Volume Rendering

Gurprit Singh

Philipp Slusalek Karol Myszkowski

Overview

Volumetric Processes:

Absorption

Scattering

Transmittance

Phase Functions

Volumetric Rendering Equation

Volumetric Path Tracing

Woodcock Tracking

Fog

Brassai (Gyula Halasz) 1899-1984

Aerial View

Gurprit Singh

Snow

Gurprit Singh

Fire

Harry Potter/Warner Brothers

Surface or Volume?

Corona Renderer / Chaos Czech a.s. / Chaos Group

Universe

lisamission.org

Defining Participating Media

Media properties are modeled as a probabilistic process

No need to consider individual interactions with particles (won't fit in the memory)

Defining Participating Media

Homoegeneous media:

- Infinite or bounded by a simple surface or simple shape

Krivanek et al. [2014]

Defining Participating Media

Heterogeneous media (spatially varying coefficients):

Disney/Pixar

Defining Participating Media

Heterogeneous media (spatially varying coefficients):

- Procedurally e.g. using a noise function

Disney/Pixar

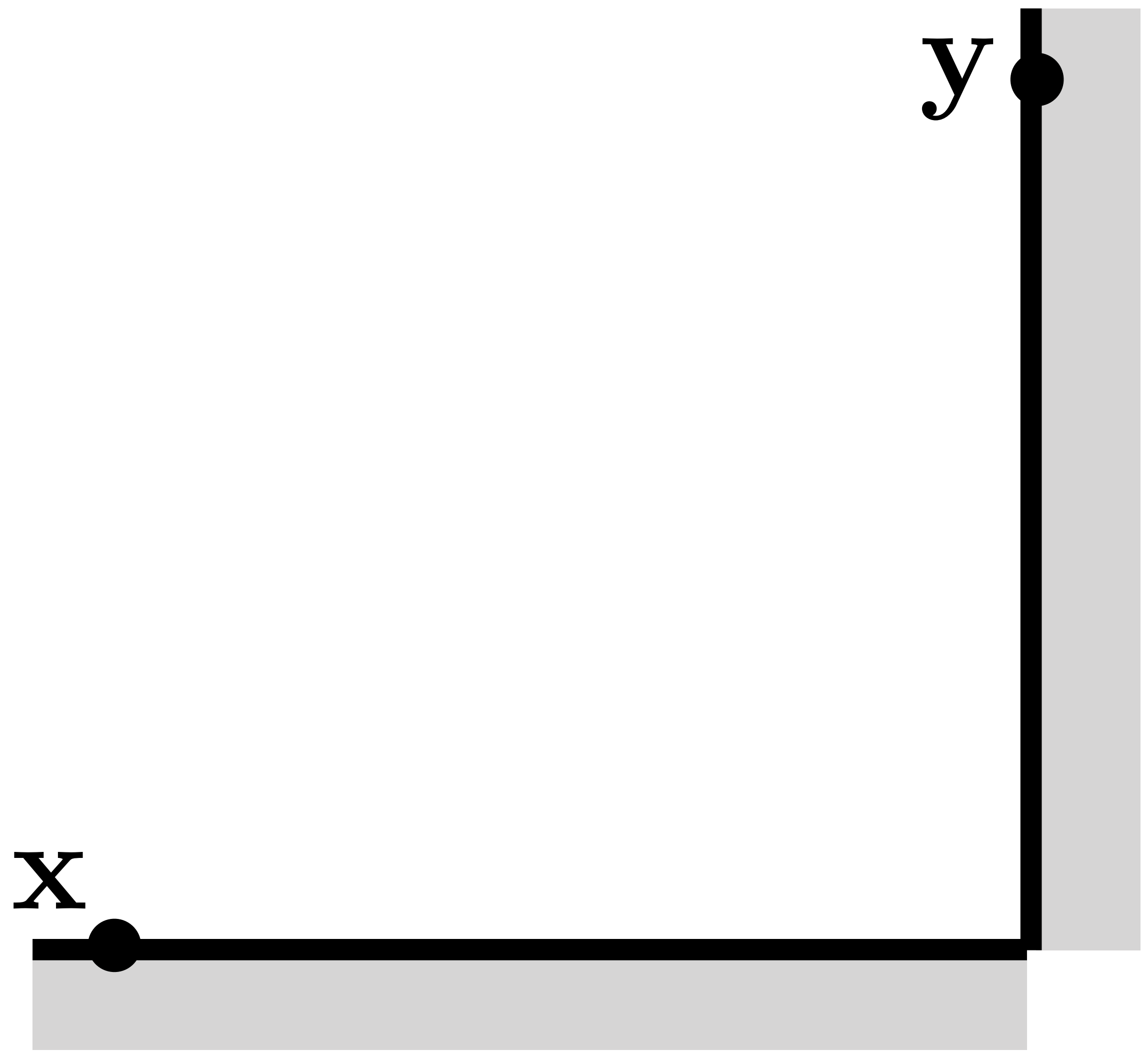
Defining Participating Media

Heterogeneous media (spatially varying coefficients):

- Procedurally e.g. using a noise function
- Simulation + volume discretization, e.g., voxel grid

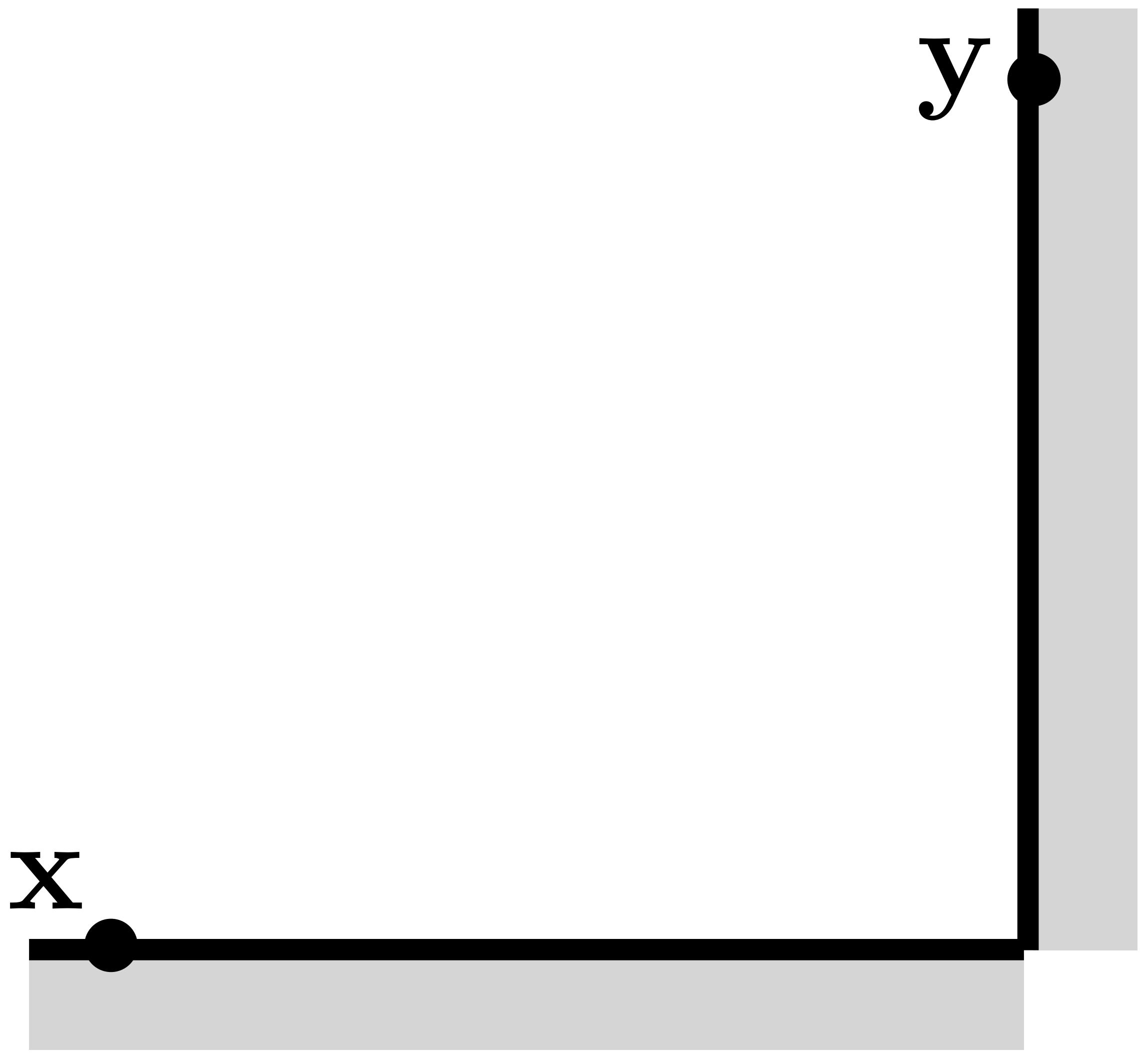
Disney/Pixar

Radiance



Radiance is the main quantity we are interested in for rendering.

Radiance



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In **vaccum**, light transport radiance remains constant along rays between surfaces

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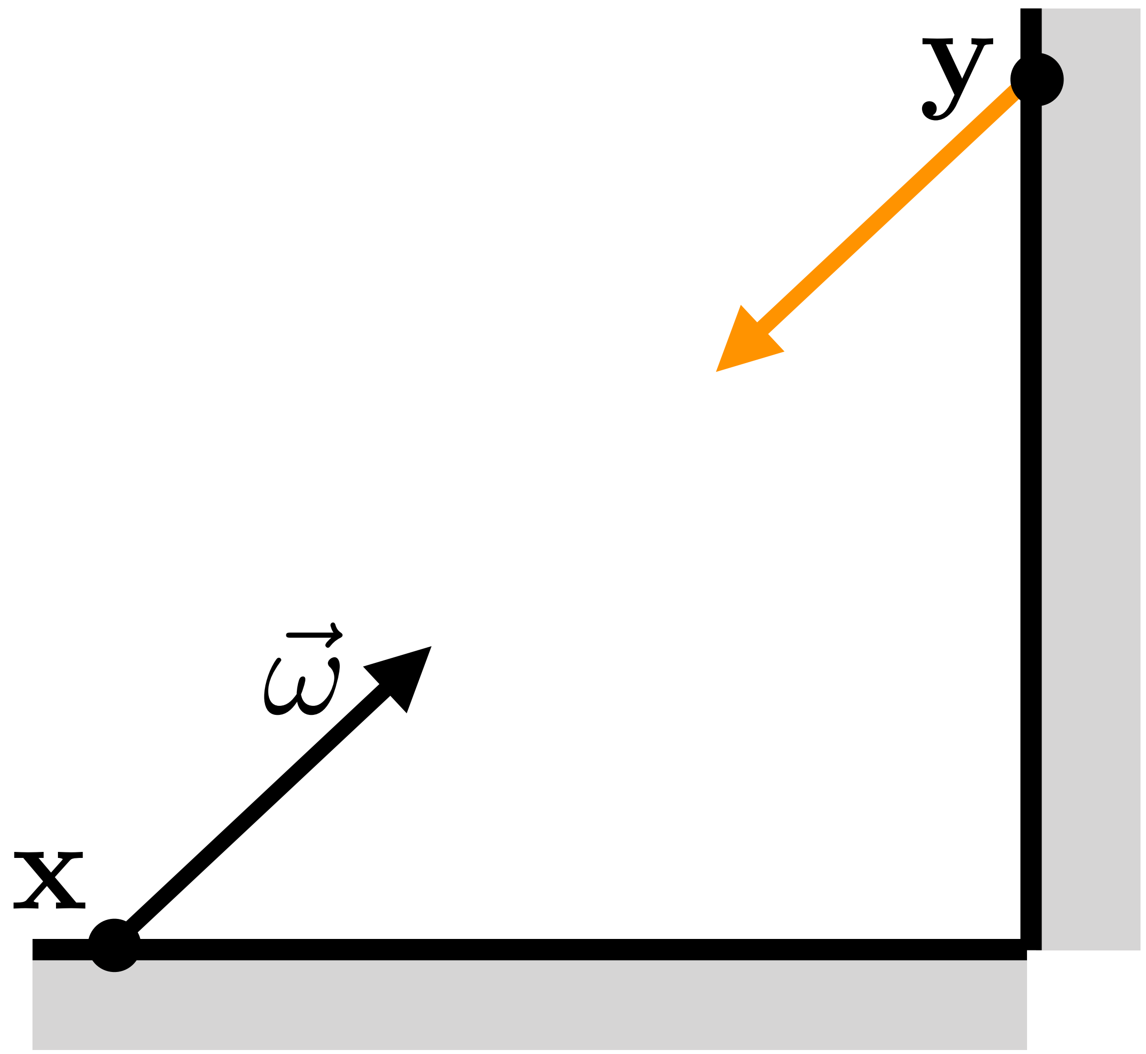

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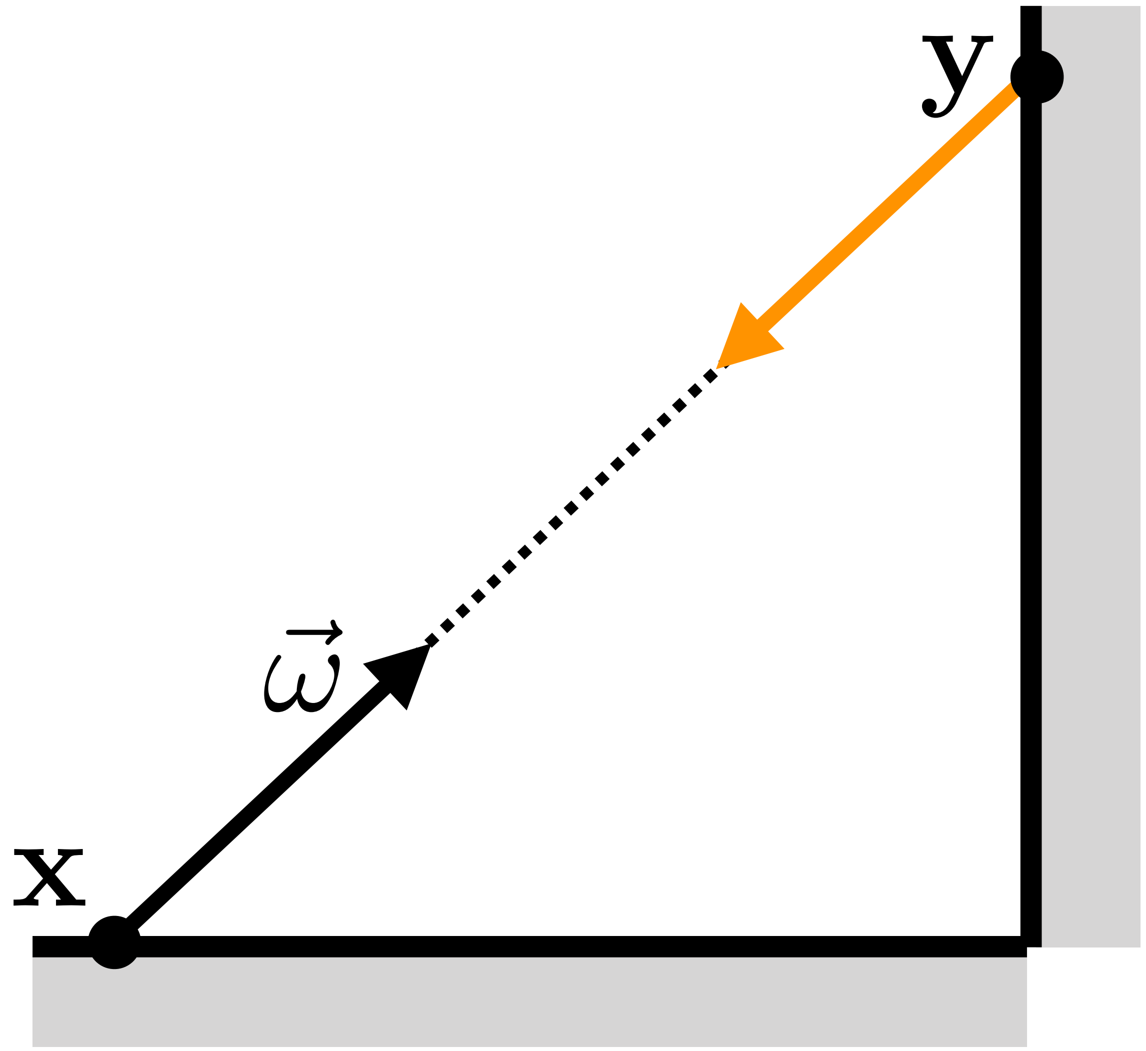

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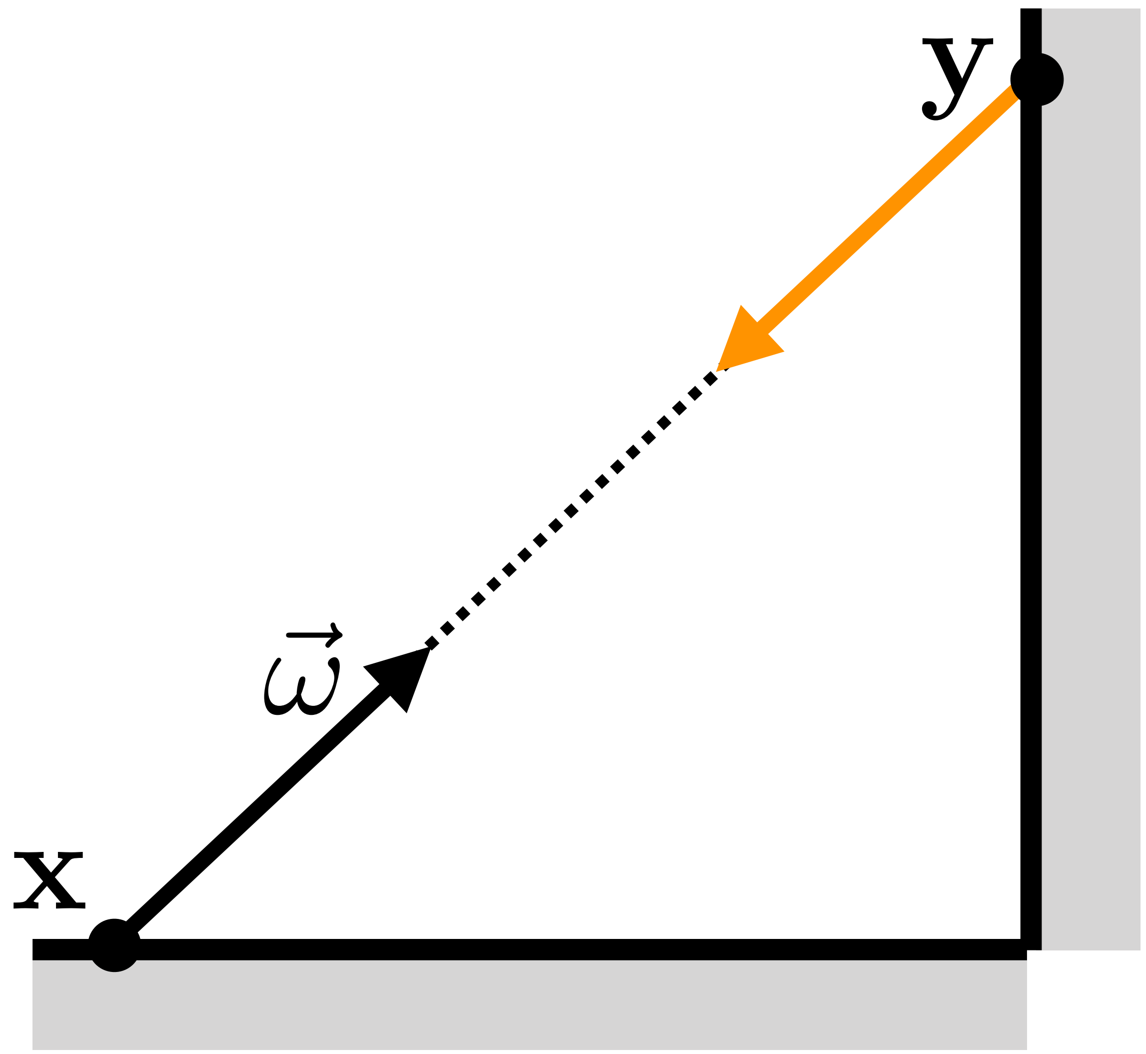

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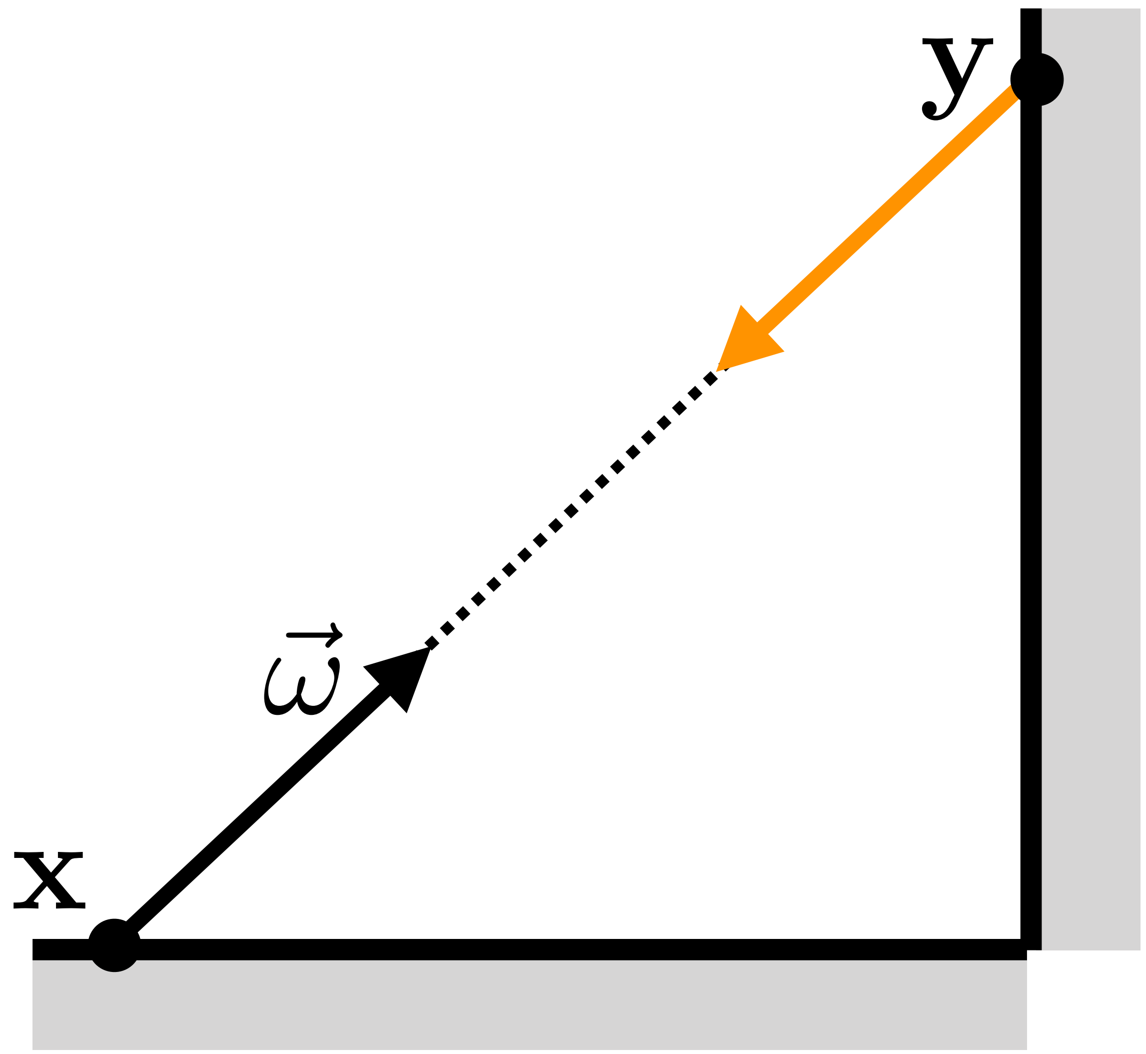
In **vaccum**, light transport radiance remains constant along rays between surfaces

$$L_i(\mathbf{x}, \vec{\omega}) = L_o(\mathbf{y}, -\vec{\omega})$$

Radiance



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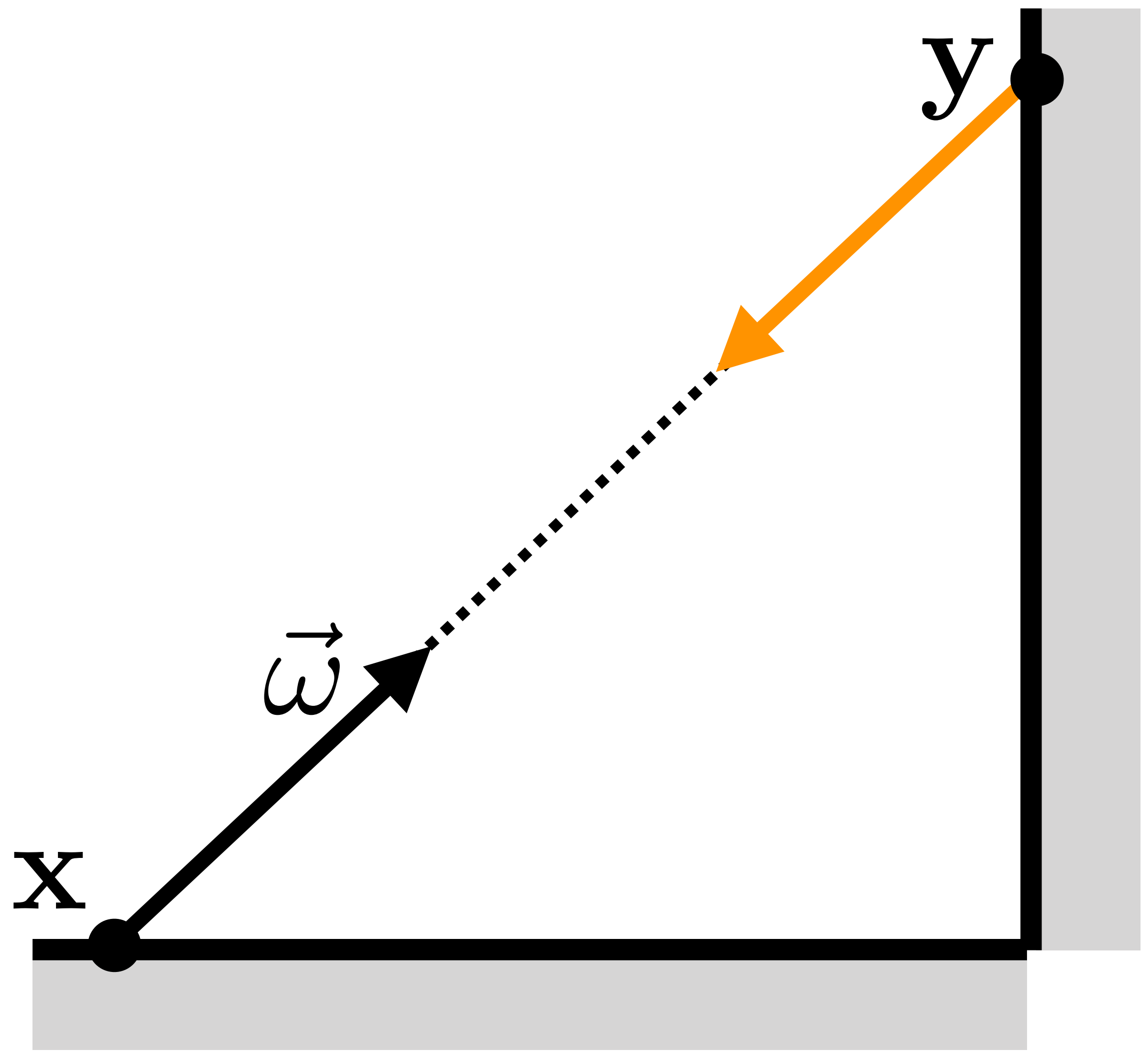
$$L_i(\mathbf{x}, \vec{\omega}) = L_o(\mathbf{y}, -\vec{\omega})$$

$$\mathbf{y} = \mathbf{r}(\mathbf{x}, \vec{\omega})$$

Radiance



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$$L_i(\mathbf{x}, \vec{\omega}) = L_o(\mathbf{y}, -\vec{\omega})$$

