Realistic Image Synthesis

- Rendering Equation -

Philipp Slusallek
Karol Myszkowski
Gurprit Singh
Angle and Solid Angle

$\theta$ the angle subtended by a curve in the plane is the length of the projected arc on the unit circle.

$\Omega, \omega$ the solid angle subtended by an object is the surface area of its projection onto the unit sphere

Solid angle units: steradians [sr]
Solid Angle for a Small Area

The solid angle subtended by an (infinitely) small surface patch $S$ with area $dA$ is obtained by dividing the projected area $dA \cos \theta$ by the square of the distance to the center:

$$d\omega, d\Omega = \frac{dA \cos \theta}{r^2}$$
Solid Angle in Spherical Coordinates

- **Infinitesimally small solid angle**
  - \( du = r \, d\theta \)
  - \( dv = r \sin \theta \, d\phi \)
  - \( dA = du \, dv = r^2 \sin \theta \, d\theta \, d\phi \)
  - \( \Rightarrow d\omega = \frac{dA}{r^2} = \sin \theta \, d\theta \, d\phi \)

- **Finite solid angle of a surface** \( S \)
  - \( \omega = \int_S \sin \theta \, d\theta \, d\phi \)

- **Definition:**
  - We denote the entire Sphere with \( \Omega \) and the (positive) hemisphere at a point \( x \) centered around its normal vector with \( \Omega_+ \)
Radiometry

• **Definition:**
  – Radiometry is the science of measuring radiant energy transfer
  – Radiometric quantities have physical meaning and can be directly measured using proper equipment such as spectral radiometers

• **Radiometric Quantities**
  – Energy [J] \( Q \) (#Photons x Energy = \( n \cdot h\nu \))
  – Radiant power [watt = J/s] \( \Phi \) (Total Flux)
  – Intensity [watt/sr] \( I \) (Flux from a point per s.angle)
  – Irradiance [watt/m\(^2\)] \( E \) ( Incoming flux per area)
  – Radiosity [watt/m\(^2\)] \( B \) (Outgoing flux per area)
  – Radiance [watt/(m\(^2\) sr)] \( L \) (Flux per area & proj. s. angle)
Radiometric Quantities: Radiance

- Radiance is used to describe radiant energy transfer
- Radiance $L$ is defined as the total flux (radiant power) traveling at some point $x$ in a specified direction $\omega$, per unit area perpendicular to the direction of travel, per unit solid angle
- Thus, the differential flux $d^2 \Phi$ radiated through the differential solid angle $d\omega$, from the projected differential area $dA \cos \theta$ is:

$$d^2 \Phi = L(x, \omega)dA \cos \theta \ d\omega$$

or

$$L(x, \omega) = \frac{d^2 \Phi}{dA \cos \theta \ d\omega}$$

- From here on we distinguish between the direction $\omega$ and the (differential) solid angle $d\omega$ !!!
Radiance in Space

Along a ray, flux leaving surface 1 must be equal to flux arriving on surface 2

\[ L_1 \cdot d\Omega_1 \cdot dA_1 = L_2 \cdot d\Omega_2 \cdot dA_2 \]

From geometry follows

\[ d\Omega_1 = \frac{dA_2}{r^2}, \quad d\Omega_2 = \frac{dA_1}{r^2} \]

Def: Ray **Throughput**

\[ T = d\Omega_1 \cdot dA_1 = d\Omega_2 \cdot dA_2 = \frac{dA_1 \cdot dA_2}{r^2} \]

\[ L_1 = L_2 \]

The radiance in the direction of a light ray remains constant as it propagates along the ray.

Sensors response is proportional to radiance (human eye, camera)
**Radiometric Quantities: Irradiance**

- Irradiance $E$ is the total radiant power per unit area (flux density) *incident* onto a surface with a fixed orientation.

