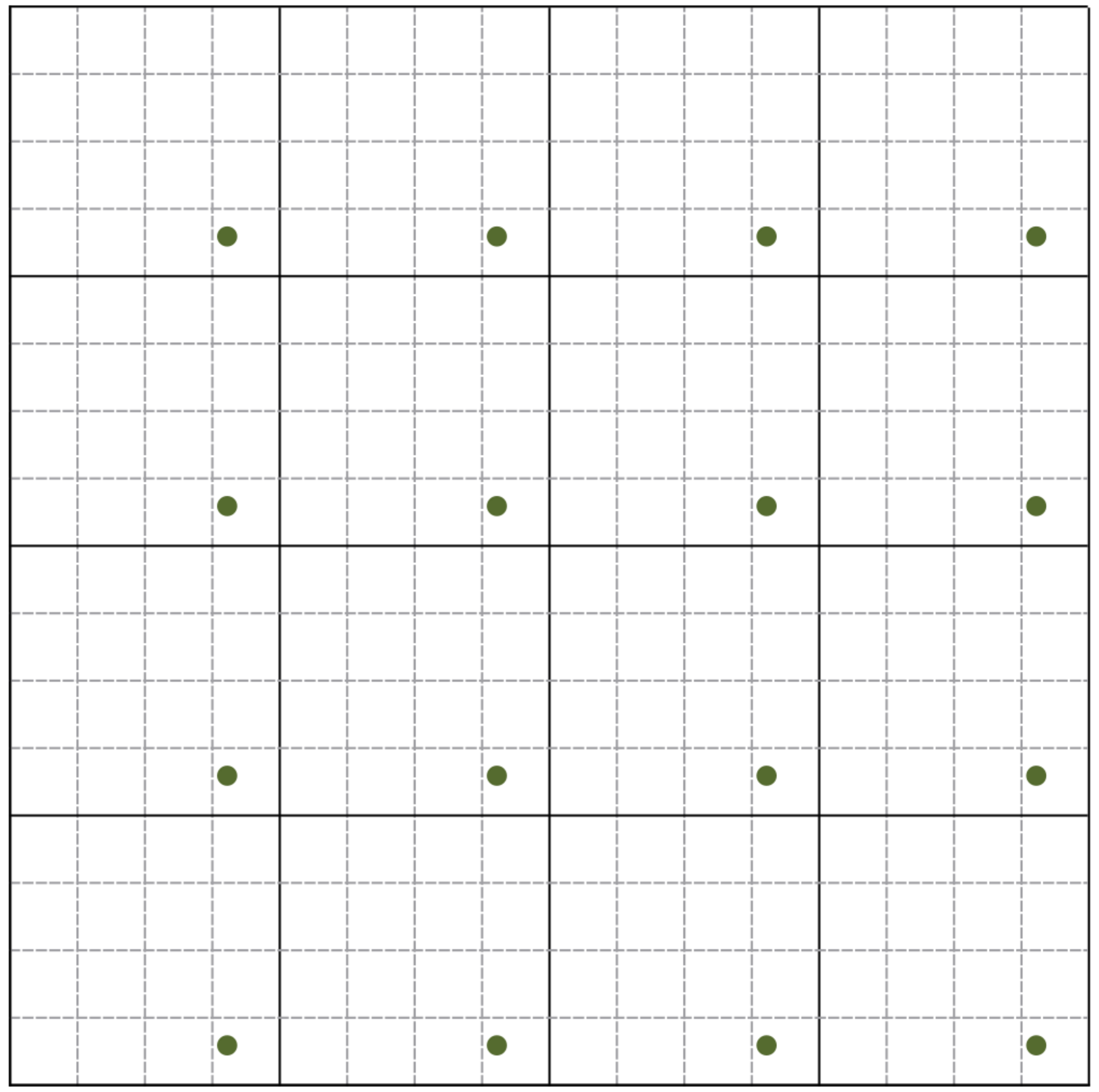
Phase change due to Random Shift



For a given frequency ω

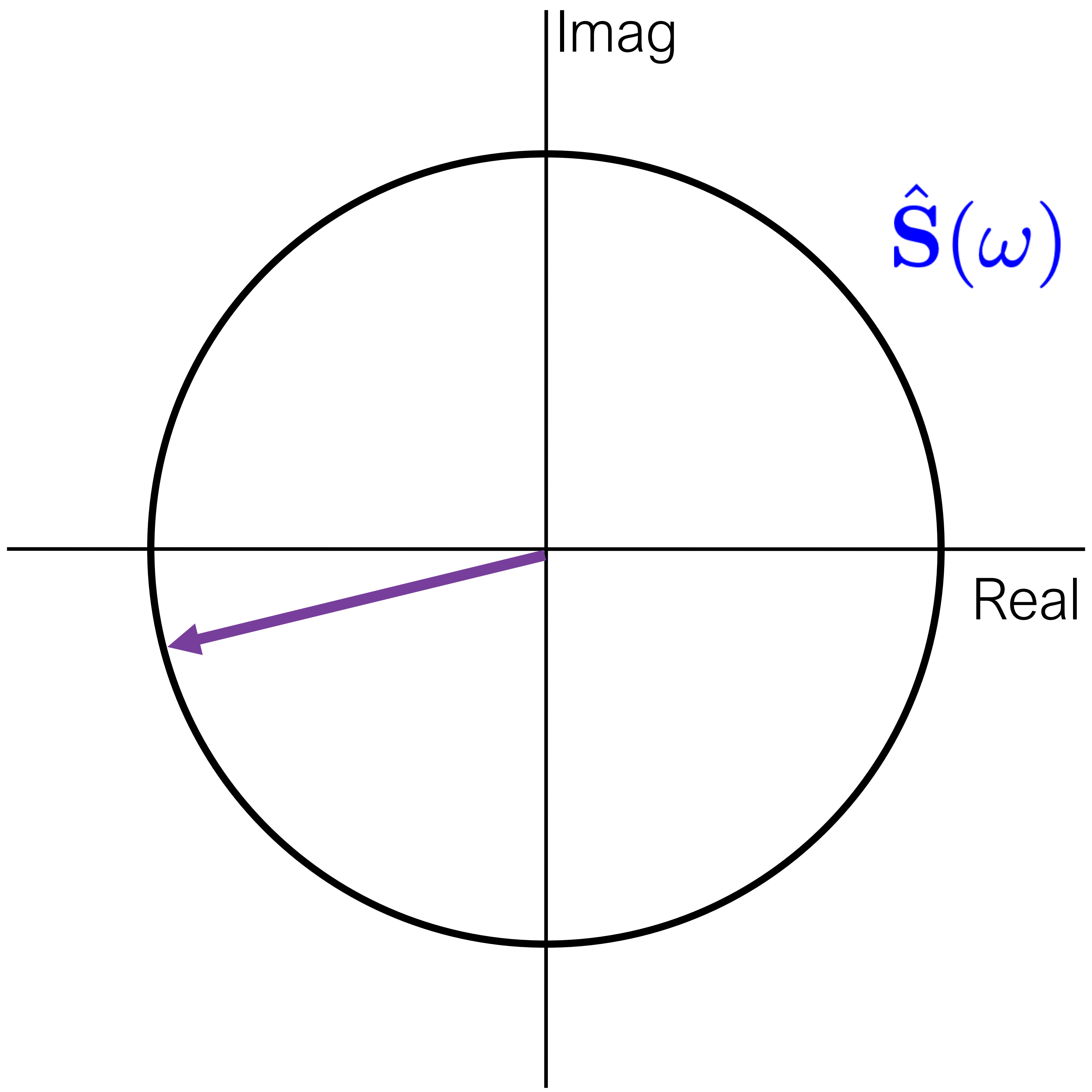


Phase change due to Random Shift

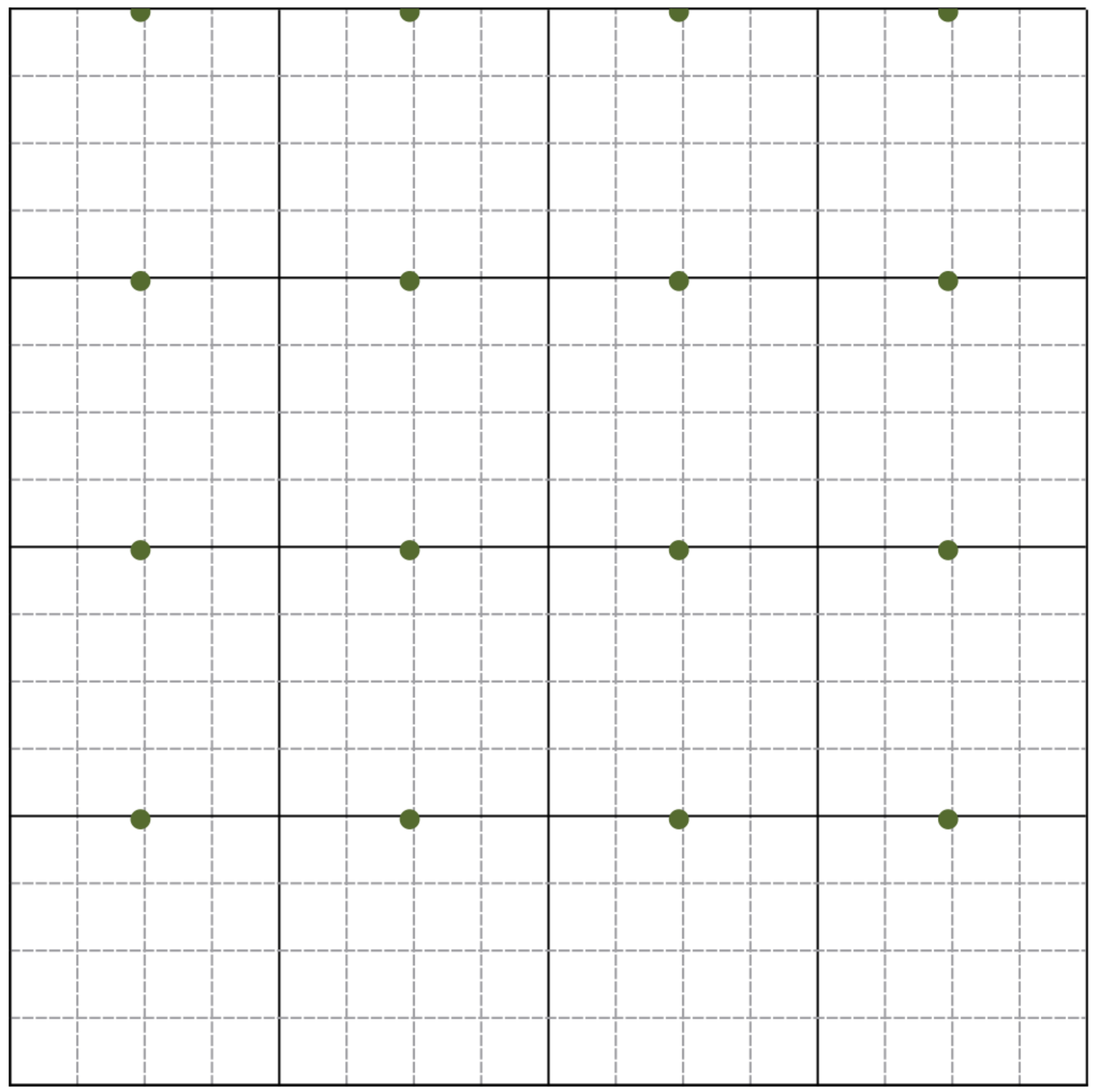


Pauly et al. [2000]
Ramamoorthi et al. [2012]