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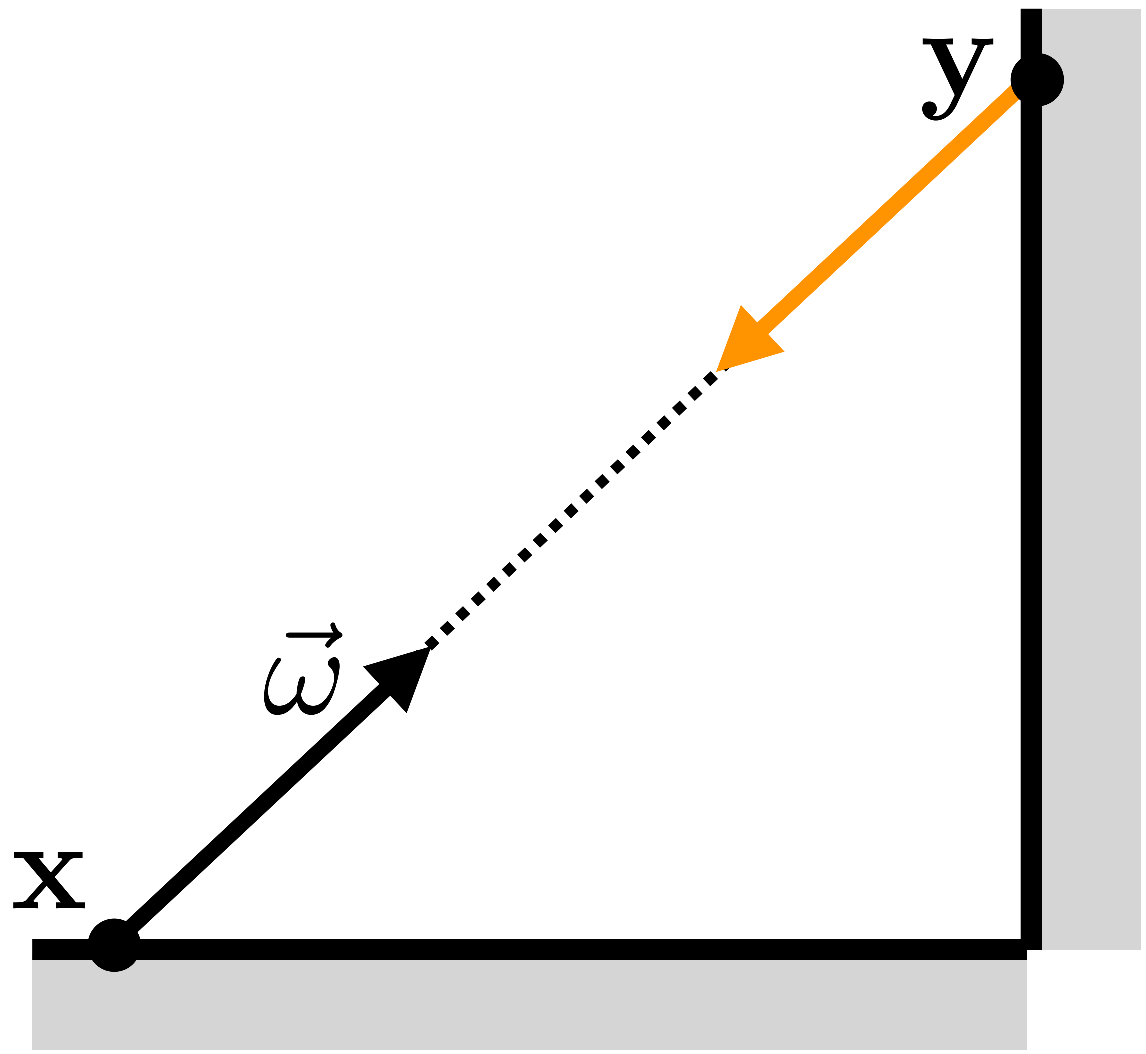
ray tracing function

Radiance



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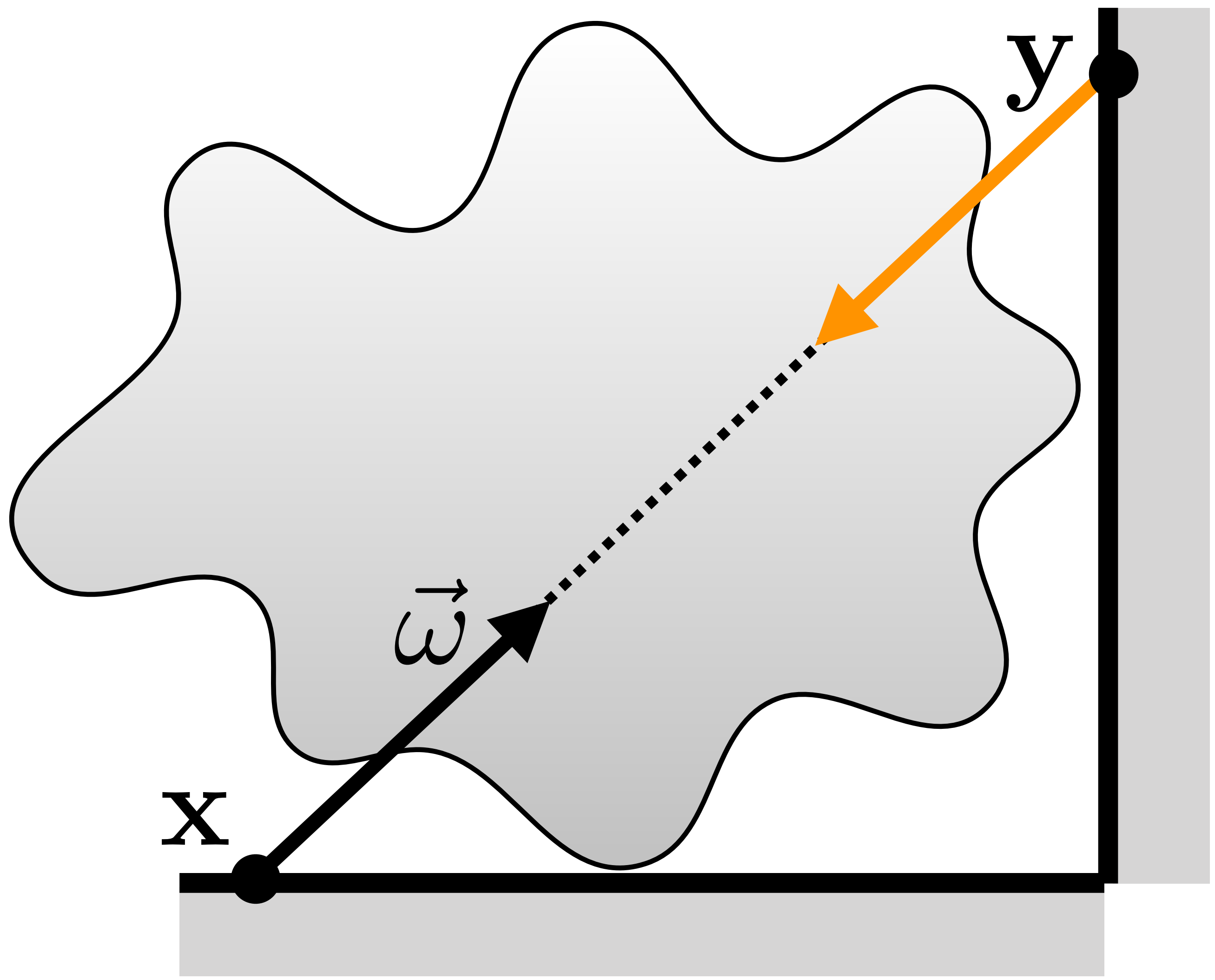
$$L_i(\mathbf{x}, \vec{\omega}) = L_o(\mathbf{y}, -\vec{\omega})$$

$$\mathbf{y} = \mathbf{r}(\mathbf{x}, \vec{\omega})$$

ray tracing function

Radiance

In **participating media**, radiance may change along rays between surfaces



$$L_i(\mathbf{x}, \vec{\omega}) \cancel{=} L_o(\mathbf{y}, -\vec{\omega})$$

$$\mathbf{y} = \mathbf{r}(\mathbf{x}, \vec{\omega})$$

ray tracing function

Volumetric Scattering Processes

Absorption

http://commons.wikimedia.org

Scattering

http://coclouds.com

Emission

http://wikipedia.org

Slide after Jan Novak

Participating Media

Participating Media

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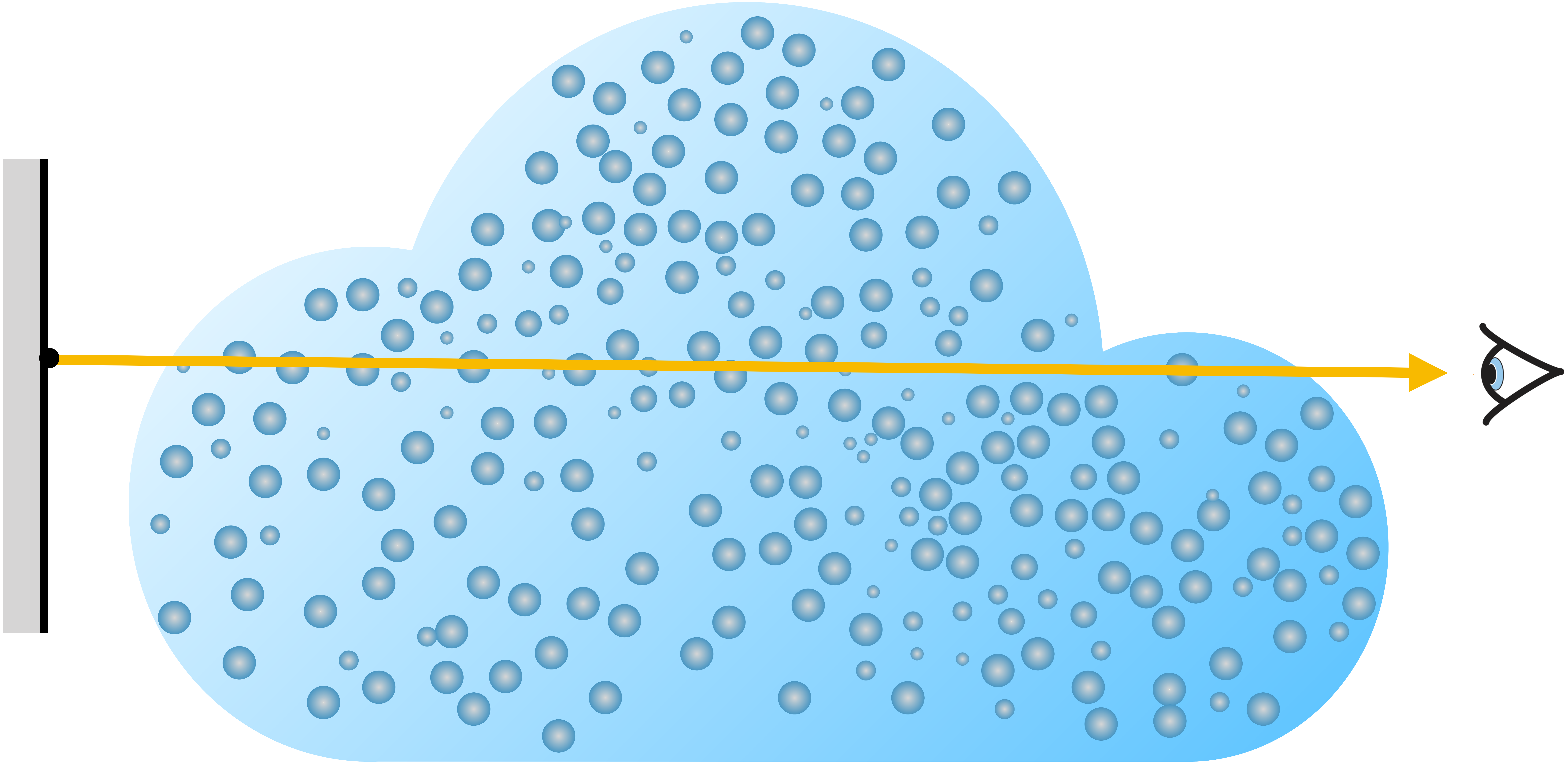
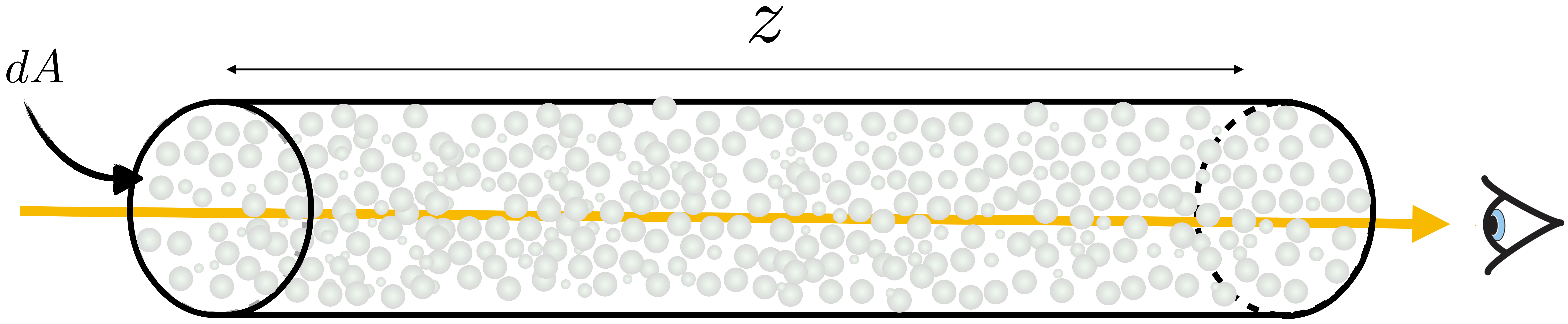
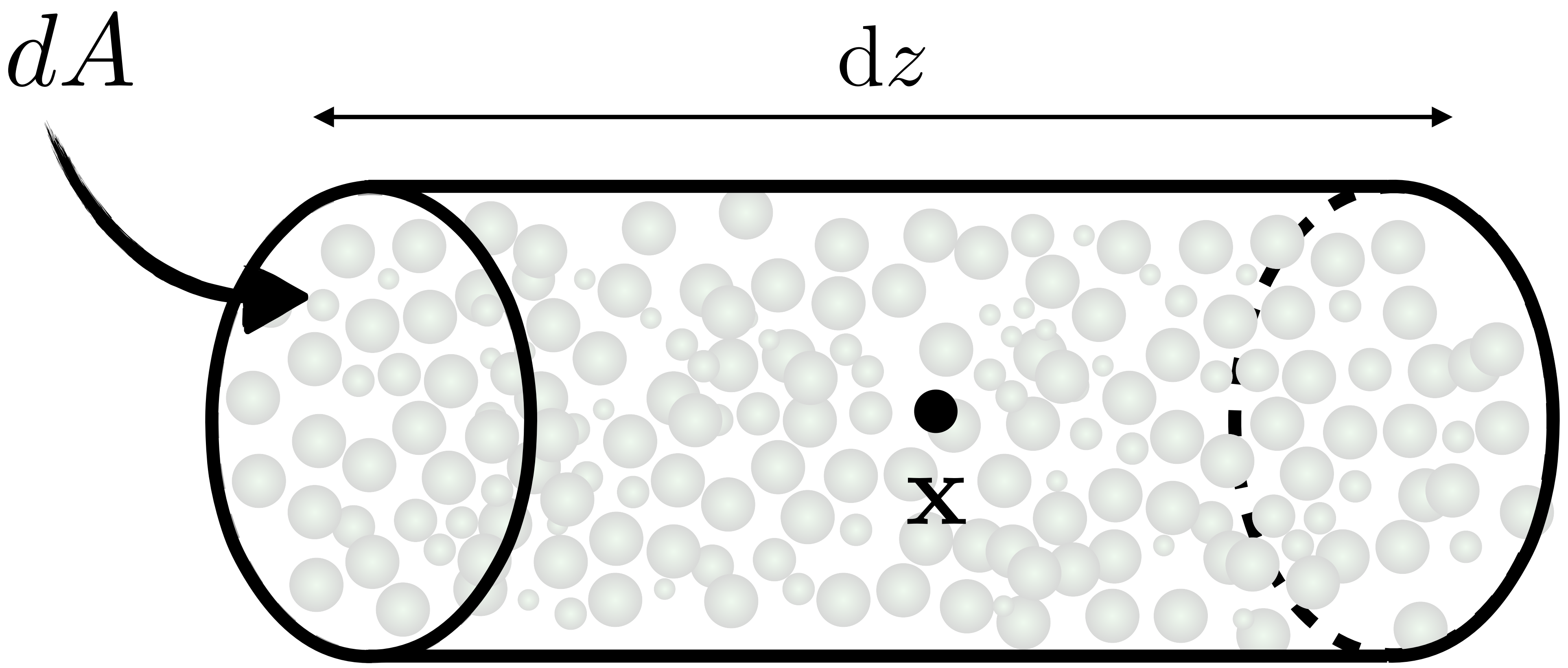


Finite distance Beam

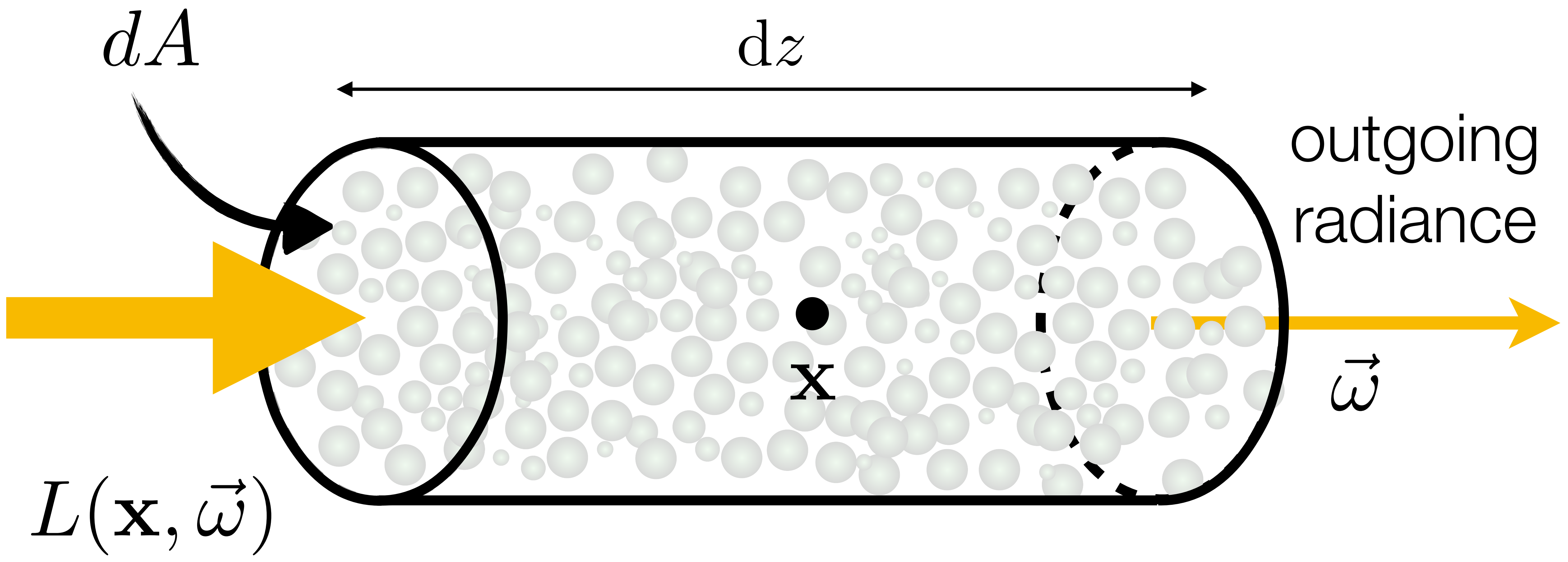


How much light is gained or lost during the travel
through this differential beam due to the interactions with the medium?