- To obtain the total flux incident to $dA$, the incoming radiance $L_i$ is integrated over the upper hemisphere $\Omega_+$ above the surface:

\[
E \equiv \frac{d\Phi}{dA} = \int_{\Omega_+} L_i(x, \theta, \phi) \cos \theta \, d\omega \, dA
\]

\[
E = \int_{\Omega_+} L_i(x, \theta, \phi) \cos \theta \, d\omega = \int_0^{2\pi} \int_0^{\pi/2} L_i(x, \theta, \phi) \cos \theta \sin \theta \, d\theta \, d\phi
\]
Radiometric Quantities: Radiosity

• Radiosity $B$ is defined as the total radiant power per unit area (flux density) *leaving* a surface.

• To obtain the total flux radiated from $dA$, the outgoing radiance $L_o$ is integrated over the upper hemisphere $\Omega_+$ above the surface:

\[
B \equiv \frac{d\Phi}{dA} = \int_{\Omega_+} L_o(x, \theta, \phi) \cos \theta \, d\omega \, dA
\]

\[
B = \int_{\Omega_+} L_o(x, \theta, \phi) \cos \theta \, d\omega = \int_{0}^{2\pi} \int_{0}^{\pi/2} L_o(x, \theta, \phi) \cos \theta \sin \theta \, d\theta \, d\phi
\]
Bidirectional Reflectance Distribution Function

- **BRDF** $f_r$ describes surface reflection at a point $x$ for light incident from direction $\omega_i = (\theta_i, \varphi_i)$ reflected into direction $\omega_o = (\theta_o, \varphi_o)$

- **Bidirectional (six-dimensional function)**
  - Depends on two directions $\omega_i$ and $\omega_o$ (2D plus 2D = 4D)
  - Also depends on location $x$ (2D)

- **Distribution function**
  - Value can be infinite but integrates to finite value
  - Strictly positive (physics!)

- **Definition of BRDF:**
  - Outgoing radiance per incident irradiance

\[
\begin{align*}
  f_r(\omega_i, x, \omega_o) &= \frac{L_o(x, \omega_o)}{dE_i} \\
  f_r(\omega_i, x, \omega_o) &= \frac{L_o(x, \omega_o)}{L_i(x, \omega_i) \cos \theta_i} d\omega_i
\end{align*}
\]
BRDF Properties

- **Helmholtz reciprocity principle**
  - BRDF remains unchanged if incident and reflected directions are interchanged
  - Due to physics (linearity)

\[ f_r(\omega_i, x, \omega_o) = f_r(\omega_o, x, \omega_i) \]

- **Smooth surface: Isotropic BRDF**
  - Reflectivity is independent of rotation around surface normal
  - BRDF directional dependence has only 3 instead of 4 degrees of freedom

\[ f_r(\omega_i, x, \omega_o) = f_r(x, \theta_i, \theta_o, \phi_i - \phi_o) \]
BRDF Properties

- **Characteristics**
  - BRDF units [sr$^{-1}$]
    - Not very intuitive
  - Range of values:
    - From 0 (complete absorption) to
    - $\infty$ (perfect mirror reflection, $\delta$-function)
      - Because it relates the density $L$ to a finite value
  - Energy conservation law
    - Integrating over all outgoing light:
      - No more energy can be reflected than was incoming
    - In other words, the directional-hemispherical reflectance must be smaller than 1
      $$\rho_{dh} = \int_{\Omega^+} f_r(\omega_i, x, \omega_o) \cos \theta \, d\omega_o \leq 1, \quad \forall \omega_i$$
- Reflection only at the point of entry ($x_i = x_o$)
  - Subsurface scattering (e.g. in skin) is not included in this formulation
More intuitive measure of reflectance is the *directional-hemispherical reflectance*:

- The fraction of the incident radiant flux density incoming from a given direction that is reflected by the surface in all possible directions.
- Dimensionless number in $[0,1]$
- Can change with the angle of incidence

$$\rho_{dh}(\omega_i) = \frac{dB}{dE(\omega_i)} = \int_{\Omega^+} \frac{L_o(x, \omega_o) \cos \theta_o}{dE(\omega_i)} d\omega_o = \int_{\Omega^+} f_r(\omega_i, x, \omega_o) \cos \theta_o \, d\omega_o$$

$$\frac{L_o(x, \omega_o)}{dE(\omega_i)} = f_r(\omega_i, x, \omega_o)$$
Lambertian Diffuse Reflection

