Volume Rendering

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Next career fair “next” on June 11, 2024 from 10:00 a.m. to 5:00 p.m.

The trade fair offers our students the opportunity to meet potential employers, make contacts and find out about career opportunities. Companies have the opportunity to offer internships, theses or entry-level positions.
Overview

Volumetric Processes:

- Absorption
- Scattering
- Transmittance
- Phase Functions

Volumetric Rendering Equation

Volumetric Path Tracing

Woodcock Tracking
Fog

Brassai (Gyula Halasz) 1899-1984
Snow

Gurprit Singh
Fire

Harry Potter/Warner Brothers
Surface or Volume?
Universe
Defining Participating Media

Media properties are modeled as a probabilistic process.

No need to consider individual interactions with particles (won't fit in the memory).
Defining Participating Media

Homoogeneous media:

- Infinite or bounded by a simple surface or simple shape

Krivanek et al. [2014]
Defining Participating Media

Heterogeneous media (spatially varying coefficients):

- Procedurally e.g. using a noise function
- Simulation + volume discretization, e.g., voxel grid
Radiance

Radiance is the main quantity we are interested in for rendering.

In *vaccum*, light transport radiance remains constant along rays between surfaces

\[ L_i(x, \bar{\omega}) = L_o(y, -\bar{\omega}) \]

\[ y = r(x, \bar{\omega}) \]

ray tracing function
In **participating media**, radiance may change along rays between surfaces

\[ L_i(x, \vec{\omega}) \neq L_o(y, -\vec{\omega}) \]

\[ y = r(x, \vec{\omega}) \]

ray tracing function
Volumetric Scattering Processes

Absorption

Scattering

Emission

Slide after Jan Novak
Participating Media
Participating Media
Participating Media
How much light is gained or lost during the travel through this differential beam due to the interactions with the medium?
Differential Beam

dA

\[ dx \]

\[ dz \]
Absorption

\[ \frac{dL(x, \vec{\omega})}{dz} = -\sigma_a L(x, \vec{\omega}) \]

\(\sigma_a\) : absorption coefficient \(m^{-1}\)
Absorption described by medium's absorption cross-section $\sigma_a$

Absorption

$$L(x, \omega)$$

$dA$

$dz$

outgoing radiance

$\omega$
Absorption

Absorption described by medium's absorption cross-section $\sigma_a$

$$\sigma_a \in [0, \infty)$$
Absorption

Absorption described by medium's absorption cross-section $\sigma_a$

$$\sigma_a \in [0, \infty)$$

It is the probability density that light is absorbed per unit distance travelled in the medium.

It can vary as a position and direction.
Out-Scattering

The probability of an out-scattering event occurring per unit distance is given by the scattering coefficient

\[
\frac{dL(x, \bar{\omega})}{dz} = -\sigma_s L(x, \bar{\omega})
\]

\(\sigma_s\) : scattering coefficient
Attenuation / Extinction

Total reduction in radiance:

\[ \sigma_a : \text{absorption coefficient} \]
\[ \sigma_s : \text{scattering coefficient} \]
Attenuation / Extinction

Total reduction in radiance:

\[ \sigma_t(x, \bar{\omega}) = \sigma_a(x, \bar{\omega}) + \sigma_s(x, \bar{\omega}) \]

- \( \sigma_a \): absorption coefficient
- \( \sigma_s \): scattering coefficient
- \( \sigma_t \): extinction coefficient

Out-scattering

Absorption
Albedo

\[ \alpha(x) = \frac{\sigma_s(x)}{\sigma_a(x) + \sigma_s(x)} = \frac{\sigma_s(x)}{\sigma_t(x)} \]

\( \sigma_s \) : scattering coefficient

\( \sigma_t \) : extinction coefficient
Albedo

\[ \alpha(x) = \frac{\sigma_s(x)}{\sigma_t(x)} \]

The albedo is always between 0 and 1

It describes the probability of scattering (versus absorption) at a scattering event

\[ \sigma_s : \text{scattering coefficient} \]

\[ \sigma_t : \text{extinction coefficient} \]
Mean-free path

\[
\frac{1}{\sigma_t}
\]

Mean free path gives the average distance travelled by the ray before interacting with a particle.

\(\sigma_t\) : extinction coefficient
In-Scattering

\[ \frac{dL(x, \bar{\omega})}{dz} = \sigma_s(x)L_s(x, \bar{\omega}) \]

\( \sigma_s(x) \): scattering coefficient

In-scattered radiance

\[ L_s(x, \bar{\omega}) = \int_{S^2} f_p(\bar{\omega}, \bar{\omega}') L(x, \bar{\omega}') d\bar{\omega}' \]
Emission

\[
\frac{dL(x, \omega)}{dz} = \sigma_a(x) L_e(x, \omega)
\]

\(L_e(x, \omega)\): emitted radiance

*sometimes modeled without the absorption coefficient term

Emitted radiance does not depend on the incoming light \(L_i\)

Here we made a choice to represent differential output radiance as a product of emitted radiance and absorption coefficient.
Radiative Transfer Equation
Radiative Transfer Equation (RTE)
Radiative Transfer Equation (RTE)
Radiative Transfer Equation (RTE)

\[ \frac{dL(x, \omega)}{dz} \]

Out-scattering \hspace{2cm} Absorption \hspace{2cm} Losses

In-scattering \hspace{2cm} Emission \hspace{2cm} Gains
Radiative Transfer Equation (RTE)

\[ \frac{dL(x, \bar{\omega})}{dz} = -\sigma_s(x) L_s(x, \bar{\omega}) + \sigma_s(x) L_s(x, \bar{\omega}) - \sigma_a(x) L(x, \bar{\omega}) + \sigma_a(x) L_e(x, \bar{\omega}) \]

Out-scattering \hspace{2cm} Absorption

Losses

In-scattering \hspace{2cm} Emission

Gains
Radiative Transfer Equation (RTE)

\[
\frac{dL(x, \bar{\omega})}{dz} = -\sigma_s(x)L(x, \bar{\omega}) - \sigma_a(x)L(x, \bar{\omega}) + \sigma_s(x)L_s(x, \bar{\omega}) + \sigma_a(x)L_e(x, \bar{\omega})
\]
Radiative Transfer Equation (RTE)

\[
\frac{dL(x, \omega)}{dz} = -\sigma_s(x)L(x, \omega) - \sigma_a(x)L(x, \omega) + \sigma_s(x)L_s(x, \omega) + \sigma_a(x)L_e(x, \omega)
\]

\[
\sigma_t(x, \omega) = \sigma_a(x, \omega) + \sigma_s(x, \omega)
\]
Radiative Transfer Equation (RTE)

\[
\frac{dL(x, \omega)}{dz} = -\sigma_t(x)L(x, \omega) + \sigma_s(x)L_s(x, \omega) + \sigma_a(x)L_e(x, \omega)
\]

What about a beam with finite-length \( z \)?
Extinction Along a Finite Beam

\[
\frac{dL(x, \omega)}{dz} = -\sigma_t(x)L(x, \omega)
\]

\[
\frac{dL(x, \omega)}{L(x, \omega)} = -\sigma_t(x)dz \quad \text{// Integrate along beam from 0 to } Z
\]

\[
\log_e L_z - \log_e L_0 = -\sigma_t(x)z
\]

\[
\log_e \left( \frac{L_z}{L_0} \right) = -\sigma_t z \quad \text{// Exponentiate}
\]

\[
\frac{L_z}{L_0} = e^{-\sigma_t z}
\]
Beer-Lambert Law

The fraction refers to as the \textit{transmittance}

\[
\frac{L_z}{L_0} = e^{-\sigma_t z}
\]

Think of this as fractional visibility loss between two points
Beer-Lambert Law

Expresses the remaining radiance after traveling a finite distance through the medium with constant extinction coefficient.