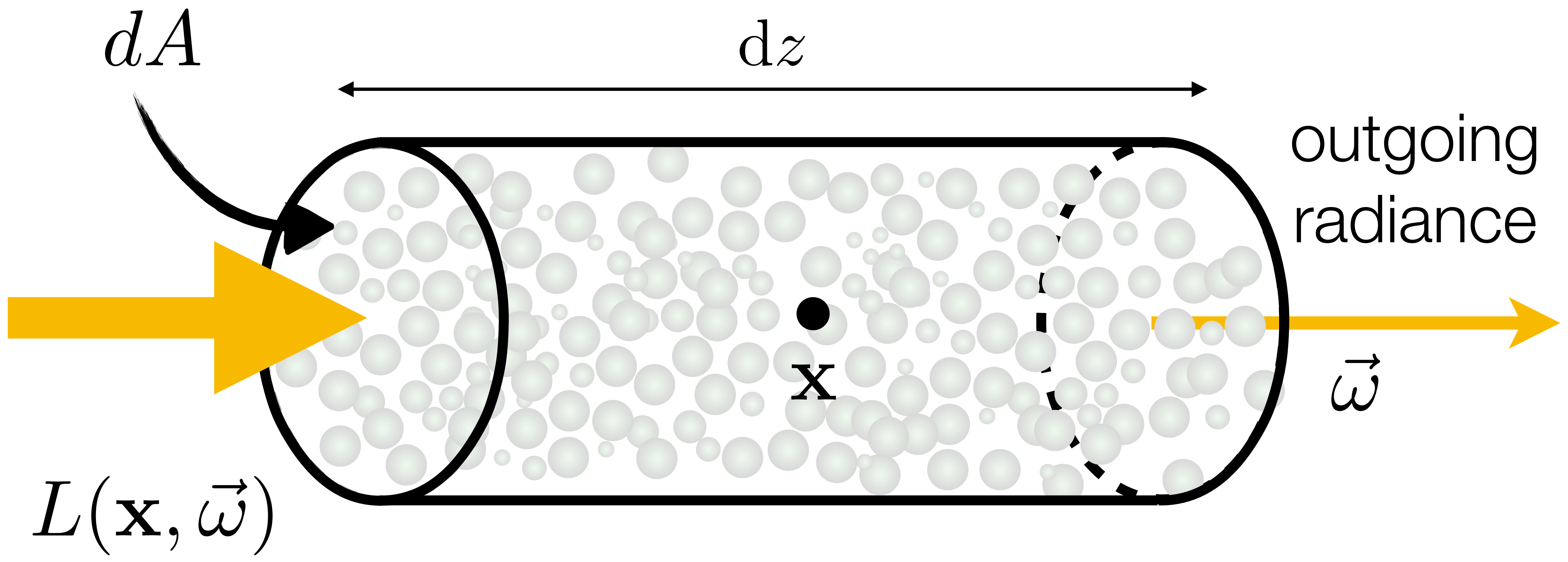
Differential Beam



Absorption

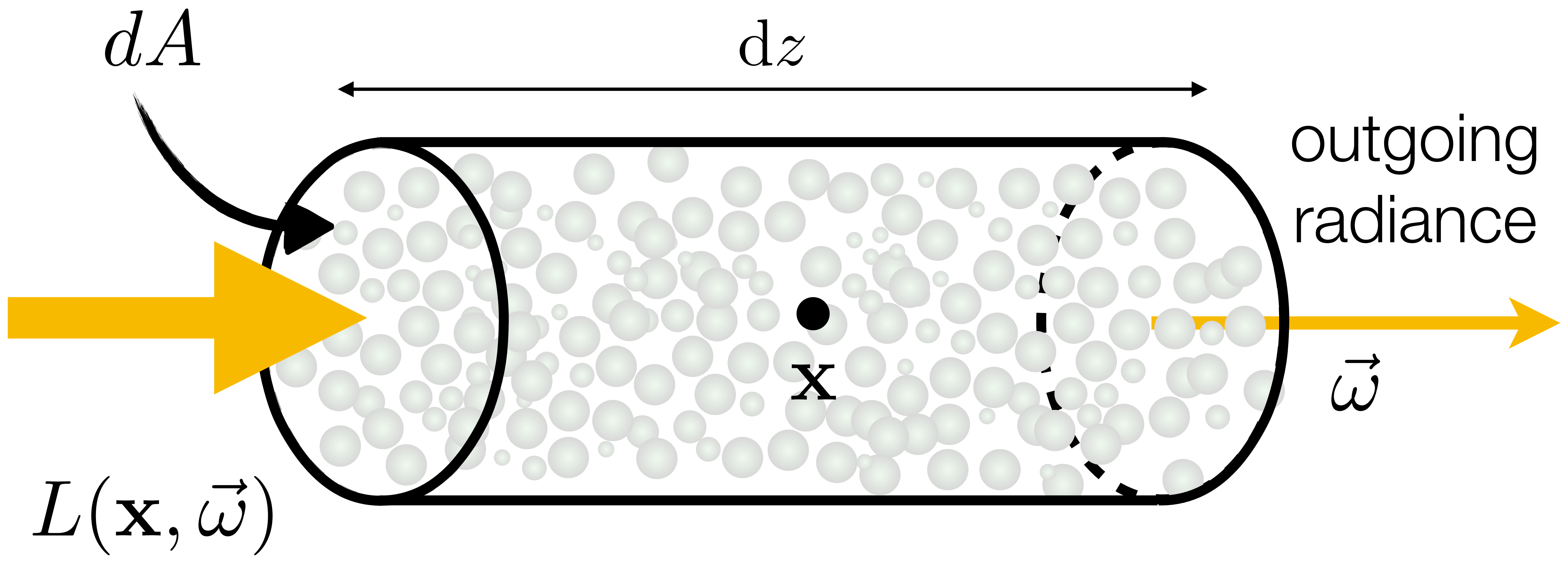

dA
$\mathrm{d}z$
outgoing
radiance
$\mathbf{x}$
$\vec{\omega}$
$L(\mathbf{x}, \vec{\omega})$

Absorption



$$\frac{dL(\mathbf{x}, \vec{\omega})}{dz} = -\sigma_a L(\mathbf{x}, \vec{\omega})$$

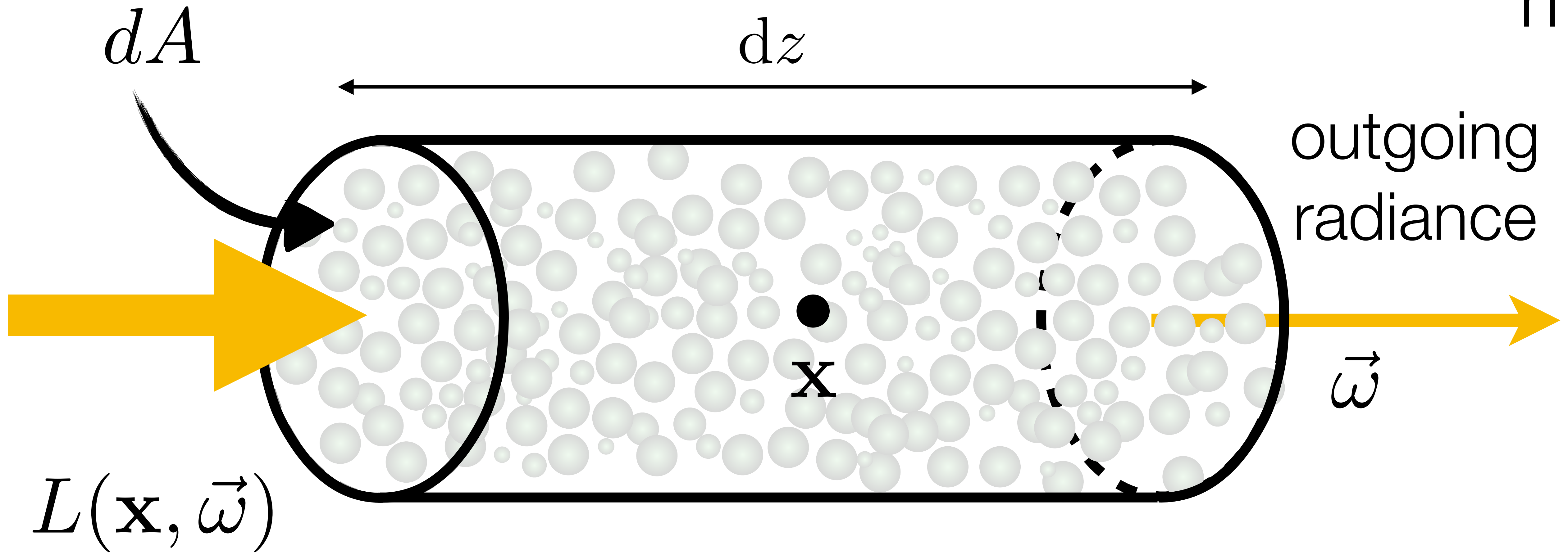
Absorption



$$\frac{dL(\mathbf{x}, \vec{\omega})}{dz} = -\sigma_a L(\mathbf{x}, \vec{\omega})$$

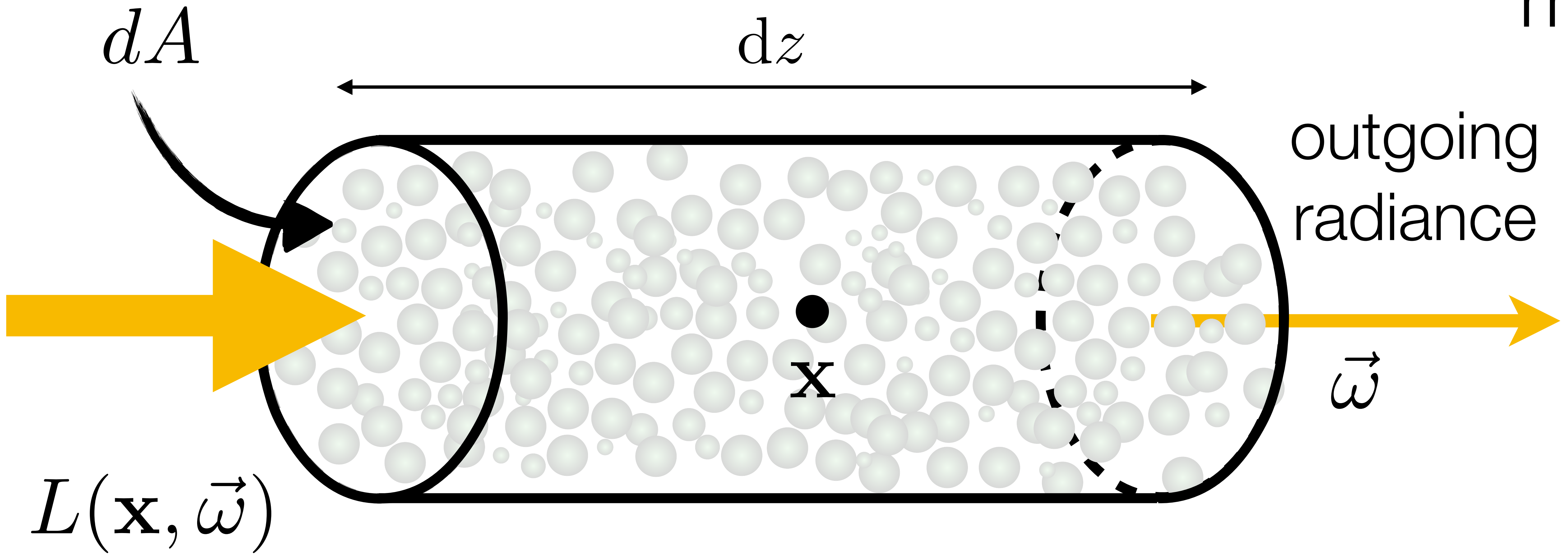
σ_a : absorption coefficient m^{-1}

Absorption



Absorption described by medium's absorption cross-section σ_a

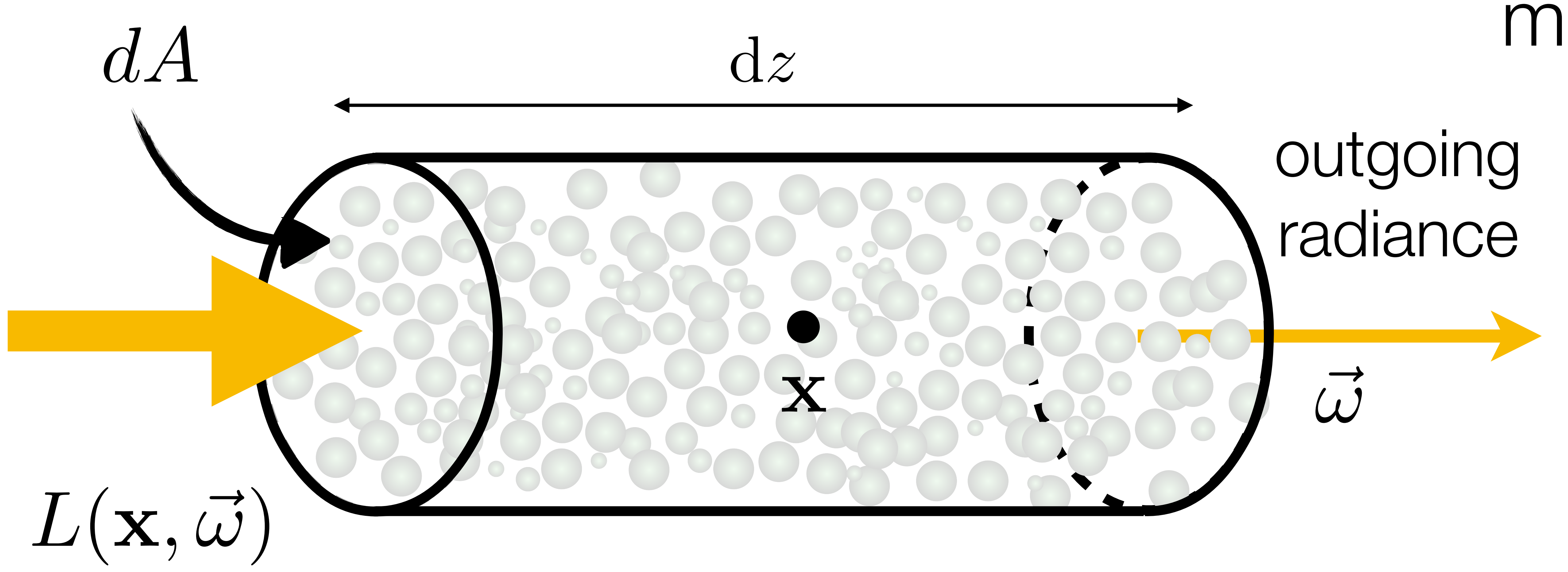
Absorption



Absorption described by
medium's absorption cross-section σ_a

$$\sigma_a \in [0, \infty)$$

Absorption



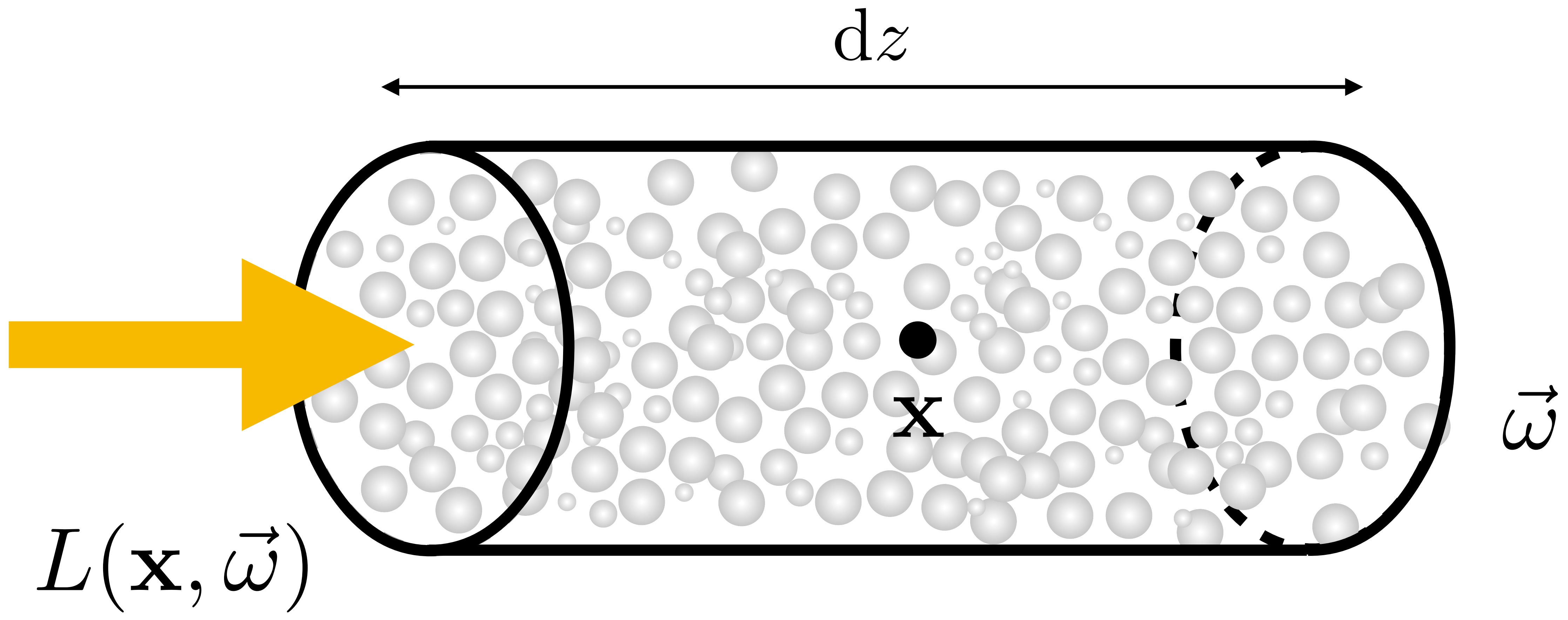
Absorption described by medium's absorption cross-section σ_a

$$\sigma_a \in [0, \infty)$$

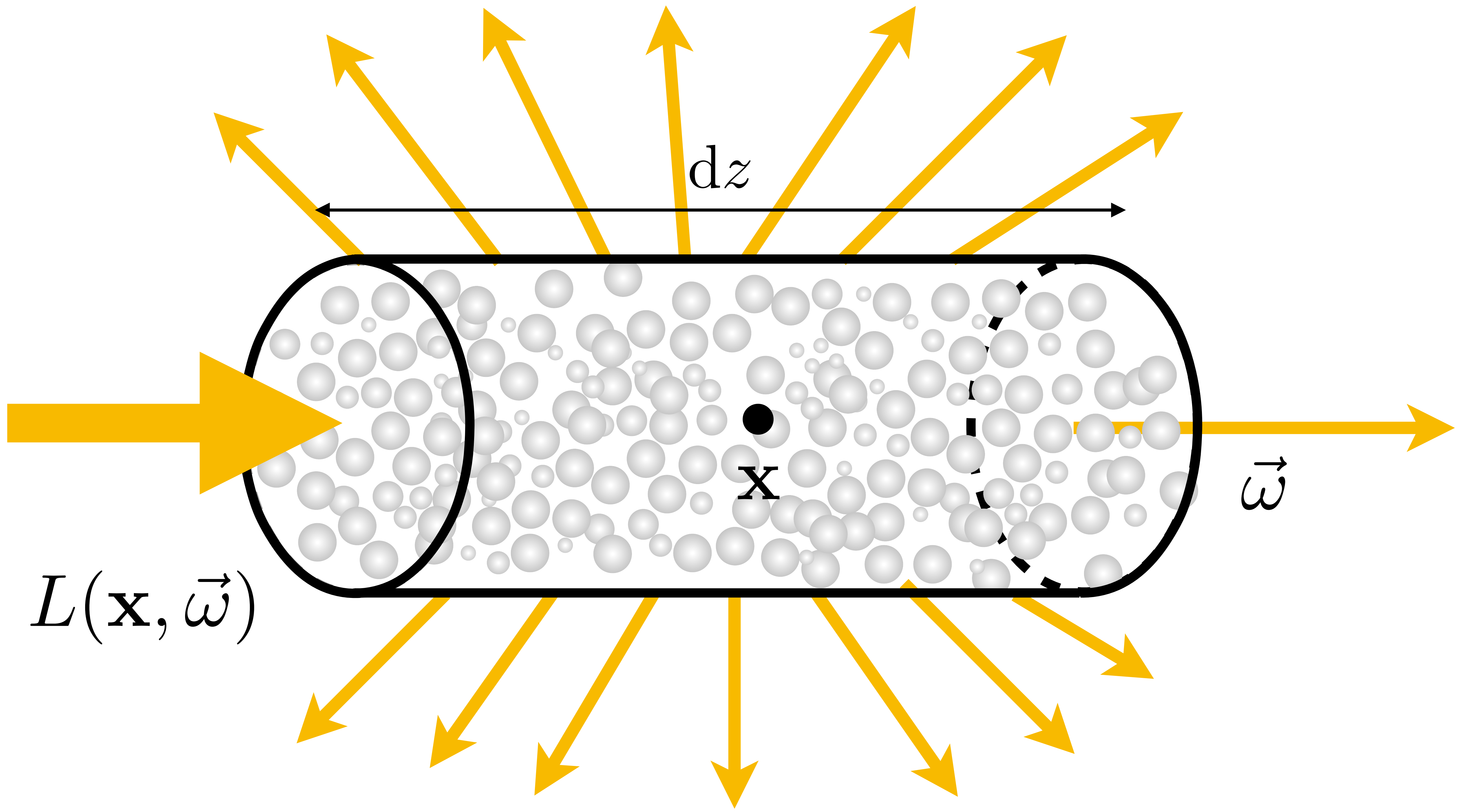
It is the probability density that light is absorbed per unit distance travelled in the medium

It can vary as a position and direction

Out-Scattering



Out-Scattering



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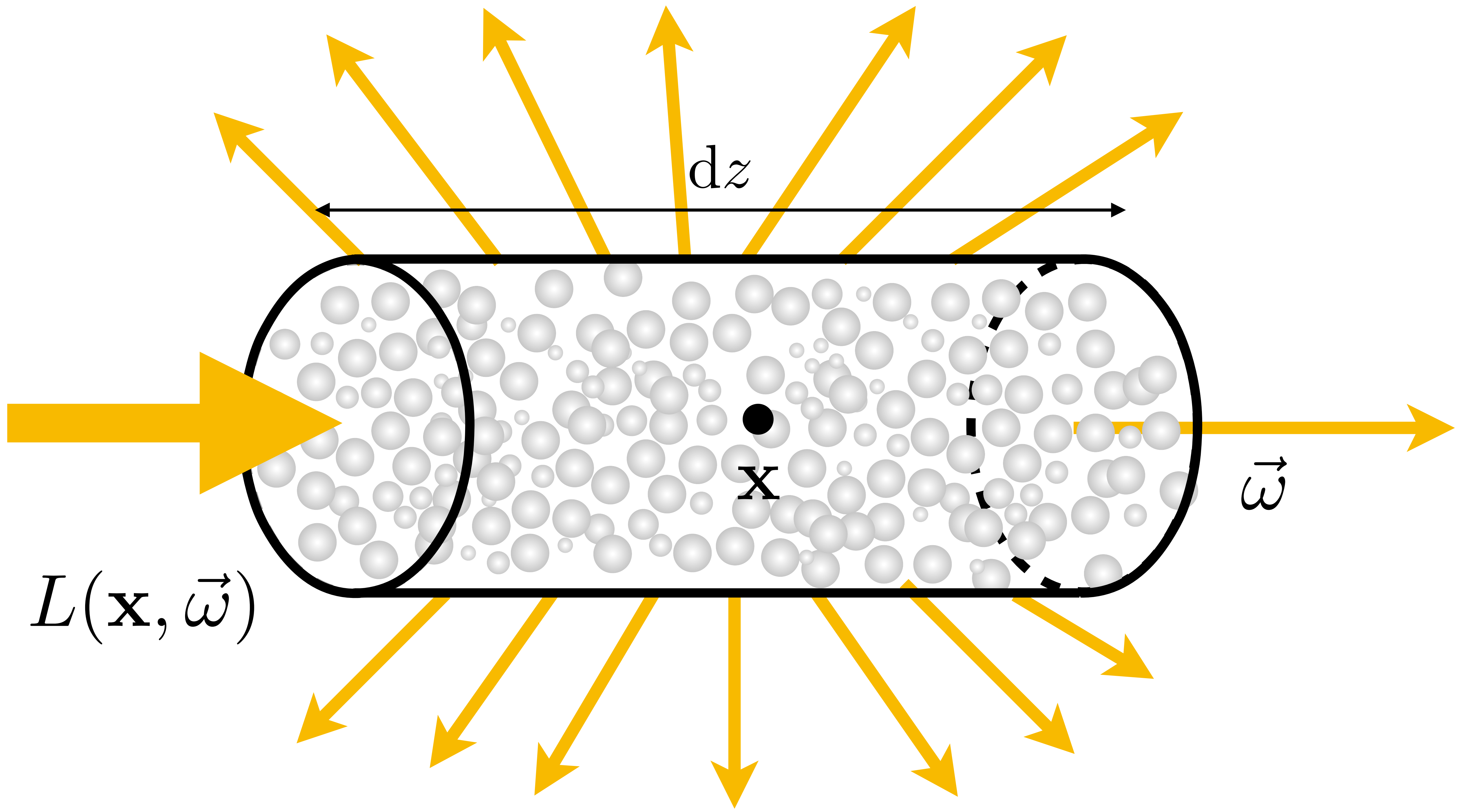
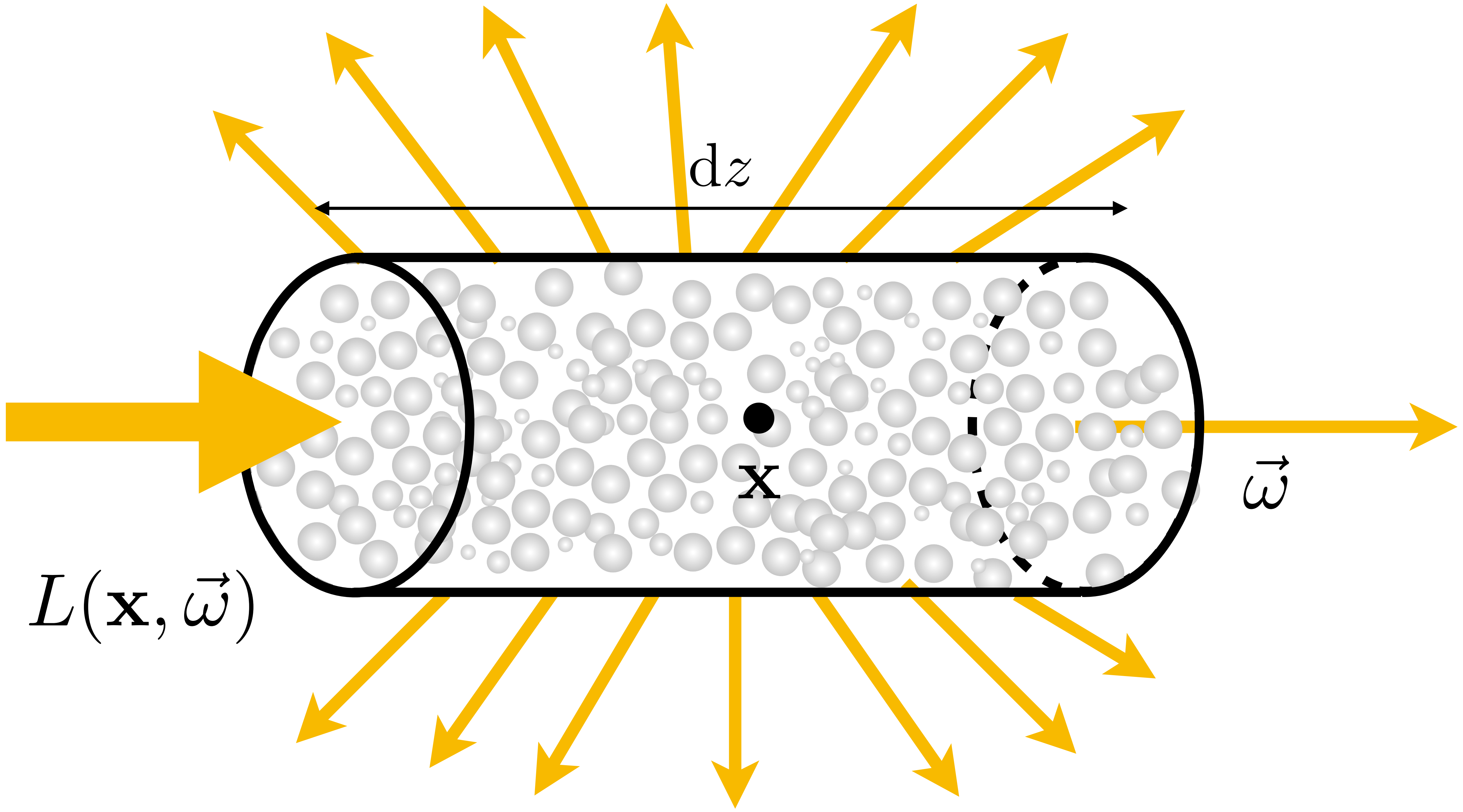


$$\frac{dL(\mathbf{x}, \vec{\omega})}{dz} = -\sigma_s L(\mathbf{x}, \vec{\omega})$$