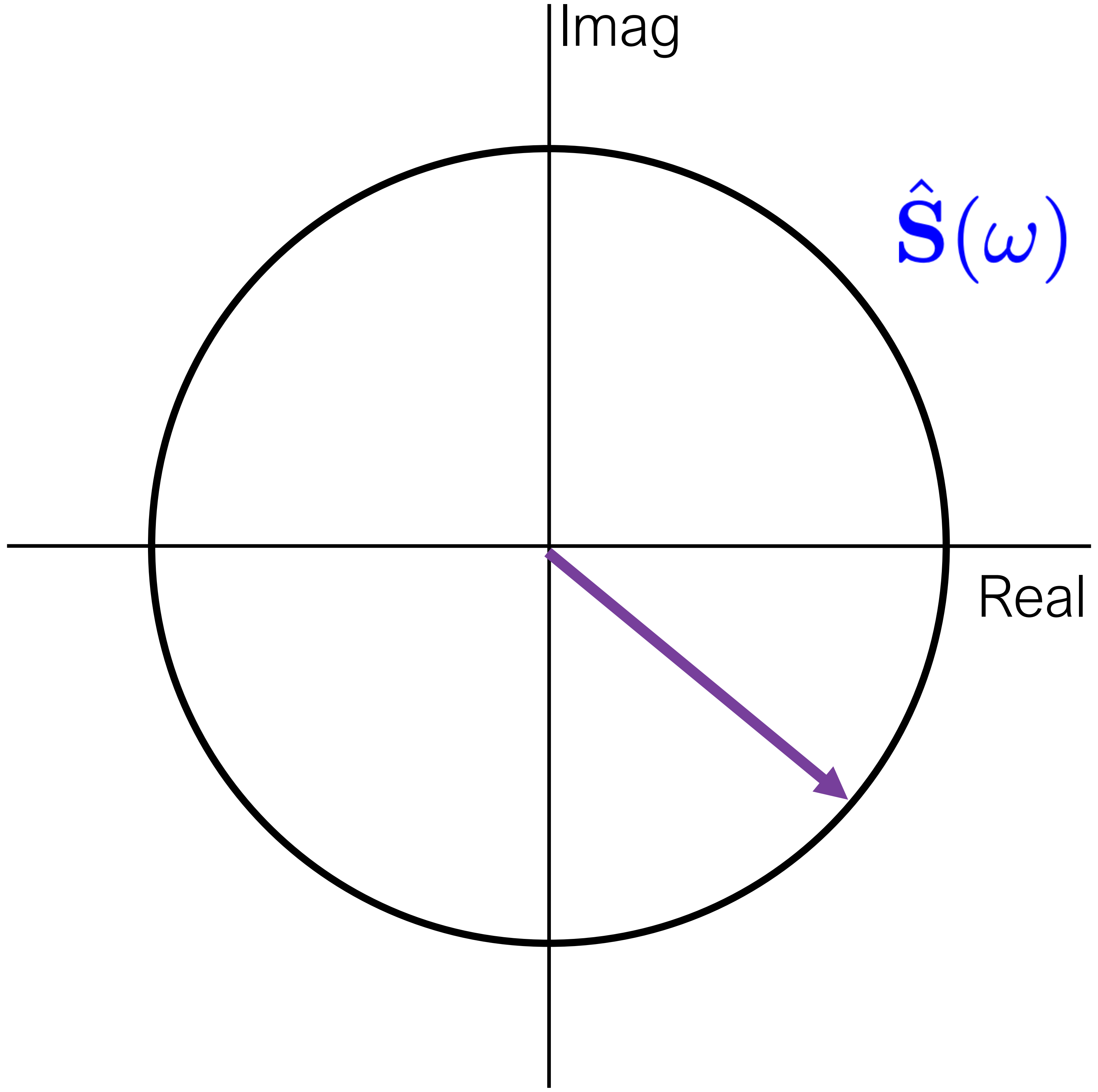
For a given frequency ω



Phase change due to Random Shift

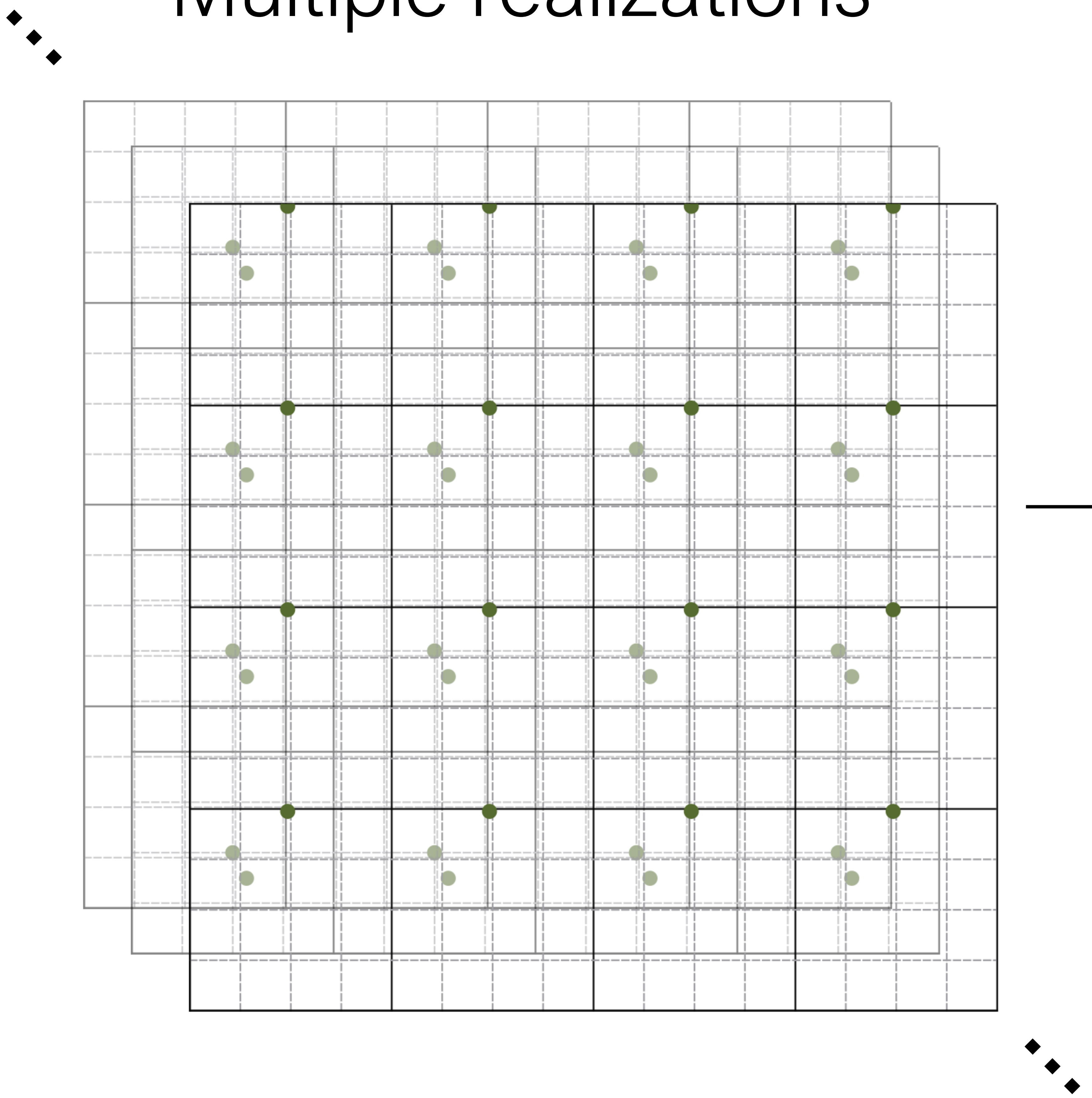


For a given frequency ω

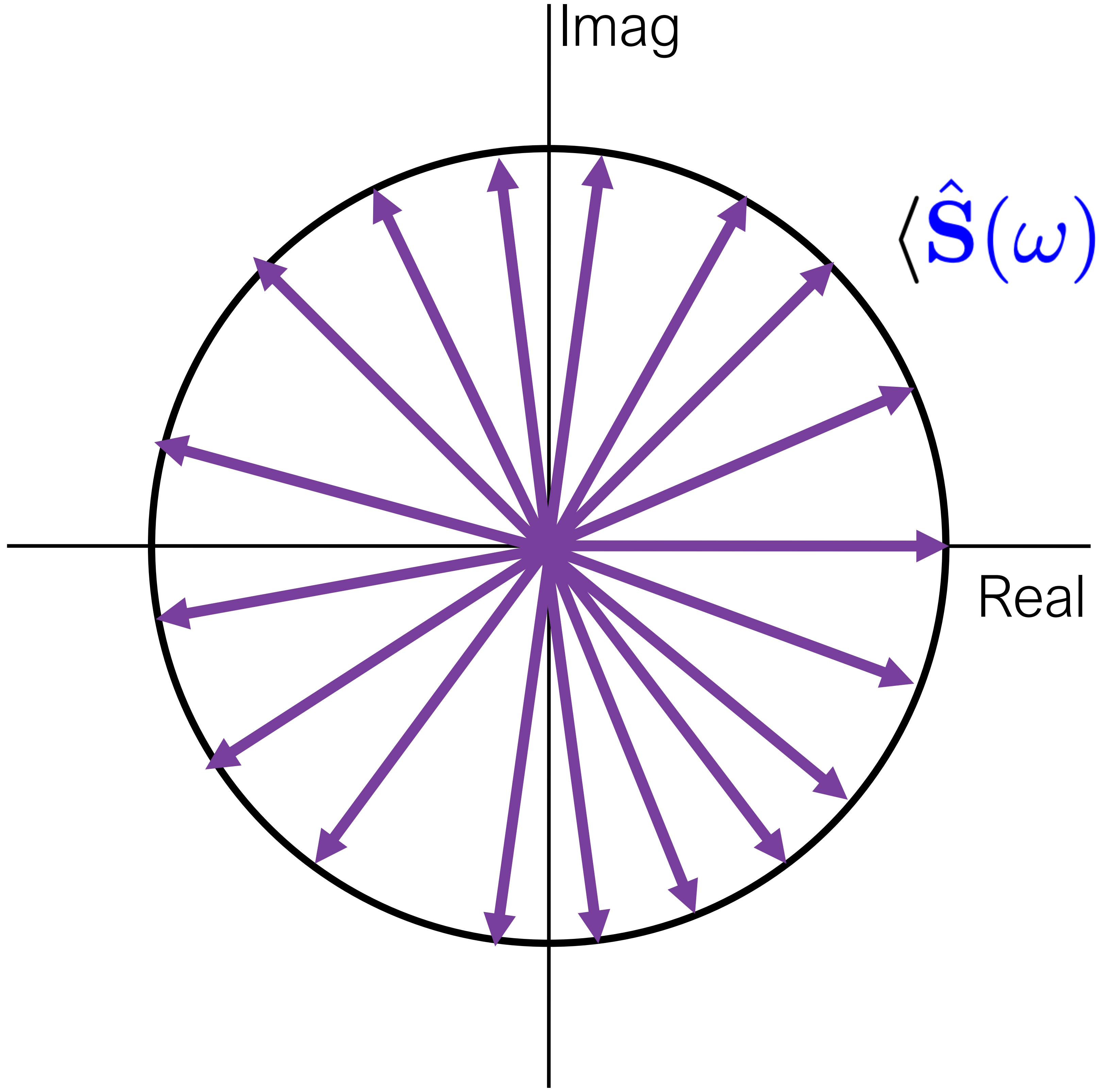


Phase change due to Random Shift

Multiple realizations



For a given frequency ω



$$\langle \hat{\mathbf{S}}(\omega) \rangle = 0 \;\; \forall \omega \neq 0$$

$$\text{Error} = \text{Bias}^2 + \text{Variance}$$

- Homogenization allows representation of error only in terms of variance
- We can take any sampling pattern and homogenize it to make the Monte Carlo estimator unbiased.

Variance in the Fourier domain

Variance in the Fourier domain

Error: $$I - \tilde{\mu}_N = \hat{f}(0) - \int_\Omega \hat{f}^*(\omega)\hat{\mathbf{S}}(\omega)d\omega$$

Variance in the Fourier domain

$$\mathrm{Var}(I - \tilde{\mu}_N) = \mathrm{Var}\left(\hat{f}(0) - \int_\Omega \hat{f}^*(\omega)\, \hat{\mathbf{S}}(\omega)\, d\omega \right)$$

Variance in the Fourier domain

$$\mathrm{Var}(I - \tilde{\mu}_N) = \mathrm{Var}\left(\hat{f}(0) - \int_\Omega \hat{f}^*(\omega)\,\hat{\mathbf{S}}(\omega)\,d\omega\right)$$

$$\mathrm{Var}(\tilde{\mu}_N) = \mathrm{Var}\left(\int_\Omega \hat{f}^*(\omega)\,\hat{\mathbf{S}}(\omega)d\omega\right)$$

Variance in the Fourier domain

$$\mathrm{Var}(\tilde{\mu}_N) = \mathrm{Var}\left(\int_\Omega \hat{f}^*(\omega)\, \hat{\mathbf{S}}(\omega) d\omega\right)$$

Variance in the Fourier domain

$$\mathrm{Var}(\tilde{\mu}_N) = \mathrm{Var}\left(\int_\Omega \hat{f}^*(\omega)\, \hat{\mathbf{S}}(\omega) d\omega\right)$$

$$\mathrm{Var}(\tilde{\mu}_N) = \int_\Omega P_f(\omega)\, \mathrm{Var}\left(\hat{\mathbf{S}}(\omega)\right) d\omega$$

where,

$$P_f(\omega) = |\hat{f}^*(\omega)|^2$$ Power Spectrum

Variance in the Fourier domain

$$\mathrm{Var}(\tilde{\mu}_N) = \int_{\Omega} P_f(\omega)\, \mathrm{Var}\left(\hat{\mathbf{S}}(\omega)\right) d\omega$$

Subr and Kautz [2013]

This is a general form, both for homogenised as well as non-homogenised sampling patterns

Variance in the Fourier domain

$$\mathrm{Var}(\tilde{\mu}_N) = \int_{\Omega} P_f(\omega)\, \mathrm{Var}\left(\hat{\mathbf{S}}(\omega)\right) d\omega$$

Variance in the Fourier domain

$$\text{Var}(\tilde{\mu}_N) = \int_\Omega P_f(\omega)\, \text{Var}\left(\hat{\mathbf{S}}(\omega)\right) d\omega$$

For purely random samples: $\langle \hat{\mathbf{S}}(\omega) \rangle = 0$

$$\text{Var}(\tilde{\mu}_N) = \int_\Omega P_f(\omega)\, \langle P_S(\omega) \rangle\, d\omega$$

Fredo Durand [2011]

where,

$$P_S(\omega) = |\hat{\mathbf{S}}(\omega)|^2$$

Variance using Homogenized Samples

Homogenizing any sampling pattern makes $\langle \hat{\mathbf{S}}(\omega) \rangle = 0$

$$\mathrm{Var}(\tilde{\mu}_N) = \int_{\Omega} P_f(\omega)\ \langle P_S(\omega) \rangle\, d\omega$$

Pilleboue et al. [2015]

where,

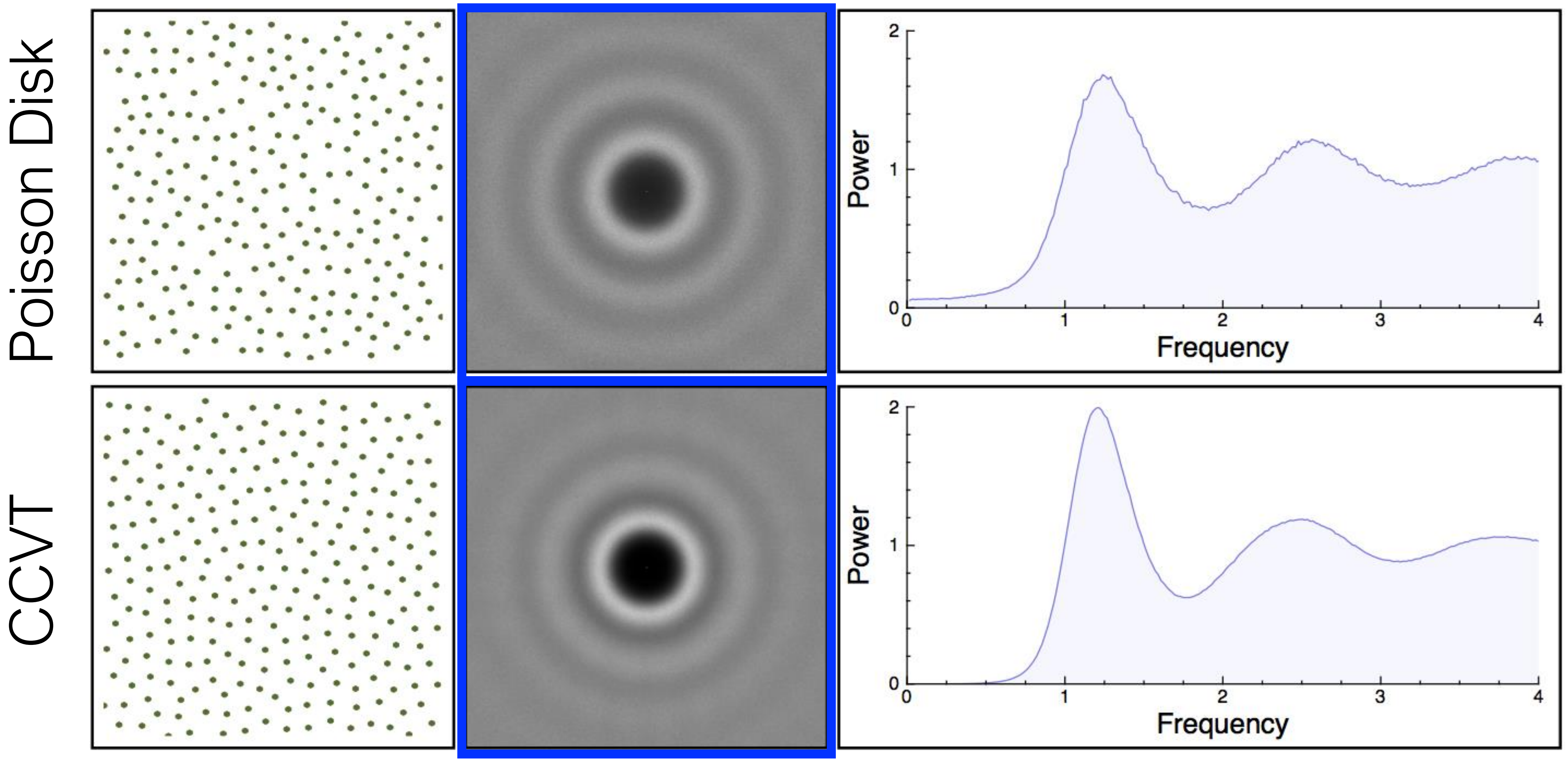
$$P_S(\omega) = |\hat{\mathbf{S}}(\omega)|^2$$

Variance using Homogenized Samples

$$\mathrm{Var}(\tilde{\mu}_N) = \int_{\Omega} P_f(\omega) \; \langle P_S(\omega) \rangle \, d\omega$$

Variance in terms of n-dimensional Power Spectra

$$\mathrm{Var}(\tilde{\mu}_N) = \int_\Omega P_f(\omega)\ \langle P_S(\omega)\rangle\ d\omega$$



Variance in the Polar Coordinates

$$\mathrm{Var}(\tilde{\mu}_N) = \int_\Omega P_f(\omega)\ \langle P_S(\omega)\rangle\, d\omega$$

In polar coordinates:

$$Var[\tilde{\mu}_N] = \mathcal{M}(\mathcal{S}^{d-1}) \int_0^\infty \int_{\mathcal{S}^{d-1}} P_f(\rho\mathbf{n})\,\langle P_{\mathbf{S}}(\rho\mathbf{n})\rangle\, d\mathbf{n}\, d\rho$$

Variance in the Polar Coordinates

$$\mathrm{Var}(\tilde{\mu}_N) = \int_{\Omega} P_f(\omega)\ \langle P_S(\omega) \rangle\, d\omega$$

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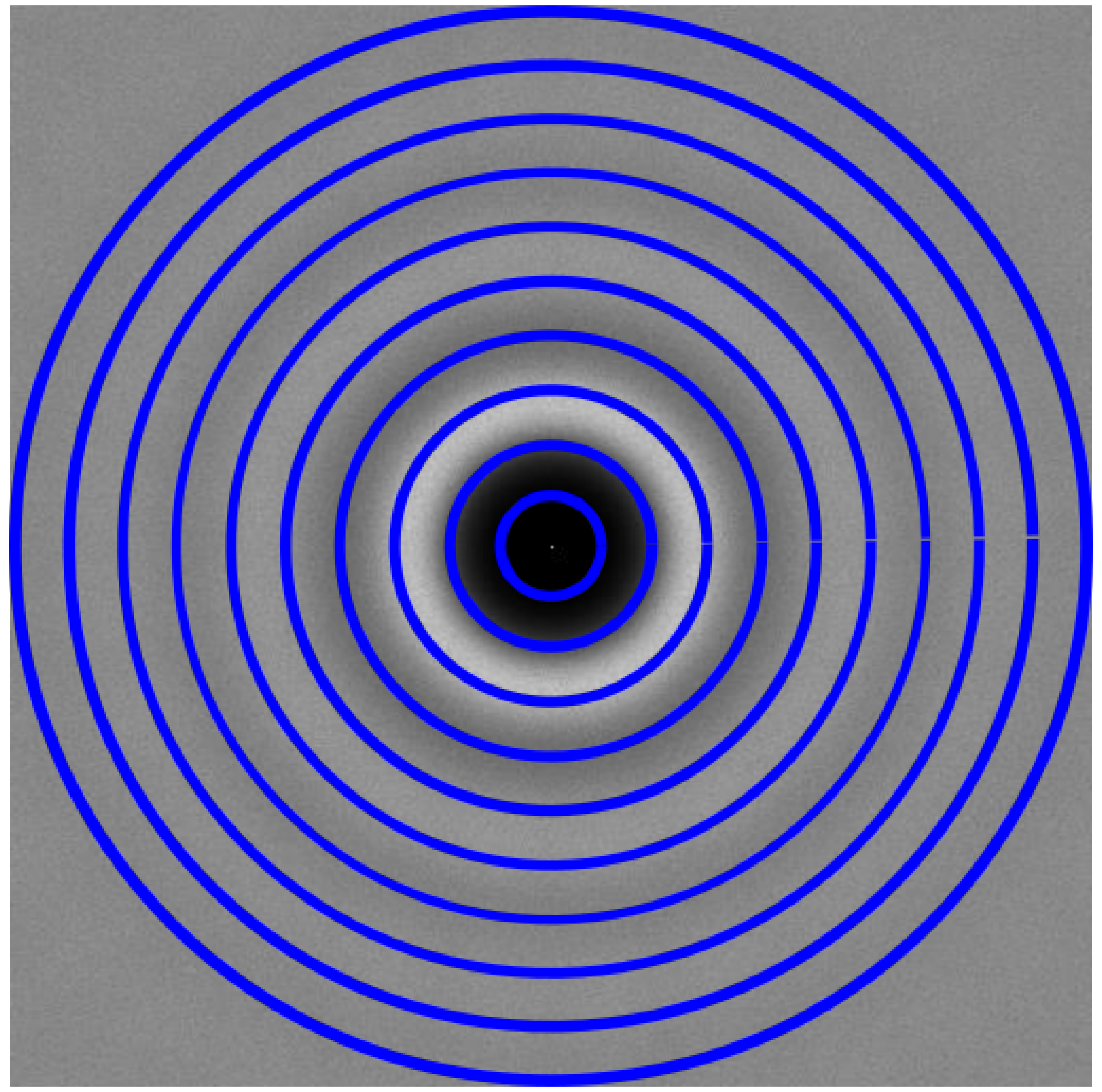
Variance for Isotropic Power Spectra

$$Var[\tilde{\mu}_N] = \mathcal{M}(\mathcal{S}^{d-1}) \int_0^\infty \int_{\mathcal{S}^{d-1}} P_f(\rho\mathbf{n}) \left\langle P_{\mathbf{S}}(\rho\mathbf{n}) \right\rangle d\mathbf{n}\, d\rho$$