The fraction refers to as the *transmittance*:

\[ \frac{L_z}{L_0} = e^{-\sigma_t z} \]

Radiances at distance 0 and z.

Think of this as fractional visibility loss between two points.
Beam Transmittance

$L_0(x, \vec{\omega})$

$x \rightarrow z \rightarrow y$

$\sigma_t$: extinction coefficient
Beam Transmittance

\[ T_r(x \to y) = e^{-\int_0^{|x-y|} \sigma_t(t) \, dt} \]

\[ T_r(x \to x') L_o(x, \vec{\omega}) \]

\( \sigma_t \): extinction coefficient
Beam Transmittance

\[ T_r(x \rightarrow y) = e^{-\int_0^{||x-y||} \sigma_t(t) dt} \]

\[ L_o(x, \omega) \]

\[ T_r(x \rightarrow x') L_o(x, \omega) \]

\( \sigma_t \) : extinction coefficient
Beam Transmittance

\[ T_r(x \rightarrow y) = e^{-\int_0^{||x-y||} \sigma_t(t) dt} \]

\( \sigma_t \) : extinction coefficient
Beam Transmittance: **Multiplicative**

\[ T_r(x \rightarrow x'') = T_r(x \rightarrow x')T_r(x' \rightarrow x'') \]

\[ L_o(x, \vec{\omega}) \]

\[ T_r(x \rightarrow x') \]

\[ T_r(x' \rightarrow x'') \]

\[ \sigma_t : \text{extinction coefficient} \]
Beam Transmittance

In Homogeneous medium $\sigma_t$ is a constant:

$$T_r(x \to y) = e^{-\sigma_t \|x-y\|}$$

In Heterogeneous medium (spatially varying $\sigma_t$):

$$T_r(x \to y) = e^{-\int_0^{\|x-y\|} \sigma_t(t) dt}$$

Optical thickness
Radiative Transfer Equation (RTE)

\[
\frac{dL(x, \omega)}{dz} = -\sigma_t(x) L(x, \omega) + \sigma_s(x) L_s(x, \omega) + \sigma_a(x) L_e(x, \omega)
\]

What about a beam with finite-length \( z \)?
Volumetric Rendering Equation

\[ L(x, \bar{\omega}) = T_r(x, x_z) L(x_z, \bar{\omega}) \]
Volumetric Rendering Equation

\[ L(x, \omega) = T_r(x, x_z) L(x_z, \omega) \]

Reduced (background) surface radiance
Volumetric Rendering Equation

\[
L(x, \vec{\omega}) = \mathcal{T}_r(x, x_z)L(x_z, \vec{\omega}) \\
+ \int_0^z \mathcal{T}_r(x, x_t)\sigma_a(x_t)L_e(x_t, \vec{\omega}) dt
\]

Accumulated emitted radiance
Volumetric Rendering Equation

\[ L(x, \tilde{\omega}) = T_r(x, x_z) L(x_z, \tilde{\omega}) \]

\[ + \int_0^z T_r(x, x_t) \sigma_a(x_t) L_e(x_t, \tilde{\omega}) dt \]

\[ + \int_0^z T_r(x, x_t) \sigma_s(x_t) L_s(x_t, \tilde{\omega}) dt \]

Accumulated in-scattered radiance
Volumetric Rendering Equation

\[ L(x, \omega) = T_r(x, x_z) L(x_z, \omega) \]

\[ + \int_{0}^{z} T_r(x, x_t) \sigma_a(x_t) L_e(x_t, \omega) dt \]

\[ + \int_{0}^{z} T_r(x, x_t) \sigma_s(x_t) \int_{S^2} f_p(x_t, \omega', \omega) L_i(x_t, \omega') d\omega' dt \]
Volumetric Rendering Equation

\[
L(x, \tilde{\omega}) = T_r(x, x_z) L(x_z, \tilde{\omega}) \\
+ \int_0^z T_r(x, x_t) \sigma_a(x_t) L_e(x_t, \tilde{\omega}) dt \\
+ \int_0^z T_r(x, x_t) \sigma_s(x_t) \int_{S^2} f_p(x_t, \tilde{\omega}', \tilde{\omega}) L_i(x_t, \tilde{\omega}') d\tilde{\omega}' dt
\]
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Scattering in Media
Phase Functions

It describes the angular distribution of scattered radiation at a point;

It is the volumetric analog to the BSDF, but it is different from the BSDF.

It has a normalization constant:

$$\int_{S^2} f_p(\vec{\omega}, \vec{\omega}') d\vec{\omega}' = 1 \quad \forall \vec{\omega}$$

This constraint means that phase functions actually define probability distributions for scattering in a particular direction.
Phase Functions

Isotropic:

\[ f_P(\vec{\omega}_o, \vec{\omega}_i) = \frac{1}{4\pi} \]

Uniform scattering, analogous to Lambertian BRDF
Phase Functions

Quantifying anisotropy by

\[ g = \int_{S^2} f_p(x, \hat{w}, \hat{w}') \cos \theta d\hat{w}' \]

where

\[ \cos \theta = -\hat{w} \cdot \hat{w}' \]

\( g = 0 \) : isotropic scattering (on average)
\( g > 0 \) : forward scattering
\( g < 0 \) : backward scattering

\( g \) is the asymmetry parameter
Henyey-Greenstein Phase Function

\[ f_P(\theta) = \frac{1}{4\pi} \frac{1 - g^2}{(1 + g^2 + 2g\cos\theta)^{3/2}} \quad g \in [-1, 1] \]

\[ g = 0 \]
Henyey-Greenstein Phase Function

\[ f_p(\theta) = \frac{1}{4\pi} \frac{1 - g^2}{(1 + g^2 + 2g\cos\theta)^{3/2}} \]

\[ g \in [-1, 1] \]

\[ g = 0 \]

\[ g > 0 \]
Henyey-Greenstein Phase Function

\[ f_P(\theta) = \frac{1}{4\pi} \frac{1 - g^2}{(1 + g^2 + 2g\cos\theta)^{3/2}} \quad g \in [-1, 1] \]