Out-Scattering



$$\frac{dL(\mathbf{x},\vec{\omega})}{dz} = -\sigma_s L(\mathbf{x},\vec{\omega})$$

σ_s : scattering coefficient

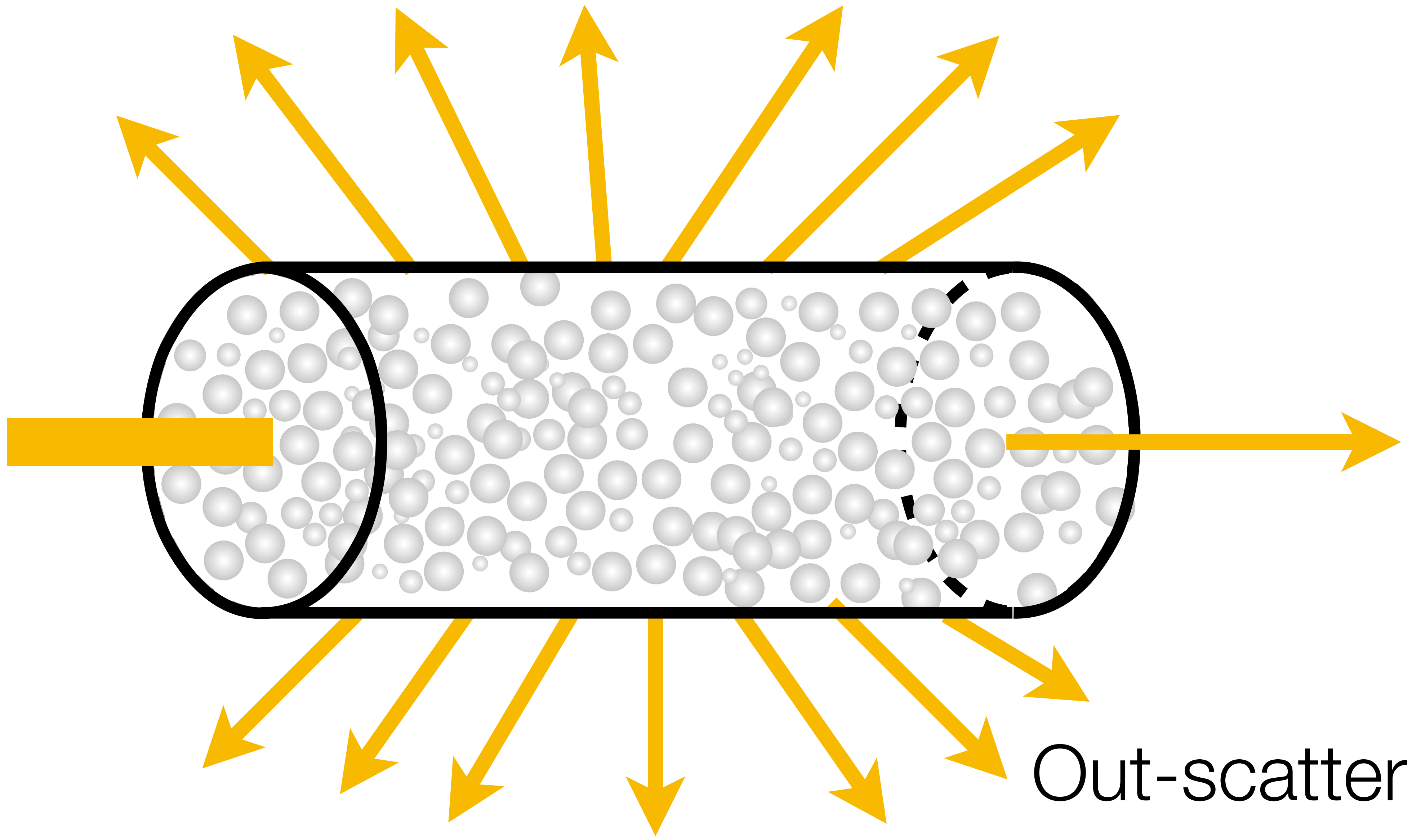
The probability of an out-scattering event occurring per unit distance is given by the scattering coefficient

Attenuation / Extinction

Total reduction in radiance:

Attenuation / Extinction

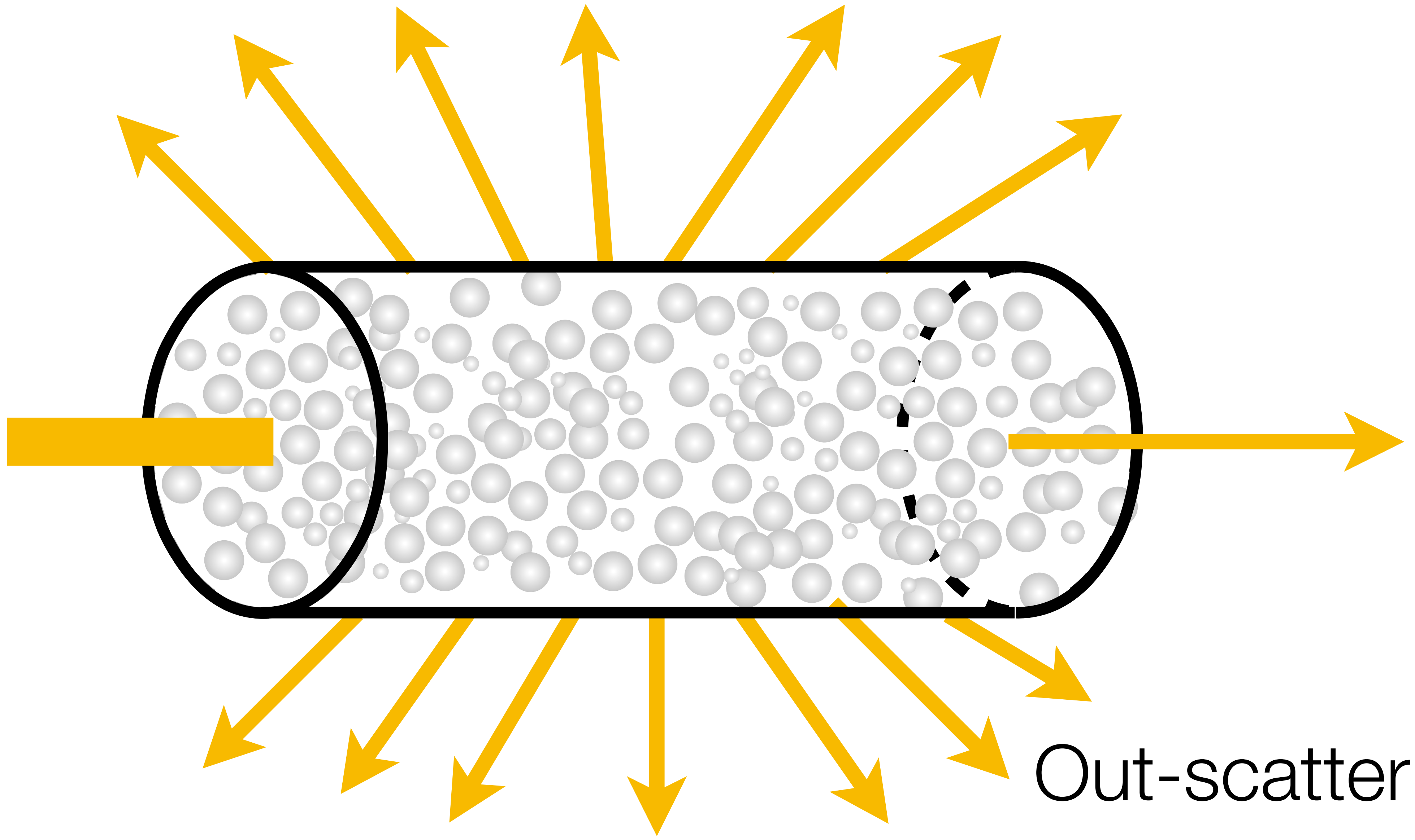
Total reduction in radiance:



Out-scattering

Attenuation / Extinction

Total reduction in radiance:



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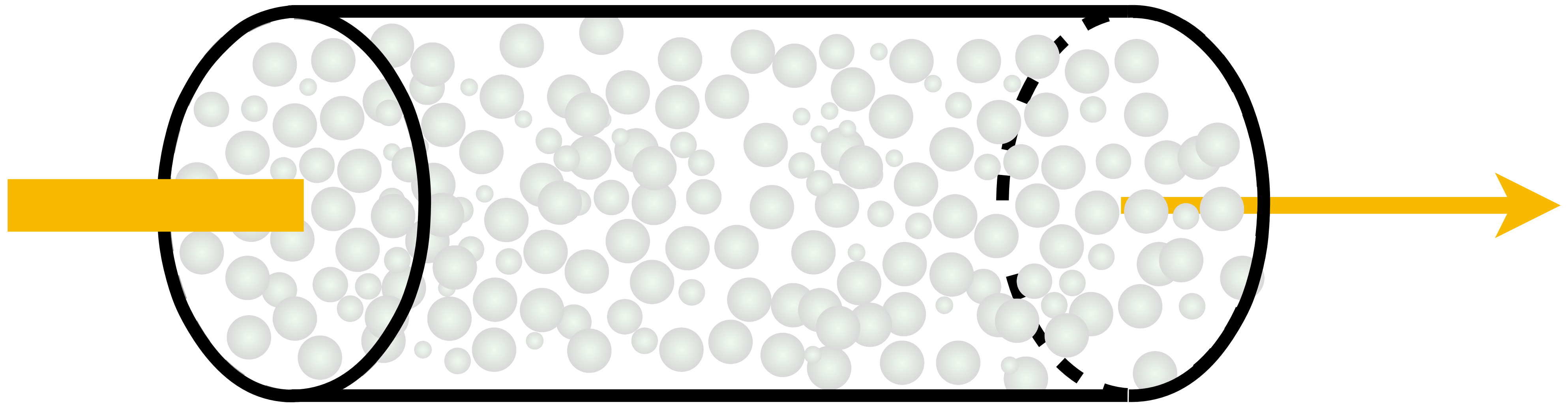


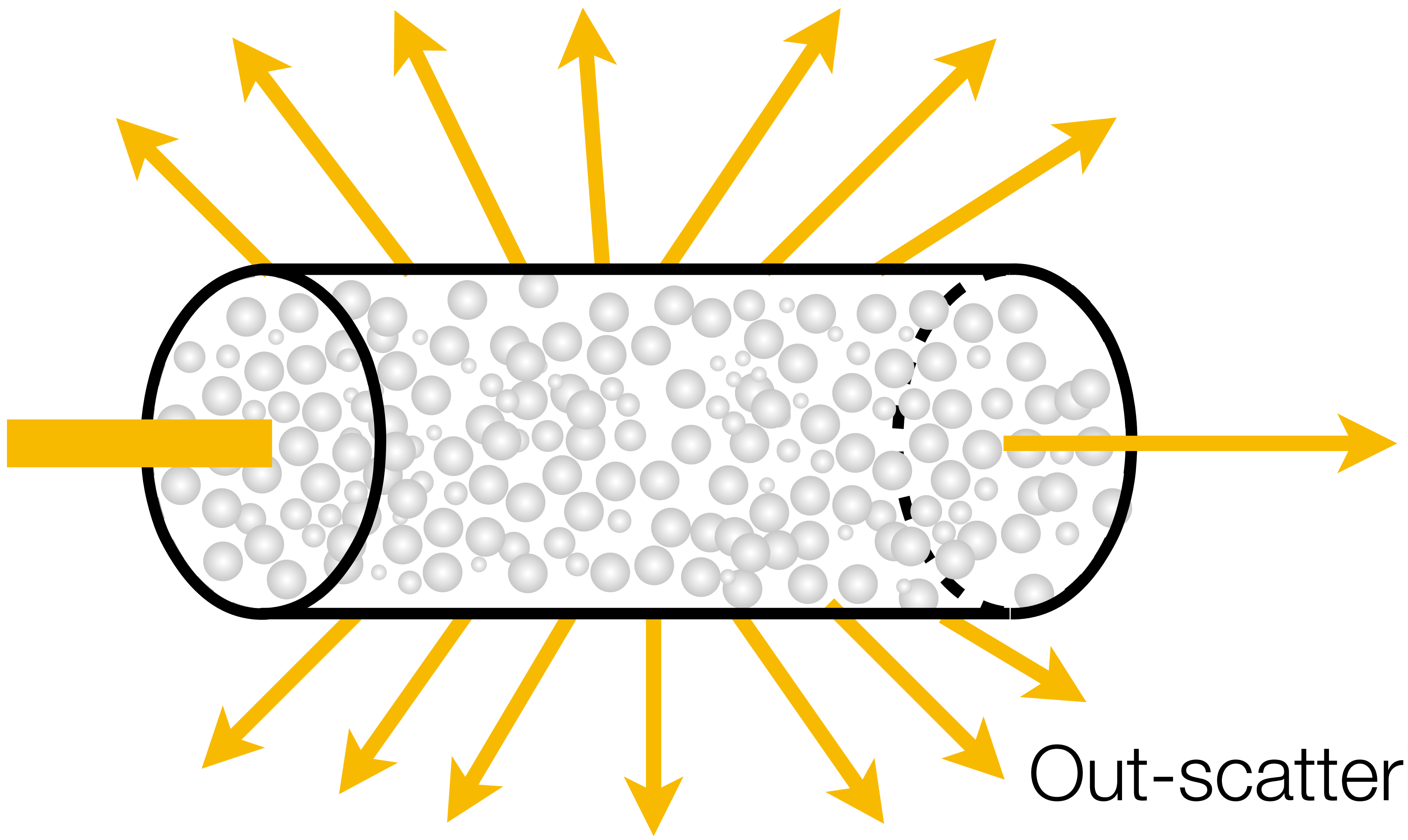
Absorption

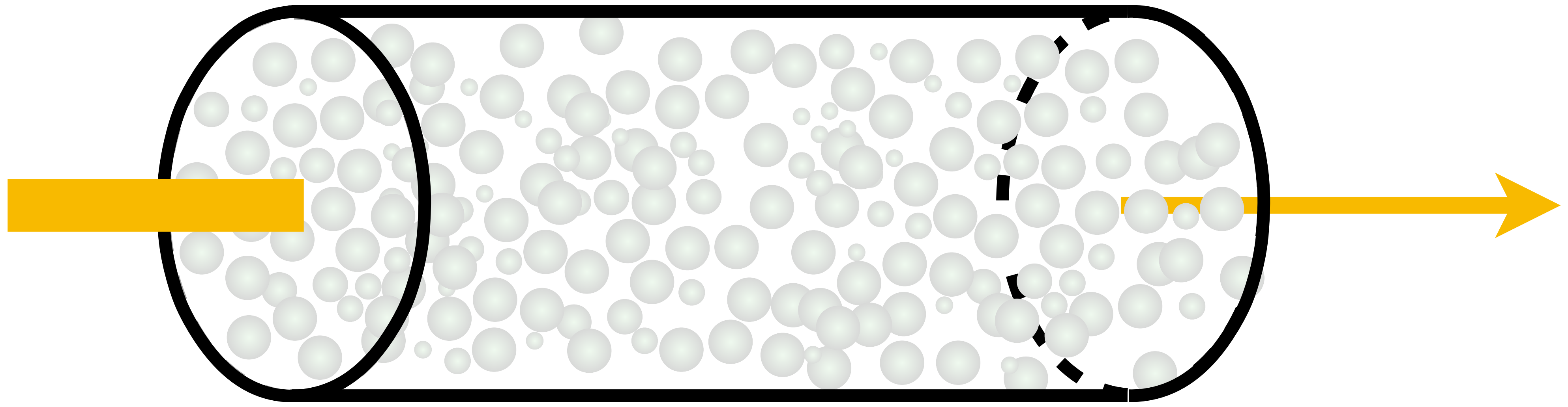
Attenuation / Extinction

Total reduction in radiance:

σ_a : absorption coefficient

σ_s : scattering coefficient





Attenuation / Extinction

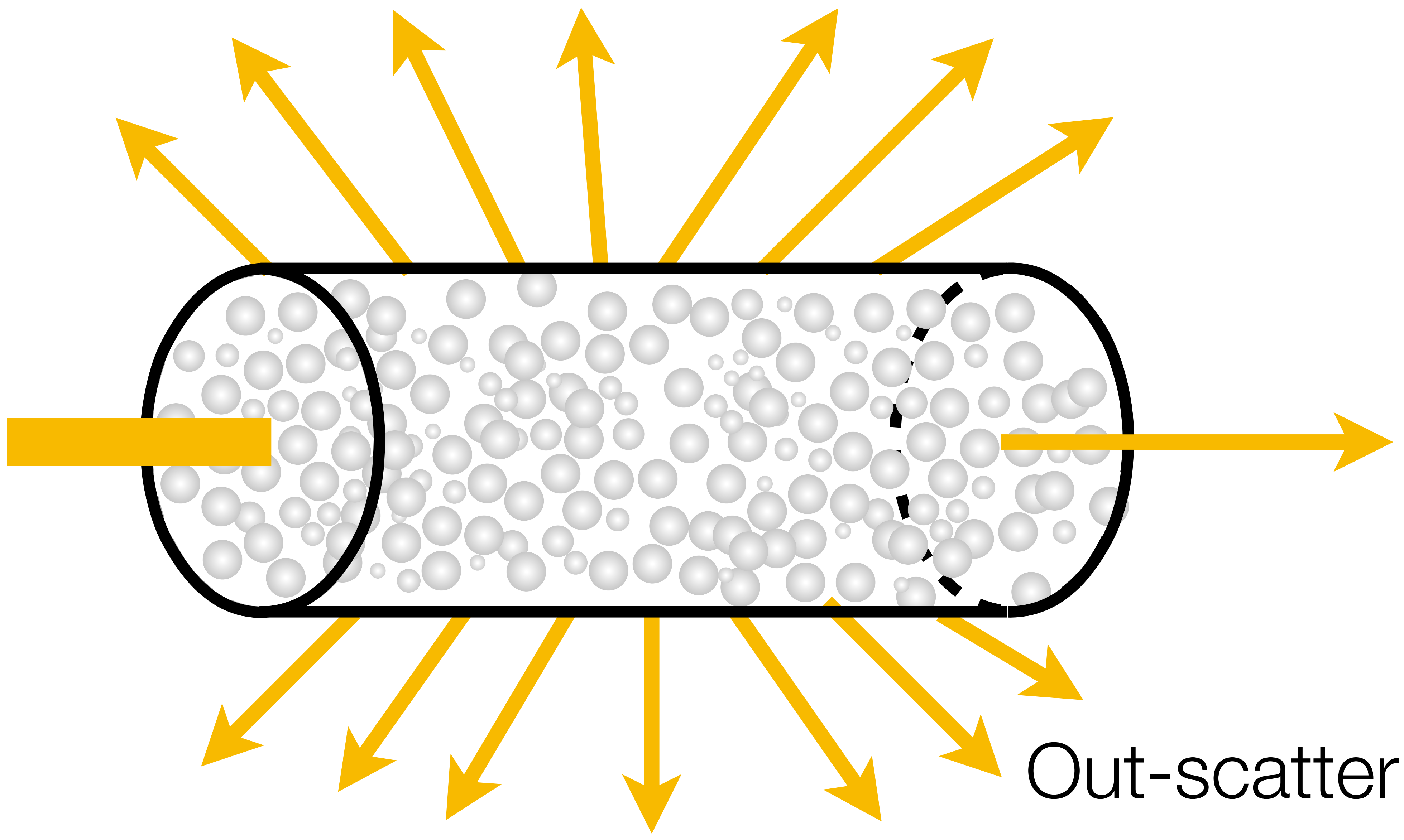
Total reduction in radiance:

$$\sigma_t(\mathbf{x},\vec{\omega}) = \sigma_a(\mathbf{x},\vec{\omega}) + \sigma_s(\mathbf{x},\vec{\omega})$$

σ_a : absorption coefficient

σ_s : scattering coefficient

σ_t : extinction coefficient



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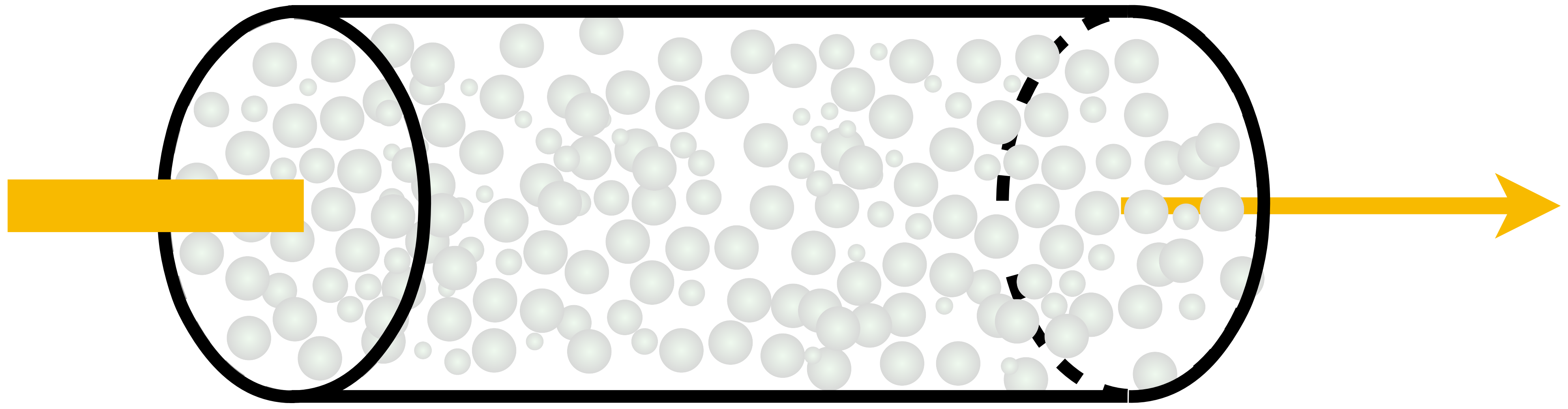

Absorption

Albedo

$$\alpha(\mathbf{x}) = \frac{\sigma_s(\mathbf{x})}{\sigma_a(\mathbf{x}) + \sigma_s(\mathbf{x})} = \frac{\sigma_s(\mathbf{x})}{\sigma_t(\mathbf{x})}$$

σ_s : scattering coefficient

σ_t : extinction coefficient

Albedo

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σ_t : extinction coefficient

Albedo

$$\alpha(\mathbf{x}) = \frac{\sigma_s(\mathbf{x})}{\sigma_t(\mathbf{x})}$$

The albedo is always between 0 and 1

It describes the probability of scattering (versus absorption) at a scattering event

σ_s : scattering coefficient

σ_t : extinction coefficient

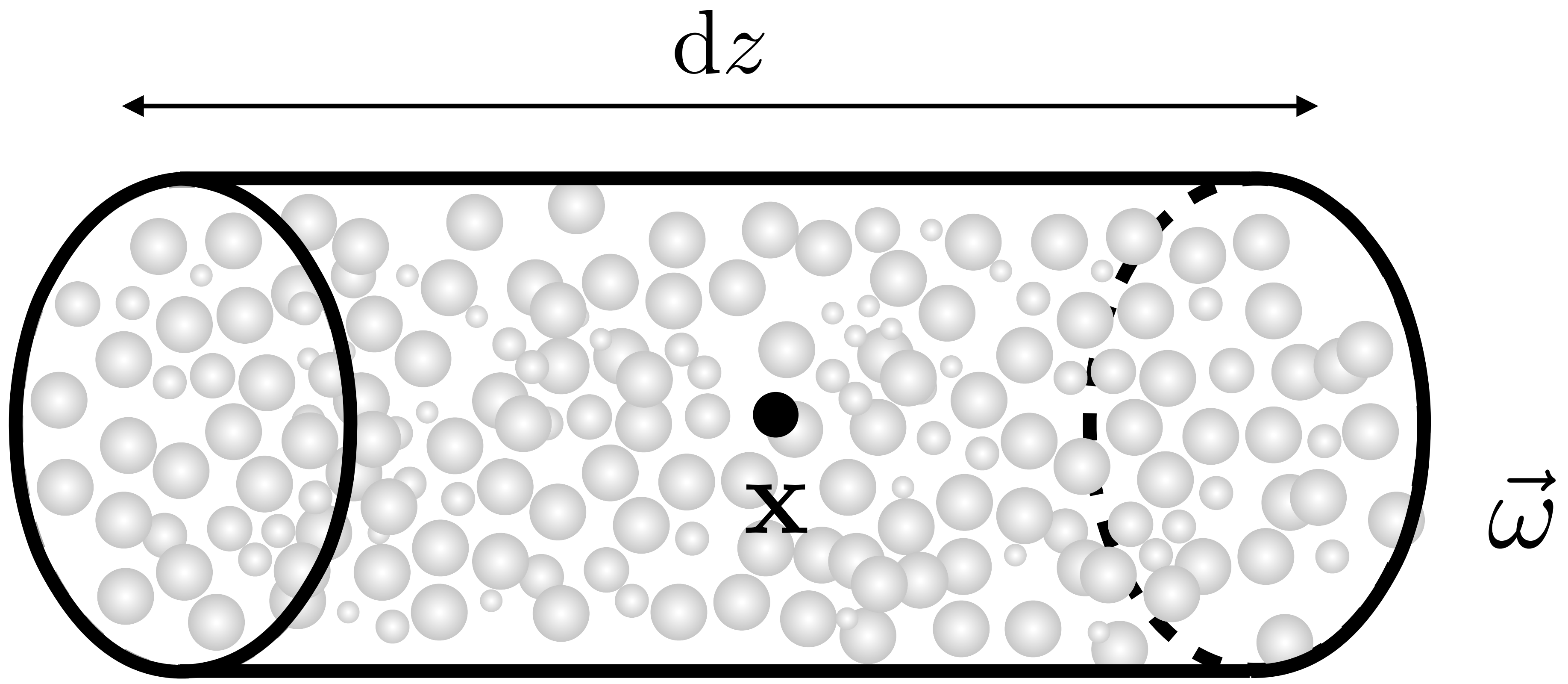
Mean-free path

$$\frac{1}{\sigma_t}$$

Mean free path gives the average distance travelled by the ray before interacting with a particle