For isotropic power spectra:

$$Var[\tilde{\mu}_N] = \mathcal{M}(\mathcal{S}^{d-1}) \int_0^\infty \tilde{P}_f(\rho) \left\langle \tilde{P}_{\mathbf{S}}(\rho) \right\rangle d\rho$$



Variance for Isotropic Power Spectra

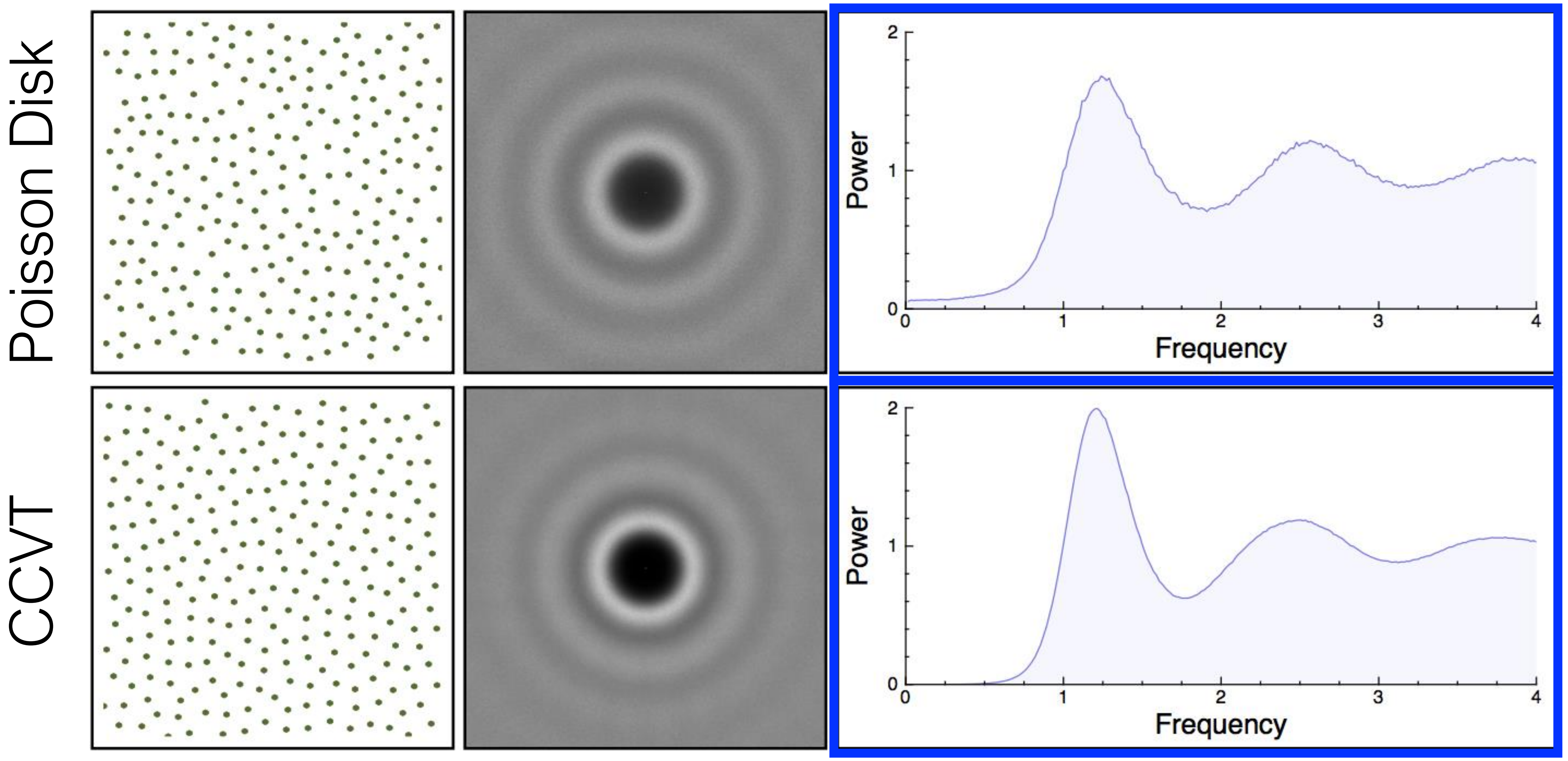
$$Var[\tilde{\mu}_N] = \mathcal{M}(\mathcal{S}^{d-1}) \int_0^\infty \int_{\mathcal{S}^{d-1}} P_f(\rho\mathbf{n}) \, \langle P_{\mathbf{S}}(\rho\mathbf{n}) \rangle \, d\mathbf{n} \, d\rho$$

For isotropic power spectra:

$$\boxed{Var[\tilde{\mu}_N] = \mathcal{M}(\mathcal{S}^{d-1}) \int_0^\infty \tilde{P}_f(\rho) \, \langle \tilde{P}_{\mathbf{S}}(\rho) \rangle \, d\rho}$$

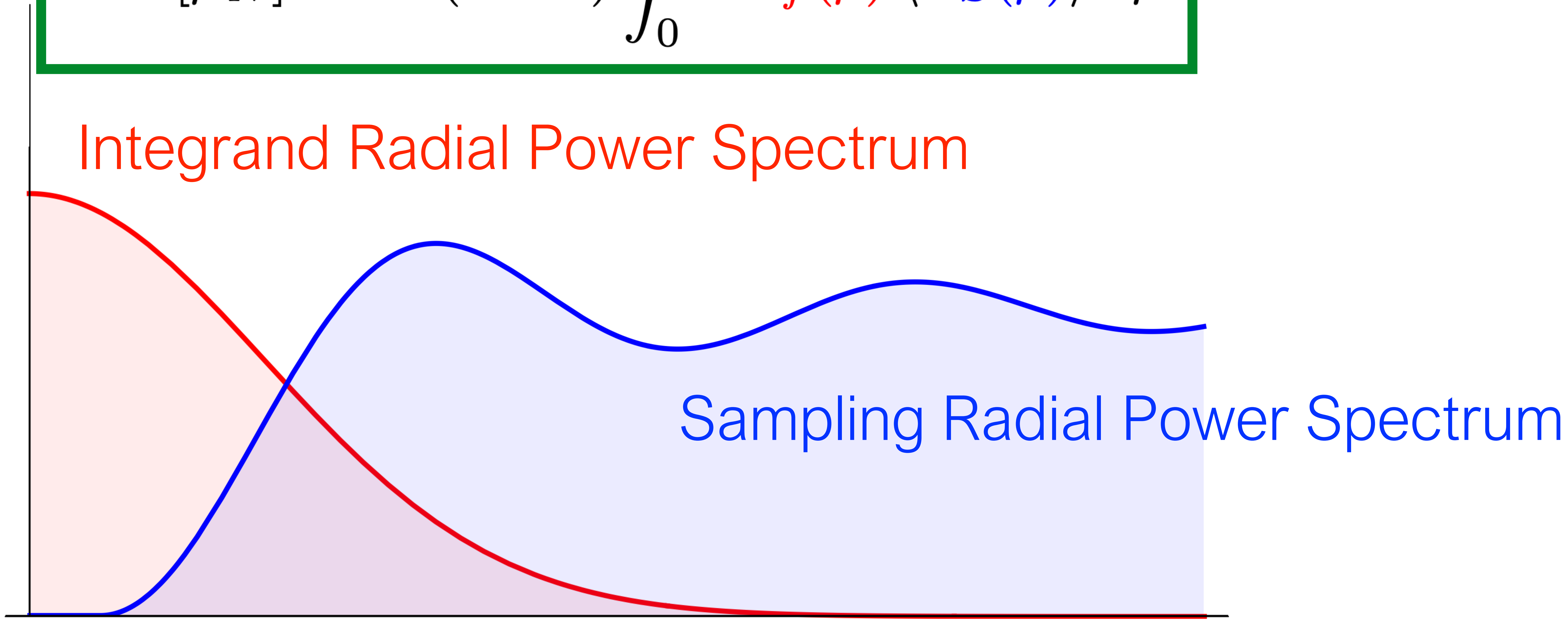
Variance in terms of 1-dimensional Power Spectra

$$Var[\tilde{\mu}_N] = \mathcal{M}(\mathcal{S}^{d-1}) \int_0^\infty \tilde{P}_f(\rho) \, \langle \tilde{P}_{\mathbf{S}}(\rho) \rangle \, d\rho$$



Variance: Integral over Product of Power Spectra

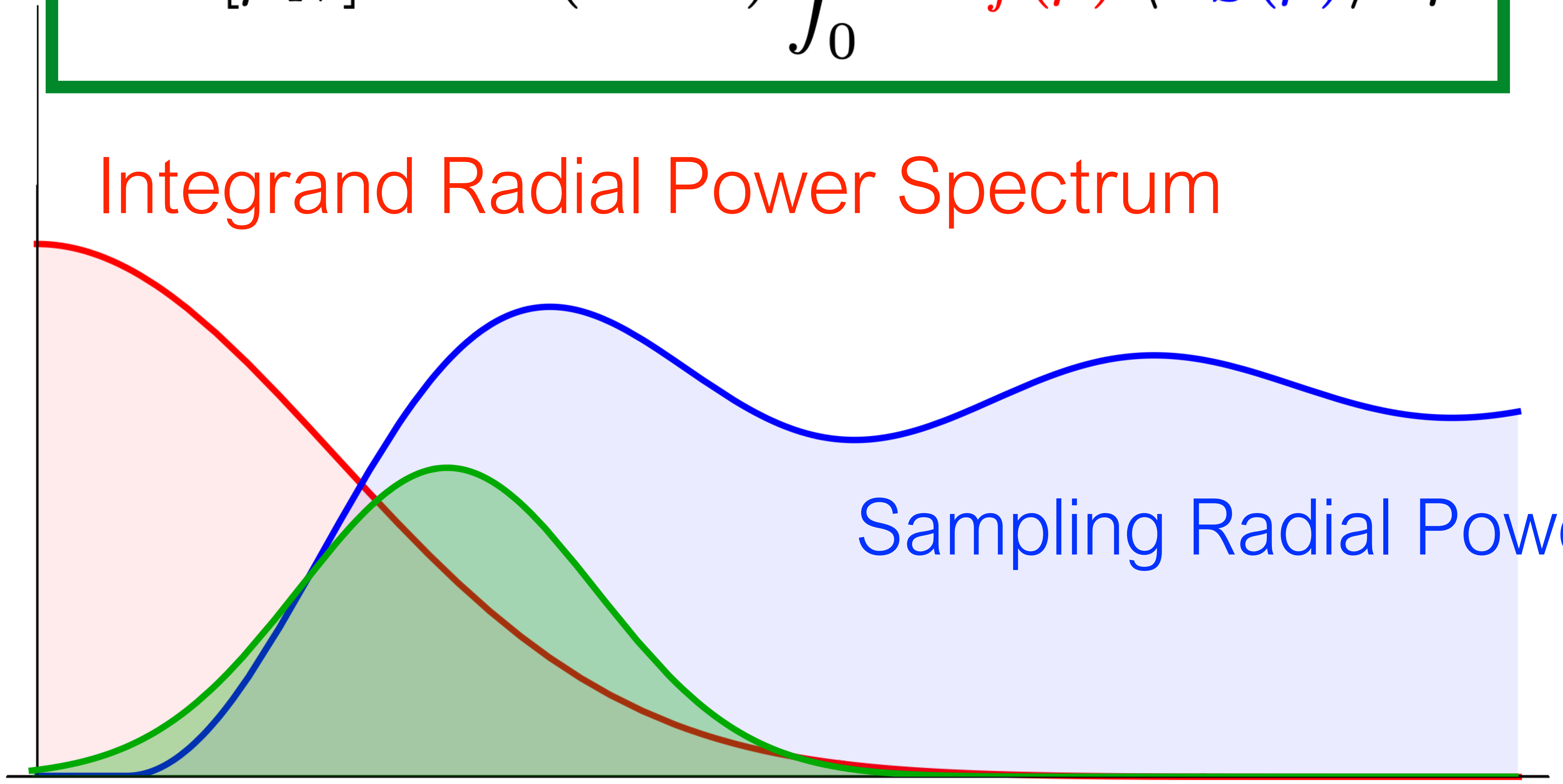
$$Var[\tilde{\mu}_N] = \mathcal{M}(\mathcal{S}^{d-1}) \int_0^\infty \tilde{P}_f(\rho) \, \langle \tilde{P}_{\mathbf{S}}(\rho) \rangle \, d\rho$$



For given number of Samples

Variance: Integral over Product of Power Spectra

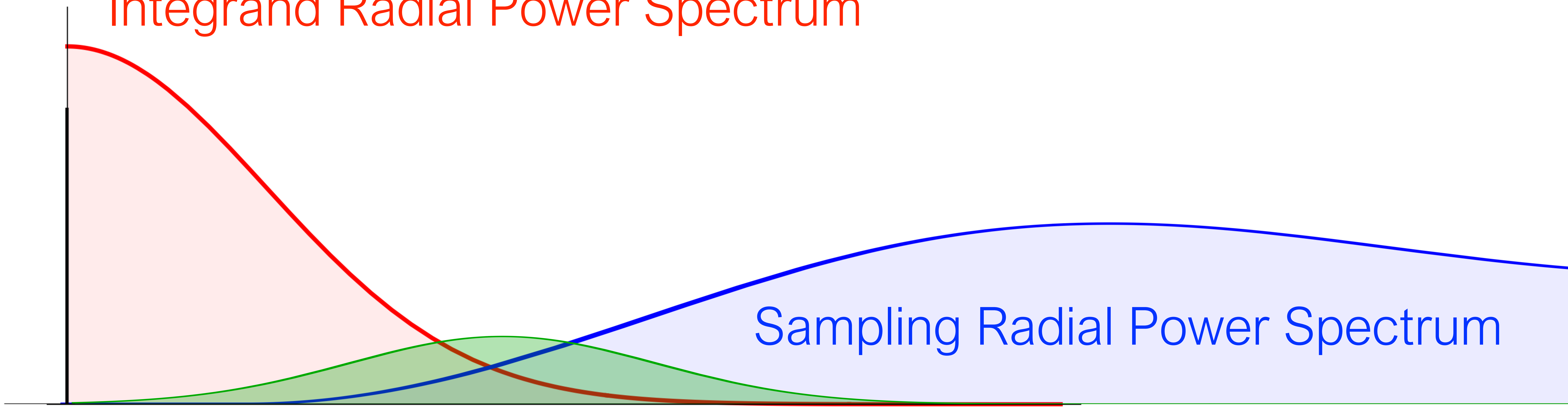
$$Var[\tilde{\mu}_N] = \mathcal{M}(\mathcal{S}^{d-1}) \int_0^\infty \tilde{P}_f(\rho) \, \langle \tilde{P}_{\mathbf{S}}(\rho) \rangle \, d\rho$$