\( g < 0 \)

\( g = 0 \)

\( g > 0 \)
Henyey-Greenstein Phase Function

\[ f_p(\theta) = \frac{1}{4\pi} \frac{1 - g^2}{(1 + g^2 + 2g(\cos \theta))^{3/2}} \]

\[ g = -0.5 \]
\[ g = 0 \]
\[ g = 0.8 \]
Henyey-Greenstein Phase Function

$g = -0.7$

Strong backward scattering

$g = 0.7$

Strong forward scattering

PBRTv3 [2016]
Schlick's Phase Function

Empirical Phase Function

Faster approximation to HG

\[ f_p(\theta) = \frac{1}{4\pi} \frac{1 - k^2}{(1 - k \cos \theta)^2} \]

\[ k = 1.55g - 0.55g^3 \]
Schlick's Phase Function

Empirical Phase Function

Faster approximation to HG

\[ g = -0.5 \quad k = -0.706 \]

\[ g = 0 \quad k = 0 \]

\[ g = 0.8 \quad k = 0.96 \]
Lorenz-Mie Scattering

For large-size particles (scatterers), we cannot ignore the wave nature of light.

Solution to Maxwell's equations for scattering from many spherical dielectric particles explains many phenomena.

Complicated: solution is an infinite analytic series.
Lorenz-Mie Scattering

Sphere diameter = 1\mu m  

Sphere diameter = 10\mu m  

Sphere diameter = 100\mu m
Lorenz-Mie Scattering

Sphere diameter = $1\mu m$  
Sphere diameter = $10\mu m$  
Sphere diameter = $100\mu m$

Linear plot

Log plot
Lorenz-Mie Approximations

\[ f^\text{hazy}_p(\theta) = \frac{1}{4\pi} \left( 5 + \left( \frac{1 + \cos \theta}{2} \right)^8 \right) \]
Lorenz-Mie Approximations

Hazy atmosphere

\[ f_p^{\text{hazy}}(\theta) = \frac{1}{4\pi} \left( 5 + \left( \frac{1 + \cos \theta}{2} \right)^8 \right) \]

Murky atmosphere

\[ f_p^{\text{murky}}(\theta) = \frac{1}{4\pi} \left( 17 + \left( \frac{1 + \cos \theta}{2} \right)^{32} \right) \]
Why is the Sky Blue?

Atmosphere
Why is the Sunset Red?
Why is the Sunset Red?
Rayleigh Scattering
Rayleigh Scattering

Approximation of Lorenz-Mie for tiny particles (scatterers) that are typically smaller than 1/10th the wavelength of visible light

Used for atmospheric scattering, gasses, transparent solids

Highly wavelength dependent
Rayleigh Phase Function

\[ f_p^{\text{Rayleigh}}(\theta) = \frac{3}{16\pi} (1 + \cos^2 \theta) \]

Scattering at right angles is half as likely as scattering forward or backward
Rayleigh Scattering

\[ \beta_s^{\text{Rayleigh}}(\lambda, d, \eta, \rho) = \rho \frac{2\pi^5 d^6}{3\lambda^4} \left( \frac{\eta^2 - 1}{\eta^2 + 2} \right)^2 \]
Rayleigh Scattering

\[ \beta_{s}^{\text{Rayleigh}}(\lambda, d, \eta, \rho) = \rho \frac{2\pi^5 d^6}{3\lambda^4} \left( \frac{\eta^2 - 1}{\eta^2 + 2} \right)^2 \]
Rayleigh Scattering

\[ \beta_{Rayleigh}(\lambda, d, \eta, \rho) = \rho \frac{2\pi^5 d^6}{3\lambda^4} \left( \frac{\eta^2 - 1}{\eta^2 + 2} \right)^2 \]

- **Wavelength**
- **Diameter of particles**
Rayleigh Scattering

\[ \beta_{\text{Rayleigh}}(\lambda, d, \eta, \rho) = \rho \frac{2\pi^5 d^6}{3 \lambda^4} \left( \frac{\eta^2 - 1}{\eta^2 + 2} \right)^2 \]

- Wavelength
- Index of refraction
- Diameter of particles
Rayleigh Scattering

\[ \beta_{s}^{\text{Rayleigh}}(\lambda, d, \eta, \rho) = \rho \frac{2\pi^5 d^6}{3\lambda^4} \left( \frac{\eta^2 - 1}{\eta^2 + 2} \right)^2 \]

- Wavelength
- Index of refraction
- Diameter of particles
- Density of particles
Recap: Phase Functions

Isotropic  Henyey-Greenstein  Rayleigh  Lorenz-Mie small particles  Lorenz-Mie large particles
Anisotropy: Phase Function vs. Medium

Isotropic Medium

| Isotropic phase function | Anisotropic phase function |

Slide after Jan Novak
Anisotropy: Phase Function vs. Medium

Isotropic Medium
- Isotropic phase function

Anisotropic Medium
- Anisotropic phase function

Slide after Jan Novak
Recap: Media Properties

Given:
- Absorption coefficient $\sigma_a(x)$ $[m^{-1}]$
- Scattering coefficient $\sigma_s(x)$ $[m^{-1}]$
- Phase function $f_p(x, \bar{\omega}, \bar{\omega}')$ $[sr^{-1}]$
Recap: Media Properties

Given:

- Absorption coefficient \( \sigma_a(x) \) \([m^{-1}]\)
- Scattering coefficient \( \sigma_s(x) \) \([m^{-1}]\)
- Phase function \( f_p(x, \vec{\omega}, \vec{\omega}') \) \([sr^{-1}]\)

Derived:

- Extinction coefficient \( \sigma_t(x) = \sigma_a(x) + \sigma_s(x) \) \([m^{-1}]\)
- Albedo \( \alpha(x) = \sigma_s(x)/\sigma_t(x) \) \([\text{None}]\)
- Mean-free path \( 1/\sigma_t(x) \) \([m]\)
- Transmittance \( T_r(x, y) = e^{-\int_0^{||x-y||} \sigma_t(t)dt} \) \([\text{None}]\)
For Homogeneous Isotropic Medium

Given:
- Absorption coefficient \( \sigma_a \) \([m^{-1}]\)
- Scattering coefficient \( \sigma_s \) \([m^{-1}]\)
- Phase function \( \frac{1}{4\pi} \) \([sr^{-1}]\)

Derived:
- Extinction coefficient \( \sigma_t = \sigma_a + \sigma_s \) \([m^{-1}]\)
- Albedo \( \alpha = \sigma_s / \sigma_t \) [None]
- Mean-free path \( 1/\sigma_t \) \([m]\)
- Transmittance \( T_r(x, y) = e^{-\sigma_t||x-y||} \) [None]
Solving the Volumetric Rendering Equation
Complexity