σ_t : extinction coefficient

In-Scattering



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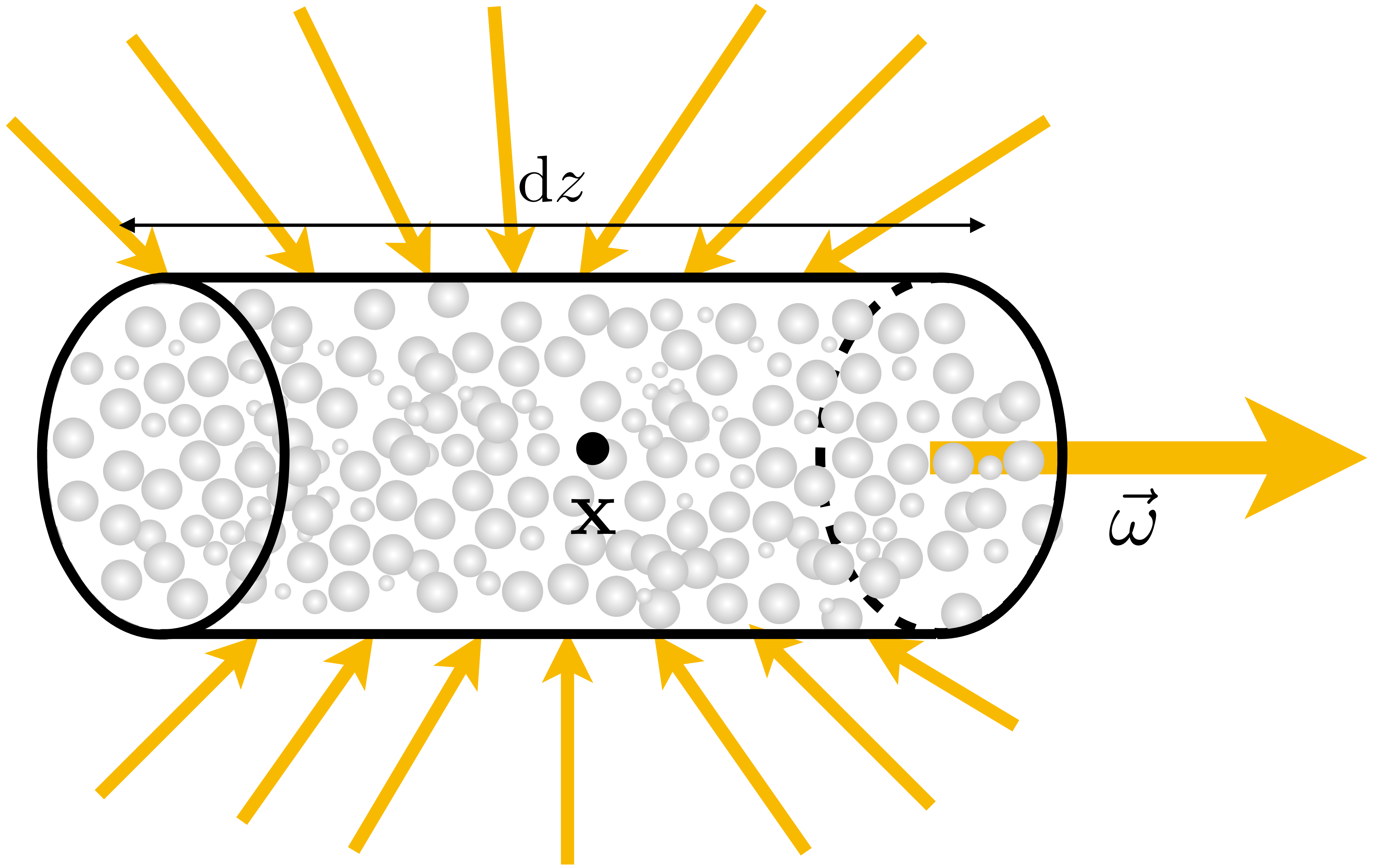
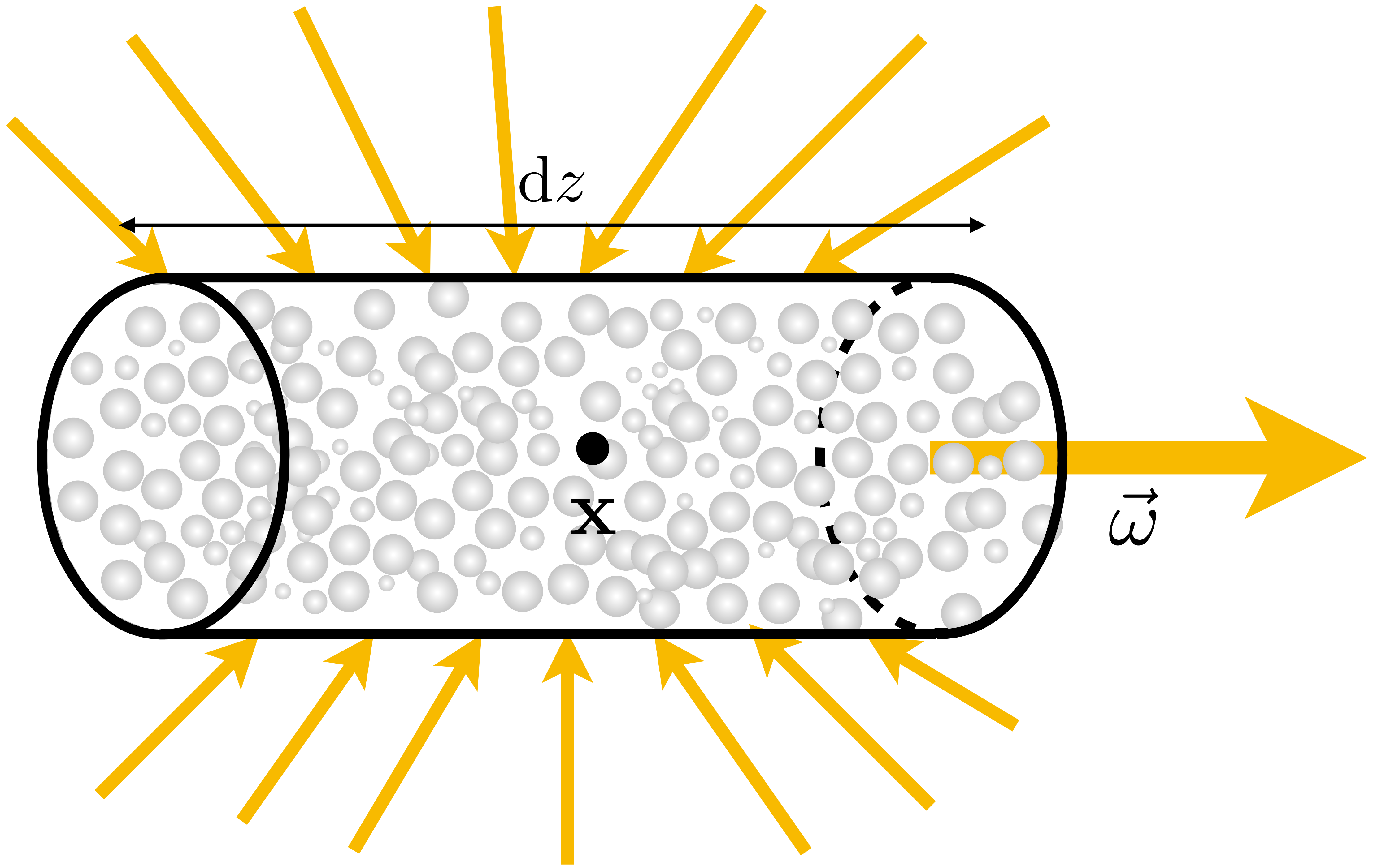
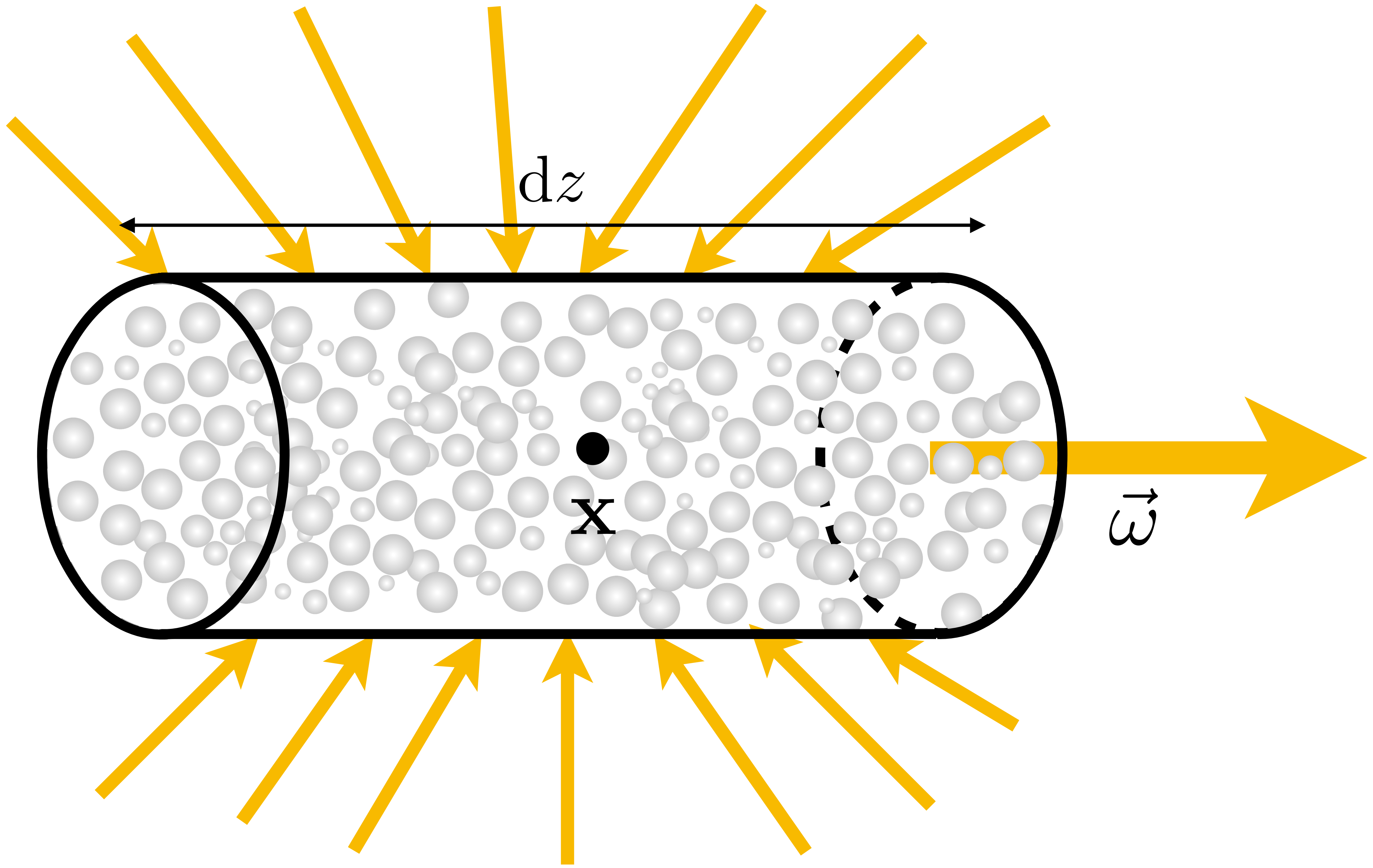


In-Scattering



$$\frac{\mathrm{d}L(\mathbf{x},\vec{\omega})}{\mathrm{d}z} = \sigma_s(\mathbf{x})L_\mathrm{s}(\mathbf{x},\vec{\omega})$$

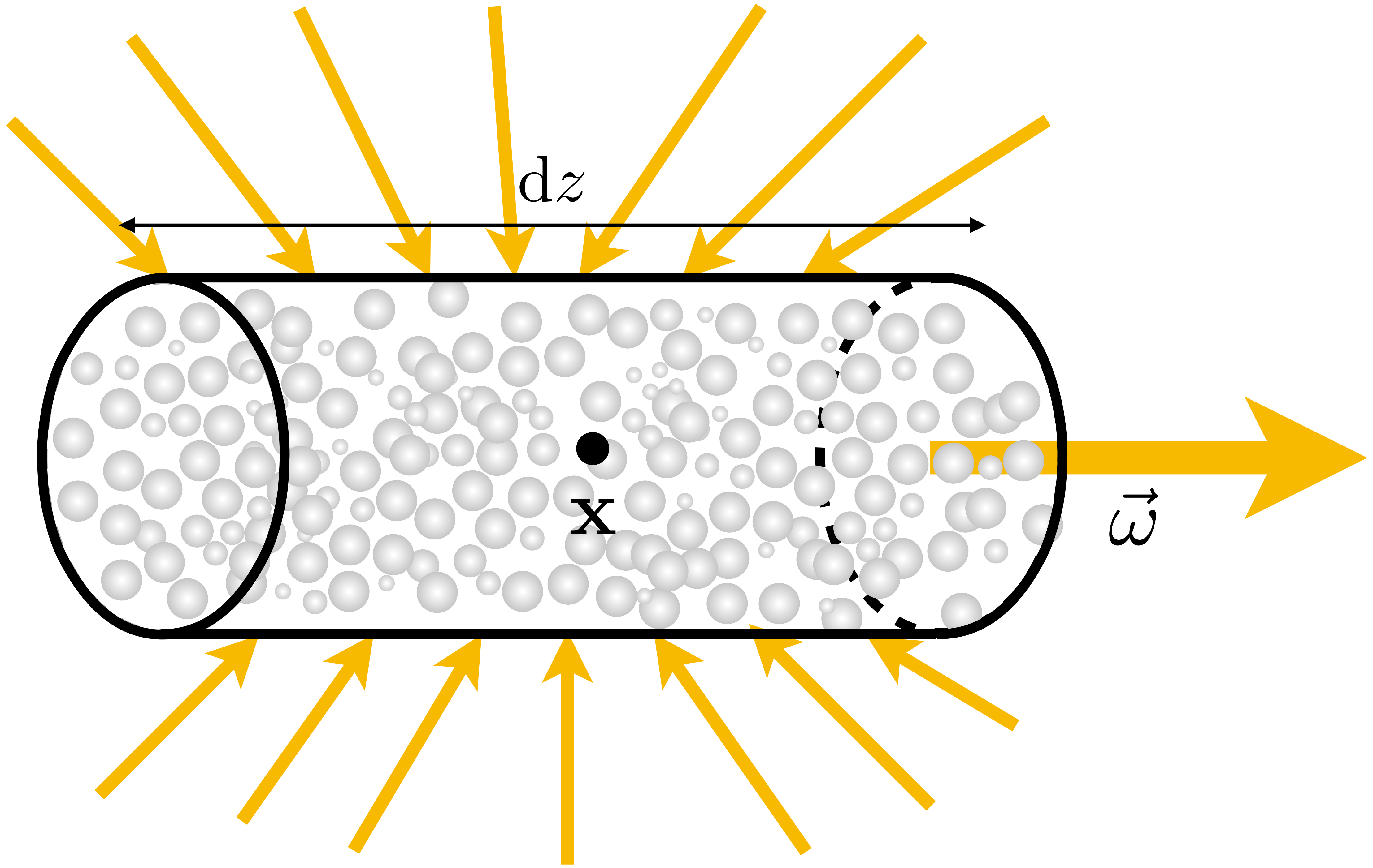
In-Scattering



$$\frac{\mathrm{d}L(\mathbf{x},\vec{\omega})}{\mathrm{d}z} = \boxed{\sigma_s(\mathbf{x})} L_\mathrm{s}(\mathbf{x},\vec{\omega})$$

$\boxed{\sigma_s(\mathbf{x})}$: scattering coefficient

In-Scattering



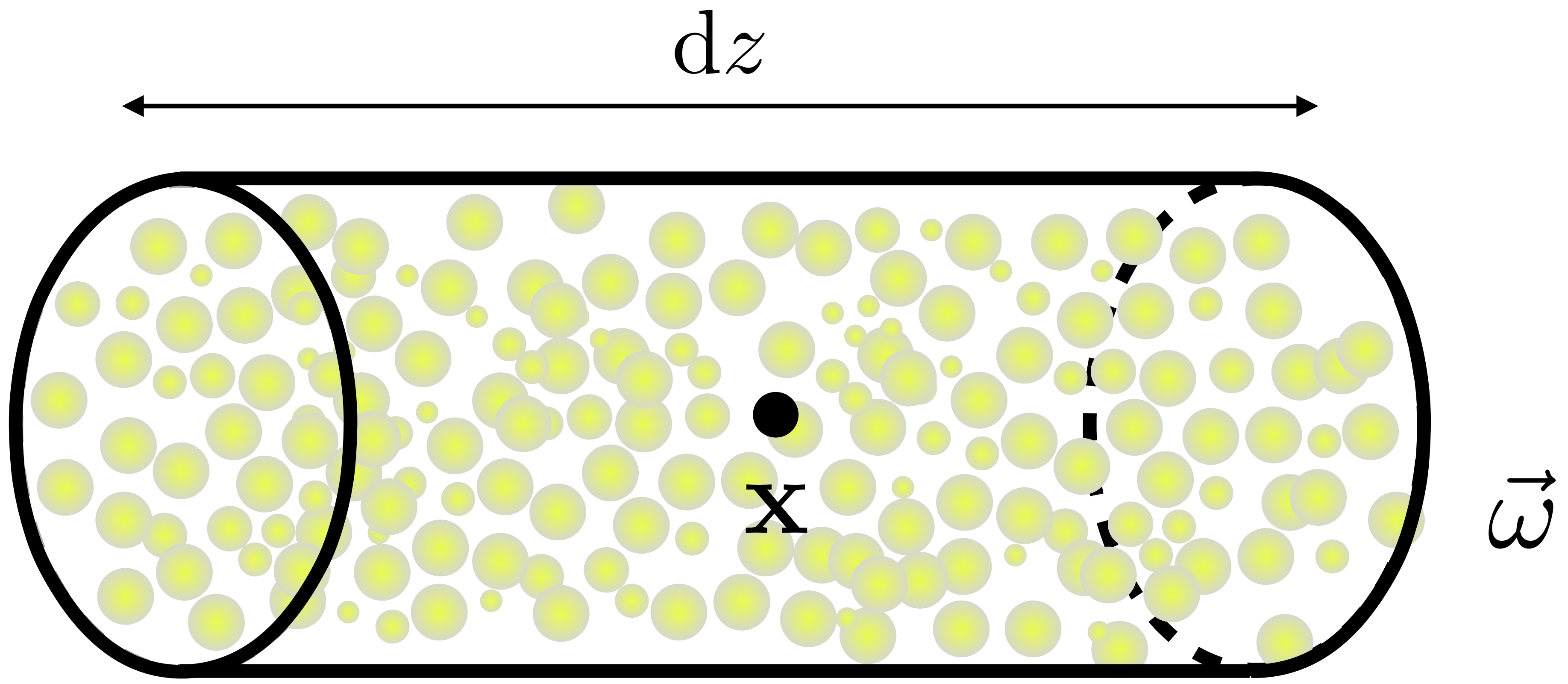
$$\frac{\mathrm{d}L(\mathbf{x},\vec{\omega})}{\mathrm{d}z} = \sigma_s(\mathbf{x})\boxed{L_{\mathrm{s}}(\mathbf{x},\vec{\omega})}$$

$\sigma_s(\mathbf{x})$: scattering coefficient

In-scattered radiance

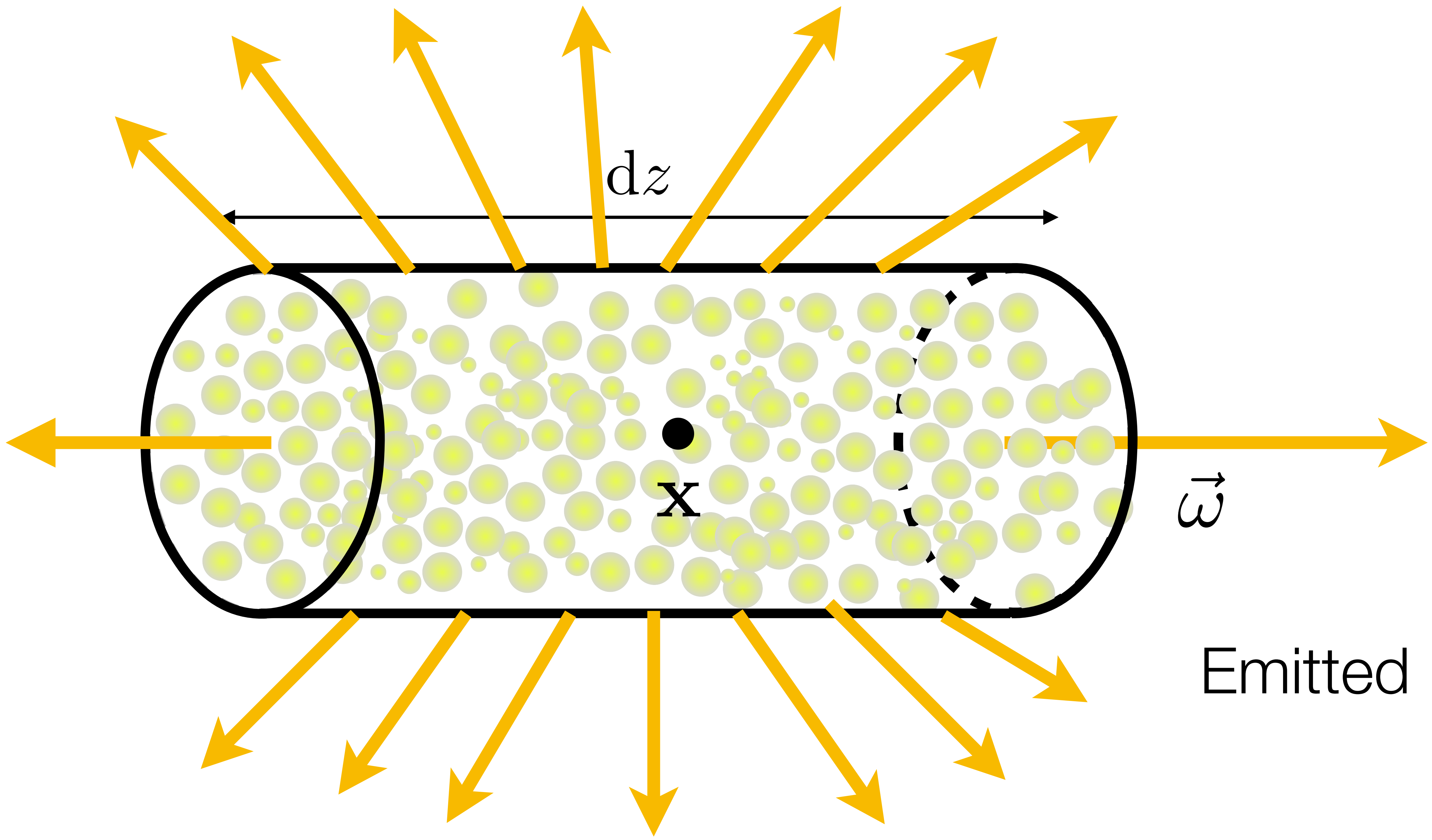
$$\boxed{L_{\mathrm{s}}(\mathbf{x},\vec{\omega})} = \int_{S^2} f_{\mathrm{p}}(\vec{\omega},\vec{\omega}')L(\mathbf{x},\vec{\omega}')\mathrm{d}\vec{\omega}'$$