For given number of Samples

Variance: Integral over Product of Power Spectra

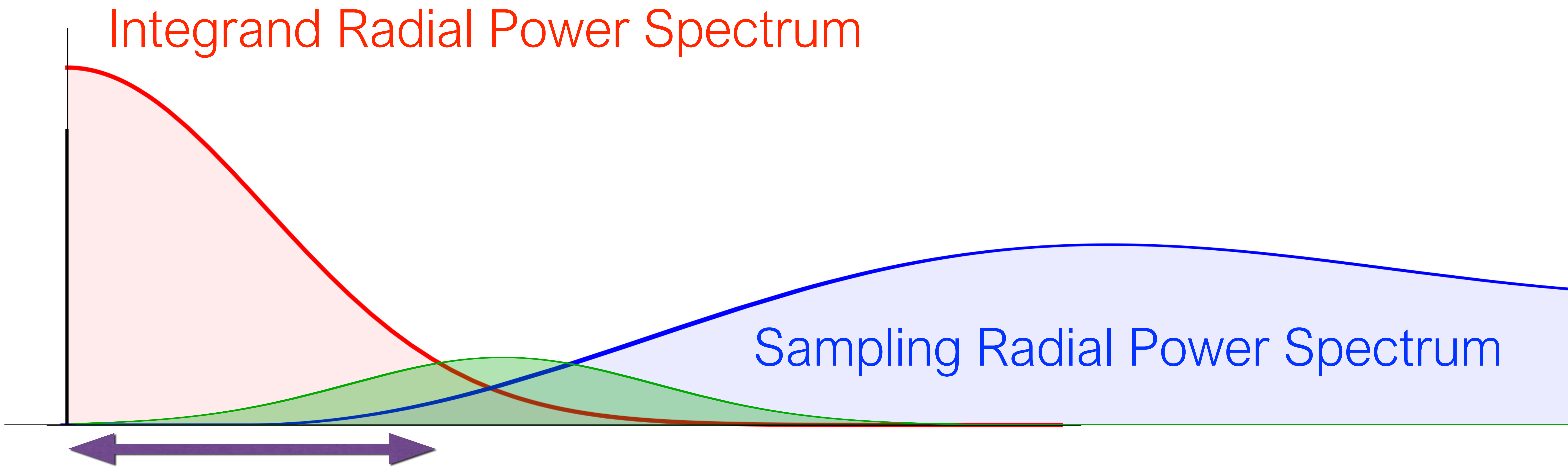
$$Var[\tilde{\mu}_N] = \mathcal{M}(\mathcal{S}^{d-1}) \int_0^\infty \tilde{P}_f(\rho) \, \langle \tilde{P}_{\mathbf{S}}(\rho) \rangle \, d\rho$$



For given number of Samples

Variance: Integral over Product of Power Spectra

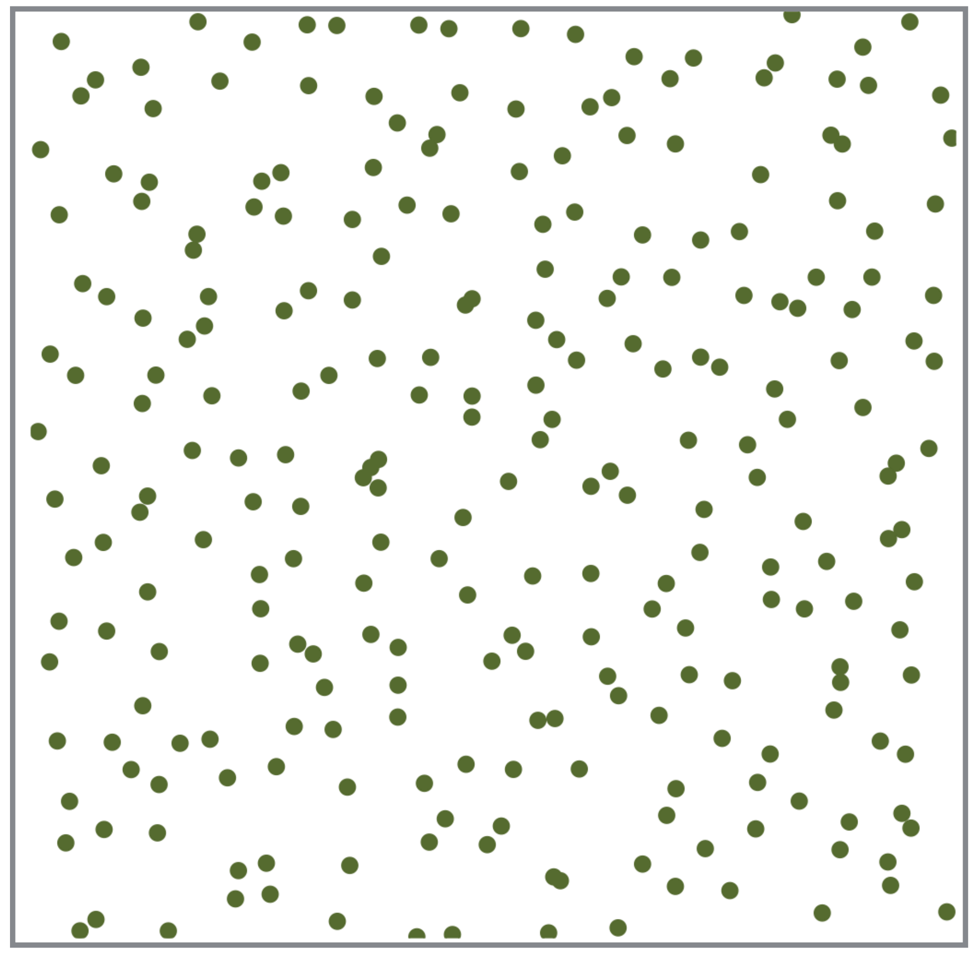
$$Var[\tilde{\mu}_N] = \mathcal{M}(\mathcal{S}^{d-1}) \int_0^{\infty} \tilde{P}_f(\rho) \, \langle \tilde{P}_{\mathbf{S}}(\rho) \rangle \, d\rho$$

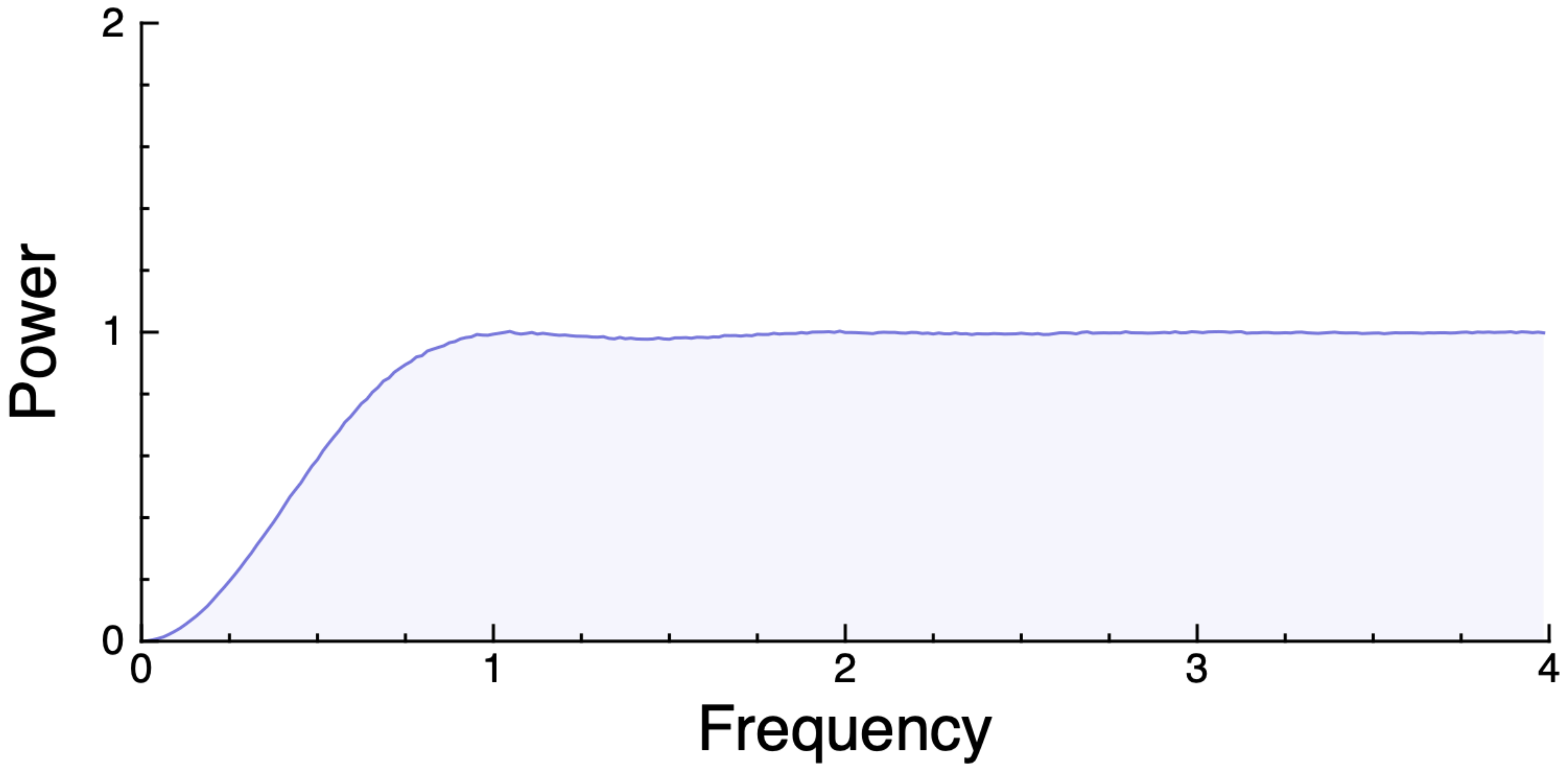


For given number of Samples

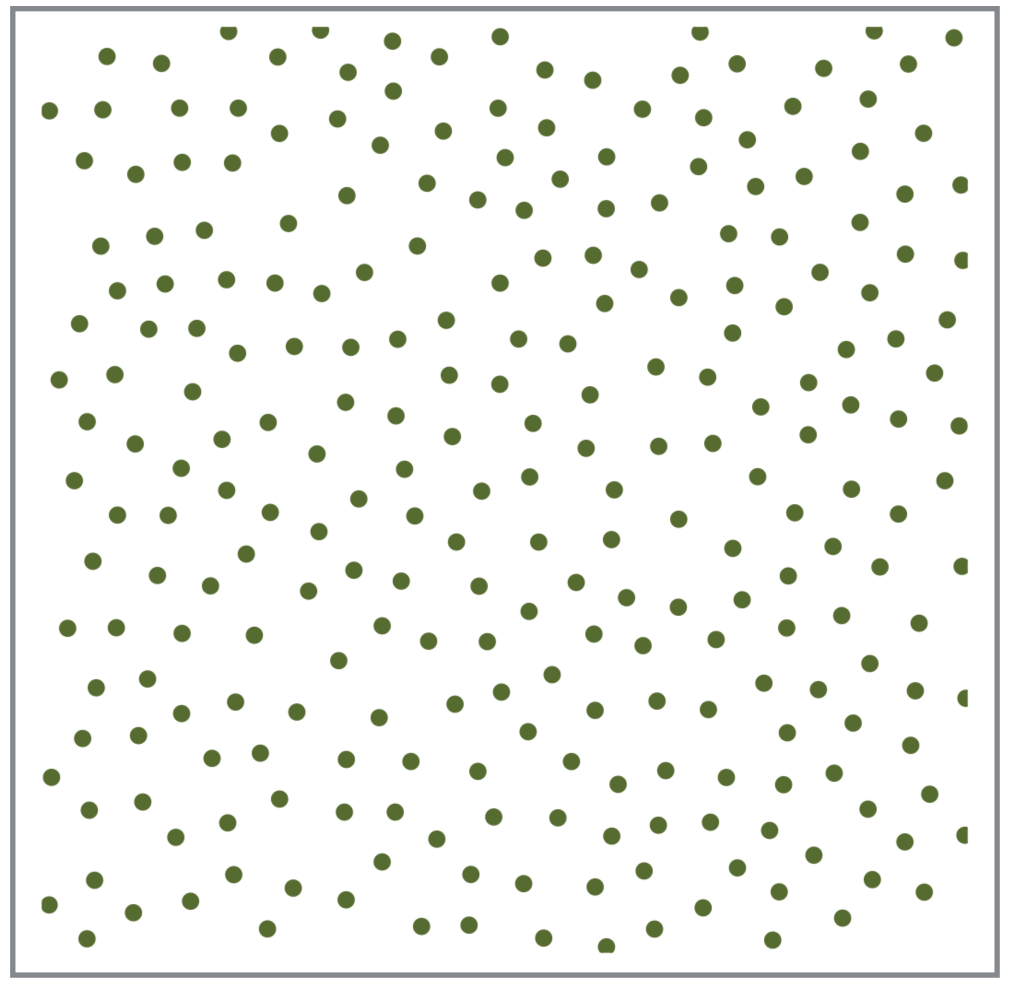
Spatial Distribution vs Radial Mean Power Spectra