Homogeneous vs. Heterogeneous

Scattering
- none
- fake
- single scattering
- multiple scattering
Volumetric Rendering Equation

\[
L(x, \bar{\omega}) = T_r(x, x_z) L(x_z, \bar{\omega}) \\
+ \int_0^z T_r(x, x_t) \sigma_a(x_t) L_e(x_t, \bar{\omega}) dt \\
+ \int_0^z T_r(x, x_t) \sigma_s(x_t) \int_{S^2} f_p(x_t, \bar{\omega}', \bar{\omega}) L_i(x_t, \bar{\omega}') d\omega' dt
\]
Volumetric Rendering Equation

\[ L(\mathbf{x}, \bar{\omega}) = T_r(\mathbf{x}, \mathbf{x}_z) L(\mathbf{x}_z, \bar{\omega}) \]

\[ + \int_0^z T_r(\mathbf{x}, \mathbf{x}_t) \sigma_a(\mathbf{x}_t) L_e(\mathbf{x}_t, \bar{\omega}) \, dt \]

\[ + \int_0^z T_r(\mathbf{x}, \mathbf{x}_t) \sigma_s(\mathbf{x}_t) \int_{S^2} f_p(\mathbf{x}_t, \bar{\omega}', \bar{\omega}) L_i(\mathbf{x}_t, \bar{\omega}') \, d\omega' \, dt \]

\[ \text{Attenuated background radiance} \]
Volumetric Rendering Equation

\[ L(\mathbf{x}, \bar{\omega}) = T_r(\mathbf{x}, \mathbf{x}_z) L(\mathbf{x}_z, \bar{\omega}) \]

\[ + \int_0^z T_r(\mathbf{x}, \mathbf{x}_t) \sigma_a(\mathbf{x}_t) L_e(\mathbf{x}_t, \bar{\omega}) \, dt \]

\[ + \int_0^z T_r(\mathbf{x}, \mathbf{x}_t) \sigma_s(\mathbf{x}_t) \int_{S^2} f_p(\mathbf{x}_t, \bar{\omega}', \bar{\omega}) L_i(\mathbf{x}_t, \bar{\omega}') \, d\omega' \, dt \]

Attenuated background radiance

Accumulated emitted radiance
Volumetric Rendering Equation

\[ L(x, \omega) = T_r(x, x_z) L(x_z, \omega) \]

**Attenuated background radiance**

\[ + \int_0^z T_r(x, x_t) \sigma_a(x_t) L_e(x_t, \omega) \, dt \]

**Accumulated emitted radiance**

\[ + \int_0^z T_r(x, x_t) \sigma_s(x_t) \int_{S^2} f_p(x_t, \omega', \omega) L_i(x_t, \omega') \, d\omega' \, dt \]

**Accumulated in-scattered radiance**
Heterogeneous/Homogeneous media
Homogeneous media
Heterogeneous media

Homogeneous media
Participating Media: Heterogeneous

\[ L(x, \bar{\omega}) = \int_0^s T_r(x \leftrightarrow x_t) \sigma_s(x_t) L_i(x_t, \bar{\omega}) dt + T_r(x \leftrightarrow x_s) L(x_s, \bar{\omega}) \]
Participating Media: Homogeneous

\[ L(x, \bar{\omega}) = \int_0^s T_r(x \leftrightarrow x_t)\sigma_s(x_t)L_i(x_t, \bar{\omega})dt + T_r(x \leftrightarrow x_s) L(x_s, \bar{\omega}) \]

\[ L(x, \bar{\omega}) = \sigma_s \int_0^s T_r(x \leftrightarrow x_t)L_i(x_t, \bar{\omega})dt + T_r(x \leftrightarrow x_s) L(x_s, \bar{\omega}) \]
Participating Media: Homogeneous

\[
L(x, \tilde{\omega}) = \int_0^s T_r(x \leftrightarrow x_t)\sigma_s(x_t)L_i(x_t, \tilde{\omega})dt + T_r(x \leftrightarrow x_s)L(x_s, \tilde{\omega})
\]

\[
L(x, \tilde{\omega}) = \sigma_s \int_0^s T_r(x \leftrightarrow x_t)L_i(x_t, \tilde{\omega})dt + T_r(x \leftrightarrow x_s)L(x_s, \tilde{\omega})
\]
Participating Media: Homogeneous

\[ L(x, \bar{\omega}) = \int_0^s T_r(x \leftrightarrow x_t) \sigma_s(x_t) L_i(x_t, \bar{\omega}) dt + T_r(x \leftrightarrow x_s) L(x_s, \bar{\omega}) \]

\[ L(x, \bar{\omega}) = \sigma_s \int_0^s T_r(x \leftrightarrow x_t) L_i(x_t, \bar{\omega}) dt + T_r(x \leftrightarrow x_s) L(x_s, \bar{\omega}) \]

\[ L(x, \bar{\omega}) = \sigma_s \int_0^s e^{-t\sigma_t} L_i(x_t, \bar{\omega}) dt + e^{-s\sigma_t} L(x_s, \bar{\omega}) \]
Participating Media: Homogeneous

\[ L(x, \omega) = \sigma_s \int_0^s e^{-t \sigma_t} L_i(x_t, \omega) dt + e^{-s \sigma_t} L(x_s, \omega) \]
Homogeneous Ambient Media

\[ L(x, \omega) = \sigma_s \int_0^s e^{-t\sigma_t} L_i(x_t, \omega) dt + e^{-s\sigma_t} L(x_s, \omega) \]
Homogeneous Ambient Media

Assume in-scattered radiance is an ambient constant

\[ L(x, \bar{\omega}) = \sigma_s \int_0^s e^{-t\sigma_t} L_i(x_t, \bar{\omega}) dt + e^{-s\sigma_t} L(x_s, \bar{\omega}) \]
Homogeneous Ambient Media

Assume in-scattered radiance is an ambient constant

\[ L(x, \bar{\omega}) = \sigma_s \int_0^s e^{-t\sigma_t} L_i(x_t, \bar{\omega}) dt + e^{-s\sigma_t} L(x_s, \bar{\omega}) \]

\[ L(x, \bar{\omega}) = \sigma_s L_i \int_0^s e^{-t\sigma_t} dt + e^{-s\sigma_t} L(x_s, \bar{\omega}) \]
Homogeneous Ambient Media

Assume in-scattered radiance is an ambient constant

\[ L(x, \bar{\omega}) = \sigma_s \int_0^s e^{-t\sigma_t} L_i(x_t, \bar{\omega}) dt + e^{-s\sigma_t} L(x_s, \bar{\omega}) \]

\[ L(x, \bar{\omega}) = \sigma_s L_i \int_0^s e^{-t\sigma_t} dt + e^{-s\sigma_t} L(x_s, \bar{\omega}) \]
Homogeneous Ambient Media