Emission



Emitted radiance does not depend on the incoming light L_i

Emission



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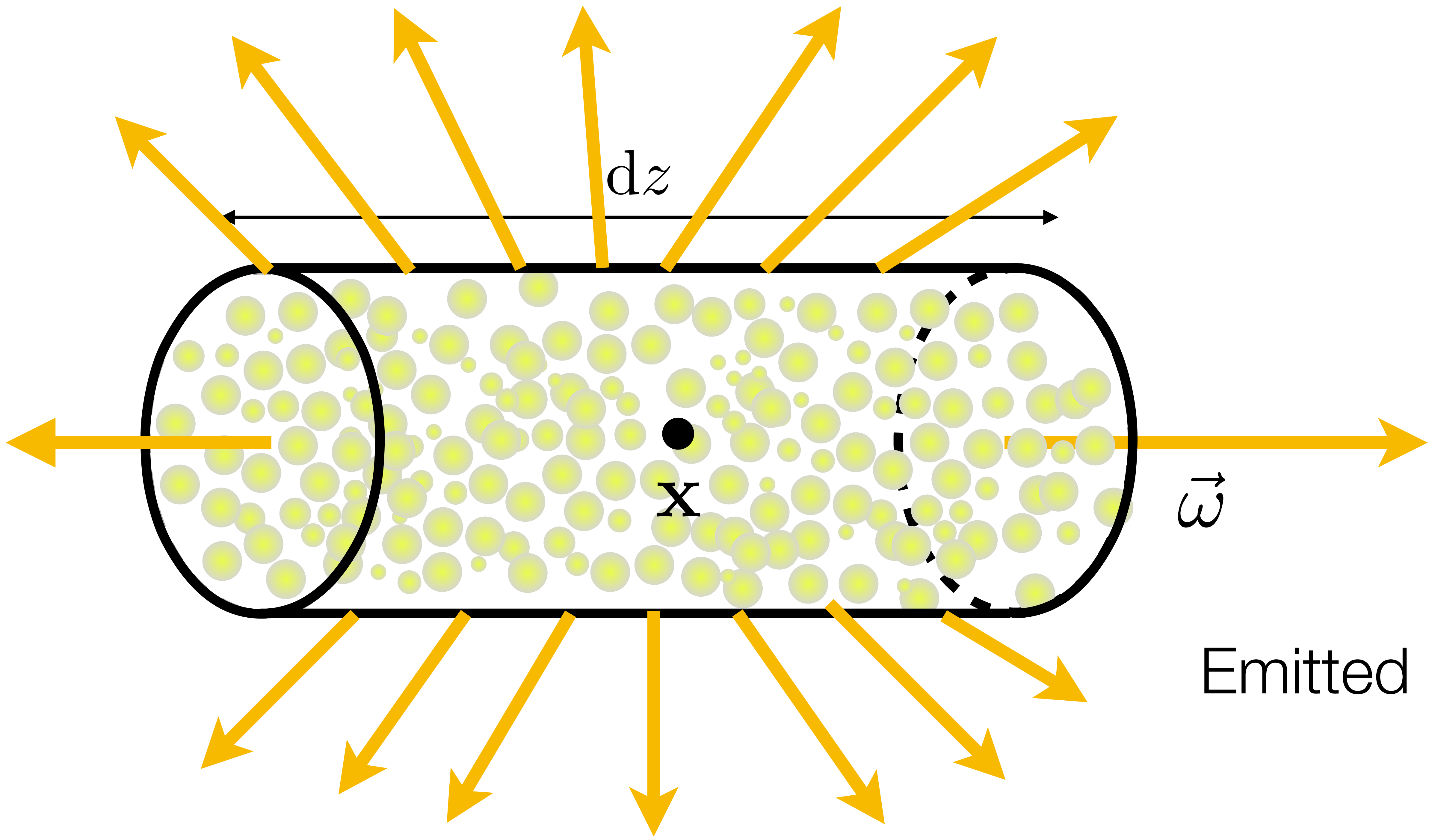
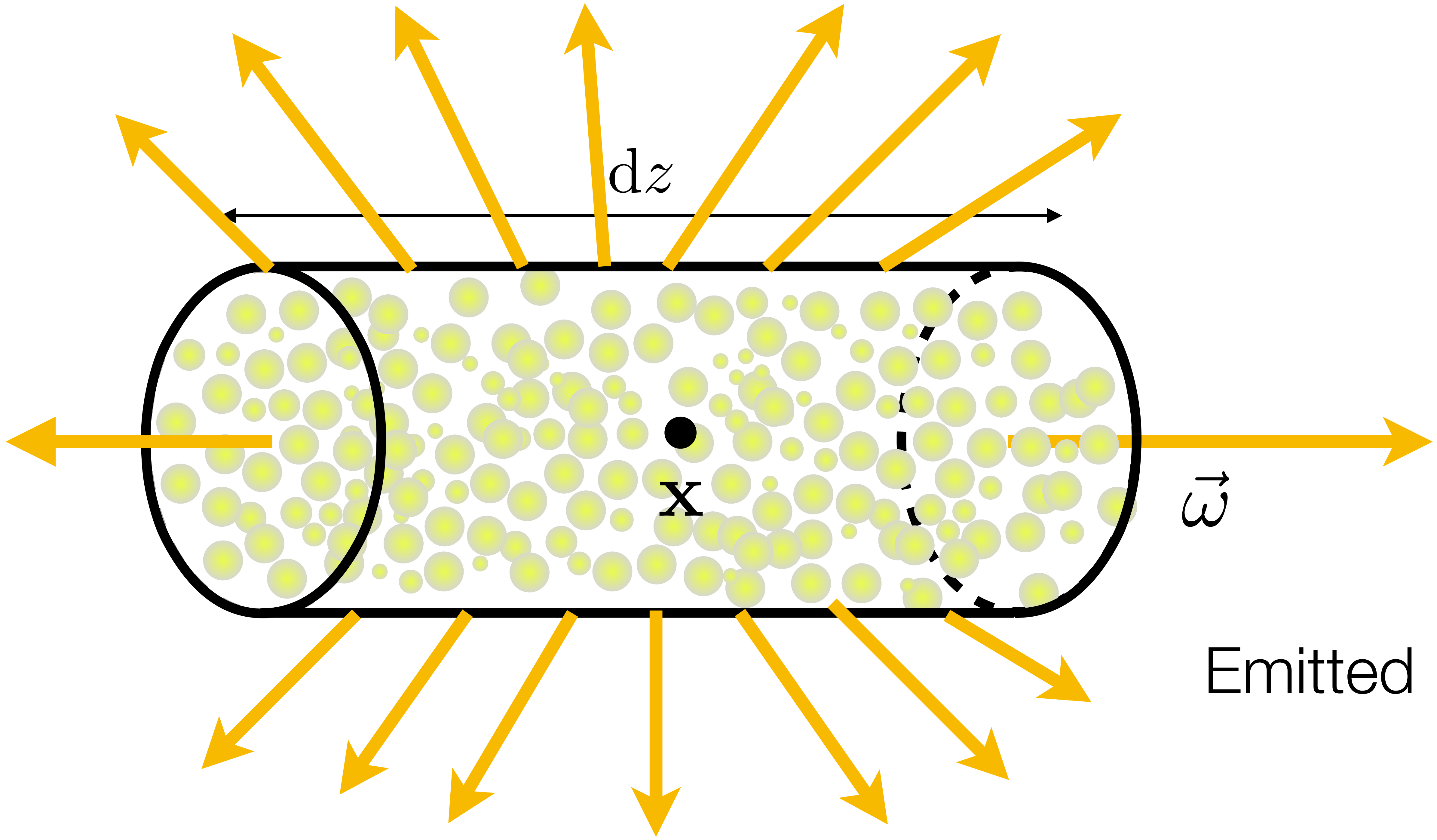


$$\frac{\mathrm{d}L(\mathbf{x},\vec{\omega})}{\mathrm{d}z} = \sigma_a(\mathbf{x})L_\mathrm{e}(\mathbf{x},\vec{\omega})$$

Emitted radiance does not depend on the incoming light L_i

Emission

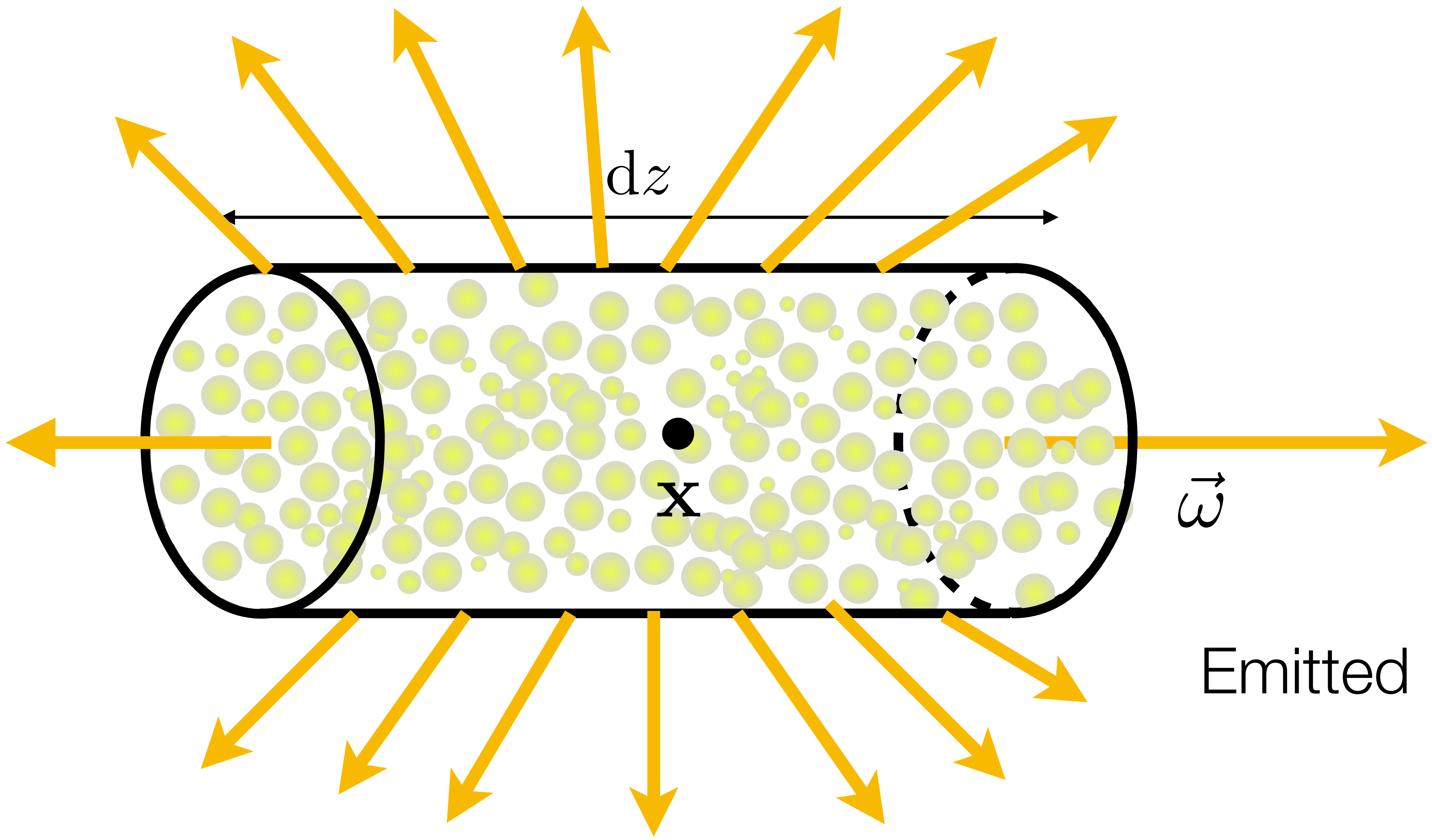


$$\frac{\mathrm{d}L(\mathbf{x},\vec{\omega})}{\mathrm{d}z} = \sigma_a(\mathbf{x})L_\mathrm{e}(\mathbf{x},\vec{\omega})$$

$L_e(\mathbf{x},\vec{\omega})$: emitted radiance

Emitted radiance does not depend on the incoming light L_i

Emission



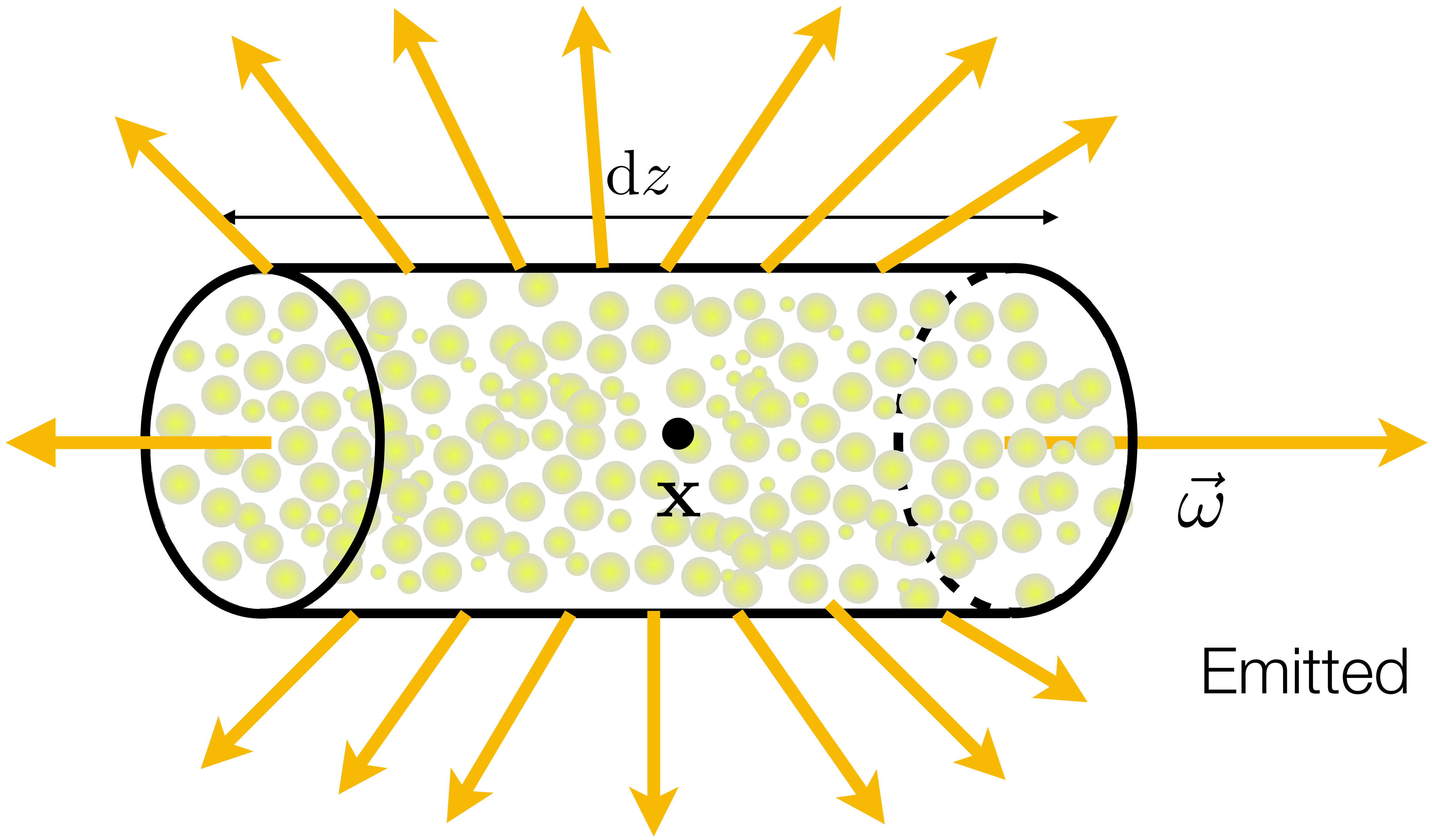
$$\frac{\mathrm{d}L(\mathbf{x},\vec{\omega})}{\mathrm{d}z} = \sigma_a(\mathbf{x})L_\mathrm{e}(\mathbf{x},\vec{\omega})$$

$L_e(\mathbf{x},\vec{\omega})$: emitted radiance

*sometimes modeled without the absorption coefficient term

Emitted radiance does not depend on the incoming light L_i

Emission



$$\frac{\mathrm{d}L(\mathbf{x},\vec{\omega})}{\mathrm{d}z} = \sigma_a(\mathbf{x})L_\mathrm{e}(\mathbf{x},\vec{\omega})$$

$L_e(\mathbf{x},\vec{\omega})$: emitted radiance

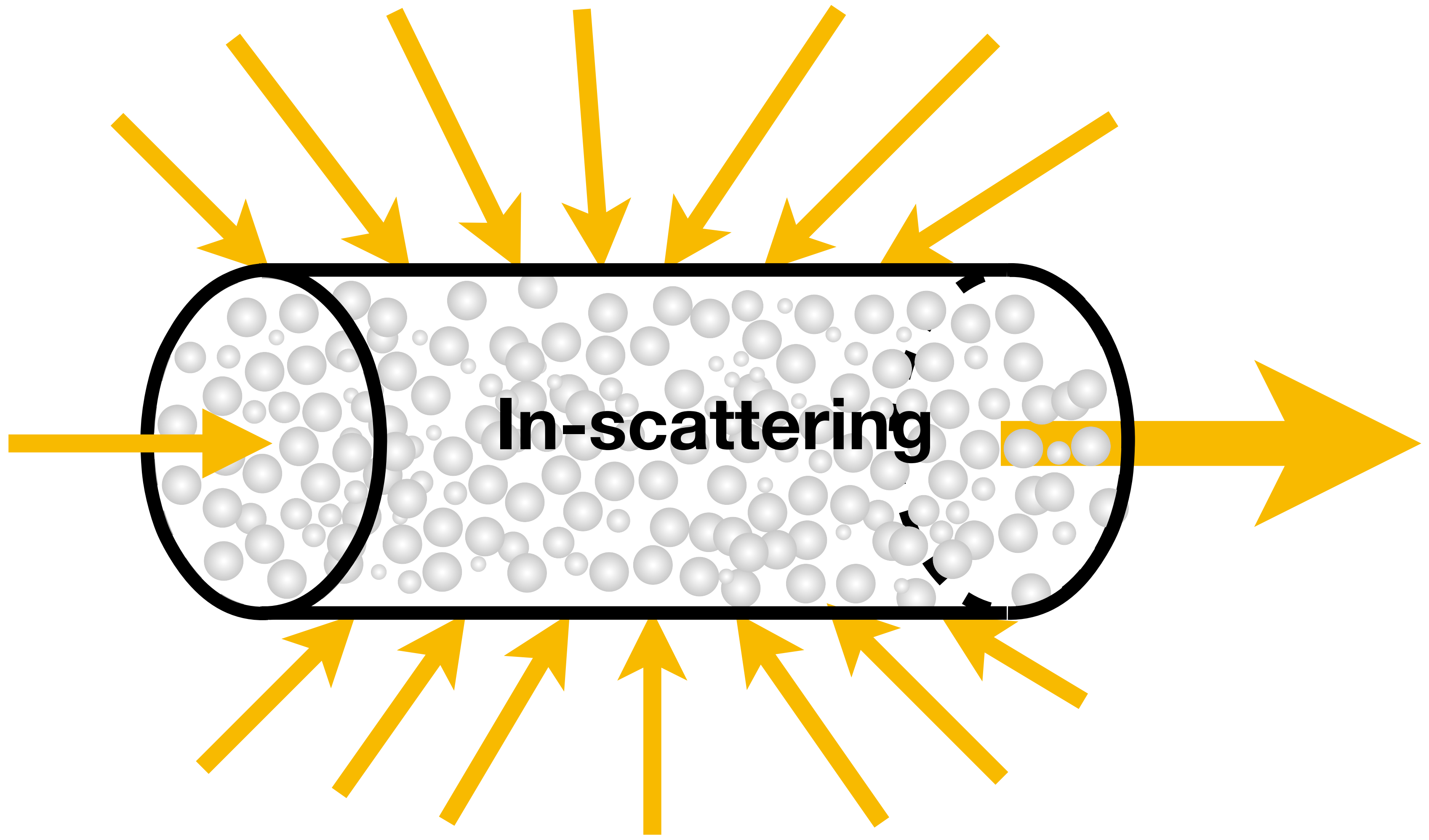
*sometimes modeled without the absorption coefficient term

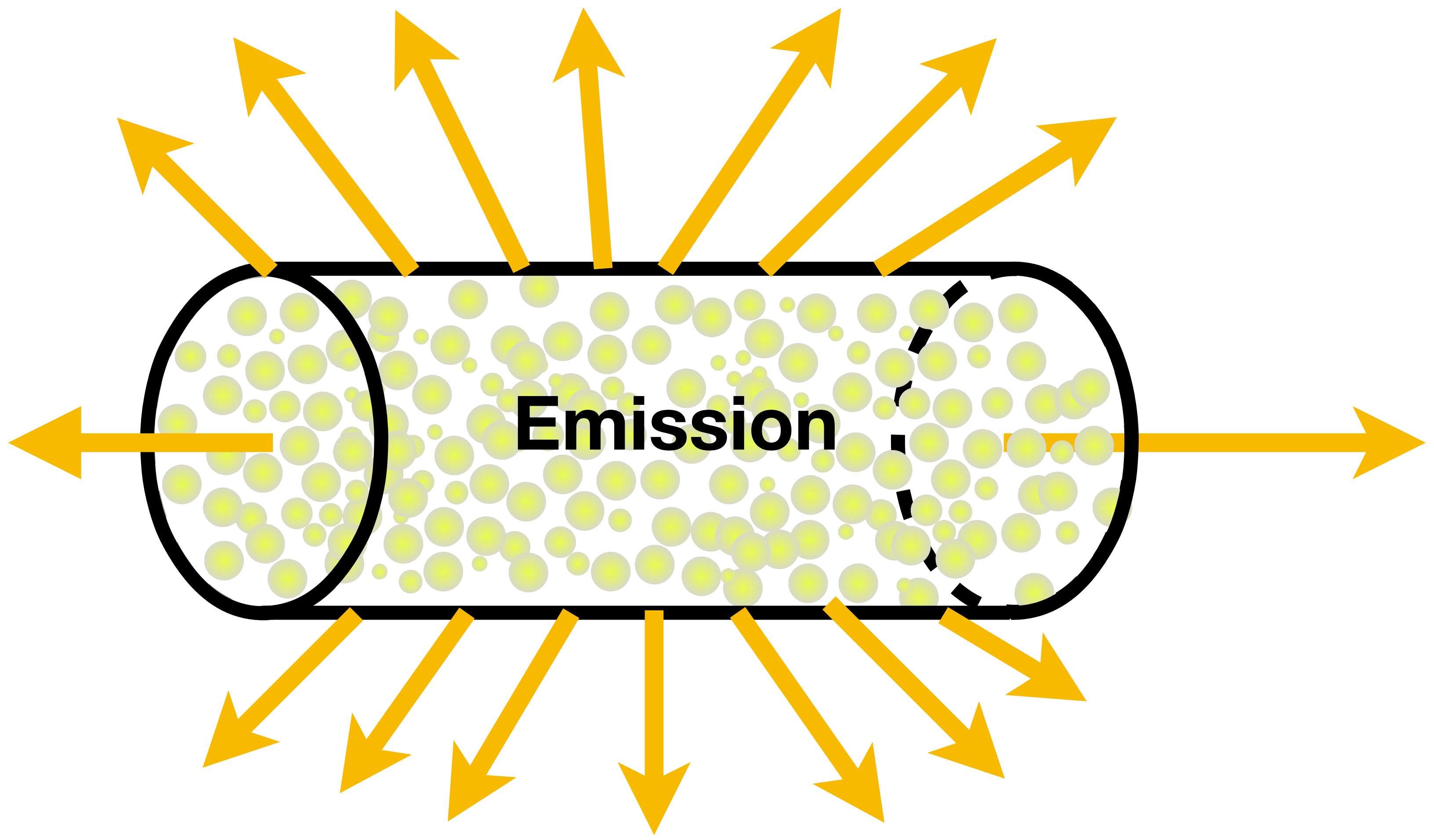
Emitted radiance does not depend on the incoming light L_i

Here we made a choice to represent differential output radiance as a product of emitted radiance and absorption coefficient.