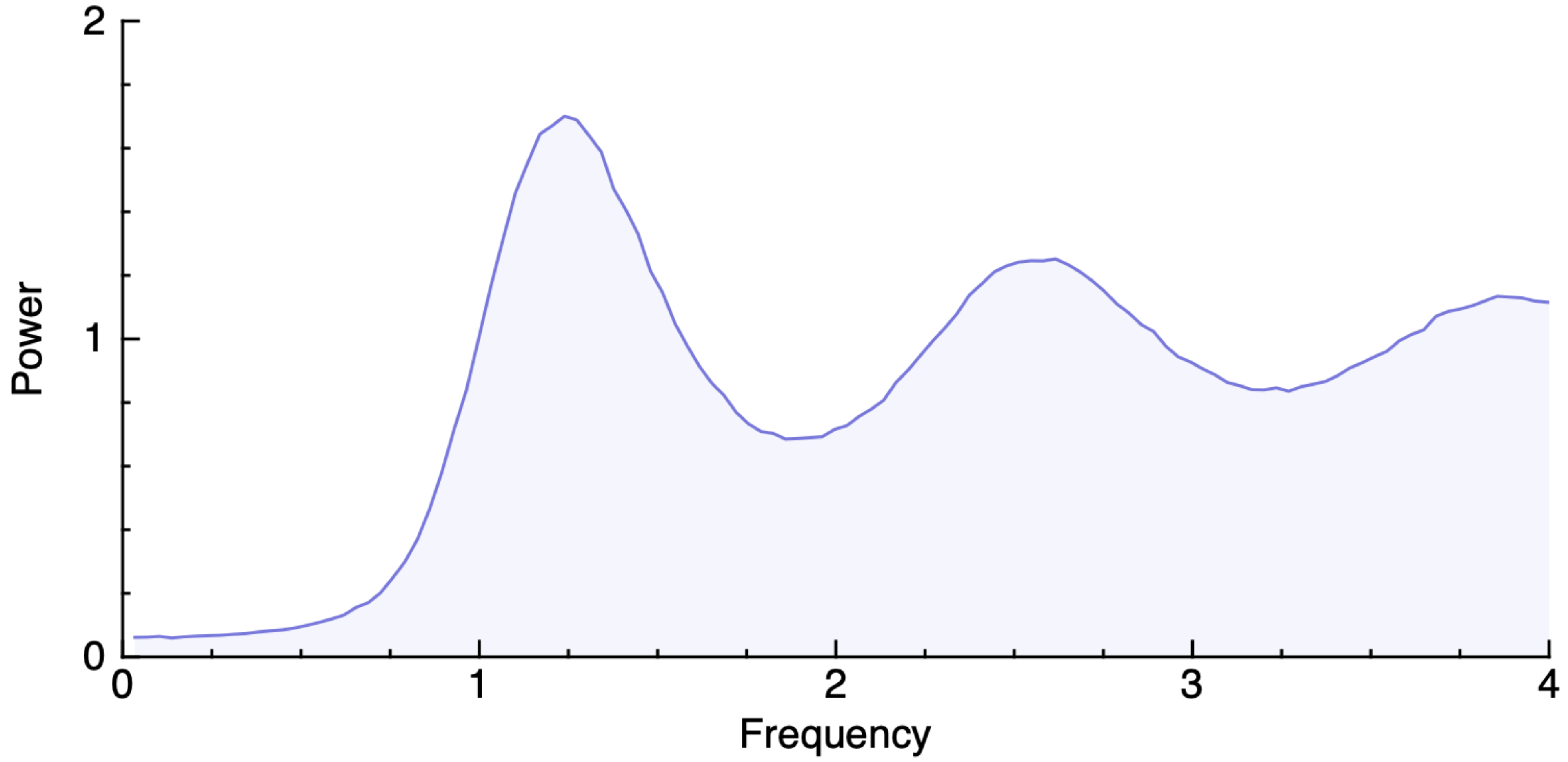
Jitter





Poisson Disk





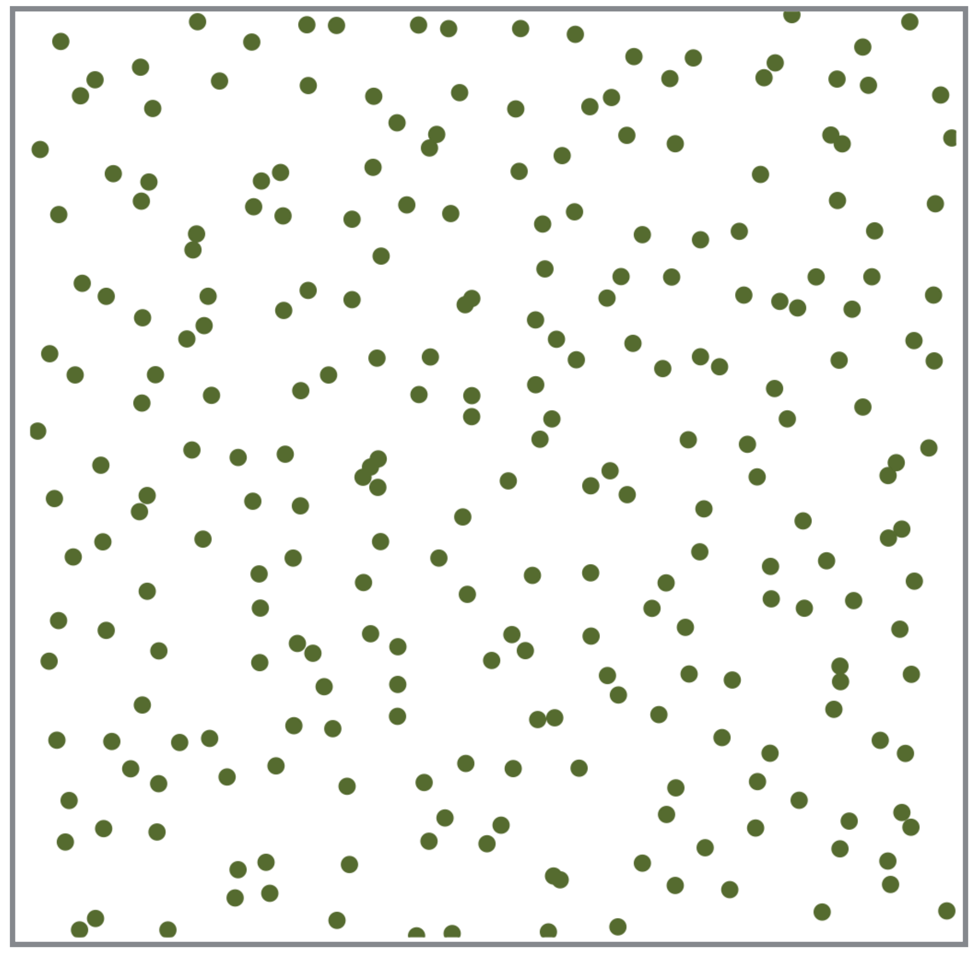
For 2-dimensions

Samplers	Worst Case	Best Case
Random	$\mathcal{O}(N^{-1})$	$\mathcal{O}(N^{-1})$
Jitter	$\mathcal{O}(N^{-1.5})$	$\mathcal{O}(N^{-2})$
Poisson Disk	$\mathcal{O}(N^{-1})$	$\mathcal{O}(N^{-1})$
CCVT	$\mathcal{O}(N^{-1.5})$	$\mathcal{O}(N^{-3})$

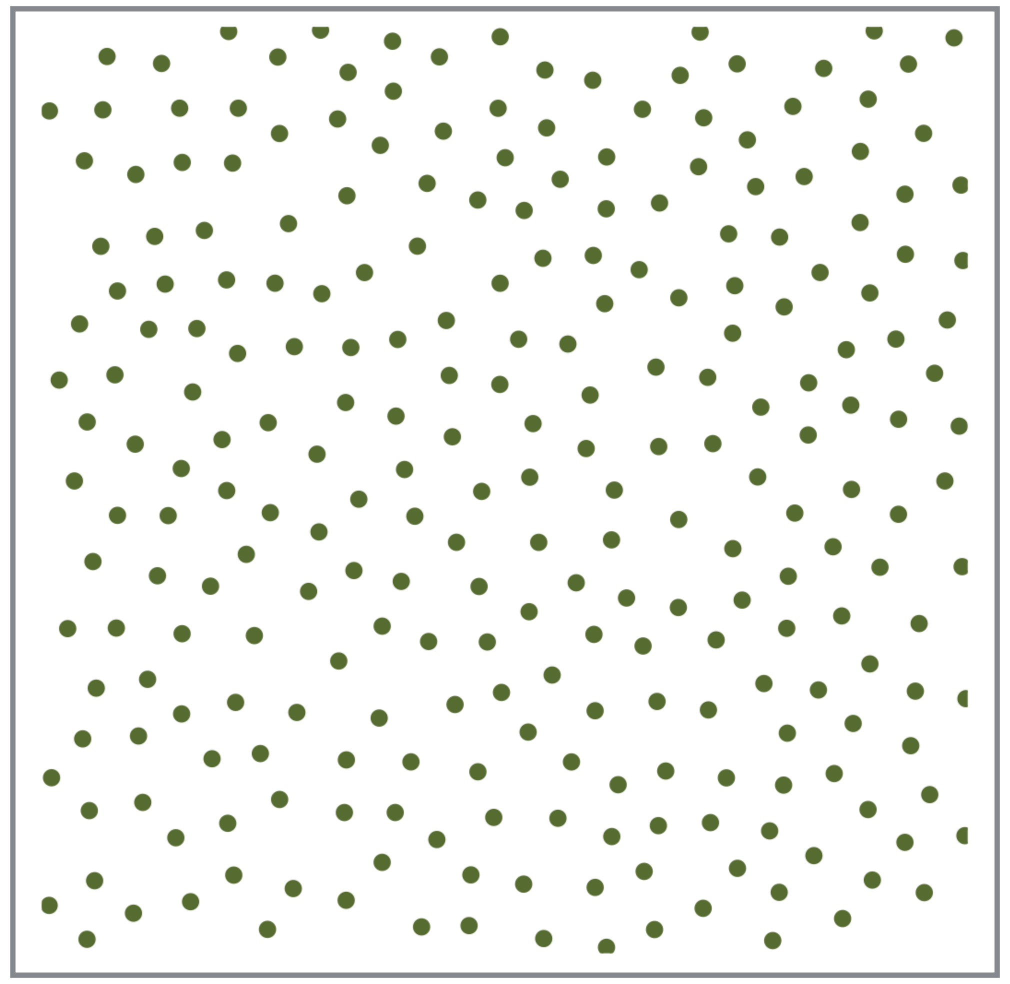
Pilleboue et al. [2015]

For 2-dimensions

Jitter



Poisson Disk



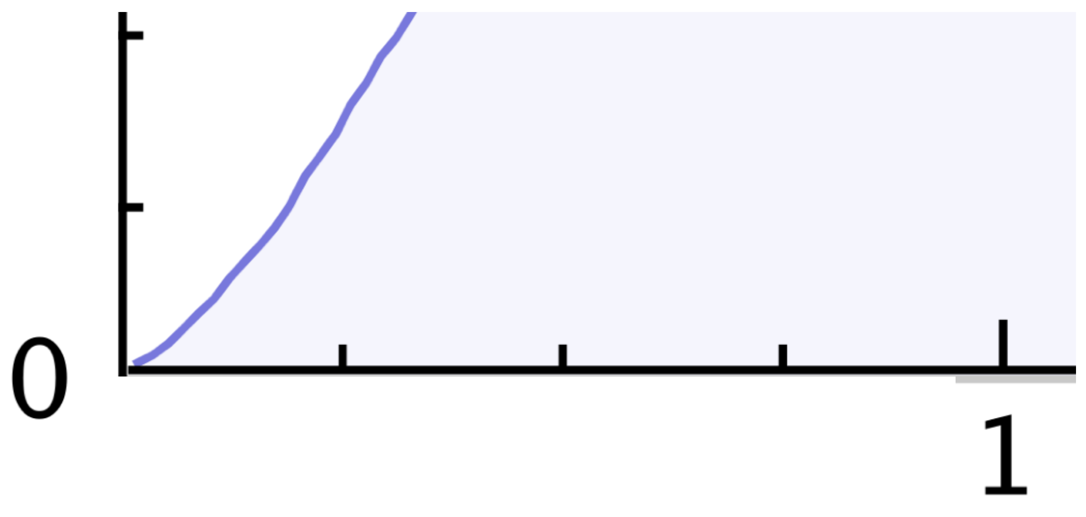
Samplers	Worst Case	Best Case
Random	$\mathcal{O}(N^{-1})$	$\mathcal{O}(N^{-1})$
Jitter	$\mathcal{O}(N^{-1.5})$	$\mathcal{O}(N^{-2})$
Poisson Disk	$\mathcal{O}(N^{-1})$	$\mathcal{O}(N^{-1})$
CCVT	$\mathcal{O}(N^{-1.5})$	$\mathcal{O}(N^{-3})$

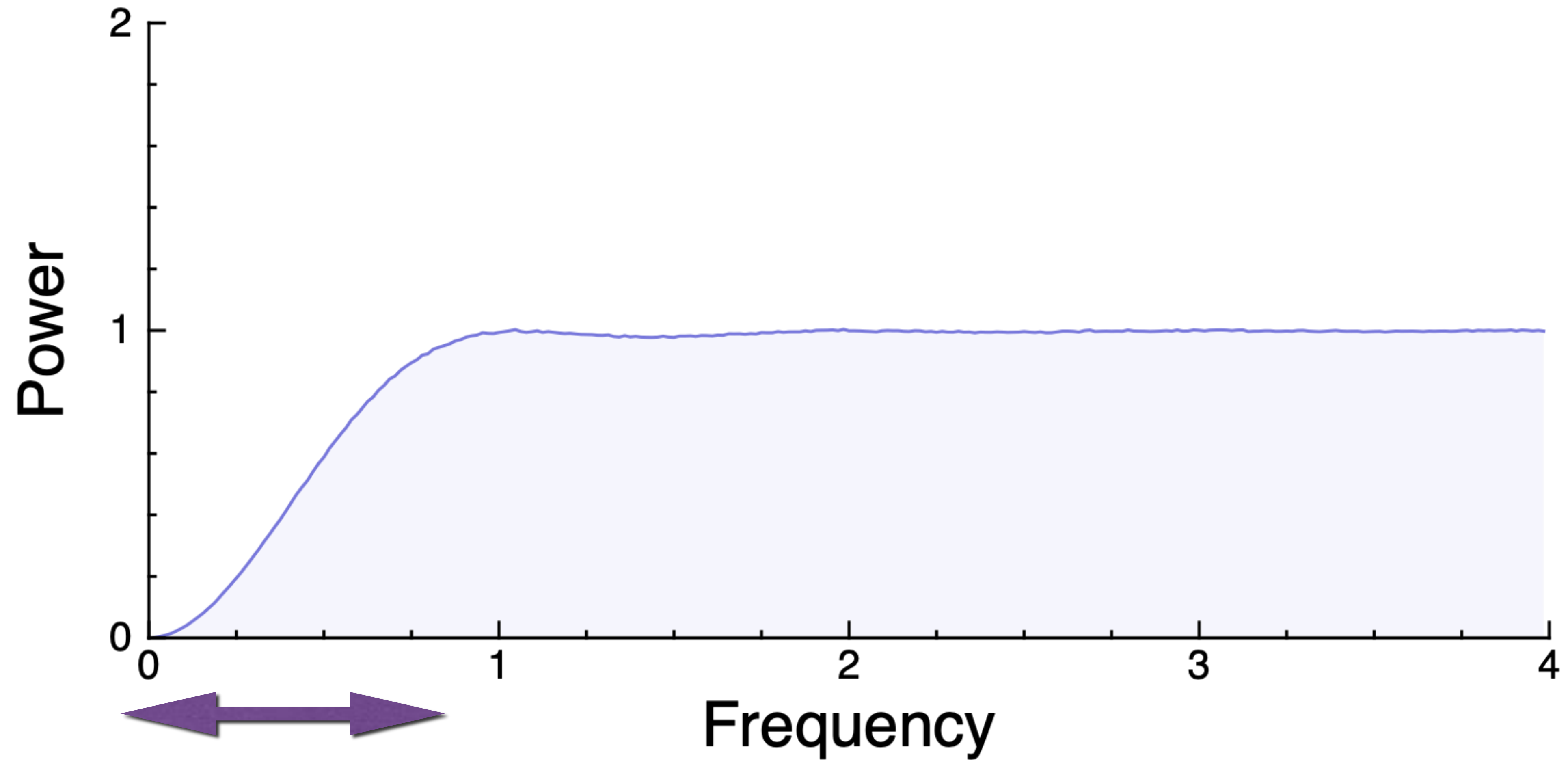
Pilleboue et al. [2015]