Assume in-scattered radiance is an ambient constant

\[ L(x, \bar{\omega}) = \sigma_s \int_0^s e^{-t\sigma_t} L_i(x_t, \bar{\omega}) dt + e^{-s\sigma_t} L(x_s, \bar{\omega}) \]

\[ L(x, \bar{\omega}) = \sigma_s L_i \int_0^s e^{-t\sigma_t} dt + e^{-s\sigma_t} L(x_s, \bar{\omega}) \]

\[ L(x, \bar{\omega}) = \sigma_s L_i \frac{1 - e^{-s\sigma_t}}{\sigma_t} + e^{-s\sigma_t} L(x_s, \bar{\omega}) \]
Homogeneous Ambient Media

Assume in-scattered radiance is an ambient constant

\[ L(x, \bar{\omega}) = \sigma_s \int_0^s e^{-t\sigma_t} L_i(x_t, \bar{\omega}) dt + e^{-s\sigma_t} L(x_s, \bar{\omega}) \]

\[ L(x, \bar{\omega}) = \sigma_s L_i \int_0^s e^{-t\sigma_t} dt + e^{-s\sigma_t} L(x_s, \bar{\omega}) \]

\[ L(x, \bar{\omega}) = \sigma_s L_i \frac{1 - e^{-s\sigma_t}}{\sigma_t} + e^{-s\sigma_t} L(x_s, \bar{\omega}) \]

\[ L(x, \bar{\omega}) = \text{lerp} \left( \frac{\sigma_s}{\sigma_t} L_i, L(x_s, \bar{\omega}), e^{-s\sigma_t} \right) \]
Homogeneous Ambient Media

Fog

Clear Day
Fog
Volumetric Rendering Equation

\[ L(x, \bar{\omega}) = T_r(x, x_z)L(x_z, \bar{\omega}) \]

\[ + \int_0^{z} T_r(x, x_t)\sigma_a(x_t)L_e(x_t, \bar{\omega})dt \]

\[ + \int_0^{z} T_r(x, x_t)\sigma_s(x_t) \int_{S^2} f_p(x_t, \bar{\omega}', \bar{\omega})L_i(x_t, \bar{\omega}')d\omega' dt \]

Accumulated in-scattered radiance
In-scattered Radiance

\[ L(x, \omega) = \int_0^z T_r(x, x_t) \sigma_s(x_t) \int_{S^2} f_p(x_t, \omega', \bar{\omega}) L_i(x_t, \bar{\omega'}) d\omega' dt \]
In-scattered Radiance

\[ L(x, \omega) = \int_0^Z T_r(x, x_t) \sigma_s(x_t) \int_{S^2} f_p(x_t, \omega', \omega') L_i(x_t, \omega') \, d\omega' \, dt \]

\[ L_s(x, \omega) = \int_{S^2} f_p(x_t, \omega', \omega) L_i(x_t, \omega') \, d\omega' \, dt \]

**Single scattering**

\[ L_i \text{ arrives directly from a light source (direct illumination)} \]

\[ L_i(x, \omega) = T_r(x, r(x, \omega)) L_e(r(x, \omega), -\omega) \]

**Multiple scattering**

arrives through multiple bounces (indirect illumination)
Single Scattering

\[ L(x, \omega) = \int_0^z T_r(x, x_t) \sigma_s(x_t) \int_{S^2} f_p(x, \omega, \omega') T_r(x_t, x_e) L_e(x_e, -\omega') d\omega' dt \]
Single Scattering

\[ L(x, \bar{\omega}) = \int_{0}^{\bar{z}} T_r(x, x_t) \sigma_s(x_t) \int_{S^2} f_p(x, \bar{\omega}, \bar{\omega}') T_r(x_t, x_e) L_e(x_e, -\bar{\omega}) d\bar{\omega}' dt \]
Single Scattering

\[ L(x, \omega) = \int_0^z T_r(x, x_t) \sigma_s(x_t) \int_{S^2} f_p(x, \omega, \omega') T_r(x_t, x_e) L_e(x_e, -\omega) d\omega' dt \]
Single Scattering

\[
L(x, \tilde{\omega}) = \int_0^z T_r(x, x_t) \sigma_s(x_t) \int_{S^2} f_p(x, \tilde{\omega}, \tilde{\omega}') T_r(x_t, x_e) L_e(x_e, -\tilde{\omega}) d\tilde{\omega}' dt
\]
Single Scattering

\[
L(x, \omega) = \int_0^z T_r(x, x_t) \sigma_s(x_t) \int_{S^2} f_p(x, \omega, \omega') T_r(x_t, x_e) L_e(x_e, -\omega) d\omega' dt
\]
Single Scattering

\[ L(x, \omega) = \int_0^z T_r(x, x_t) \sigma_s(x_t) \int_{S^2} f_p(x, \omega, \omega') T_r(x_t, x_e)L_e(x_e, -\omega)d\omega'dt \]
Single Scattering

\[ L(x, \omega) = \int_{0}^{\bar{z}} T_r(x, x_t) \sigma_s(x_t) \int_{S^2} f_p(x, \omega, \omega') T_r(x_t, x_e) L_e(x_e, -\omega) \, d\omega' \, dt \]
Single Scattering

\[ L(x, \bar{\omega}) = \int_0^{
abla} T_r(x, x_t) \sigma_s(x_t) \int_{S^2} f_p(x, \bar{\omega}, \bar{\omega}') T_r(x_t, x_e) L_e(x_e, -\bar{\omega}) d\bar{\omega}' dt \]
Single Scattering

\[
L(x, \bar{\omega}) = \int_{0}^{z} T_{r}(x, x_{t}) \sigma_s(x_{t}) \int_{S^2} f_{p}(x, \bar{\omega}, \bar{\omega}') T_{r}(x_{t}, x_{e}) L_{e}(x_{e}, -\bar{\omega}) d\bar{\omega}' dt
\]

Semi-analytic solutions

Sun et al. [2005]

Pegoraro et al. [2009, 2010]

Numerical solutions

Ray marching

Equiangular sampling
Analytic Single Scattering

\[ L(\mathbf{x}, \mathbf{\omega}) = \int_0^Z T_r(\mathbf{x}, \mathbf{x}_t) \sigma_s(\mathbf{x}_t) \int_{S^2} f_p(\mathbf{x}, \mathbf{\omega}, \mathbf{\omega}') T_r(\mathbf{x}_t, \mathbf{x}_e) L_e(\mathbf{x}_e, -\mathbf{\omega}) d\mathbf{\omega}' dt \]

Assumptions:

- Homogeneous
- Point or spot light
- Relatively simple phase function
- No occlusion

\[ L(\mathbf{x}, \mathbf{\omega}) = \frac{\Phi}{4\pi} \frac{1}{4\pi} \int_0^Z e^{-\sigma_t ||\mathbf{x}, \mathbf{x}_t||} \frac{e^{-\sigma_t ||\mathbf{x}_t, \mathbf{x}_p||}}{e^{-\sigma_t ||\mathbf{x}_t, \mathbf{x}_p||^2} dt} \]
OpenGL Fog

Sun et al., 2005
Analytic Single Scattering
Analytic Single Scattering
Analytic Single Scattering

\[
L_m(x_a, x_b, \tilde{\omega}) = \frac{k_s}{h} e^{\kappa_s (x_a - x_h)} 2 \sum_{n=0}^{N-1} c(n) \sum_{k=0}^{2n} d(n, k) \int_{\nu_a}^{\nu_b} \frac{e^{-H \nu}}{(\nu^2 + 1)^{n+1}} \nu^k d\nu
\]

\[
\int \frac{e^{av}}{(\nu^2 + 1)^m} \nu^n d\nu = \frac{1}{2^{m-1}} \sum_{i=0}^{m-1} \frac{1}{2^i} \binom{m-1+l}{m-1} \left( \sum_{k=0}^{\min\{m-1-l,n\}} \binom{n}{k} \frac{a^{m-1-l-k}}{(m-1-l-k)!} E(a, v, m-n-l+k) \right)
\]

No shadows, implementation nightmare, computationally intensive,...

Let's try brute force!
Ray Marching

\[ L(x, \vec{\omega}) = \int_0^z T_r(x, x_t) \sigma_s(x_t) L_s(x_t, \vec{\omega}) dt \]

Approximate with Riemann summation
Ray Marching

\[ L(x, \bar{\omega}) \approx \sum_{k=0}^{N} T_r(x, x_t, k) \sigma_s(x_t, k) L_s(x_t, k, \bar{\omega}) \Delta t \]
Ray Marching

\[ L(x, \omega) \approx \sum_{k=0}^{N} T_r(x, x_{t,k}) \sigma_s(x_{t,k}) L_s(x_{t,k}, \omega) \Delta t \]
Ray Marching

\[ L(x, \bar{\omega}) \approx \sum_{k=0}^{N} T_r(x, x_{t,k}) \sigma_s(x_{t,k}) L_s(x_{t,k}, \bar{\omega}) \Delta t \]
Ray Marching

\[ L(x, \omega) \approx \sum_{k=0}^{N} T_r(x, x_{t,k}) \sigma_s(x_{t,k}) L_s(x_{t,k}, \omega) \Delta t \]

Homogeneous volume: \[ T_r(x, x_{t,k}) = e^{-\sigma t \|x, x_{t,k}\|} \]
Ray Marching

\[ L(x, \bar{\omega}) \approx \sum_{k=0}^{N} T_r(x, x_t, k) \sigma_s(x_t, k) L_s(x_t, k, \bar{\omega}) \Delta t \]

Heterogeneous volume: \( T_r(x, x_t, k) = T_r(x, x_t, k-1) e^{-\sigma_t(x_t, k) \Delta t} \)
Ray Marching

\[ L(x, \bar{\omega}) \approx \sum_{k=0}^{N} T_r(x, x_{t,k}) \sigma_s(x_{t,k}) L_s(x_{t,k}, \bar{\omega}) \Delta t \]
Ray Marching

\[ L_s(x_t, \bar{\omega}) = \int_{S^2} f_p(x_t, \bar{\omega}, \bar{\omega}') L_i(x_t, \bar{\omega}') d\bar{\omega}' \]
Ray Marching

\[ L_s(x_t, \bar{\omega}) \approx \frac{1}{M} \sum_{j=1}^{M} \frac{f_p(x_t, \bar{\omega}, \bar{\omega}_j') L_i(x_t, \bar{\omega}_j')}{p(\bar{\omega}_j')} \]
Ray Marching

\[ L_s(x_t, \bar{\omega}) \approx \frac{1}{M} \sum_{j=1}^{M} \frac{f_p(x_t, \bar{\omega}, \bar{\omega}_j') L_i(x_t, \bar{\omega}_j')}{p(\bar{\omega}_j')} \]
Ray Marching

\[ L_s(x_t, \bar{\omega}) \approx \frac{1}{M} \sum_{j=1}^{M} \frac{f_p(x_t, \bar{\omega}, \bar{\omega}_j')}{p(\bar{\omega}_j')} L_i(x_t, \bar{\omega}_j') \]
Ray Marching

\[ L_s(x_t, \widetilde{\omega}) \approx \frac{1}{M} \sum_{j=1}^{M} \frac{f_p(x_t, \widetilde{\omega}, \widetilde{\omega}'_j) L_i(x_t, \widetilde{\omega}'_j)}{p(\widetilde{\omega}'_j)} \]

Another ray marching needed to estimate the transmittance along the connection ray (in the heterogeneous media)
Ray Marching in Heterogeneous Media

Marching towards the light source

- Connections are expensive, many, and uniformly distributed along the primary ray
Decoupled Transmittance and in-scattering

1. Ray march and cache transmittance

   - Choose step-size w.r.t. frequency content to accurately capture variations
Decoupled Transmittance and in-scattering

1. Ray march and cache transmittance

- Choose step-size w.r.t. frequency content to accurately capture variations
2. Estimate in-scattering using MC integration

- Distribute samples proportional to (part of) the integrand
Decoupled Transmittance and in-scattering

2. Estimate in-scattering using MC integration

- Distribute samples proportional to (part of) the integrand

\[ p(x_t) \propto T_r(x, x_t) \]
Decoupled Transmittance and in-scattering

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Decoupled Transmittance and in-scattering

2. Estimate in-scattering using MC integration

- Distribute samples proportional to (part of) the integrand

\[ p(x_t) \propto \frac{1}{d^2} \]

\( d \): distance to light
Decoupled Transmittance and in-scattering

2. Estimate in-scattering using MC integration

- Distribute samples proportional to (part of) the integrand

\[ p(x_t) \propto \frac{1}{d^2} \]

\( d : \) distance to light
Decoupled Transmittance and in-scattering

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Decoupled Transmittance and in-scattering

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Decoupled Transmittance and in-scattering

2. Estimate in-scattering using MC integration

- Distribute samples proportional to (part of) the integrand

\[ p(x_t) \propto \frac{1}{d^2} \]

\( d \): distance to light

Equiangular sampling

Kulla and Fajardo [2012]
Decoupled Transmittance and in-scattering

Ray Marching

Equi-angular sampling
Motivation

Same as with path tracing: avoid the exponential growth
Volumetric Path Tracing
Volumetric Path Tracing

Motivation

Same as with path tracing: avoid the exponential growth

Paths can:

Reflect / Refract off surfaces

Scatter inside a volume
Volumetric Rendering Equation

\[ L(x, \omega) = \int_0^Z T_r(x, x_t) \sigma_a(x_t) L_e(x_t, \omega) \, dt + \int_0^Z T_r(x, x_t) \sigma_s(x_t) L_s(x_t, \omega) \, dt + T_r(x, x_Z) L(x_Z, \omega) \]