- Diffuse reflectance is modeled by assuming that light is equally likely to be scattered in any direction, regardless of the incident direction.
- Therefore, the BRDF and reflected radiance are constant:
  \[ f_r(\omega_i, x, \omega_o) = \rho \quad \text{and} \quad L_o = \text{const} \]
- Also, directional-hemispherical reflectance \( \rho_d \) becomes independent of direction. This dimensionless constant, which corresponds to the intuitive meaning of reflectance, is then called the diffuse reflectance \( \rho_d \):
  \[ \rho_d = \int_{\Omega_+} \rho \cos \theta_o d\omega_o = \rho \int_0^{2\pi} \int_0^{\pi/2} \cos \theta_o \sin \theta_o d\theta_o d\phi_o = \pi \rho \]
- Irradiance \( E \) and radiosity \( B \) for the Lambertian surface are related as:
  \[ \rho_d = \frac{B}{E} \]

\[ B = \int_{\Omega} L_o(x, \theta, \phi) \cos \theta d\omega = L_o \cdot \pi \]
Reflection Equation

- **Putting at all together:**
  - The light reflected at a point $x$ in direction $\omega$ is given as
  \[
  L_r(x, \omega_o) = \int_{\omega_i \in \Omega_+} f_r(\omega_i, x, \omega_o) L_i(x, \omega_i) \cos \theta_i \, d\omega_i
  \]

- **Visible surface radiance** $L_r(x, \omega_o)$
  - Surface position $x$
  - Outgoing direction $\omega_o$
  - Incoming illumination direction $\omega_i$

- **Reflected light**
  - Incoming radiance $L_i(x, \omega_i)$
  - Direction-dependent reflectance $f_r(\omega_i, x, \omega_o)$
Reflection Equation: Properties

- **Reflection operator is linear**
  - Superposition holds
  - Solution could be computed separately for each light source
    - And be accumulated

- **BRDF is a six-dimensional function**
  - Difficult to represent and compute accurately
    - But we have various representations for BRDFs (more or less physically accurate)
  - Measurements are expensive and need much storage
    - But often compresses well
Light Transport in a Scene

• **Scene**
  – Lights (emitters)
  – Object surfaces (partially absorbing)

• **Illuminated object surfaces become emitters, too!**
  – Radiosity = Irradiance minus absorbed photon flux
    • Radiosity: photons per second per m^2 leaving surface
    • Irradiance: photons per second per m^2 incident on surface

• **Light bounces between all mutually visible surfaces**

• **Invariance of radiance in free space (vacuum)**
  – No absorption in-between objects
  – Hold also in clean air (approximately!)

• **Dynamic Energy Equilibrium**
  – Emitted photons = absorbed photons (+ photons escaping scene)

Global Illumination Problem
Definition: Rendering Equation

- **Light exiting at some point**
  - Given by emitted light plus reflected incoming light at \( x \)
  
  \[
  L_o(x, \omega_o) = L_e(x, \omega_o) + L_r(x, \omega_o) 
  = L_e(x, \omega_o) + \int_{\omega_i \in \Omega_+} f_r(\omega_i, x, \omega_o) L_i(x, \omega_i) \cos \theta_i \, d\omega_i
  \]

- **Coupling output back to input**
  - Light incident at \( x \) is the light exiting at some other point \( y \)
    
    \[
    L_i(x, \omega_i) = L_o(y, -\omega_i) = L_o(RT(x, \omega_i), -\omega_i)
    \]

  - With the visibility or ray-tracing operator RT
    
    \[
    y = RT(x, \omega_i)
    \]
Definition: Rendering Equation

- **Rendering Equation**
  
  Parameterized by direction
  
  \[ L_o(x, \omega_o) = L_e(x, \omega_o) + \int_{\omega_i \in \Omega_+} f_r(\omega_i, x, \omega_o) L_i(x, \omega_i) \cos \theta_i \, d\omega_i \]
  