Low Frequency Region

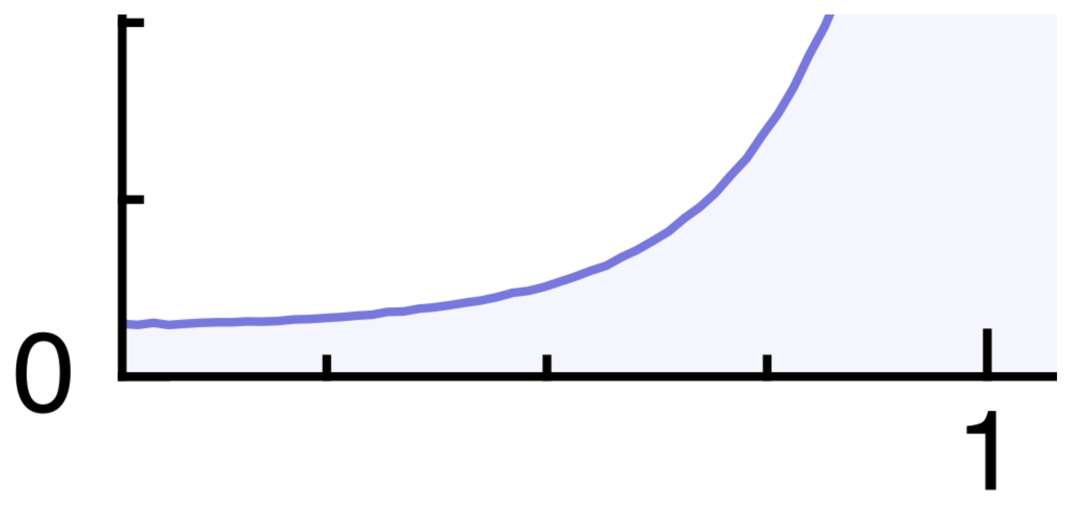
Zoom-in

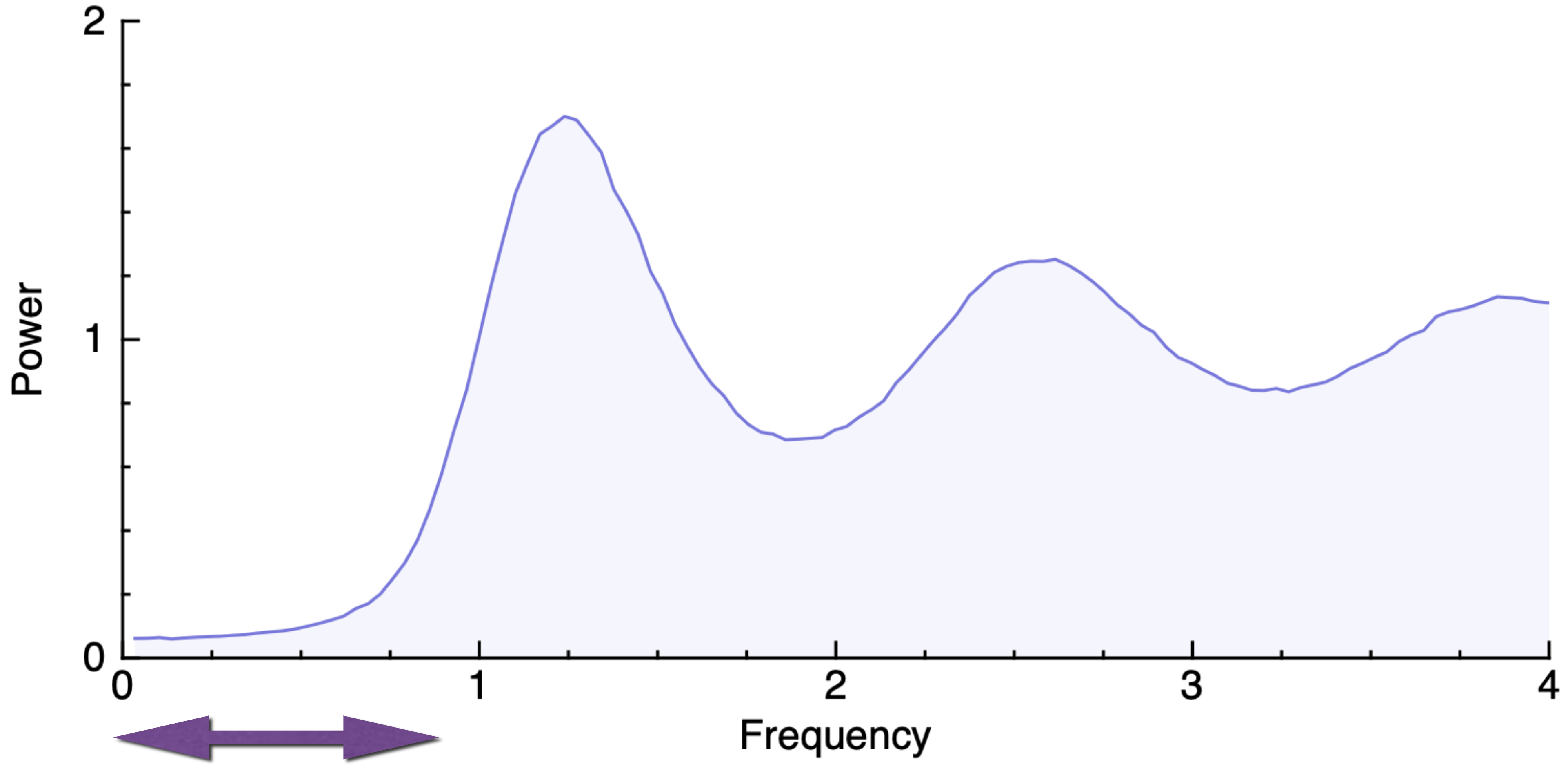
Jitter



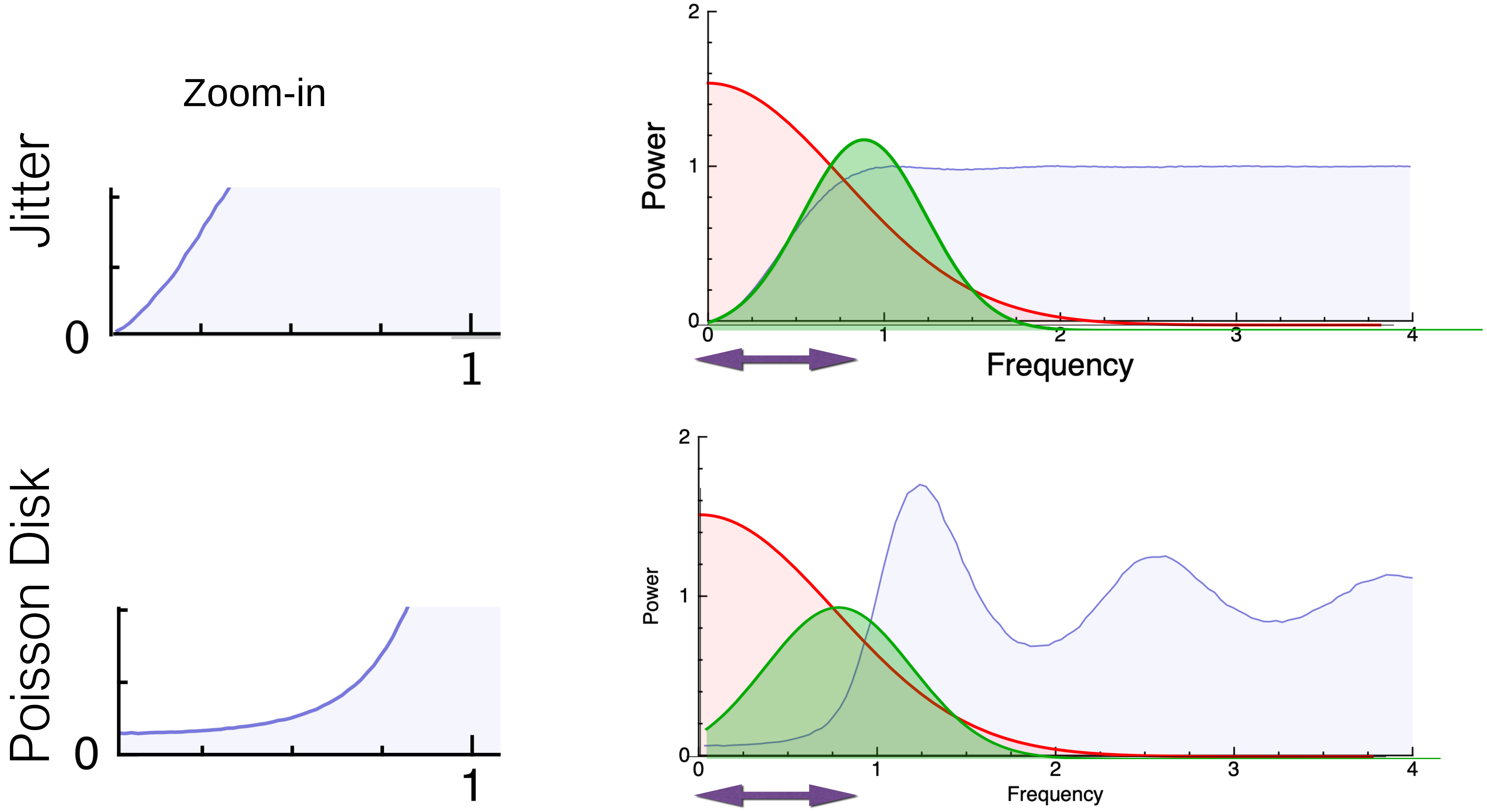


Poisson Disk

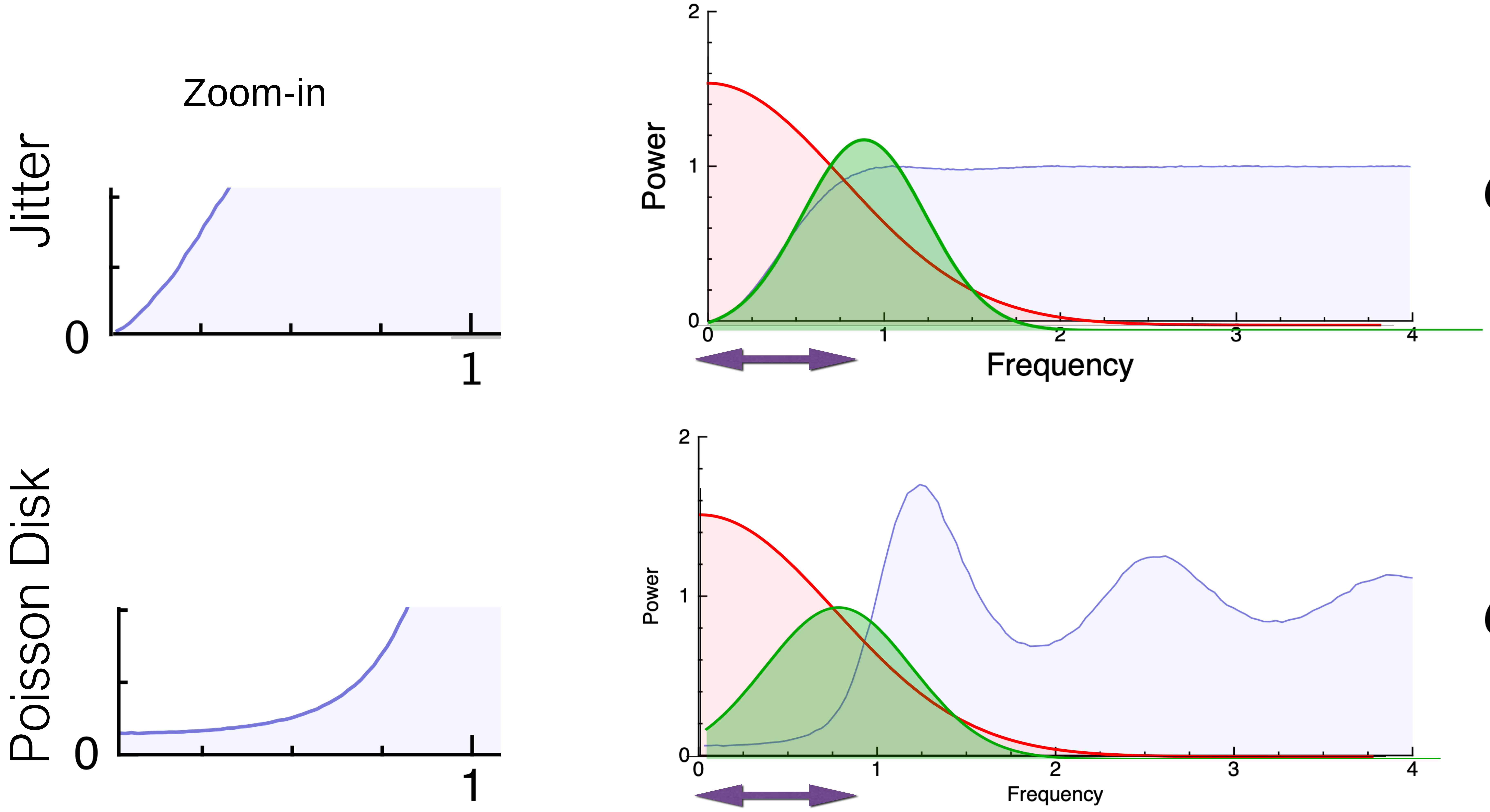




Variance for Low Sample Count



Variance for Increasing Sample Count

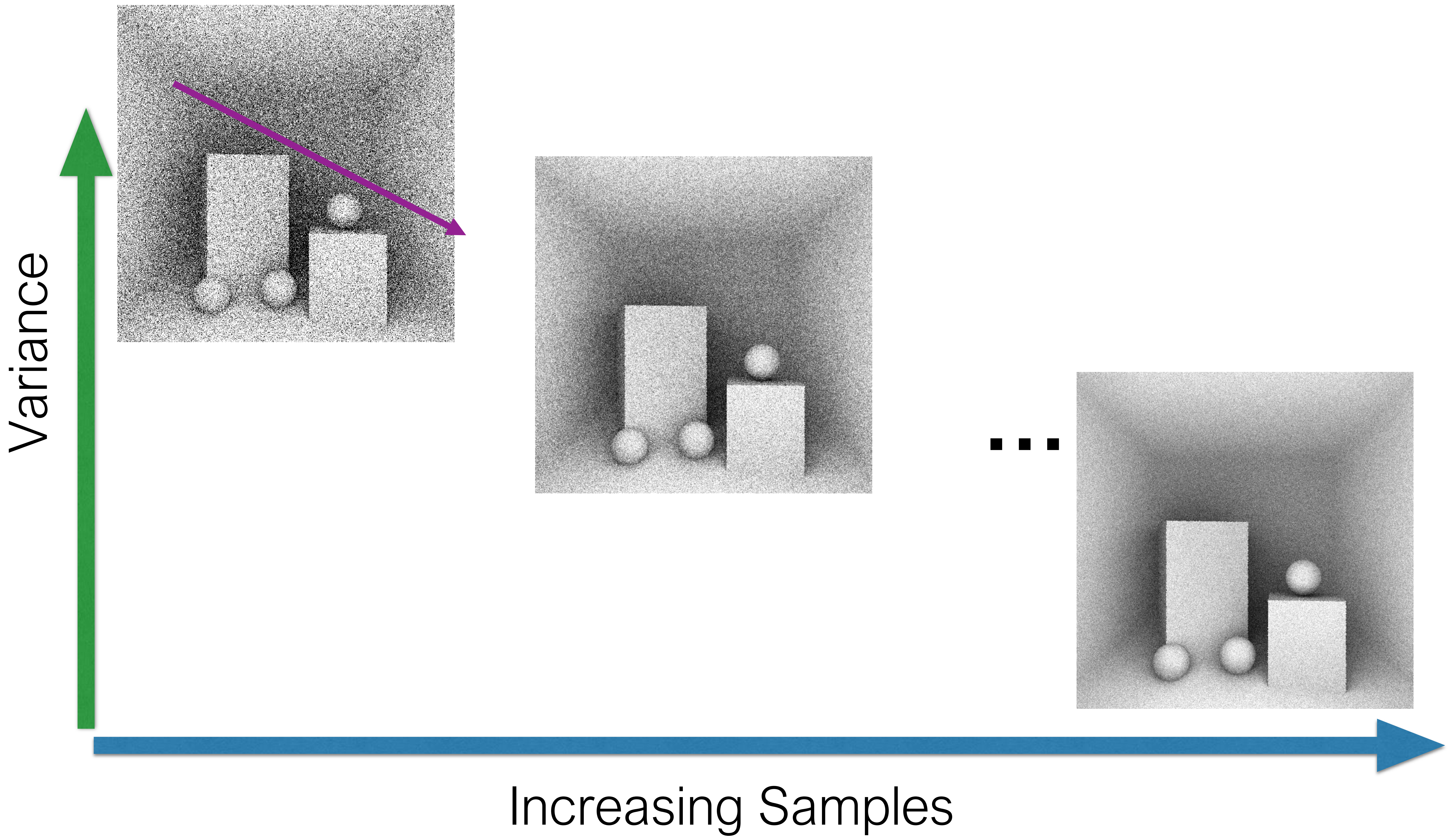


$\mathcal{O}(N^{-2})$

$\mathcal{O}(N^{-1})$

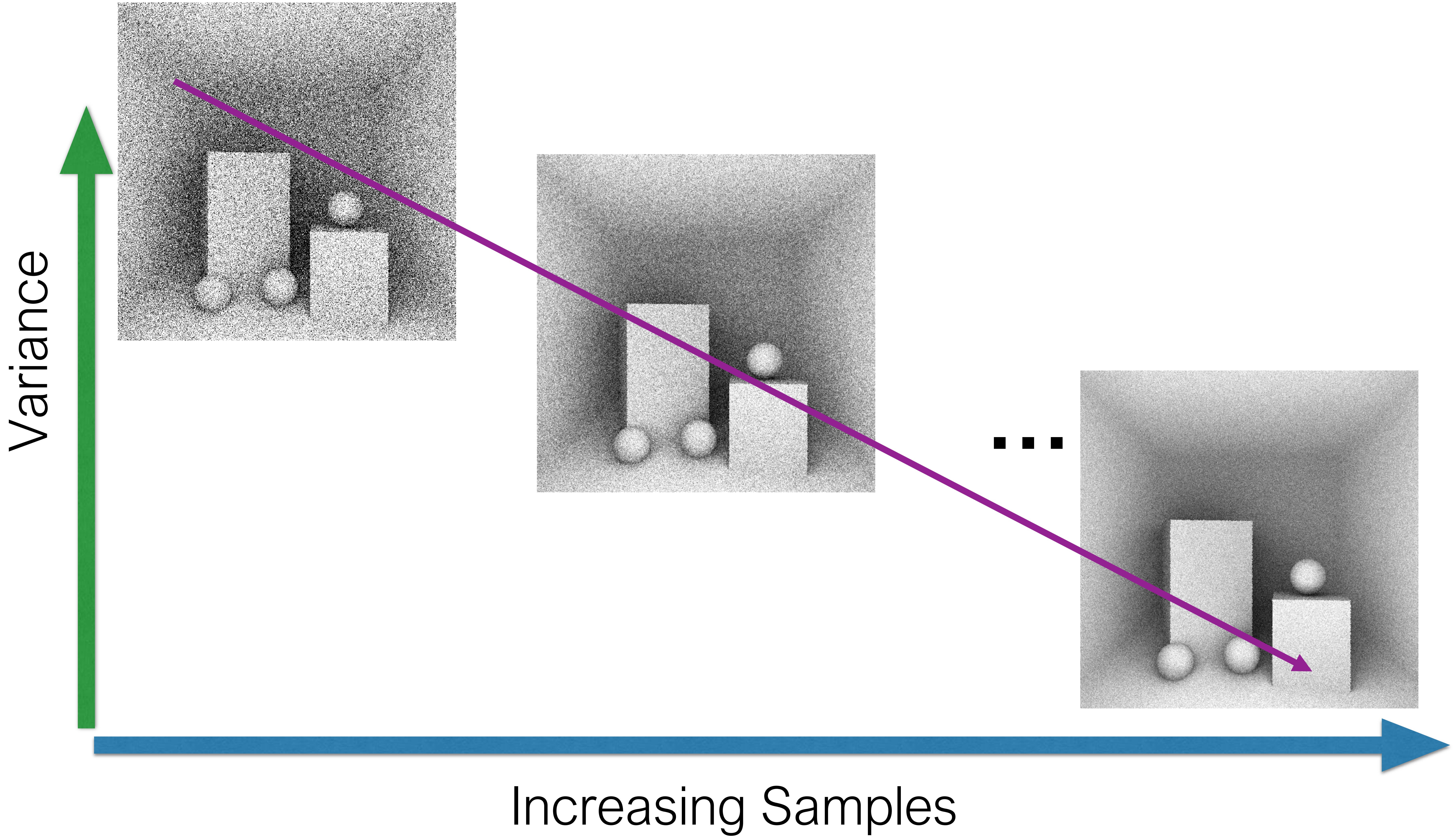
Experimental Verification

Convergence rate

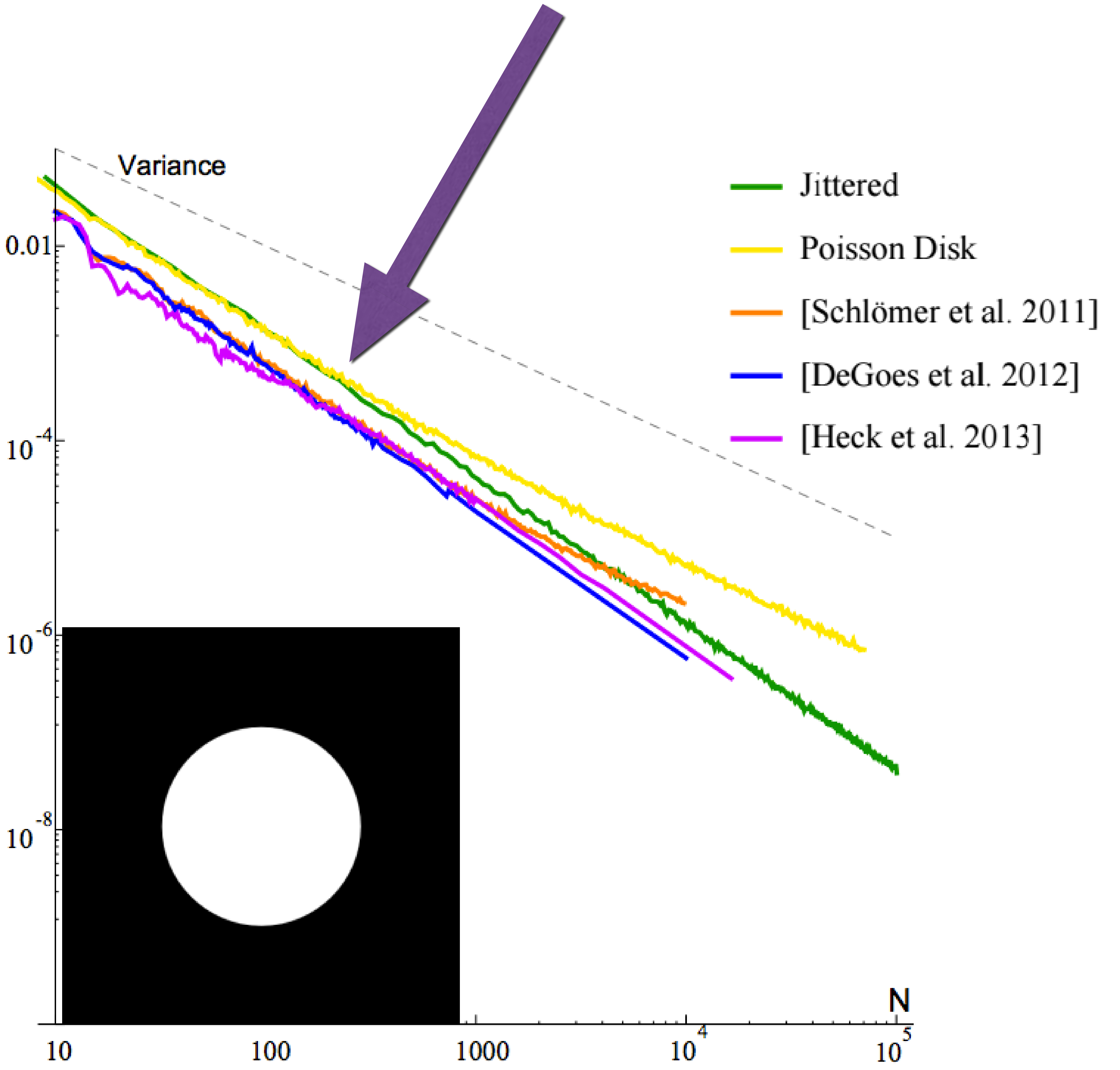


165

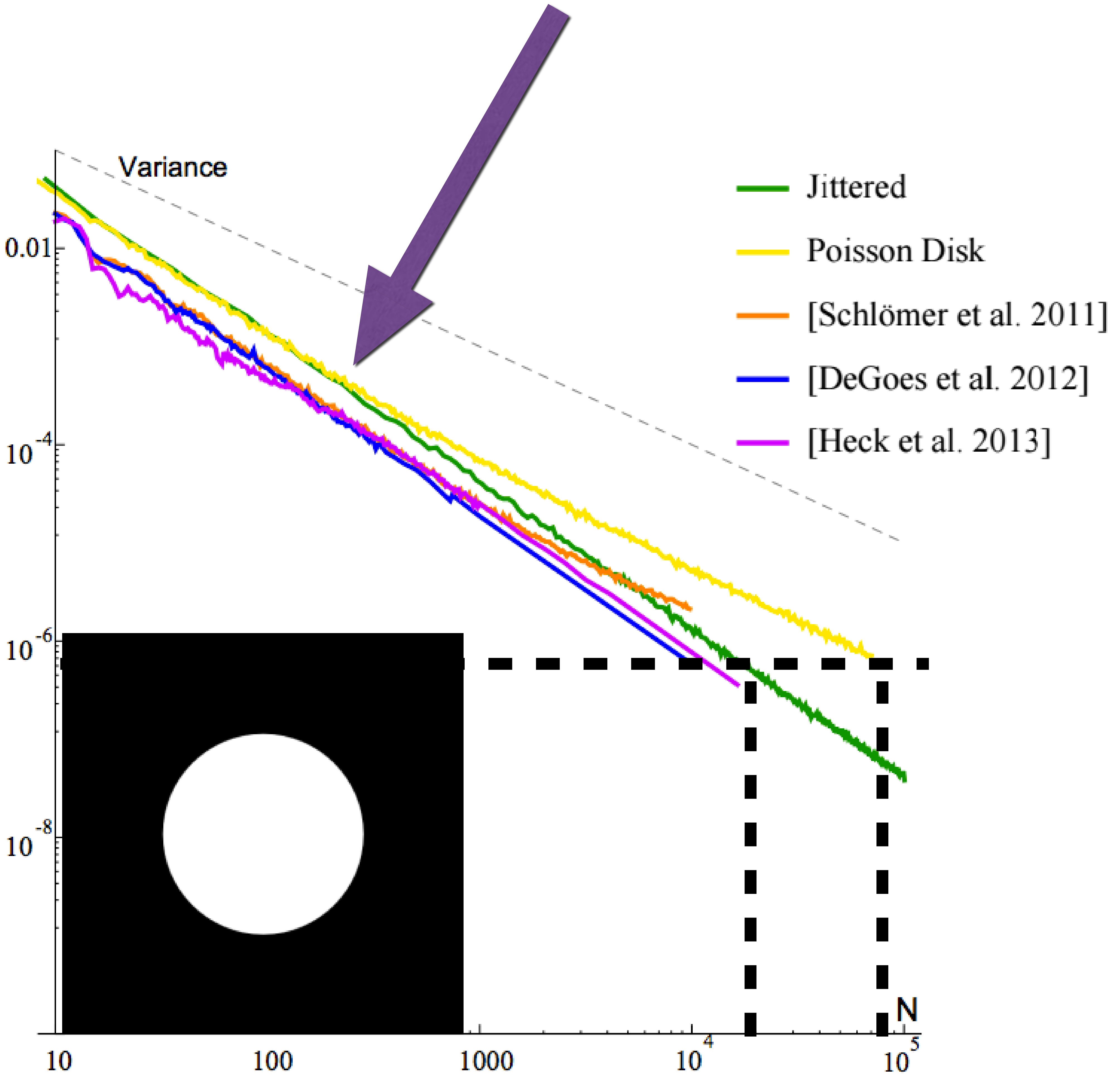