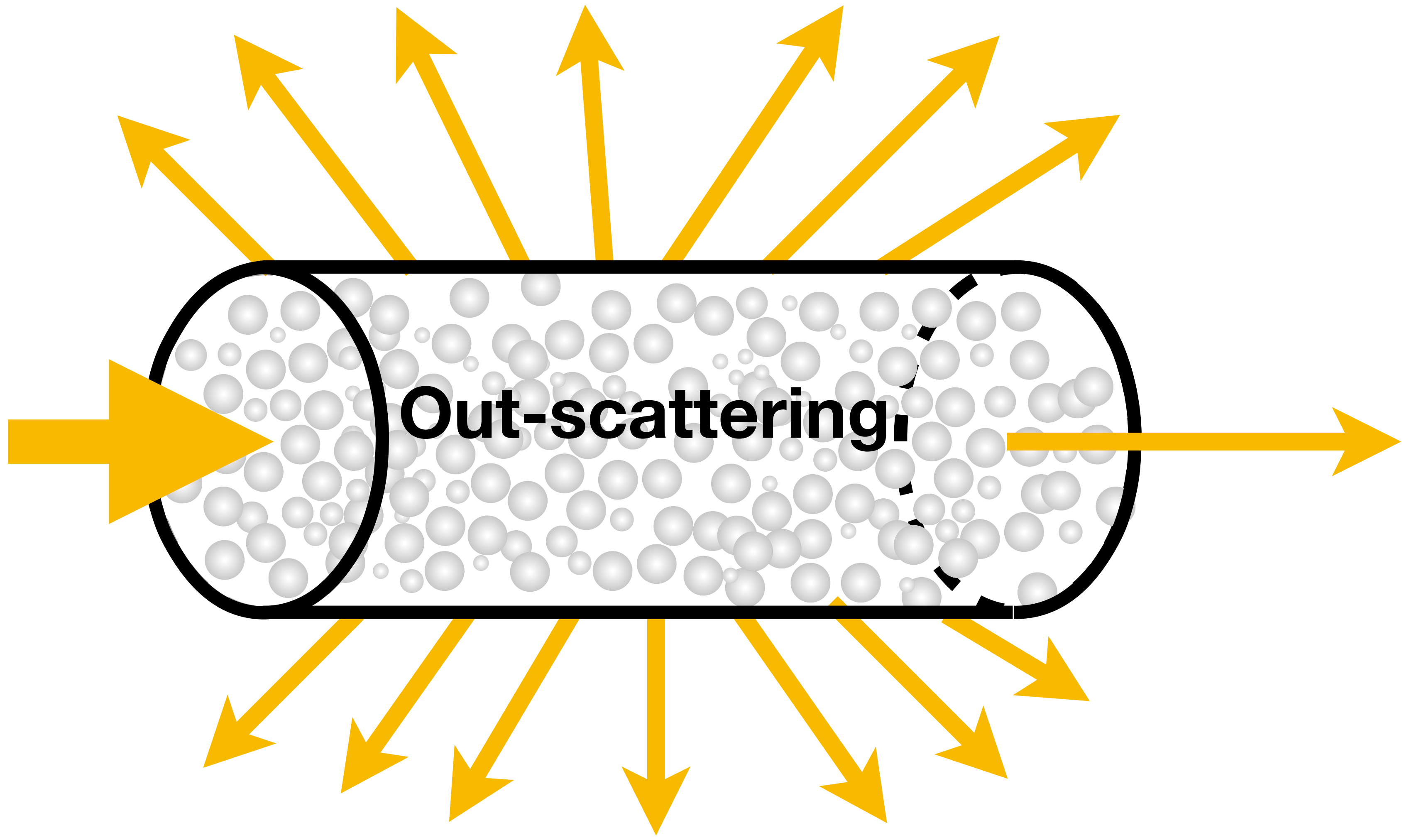
Radiative Transfer Equation

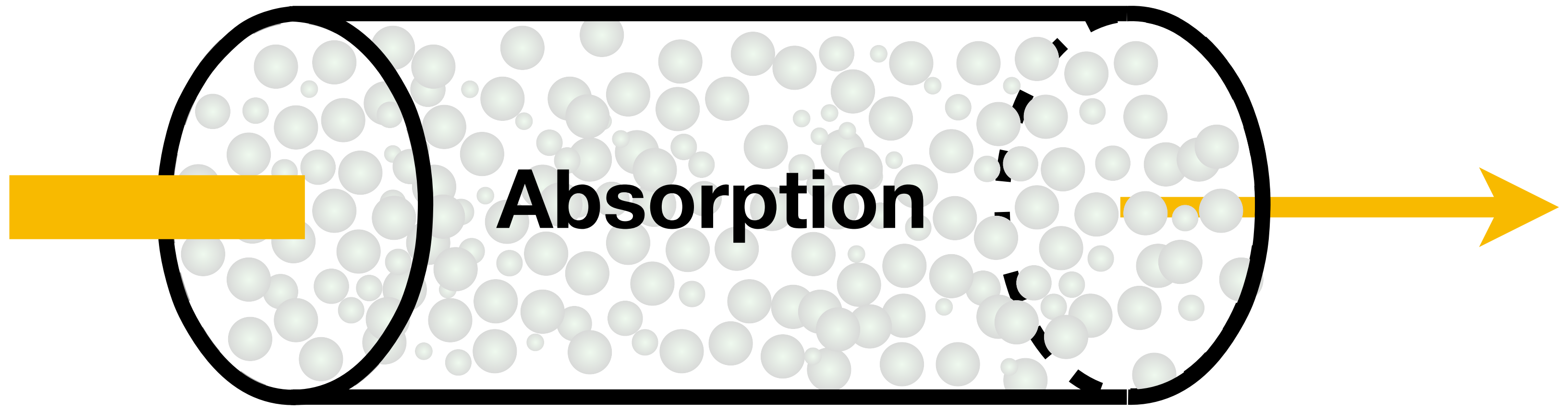
Radiative Transfer Equation (RTE)

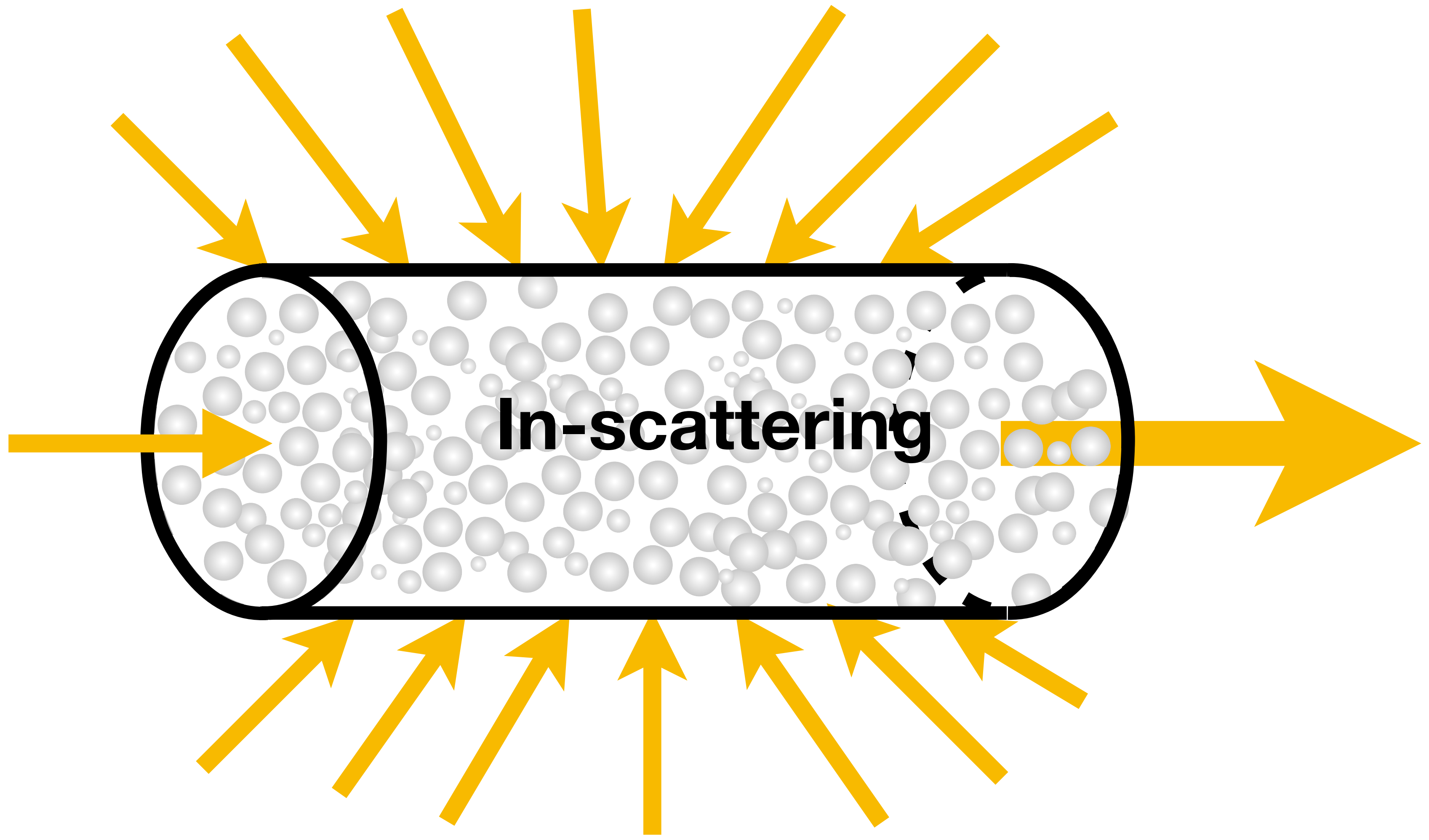


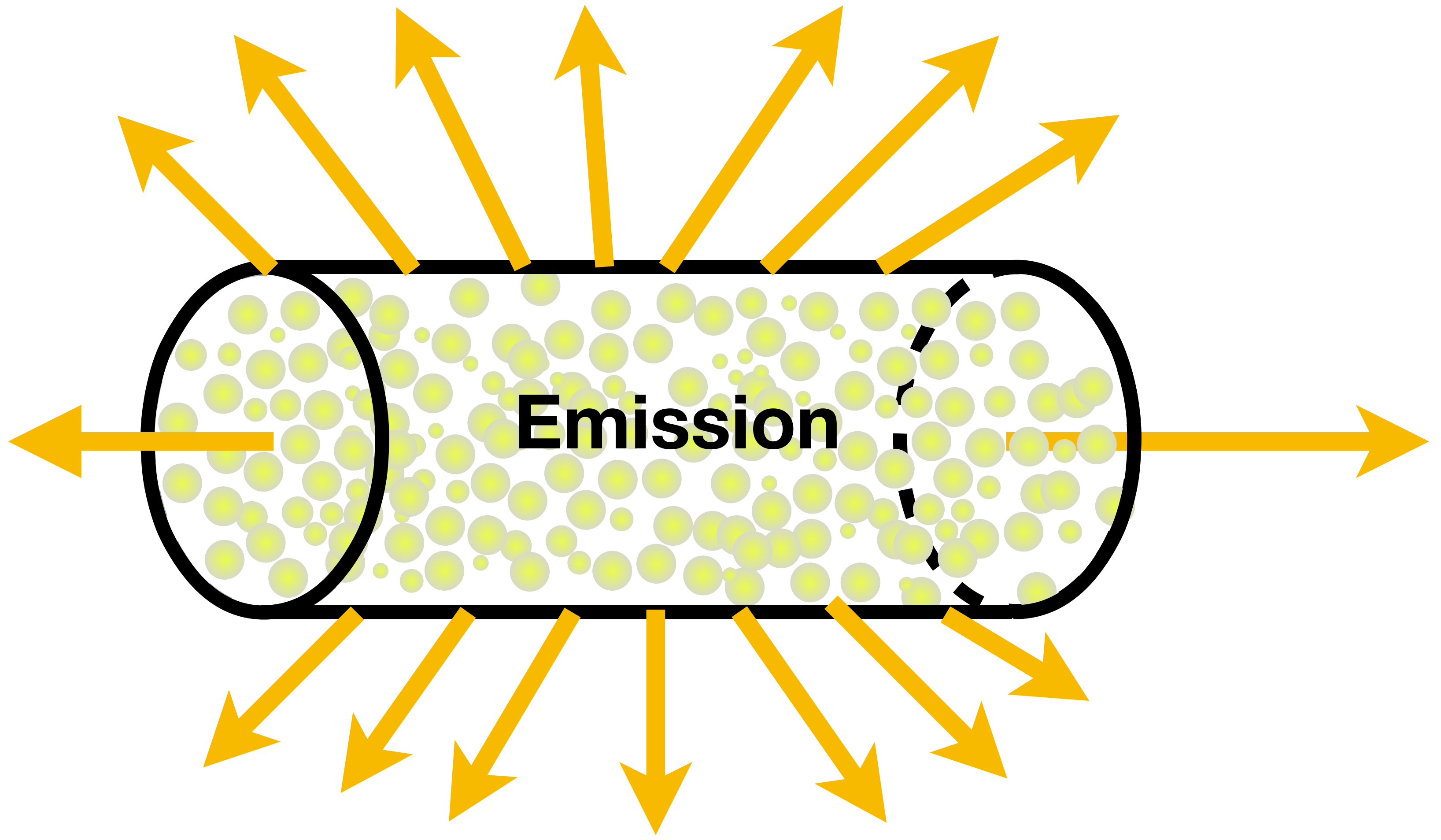


Radiative Transfer Equation (RTE)









Radiative Transfer Equation (RTE)

Out-scattering **Absorption** Losses

$$\frac{\mathrm{d}L(\mathbf{x},\vec{\omega})}{\mathrm{d}z}$$

In-scattering **Emission** Gains

Radiative Transfer Equation (RTE)

$$\frac{\mathrm{d}L(\mathbf{x},\vec{\omega})}{\mathrm{d}z}$$

$-\sigma_s(\mathbf{x})L(\mathbf{x},\vec{\omega})$ $-\sigma_a(\mathbf{x})L(\mathbf{x},\vec{\omega})$

Out-scattering **Absorption** Losses

$\sigma_s(\mathbf{x})L_s(\mathbf{x},\vec{\omega})$ $\sigma_a(\mathbf{x})L_e(\mathbf{x},\vec{\omega})$

In-scattering **Emission** Gains

Radiative Transfer Equation (RTE)

Out-scattering **Absorption**

$$\frac{\mathrm{d}L(\mathbf{x},\vec{\omega})}{\mathrm{d}z} \quad -\sigma_s(\mathbf{x})L(\mathbf{x},\vec{\omega}) -\sigma_a(\mathbf{x})L(\mathbf{x},\vec{\omega}) + \sigma_s(\mathbf{x})L_s(\mathbf{x},\vec{\omega}) + \sigma_a(\mathbf{x})L_e(\mathbf{x},\vec{\omega})$$

In-scattering **Emission**

Radiative Transfer Equation (RTE)

Out-scattering **Absorption**

$$\frac{\mathrm{d}L(\mathbf{x},\vec{\omega})}{\mathrm{d}z} \quad -\sigma_s(\mathbf{x})L(\mathbf{x},\vec{\omega}) - \sigma_a(\mathbf{x})L(\mathbf{x},\vec{\omega}) + \sigma_s(\mathbf{x})L_s(\mathbf{x},\vec{\omega}) + \sigma_a(\mathbf{x})L_e(\mathbf{x},\vec{\omega})$$

In-scattering **Emission**

$$\sigma_t(\mathbf{x},\vec{\omega}) = \sigma_a(\mathbf{x},\vec{\omega}) + \sigma_s(\mathbf{x},\vec{\omega})$$

Radiative Transfer Equation (RTE)

Attenuation

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In-scattering **Emission**

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What about a beam with finite-length z?

Extinction Along a Finite Beam

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// Exponentiate

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Beer-Lambert Law

The fraction refers to as the *transmittance*

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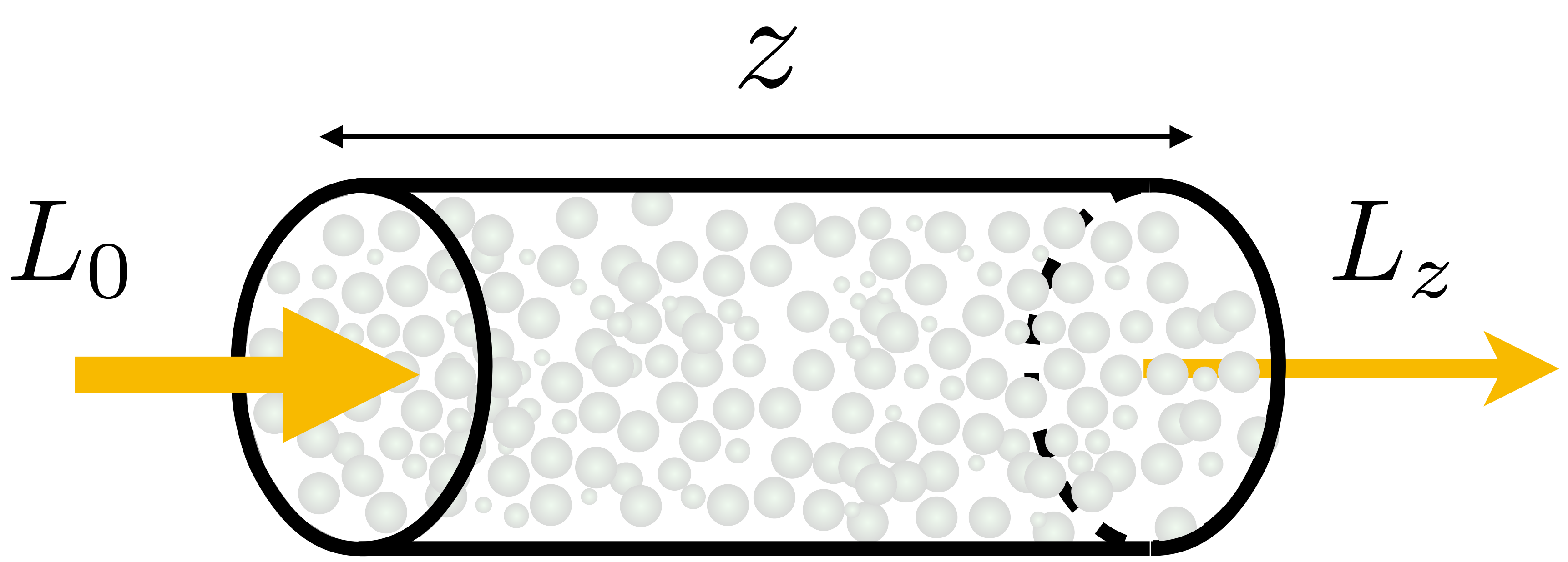

Think of this as fractional visibility loss between two points

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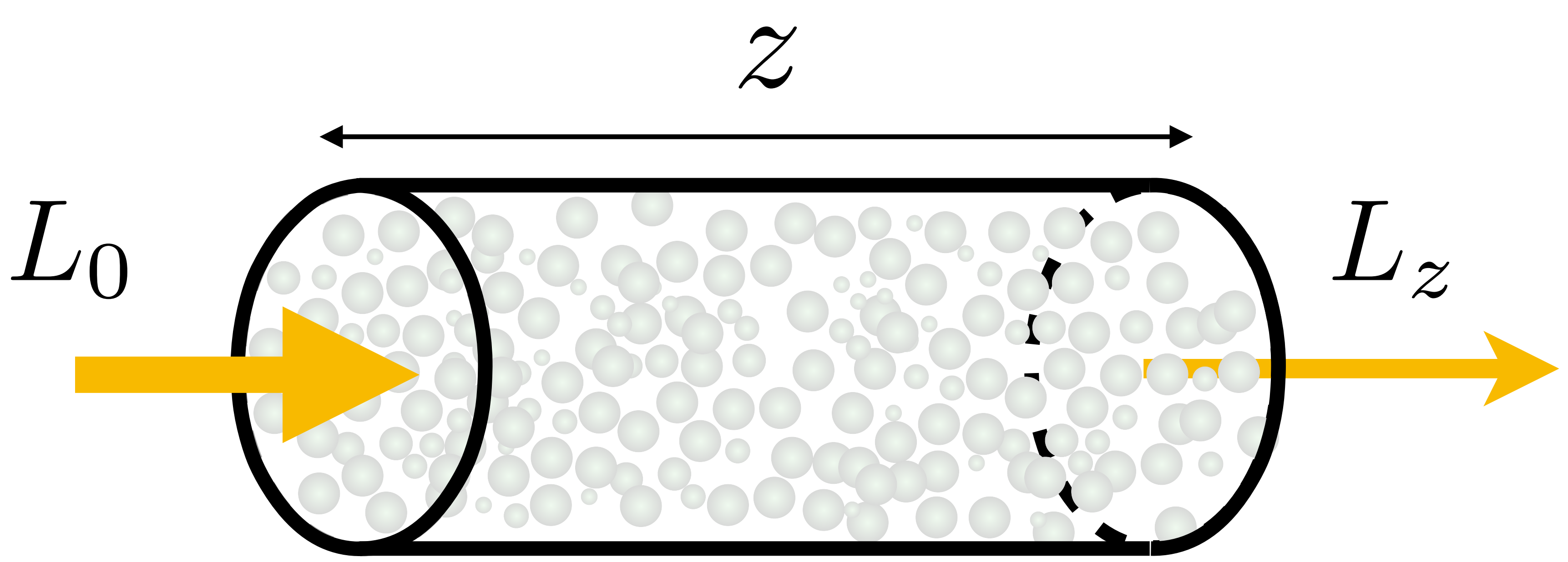

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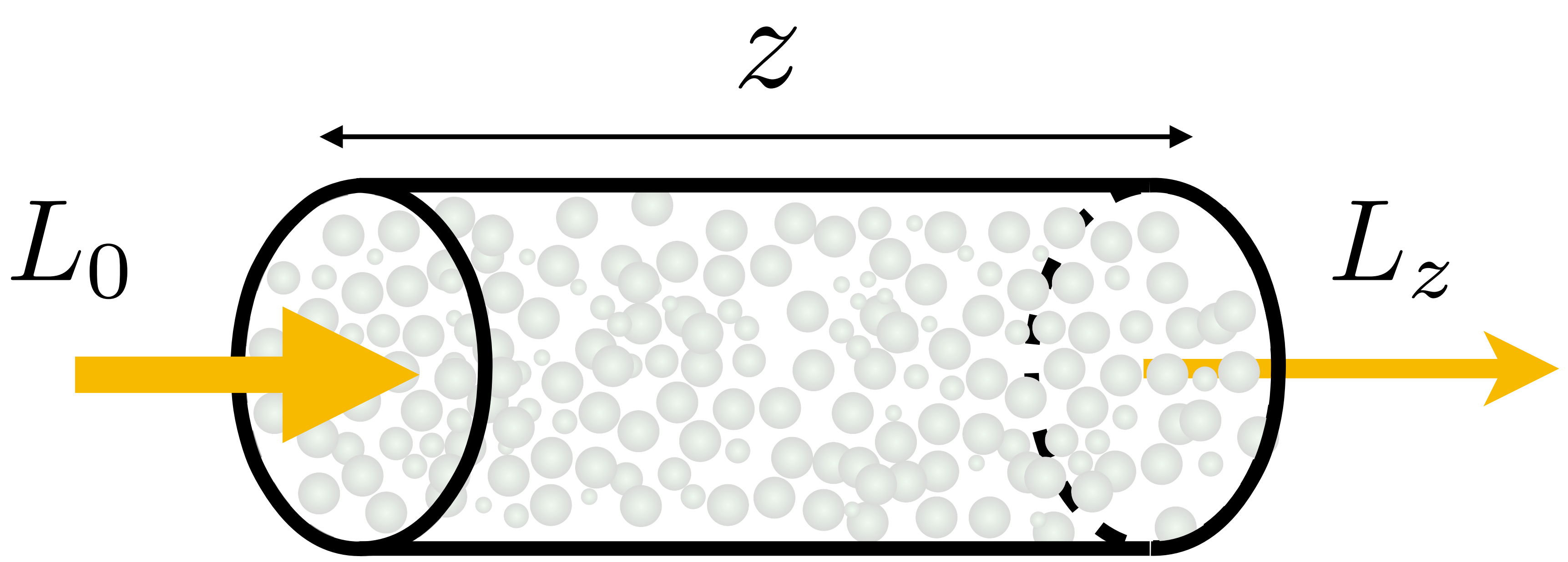


Think of this as fractional visibility loss between two points

Beer-Lambert Law

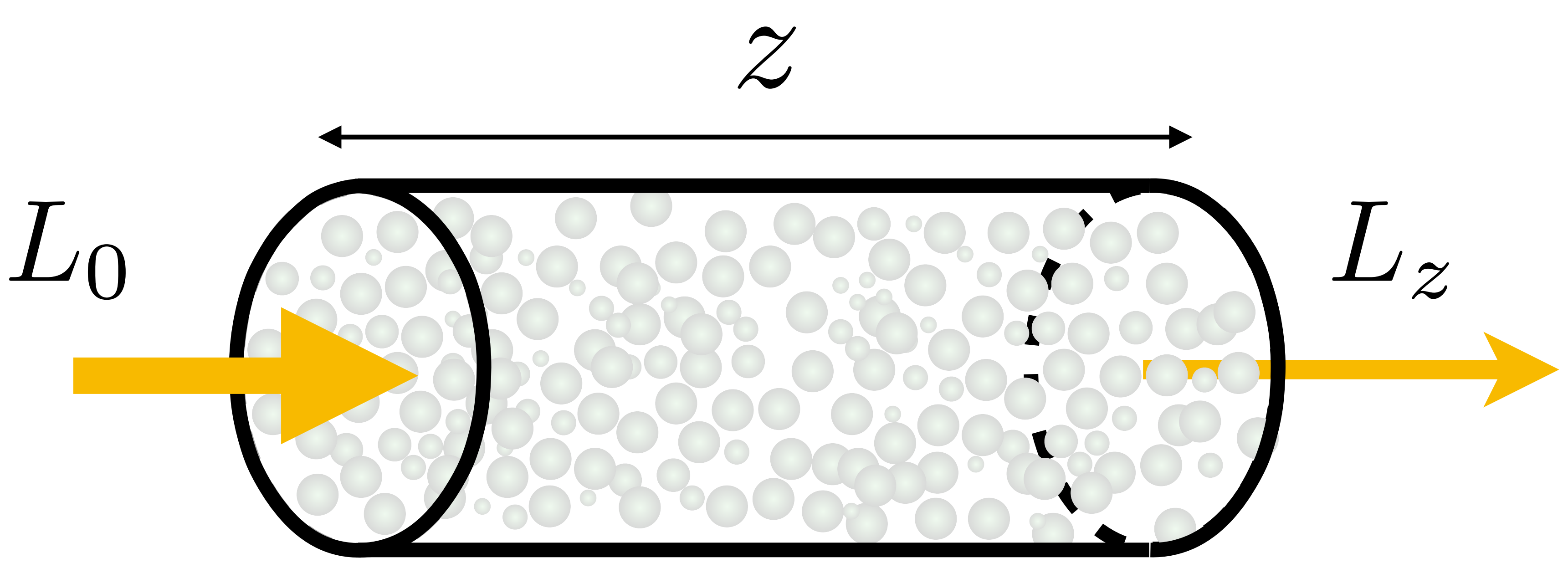
Expresses the remaining radiance after traveling a finite distance through the medium with constant extinction coefficient

The fraction refers to as the *transmittance*

Radiance at distance z

$$\frac{L_z}{L_0} = e^{-\sigma_t z}$$

Radiance at distance 0



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Beam Transmittance

σ_t : extinction coefficient

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σ_t : extinction coefficient

Beam Transmittance

$L_o(\mathbf{x}, \vec{\omega})$

$\mathbf{x}$

Radiance at $\mathbf{y}$

$\mathbf{y}$

$T_r(\mathbf{x} \to \mathbf{x}')L_o(\mathbf{x}, \vec{\omega})$

σ_t : extinction coefficient

Beam Transmittance

$$T_r(\mathbf{x} \to \mathbf{y}) = e^{-\int_0^{||\mathbf{x}-\mathbf{y}||} \sigma_t(t)dt}$$

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$\mathbf{y}$

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Beam Transmittance: **Multiplicative**

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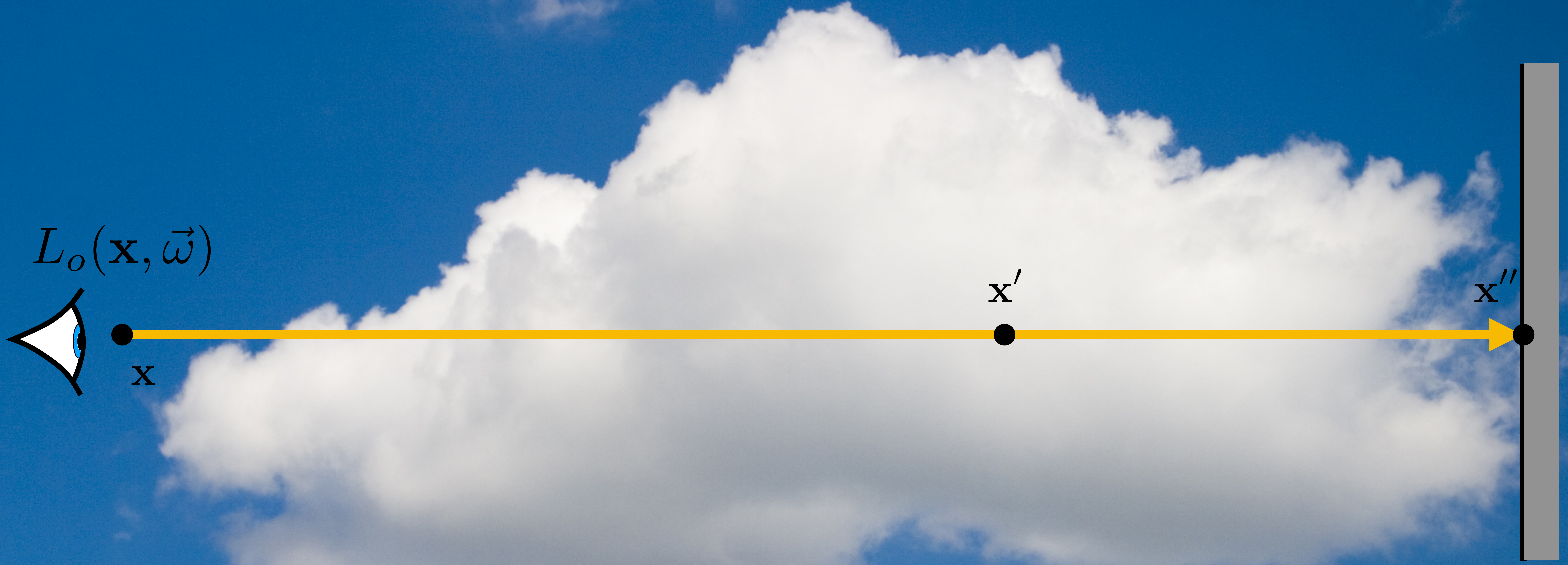