  Parameterized by position over all surfaces \( S \)
  
  \[ L_o(x, \omega_o) = L_e + \int_{y \in S} f_r(\omega_i, x, \omega_o) L_o \left( y, \frac{x - y}{\|x - y\|} \right) V(x, y) G(x, y) dA_y \]
  
  - with \( V(x, y) \) giving visibility between \( x \) and \( y \),
  
  - and the Geometric Term \( G \) given by

  \[ d\omega_i = dA_y \frac{\cos \theta_y}{\|x - y\|^2} \]
  
  \[ G(x, y) = \frac{\cos \theta_x \cos \theta_y}{\|x - y\|^2} \]
Rendering Equation

\[ L_o(x, \omega_o) = L_e(x, \omega_o) + \int_{y \in S} f_r(\omega_i, x, \omega_o) L_o(y(x, \omega_i), -\omega_i) V(x, y) G(x, y) dA_y \]

- **Properties**
  - Mathematical: Fredholm equation of the 2-nd kind
  - Global coupling of illumination
    - Each point potentially influences each other point
    - Often still a sparse operator due to occlusion
  - Linear transport operator \( T \)
    - Solution can be computed separately for each light source
      - And accumulated
      - Dimmed lights result in dimmed solutions
  - Volume effects are not considered !!

► **Lighting Simulation == Solving the Rendering Equation**
**RE: In Operator Form**

- **Transport operator** \( T \)
  - Built from **reflection operator** \( S \) and **propagation operator** \( H \)
    - \( L = L_o = L_e + TL = L_e + (S \circ H)L \)

- **Propagation operator** \( H \)
  - Computes light incident at points \( L_i(x_i, w_i) \) from excitant light at other locations \( L(y_i, w_i) \)
  - Evaluation of ray tracing operator
  - **Global operator**: needs essentially entire scene

- **Reflection (scattering) operator** \( S \)
  - Computes reflected light field \( L(y_i, w_i) \) from incident light \( L_i(x_i, w_i) \) evaluating reflection equation
  - Evaluates BRDF for entire incident light field
  - **Local operator**: operates at one point only
Rendering Equation

• Solution Approaches
  – Monte Carlo technique (and extensions)
    • Point-wise evaluation of multi-dimensional integral equation
    • Efficient solution for the general case
    • Can cause noise through variance of random evaluation
    • No bias (in approach, if done right)
  – Finite Element technique
    • Projection of infinite dimensional equation into function space with finite dimensions
      – Solution is represented as combination of basis functions
      – Constant basis functions in the simplest case
    • Leads to solution of a linear system of equations
    • Efficient for smooth, slowly varying illumination and reflection
    • Causes bias through correlation between solution of neighboring points
Discretization of Rendering Equation

• **Simplification of the rendering equation**

\[ L_o(x, \theta_o, \phi_o) = L_e(x, \theta_o, \phi_o) + \int_{\Omega} \rho_{bd}(x, \theta_o, \phi_o, \theta, \phi) L_i(x, \theta, \phi) \cos \theta d\omega \]

\[ \downarrow \]

\[ B_i = E_i + \rho_i \sum_{j=1}^{N} B_j F_{ij} \]

– Assumption: All surfaces in the scene are Lambertian
– Equation expressed in terms of the radiosity quantities
– Integration domain split into \( N \) pieces corresponding to discrete patches in the scene
– Constant radiosity and reflectance assumptions for each patch

• **We are going to discuss all these steps in detail**
Lambertian Diffuse Reflection

- Diffuse reflectance is modeled by assuming that light is equally likely to be scattered in any direction, regardless of the incident direction:

\[ \rho_{bd}(x, \theta_o, \varphi_o, \theta, \varphi) = \rho(x) \]