- Accumulated emitted radiance
- Accumulated in-scattered radiance
- Attenuated background radiance
Volumetric Rendering Equation

\[
L(x, \bar{\omega}) = \int_0^z T_r(x, x_t) \left[ \sigma_a(x_t)L_e(x_t, \bar{\omega}) + \sigma_s(x_t)L_s(x_t, \bar{\omega}) \right] dt \\
+ T_r(x, x_z)L(x_z, \bar{\omega})
\]

Accumulated emitted + in-scattered radiance

Attenuated background radiance
Volumetric Rendering Equation

\[
L(x, \bar{\omega}) = \int_0^Z T_r(x, x_t) \left[ \sigma_a(x_t) L_e(x_t, \bar{\omega}) + \sigma_s(x_t) L_s(x_t, \bar{\omega}) \right] dt \\
+ T_r(x, x_z) L(x_z, \bar{\omega})
\]
1-Sample Monte Carlo Estimator

\[
\langle L(x, \bar{w}) \rangle = \frac{T_r(x, x_t)}{p(t)} \left[ \sigma_a(x_t)L_e(x_t, \bar{w}) + \sigma_s(x_t)L_s(x_t, \bar{w}) \right] \\
+ \frac{T_r(x, x_z)}{P(\bar{z})} L(x_z, \bar{w})
\]
1-Sample Monte Carlo Estimator

\[
\langle L(x, \omega) \rangle = \frac{T_r(x, x_t)}{p(t)} \left[ \sigma_a(x_t) L_e(x_t, \omega) + \sigma_s(x_t) L_s(x_t, \omega) \right]
+ \frac{T_r(x, x_z)}{P(z)} L(x_z, \omega)
\]

- \( p(t) \) Probability density of distance \( t \)
- \( P(z) \) Probability of exceeding distance \( z \)
1-Sample Monte Carlo Estimator

\[
\langle L(\mathbf{x}, \mathbf{\omega}) \rangle = \frac{T_r(\mathbf{x}, \mathbf{x}_t)}{p(t)} \left[ \sigma_a(\mathbf{x}_t) L_e(\mathbf{x}_t, \mathbf{\omega}) + \sigma_s(\mathbf{x}_t) \frac{f_p(\mathbf{x}, \mathbf{\omega}, \mathbf{\omega}_i) L(\mathbf{x}_t, \mathbf{\omega})}{p(\mathbf{\omega}_i)} \right] \\
+ \frac{T_r(\mathbf{x}, \mathbf{x}_z)}{P(z)} L(\mathbf{x}_z, \mathbf{\omega})
\]

- \(p(t)\) Probability density of distance \(t\)
- \(P(z)\) Probability of exceeding distance \(z\)
- \(p(\mathbf{\omega}_i)\) Probability density of direction \(\mathbf{\omega}_i\)
Volumetric Path Tracing

1. Sample distance to next interaction

2. Scatter in the volume or bounce off a surface
1. Sample distance to next interaction
2. Scatter in the volume or bounce off a surface
Sampling the Phase Function

**Isotropic:** Uniform sphere sampling

**Henyey-Greenstein:** Using the inversion method we can derive

\[
\cos \theta = \frac{1}{2g} \left( 1 + g^2 - \left( \frac{1 - g^2}{1 - g + 2g\xi_1} \right)^2 \right)
\]

\[
\phi = 2\pi\xi_2
\]

PDF is the value of the HG phase function
Free-path Sampling

Free-path or free-flight distance:

- Distance to the next interaction in the medium
- Dense media (e.g. milk): short mean-free path
- Thin media (e.g. atmosphere): long mean-free path
Free-path Sampling

Free-path or free-flight distance:

- Distance to the next interaction in the medium
- Dense media (e.g. milk): short mean-free path
- Thin media (e.g. atmosphere): long mean-free path

Ideally, we want to sample according to (part of) of the integrand:

$$p(x_t | (x, \omega)) \propto T_r(x, x_t)$$

simplified notation

$$p(t) \propto T_r(t)$$
Free-path Sampling

Homogeneous media: \[ T_r(t) = e^{-\sigma_t t} \]

PDF: \[ p(t) \propto e^{-\sigma_t t} \]

\[ p(t) = \frac{e^{-\sigma_t t}}{\int_0^\infty e^{-\sigma_t s} ds} = \sigma_t e^{-\sigma_t t} \]

CDF: \[ P(t) = \int_0^t e^{-\sigma_t s} ds = 1 - e^{-\sigma_t t} \]

Inverted CDF: \[ P^{-1}(\xi) = -\frac{\log_e(1 - \xi)}{\sigma_t} \]
Free-path Sampling

Homogeneous media: \( T_r(t) = e^{-\sigma_t t} \)

Recipe:

- Generate a random number \( \xi \)
- Sample distance \( t = -\frac{\log_e(1 - \xi)}{\sigma_t} \)
- Compute PDF \( p(t) = \sigma_t e^{-\sigma_t t} \)

\[ t \]

\[ x \quad x_t \]
Free-path Sampling

Homogeneous media: \( T_r(t) = e^{-\sigma t t} \)

Recipe:

- Generate a random number \( \xi \)
- Sample distance \( t = -\frac{\log_e (1 - \xi)}{\sigma t} = s \)
- Compute PDF \( p(t) = \sigma t e^{-\sigma t t} = e^{-\sigma t s} \)

Note: This is now a probability, not a probability density.

Surface hit before reaching \( t \)
What about heterogeneous media?
Free-path Sampling

Heterogeneous medium: 

$$T_r(t) = e^{\int_0^t -\sigma_t(s)ds}$$

- Closed form solutions exist but for only simple media
  - e.g., linearly or exponentially varying extinction

- Other solutions:
  - Regular tracking (3D DDA)
  - Ray marching
  - Delta tracking
Free-path Sampling

How to sample the flight distance to the next interaction?

\[ T(t) = e^{-\int_0^t \mu_t(s)ds} = P(X > t) \]

\[
\begin{cases} 
  P(X \leq t) = F(t) \\
  P(X > t) = 1 - F(t)
\end{cases}
\]

Recipe for generating samples
Free-path Sampling

Cumulative distribution function (CDF)

\[ F(t) = 1 - T(t) = 1 - e^{-\tau(t)} \]

Probability density function (PDF)

\[ p(t) = \frac{dF(t)}{dt} = \frac{d}{dt} \left(1 - e^{-\tau(t)}\right) = \mu_t(t)e^{-\tau(t)} \]

Inverted cumulative distr. function (CDF⁻¹)

\[ \xi = 1 - e^{-\tau(t)} \]

\[ \int_0^t \mu_t(s)ds = -\ln(1 - \xi) \]