Convergence rate



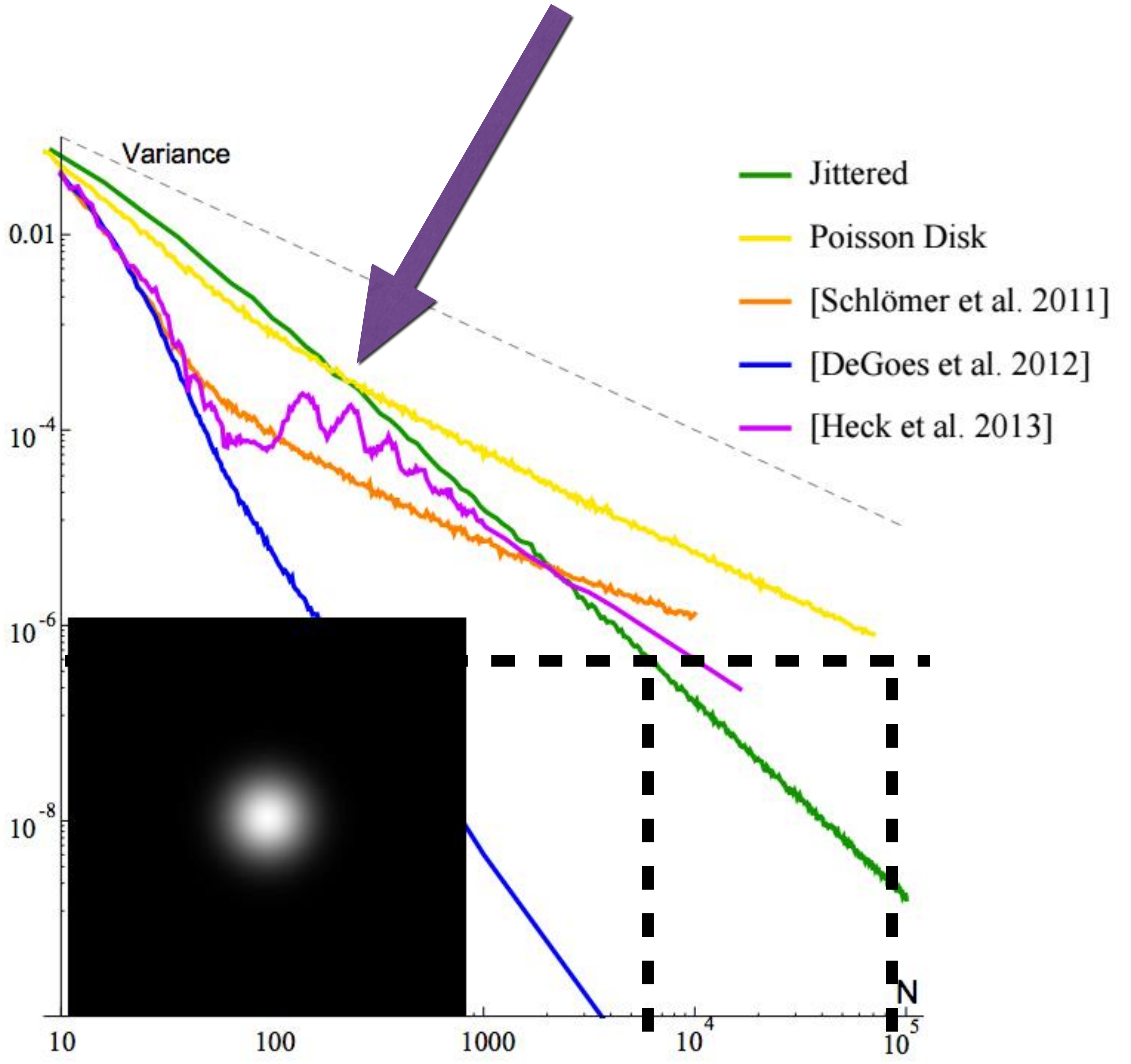
Disk Function as Worst Case



Disk Function as Worst Case



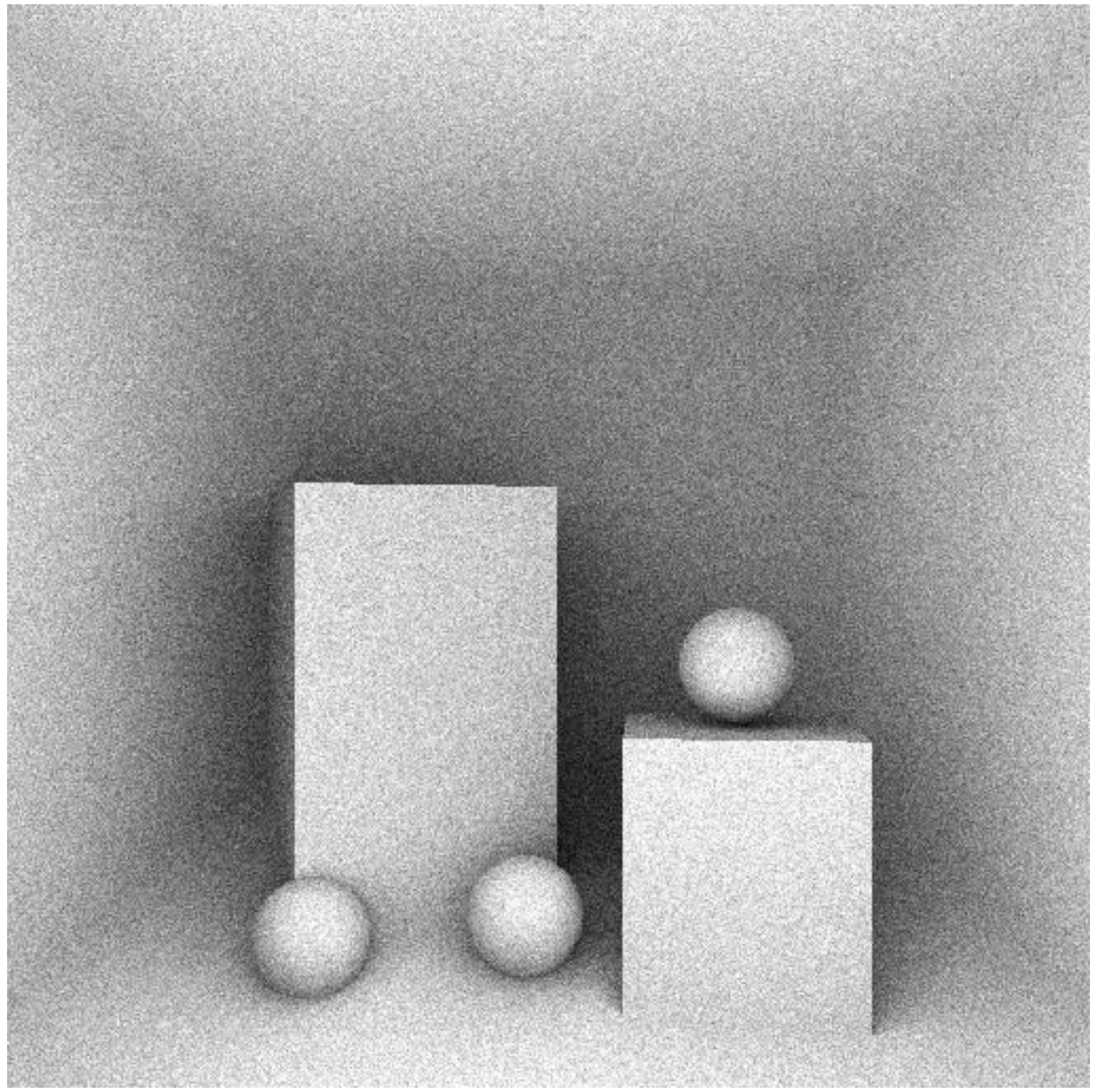
Gaussian as Best Case



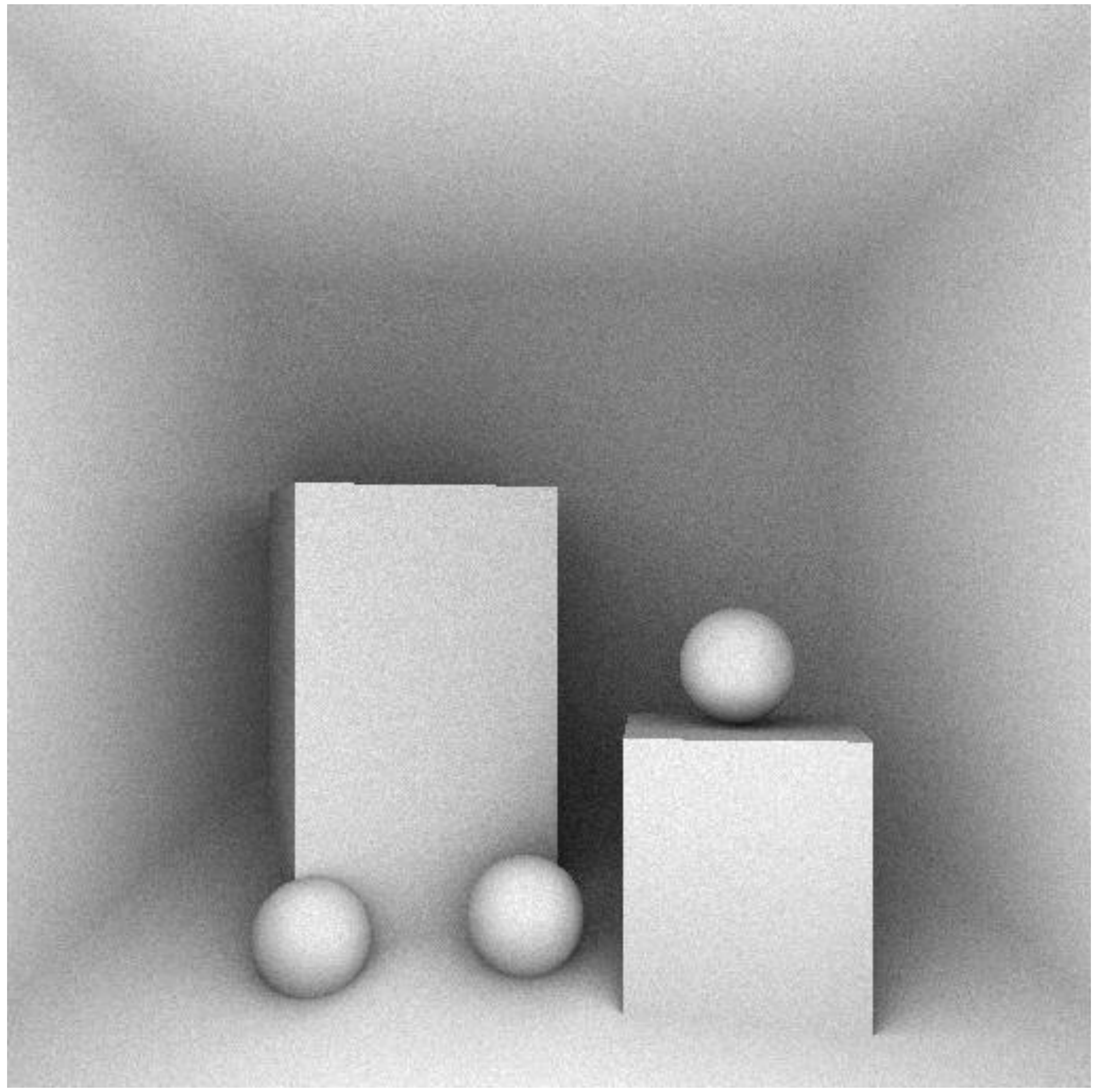
Ambient Occlusion Examples

Random vs Jittered

96 Secondary Rays



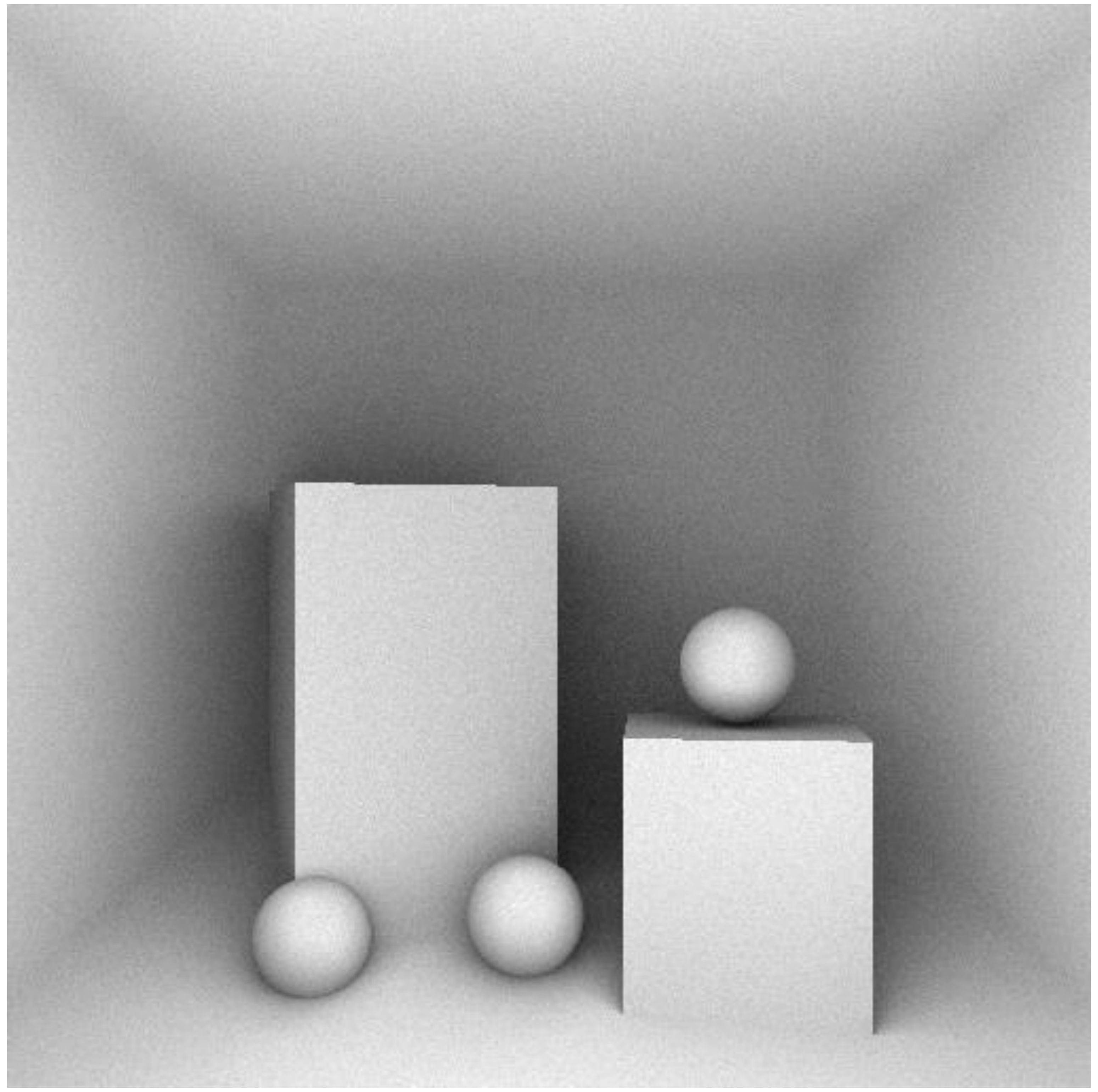
MSE: 4.74 x 10e-3



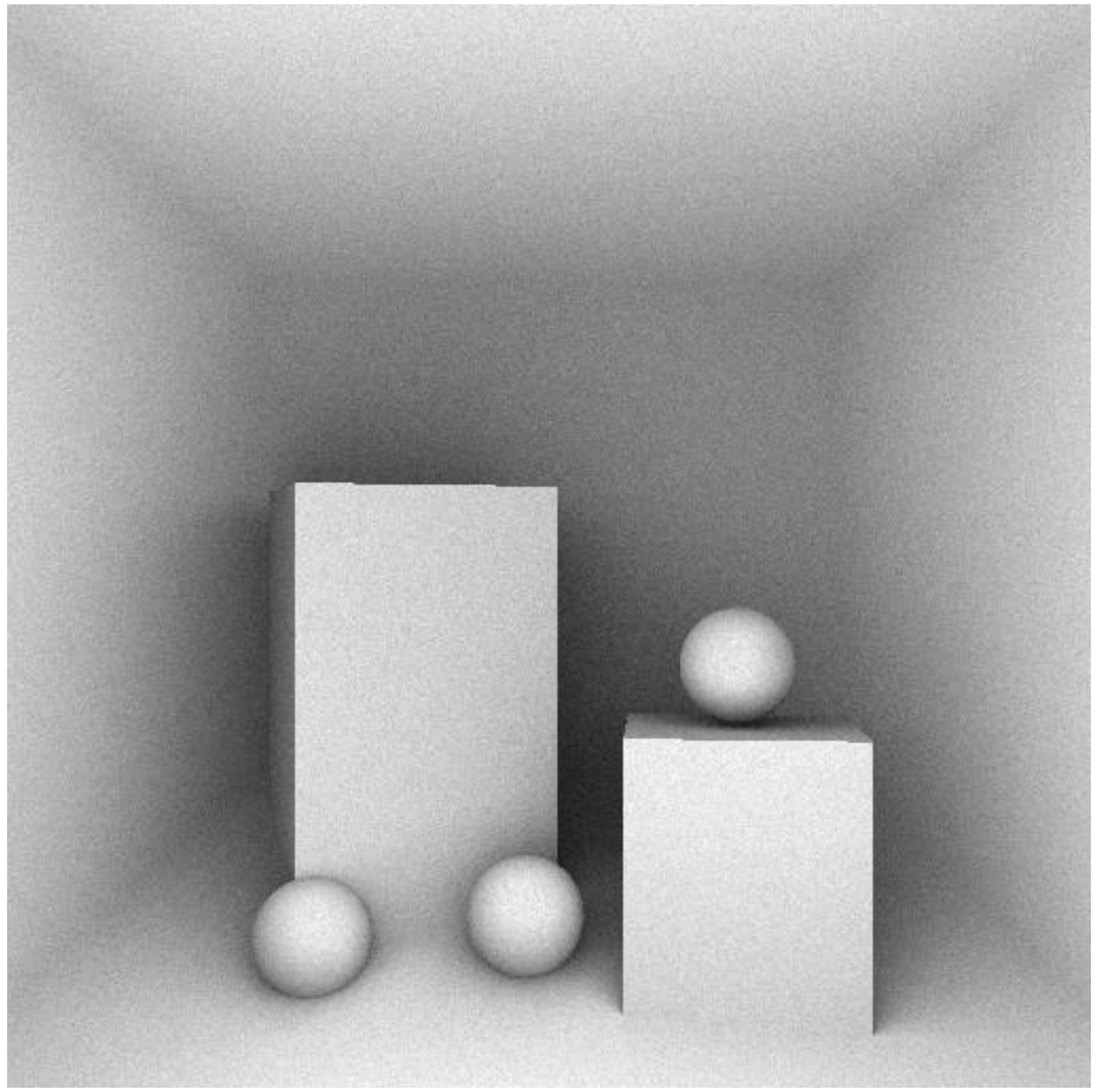
MSE: 8.56 x 10e-4

CCVT vs. Poisson Disk

96 Secondary Rays

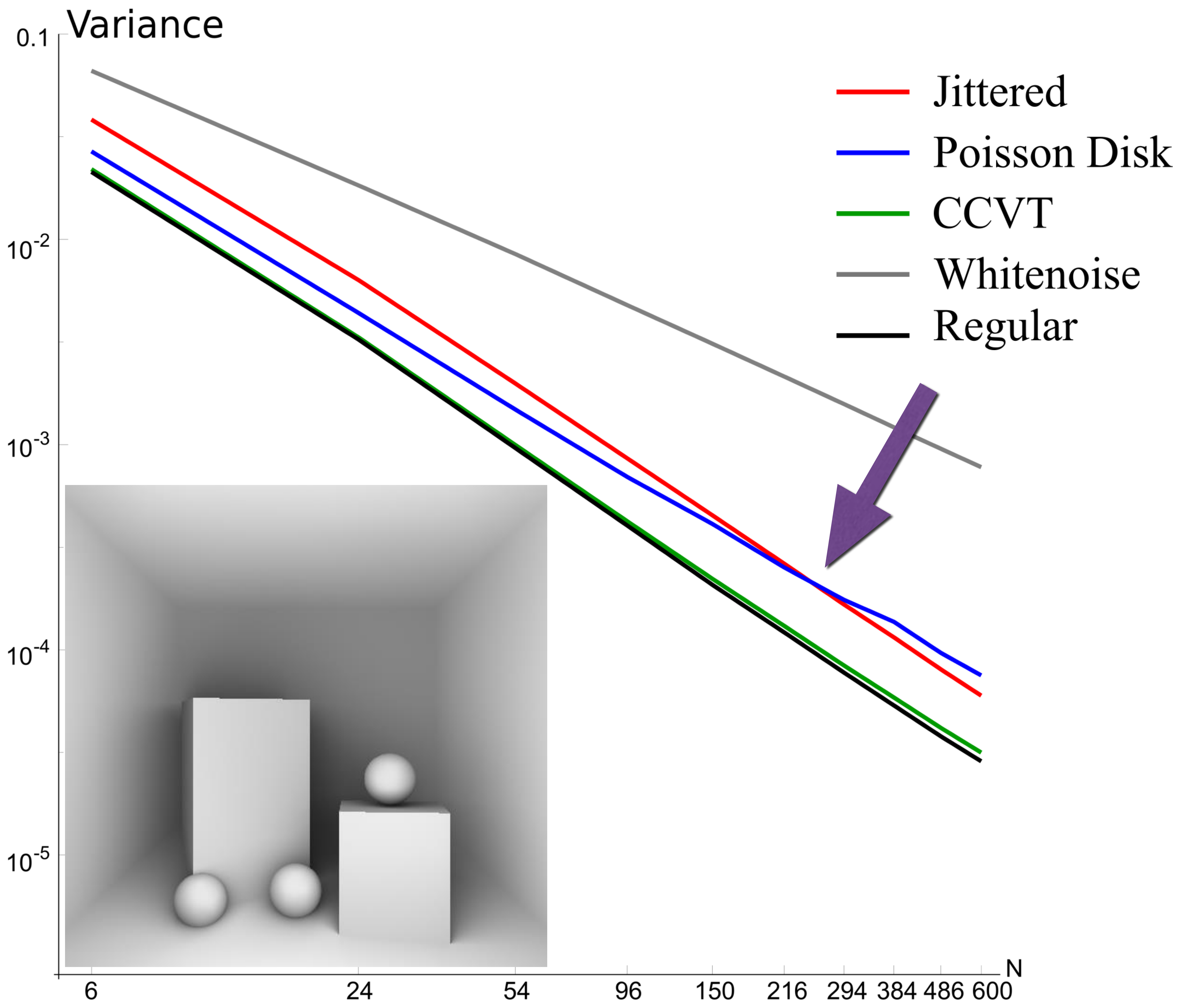


MSE: 4.24 x 10e-4