- Directional-hemispherical reflectance \( \rho_d \) becomes independent of direction:

\[
\rho_d(x) = \int_{\Omega} \rho(x) \cos \theta_o d\omega_o = \rho(x) \int_{0}^{2\pi} \int_{0}^{\pi/2} \cos \theta_o \sin \theta_o d\theta_o d\varphi_o = \pi \rho(x)
\]

\[
\rho(x) = \frac{\rho_d(x)}{\pi}
\]

- Then the rendering equation simplifies to:

\[
L_o(x, \theta_o, \varphi_o) = L_e(x, \theta_o, \varphi_o) + \frac{\rho_d(x)}{\pi} \int_{\Omega} L_i(x, \theta, \varphi) \cos \theta d\omega
\]
Further Simplifications

• For diffuse surfaces
  – the radiance $L_o(x, \theta_o, \varphi_o) \equiv L_o(x)$ does not depend on the outgoing direction,
  – the incoming radiance $L_i$ still depends on the incoming direction

$$L_o(x) = L_e(x) + \frac{\rho_d(x)}{\pi} \int_{\Omega} L_i(x, \theta, \varphi) \cos \theta d\omega$$

• Now let us replace radiances by radiosities:

$$B(x) = \int_{\Omega} L(x) \cos \theta \ d\omega = \int_0^{2\pi} \int_0^{\pi/2} L(x) \cos \theta \sin \theta \ d\theta \ d\phi = \pi L(x)$$

$$\pi L_o(x) = \pi L_e(x) + \pi \frac{\rho_d(x)}{\pi} \int_{\Omega} L_i(x, \theta, \varphi) \cos \theta d\omega$$

$$B(x) = E(x) + \rho_d(x) \int_{\Omega} L_i(x, \theta, \varphi) \cos \theta d\omega$$
Transforming the Hemispherical Integral into a Surface Integral

- The invariance of radiance along a line of sight states that:

\[ L_i(x, \theta, \varphi) = L(y, \theta', \varphi') \]

\[ L(y, \theta', \varphi') = \frac{B(y)}{\pi} \]

- Now let us replace integration over the hemisphere by integration over all surfaces \( y \) taking into account their visibility from \( x \):

\[ d\omega = \frac{\cos \theta' dy}{r^2} \]

\[ V(x, y) = \begin{cases} 
1 & \text{if } x \text{ and } y \text{ are mutually visible} \\
0 & \text{otherwise} 
\end{cases} \]

\[ B(x) = E(x) + \rho_d(x) \int_{y \in S} B(y) \frac{\cos \theta \cos \theta'}{\pi r^2} V(x, y) dy \]
Discrete Formulation

• The integral over all surfaces in the scene in the previous slide is broken into $N$ pieces, each corresponding to a discrete patch.

• It is assumed that each patch has a uniform radiosity at each point $y$ in patch $P_j$.

$$B(x) = E(x) + \rho_d(x) \sum_{j=1}^{N} B_j \int_{y \in P_j} \frac{\cos \theta \cos \theta'}{\pi r^2} V(x, y) dy$$
Radiosity Equation and Form Factors

- The constant radiosity value for each patch is computed as an area-weighted average of radiosity:

\[
B_i = \frac{1}{A_i} \int_{x \in P_i} B(x) \, dx \quad E_i = \frac{1}{A_i} \int_{x \in P_i} E(x) \, dx
\]

- Then assuming also that reflectance is constant across each patch \( \rho_d(x) = \rho_i \), the radiosity equation can be formulated as:

\[
B_i = E_i + \rho_i \sum_{j=1}^{N} B_j \frac{1}{A_i} \int_{x \in P_i} \int_{y \in P_j} \frac{\cos \theta \cos \theta'}{\pi r^2} V(x, y) \, dx \, dy
\]

\[
B_i = E_i + \rho_i \sum_{j=1}^{N} B_j F_{ij}
\]

- where \( F_{ij} \) is the form factor:

\[
F_{ij} = \frac{1}{A_i} \int_{x \in P_i} \int_{y \in P_j} \frac{\cos \theta \cos \theta'}{\pi r^2} V(x, y) \, dx \, dy
\]