Approaches for finding t:
1) ANALYTIC (closed-form CDF⁻¹)
2) SEMI-ANALYTIC (regular tracking)
3) APPROXIMATE (ray marching)
Free-path Sampling

Inverted cumulative distr. function ($CDF^{-1}$)

$$\int_{0}^{t} \mu_t(s)ds = -\ln(1 - \xi)$$

Some simple volumes permit closed-form solutions

Example: **homogeneous** medium ($\mu_t(x) = \mu_t$)

**Opt. thickness**

$$\int_{0}^{t} \mu_t(s)ds = t\mu_t$$

$$\Rightarrow \quad F^{-1}(\xi) = -\frac{\ln(1 - \xi)}{\mu_t}$$
Analytic Approach

Inverted cumulative distr. function ($CDF^{-1}$)

\[
\int_0^t \mu_t(s) ds = -\ln(1 - \xi)
\]

Sampling in homogeneous vol:

1) Draw a random number $\xi$
2) Set $t = -\frac{\ln(1 - \xi)}{\mu_t}$
3) Set $p(t) = \mu_t e^{-t \mu_t}$
Regular Tracking (Semi-Analytic)

For piecewise-simple (e.g. piecewise-constant), summation replaces integration

\[
\int_0^t \mu_t(s)ds = -\ln(1 - \xi)
\]

\[
\sum_{i=1}^k \mu_{t,i} \Delta_i = -\ln(1 - \xi)
\]

Regular tracking:
1) Draw a random number \( \xi \)
2) While LHS < RHS, move to the next intersection
3) Find the exact location in the last segment analytically

(Hierarchical) voxel grid
Ray Marching

Find the collision distance approximately

\[ \int_{0}^{t} \mu_t(s) \, ds = -\ln(1 - \xi) \]
\[ \sum_{i=1}^{k} \mu_{t,i} \Delta = -\ln(1 - \xi) \]

Constant step

Ray marching:
1) Draw a random number \( \xi \)
2) While LHS < RHS
   make a (fixed-size) step
3) Find the exact location
   in the last segment analytically
Ray Marching

Find the collision distance approximately

\[ \int_0^t \mu_t(s) \, ds = -\ln(1 - \xi) \]

\[ \sum_{i=1}^{k} \mu_{t,i} \Delta = -\ln(1 - \xi) \]

**Constant step**

Ray marching:
1) Draw a random number \(\xi\)
2) While LHS < RHS
   make a (fixed-size) step
3) Find the exact location in the last segment analytically
Ray Marching

Find the collision distance approximately

\[ \int_0^t \mu_t(s) ds = -\ln(1 - \xi) \]

\[ \sum_{i=1}^k \mu_{t,i} \Delta = -\ln(1 - \xi) \]

**Constant step**

Ray marching:

1) Draw a random number \( \xi \)
2) While LHS < RHS
   make a (fixed-size) step
3) Find the exact location
   in the last segment analytically

General volume

LHS > RHS

Sampled collision LHS=RHS
Free-path Sampling

<table>
<thead>
<tr>
<th>ANALYTIC CDF$^{-1}$</th>
<th>REGULAR TRACKING</th>
<th>RAY MARCHING</th>
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<tr>
<td>Efficient &amp; simple, limited to few volumes</td>
<td>Iterative, inefficient if free paths cross many boundaries</td>
<td>Iterative, inaccurate (or inefficient) for media with high frequencies</td>
</tr>
<tr>
<td>Simple volumes (e.g. homogeneous)</td>
<td>Piecewise-simple volumes</td>
<td>Any volume</td>
</tr>
<tr>
<td>Unbiased</td>
<td>Unbiased</td>
<td>Biased</td>
</tr>
</tbody>
</table>

Common approach: sample optical thickness, find corresponding distance
Delta Tracking

a.k.a. Woodcock tracking, pseudo scattering, hole tracking, null-collision method, …
Physical Interpretation

Add **FICTITIOUS MATTER** to homogenize heterogeneous extinction

- albedo $\alpha(x) = 1$
- phase function $f_P(\omega, \bar{\omega}) = \delta(\omega - \bar{\omega})$

**Incident light** $ightarrow$ **Fictitious particle** $ightarrow$ **Outgoing light**

*Presence of fictitious matter does not impact light transport*
Physical Interpretation

HOMOGENIZATION

Volume bounds

Real particle
Physical Interpretation

HOMOGENIZATION

Volume bounds

Fictitious particle

Real particle
Physical Interpretation

HOMOGENIZATION
Physical Interpretation

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Real particle
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Volume bounds

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Real particle

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Physical Interpretation

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Volume bounds

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Volume bounds
Fictitious particle
Real particle
Physical Interpretation

HOMOGENIZATION

Volume bounds

Real particle
Stochastic Sampling

Volume bounds

Real medium
Stochastic Sampling

\[ \bar{\mu} = \mu_t(x) + \mu_n(x) \]
Stochastic Sampling

\[ P_r(x) = \frac{\mu_t(x)}{\bar{\mu}} \quad P_n(x) = \frac{\mu_n(x)}{\bar{\mu}} \]

Majorant

\[ \bar{\mu} = \mu_t(x) + \mu_n(x) \]

Tentative collision

Extinction

Distance

\[ \frac{\ln(1 - \xi)}{\bar{\mu}} \]
Stochastic Sampling

\[ P_r(x) = \frac{\mu_t(x)}{\bar{\mu}} \quad \text{Majorant} \quad \bar{\mu} = \mu_t(x) + \mu_n(x) \]

\[ P_n(x) = \frac{\mu_n(x)}{\bar{\mu}} \]
Stochastic Sampling

\[ P_r(x) = \frac{\mu_t(x)}{\bar{\mu}} \quad P_n(x) = \frac{\mu_n(x)}{\bar{\mu}} \]

Majorant

\[ \bar{\mu} = \mu_t(x) + \mu_n(x) \]
Stochastic Sampling

\[ P_r(x) = \frac{\mu_t(x)}{\bar{\mu}} \quad P_n(x) = \frac{\mu_n(x)}{\bar{\mu}} \]

Majorant \( \bar{\mu} = \mu_t(x) + \mu_n(x) \)
Impact of Majorant

\[ \bar{\mu}(x) = \mu_t(x) + \mu_n(x) \]

Majorant

Extinction

Distance

Sampled free path
Impact of Majorant

Tight majorant = GOOD (few rejected collisions)

\[
\bar{\mu} = \mu_t(x) + \mu_n(x)
\]
Impact of Majorant

Loose majorant = BAD (many expensive rejected collisions)
Acknowledgements

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Next career fair “next” on June 11, 2024 from 10:00 a.m. to 5:00 p.m.

The trade fair offers our students the opportunity to meet potential employers, make contacts and find out about career opportunities. Companies have the opportunity to offer internships, theses or entry-level positions.