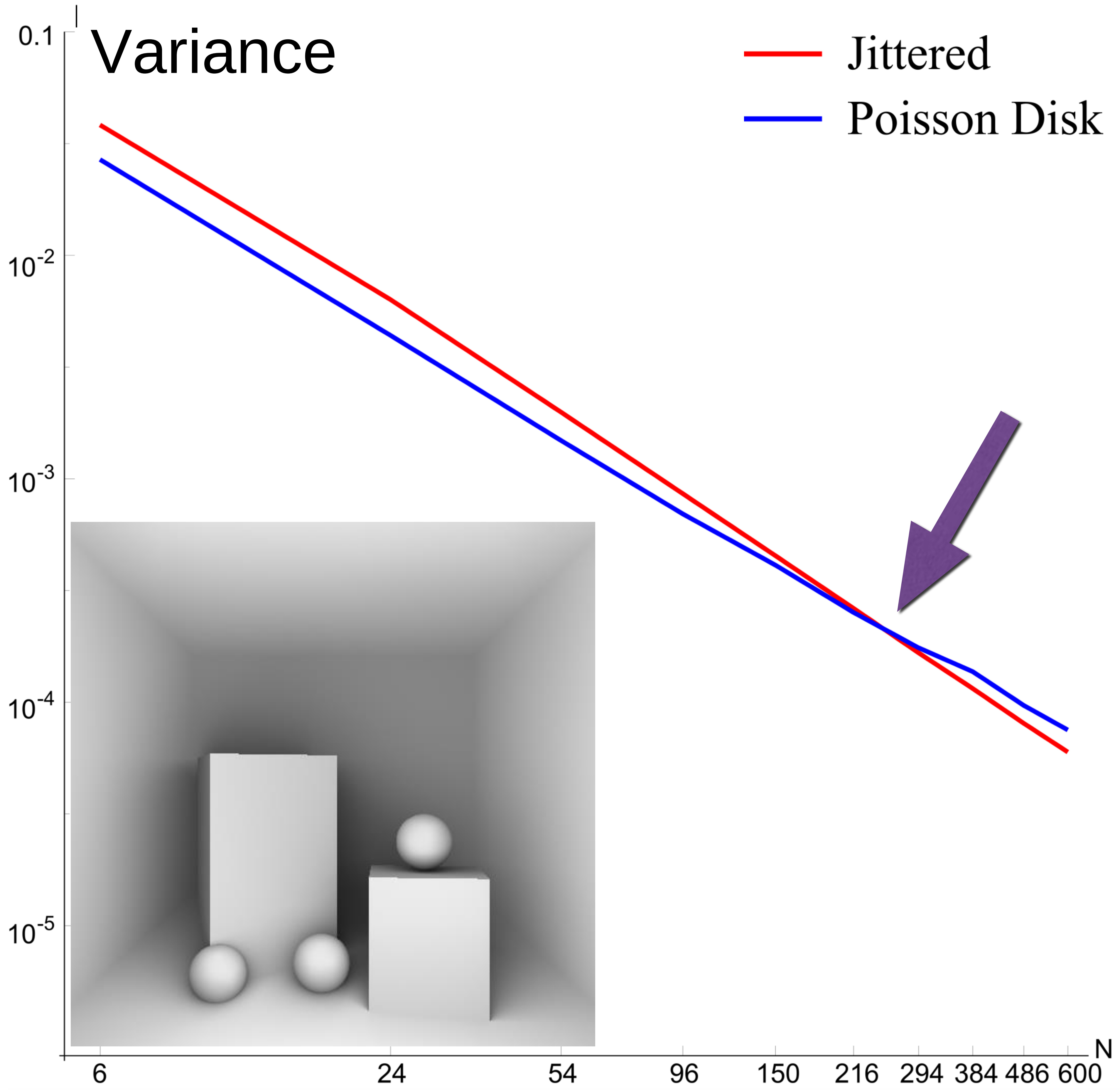


MSE: 6.95 x 10e-4

Convergence rates



Jittered vs Poisson Disk



What are the benefits of this analysis ?

- For offline rendering, analysis tells which samplers would converge faster.
- For real time rendering, blue noise samples are more effective in reducing variance for a given number of samples

Acknowledgements

Fourier Analysis of Numerical Integration in Monte Carlo Rendering

Kartic Subr

Gurprit Singh

*Wojciech Jarosz

Render the Possibilities
SIGGRAPH2016

*First part of slides are from Wojciech Jarosz