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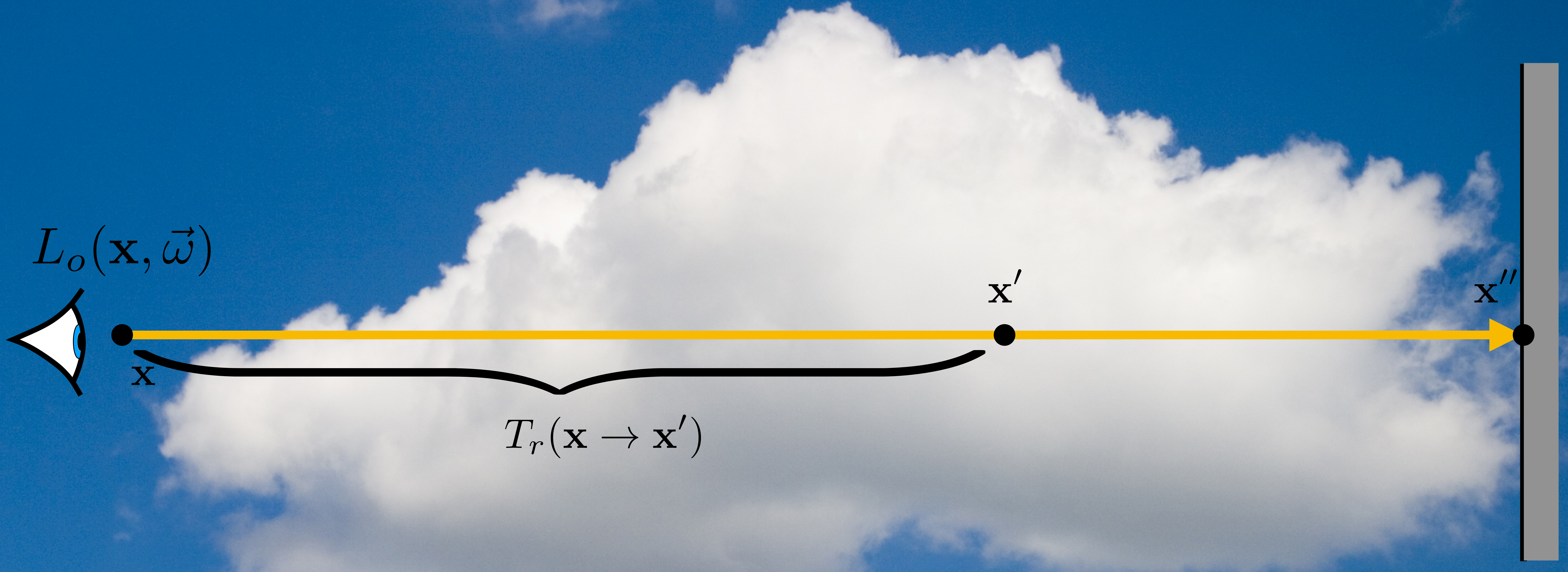
Beam Transmittance: **Multiplicative**



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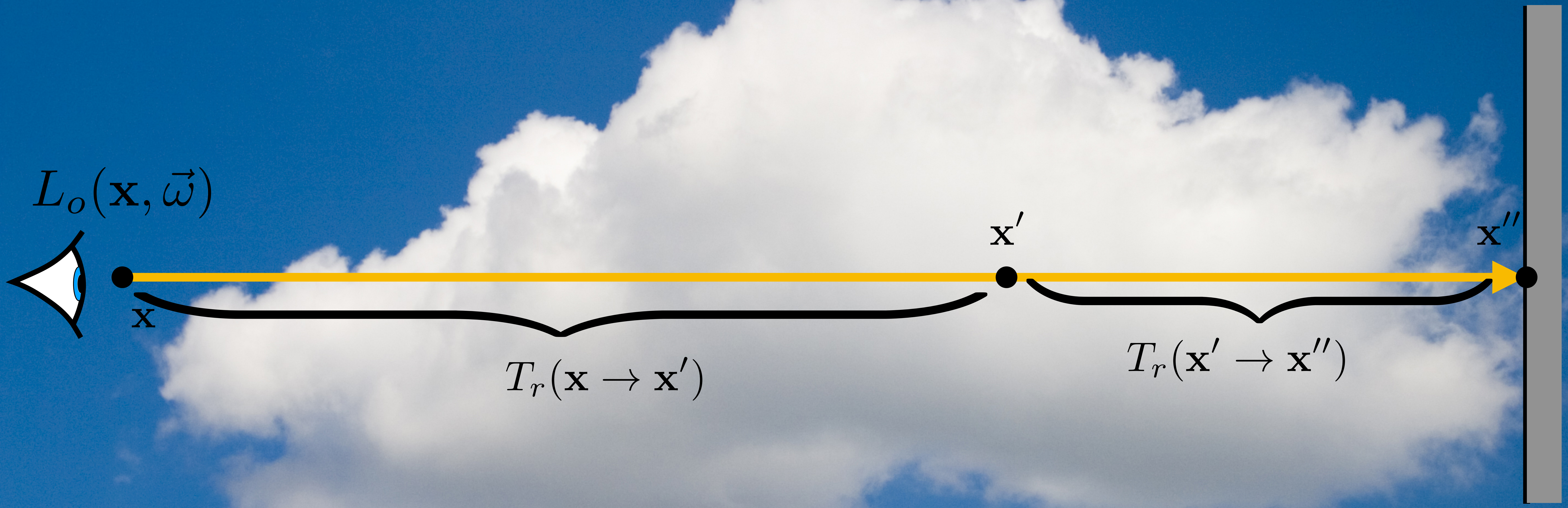
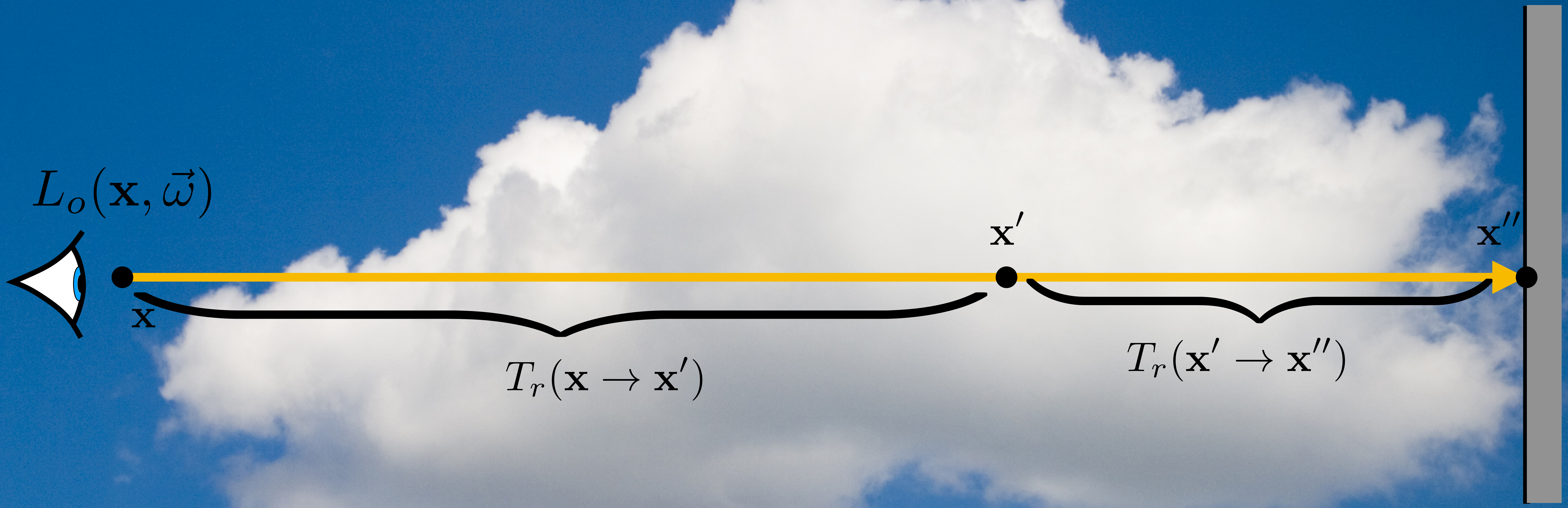


σ_t : extinction coefficient

Beam Transmittance: **Multiplicative**

$$T_r(\mathbf{x} \to \mathbf{x}'') = T_r(\mathbf{x} \to \mathbf{x}')T_r(\mathbf{x}' \to \mathbf{x}'')$$



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Optical thickness

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In-scattering **Emission**

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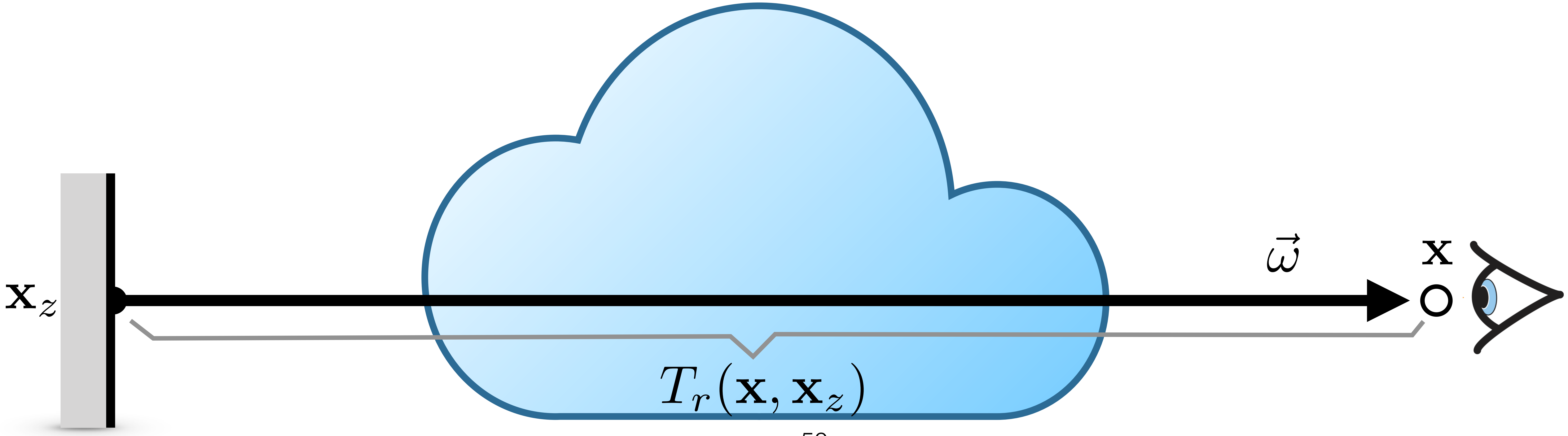
In-scattering **Emission**

What about a beam with finite-length z?

Volumetric Rendering Equation

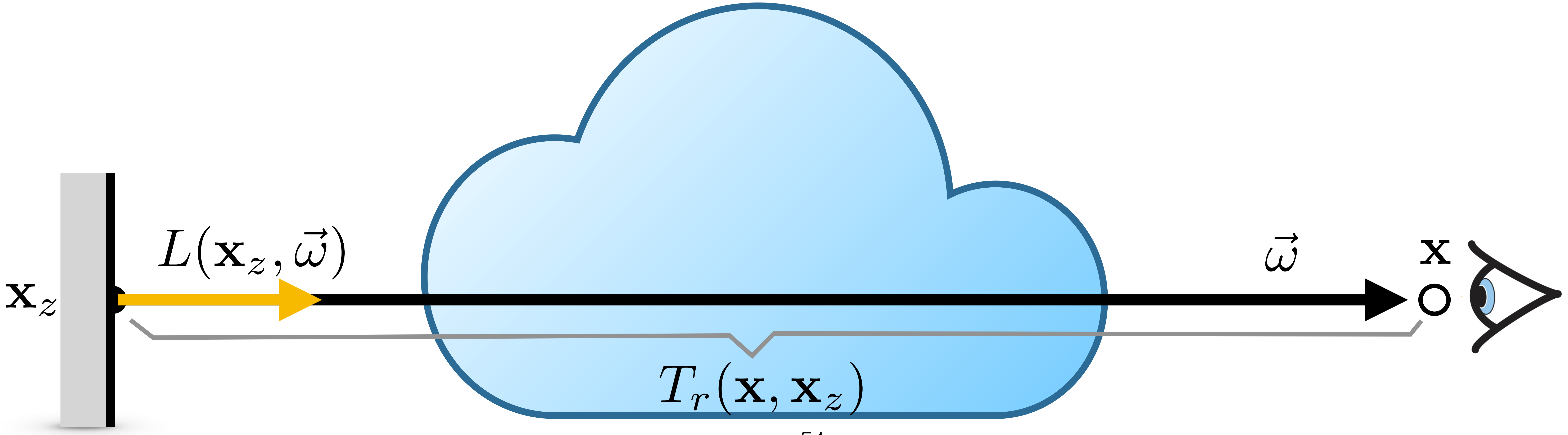
$$L(\mathbf{x}, \vec{\omega}) = T_r(\mathbf{x}, \mathbf{x}_z) L(\mathbf{x}_z, \vec{\omega})$$



Volumetric Rendering Equation

$$L(\mathbf{x}, \vec{\omega}) = \boxed{T_r(\mathbf{x}, \mathbf{x}_z) L(\mathbf{x}_z, \vec{\omega})}$$

Reduced (background) surface radiance

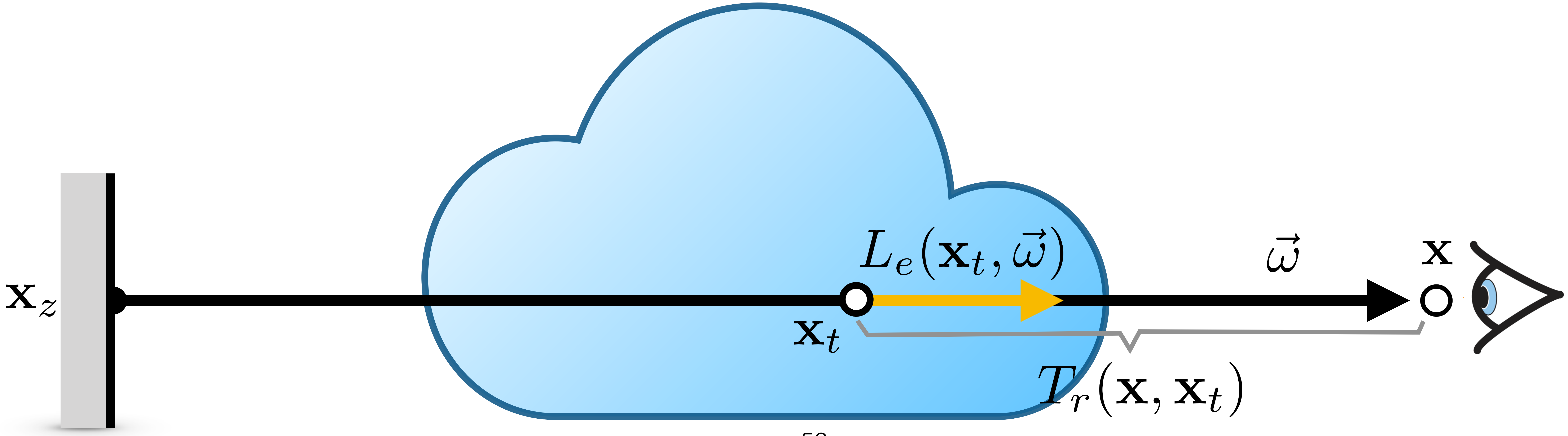


Volumetric Rendering Equation

$$L(\mathbf{x}, \vec{\omega}) = T_r(\mathbf{x}, \mathbf{x}_z) L(\mathbf{x}_z, \vec{\omega})$$

$$+ \boxed{\int_0^z T_r(\mathbf{x}, \mathbf{x}_t) \sigma_a(\mathbf{x}_t) L_e(\mathbf{x}_t, \vec{\omega}) dt}$$

Accumulated emitted radiance



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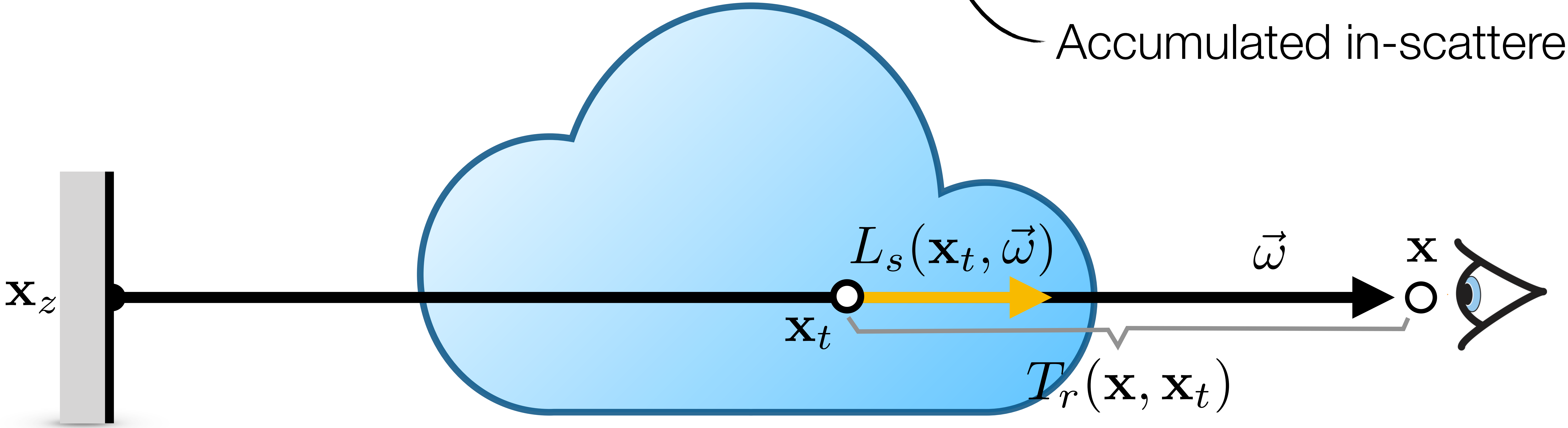
Accumulated in-scattered radiance



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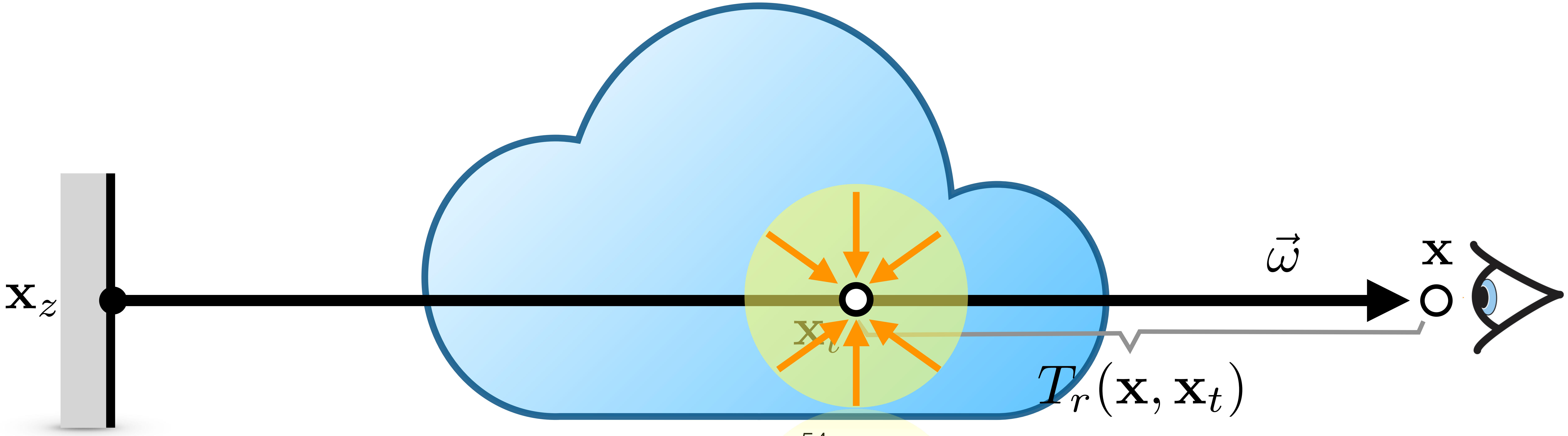
$$+ \int_0^z T_r(\mathbf{x},\mathbf{x}_t)\sigma_s(\mathbf{x}_t)\int_{S^2} f_p(\mathbf{x}_t,\vec{\omega}',\vec{\omega})L_i(\mathbf{x}_t,\vec{\omega}')d\omega' dt$$



Volumetric Rendering Equation

$$\begin{aligned} L(\mathbf{x},\vec{\omega}) &= T_r(\mathbf{x},\mathbf{x}_z)L(\mathbf{x}_z,\vec{\omega}) \\ &\quad + \int_0^z T_r(\mathbf{x},\mathbf{x}_t)\sigma_a(\mathbf{x}_t)L_e(\mathbf{x}_t,\vec{\omega})dt \\ &\quad + \int_0^z T_r(\mathbf{x},\mathbf{x}_t)\sigma_s(\mathbf{x}_t)\int_{S^2} f_p(\mathbf{x}_t,\vec{\omega}',\vec{\omega})L_i(\mathbf{x}_t,\vec{\omega}')d\omega' dt \end{aligned}$$

Scattering in Media

Phase Functions

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This constraint means that phase functions actually define probability distributions for scattering in a particular direction.

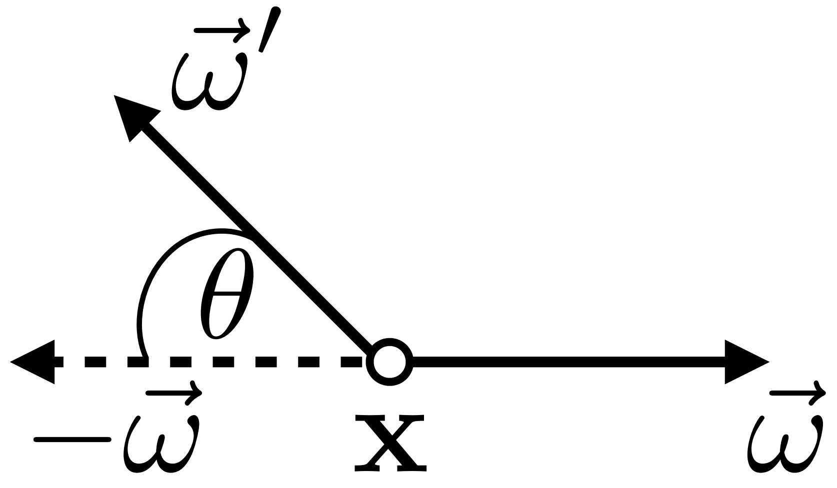
Phase Functions

Isotropic:

$$f_{\mathrm{p}}(\vec{\omega}_o, \vec{\omega}_i) = \frac{1}{4\pi}$$

Uniform scattering, analogous to Lambertian BRDF

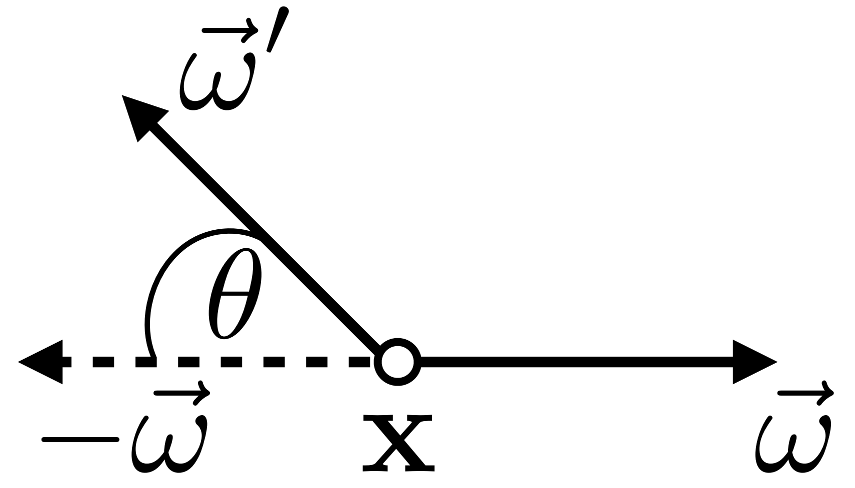
Phase Functions



Phase Functions

Quantifying anisotropy by

$$g = \int_{S^2} f_{\mathrm{p}}(\mathbf{x}, \vec{\omega}, \vec{\omega}') \cos\theta d\vec{\omega}'$$



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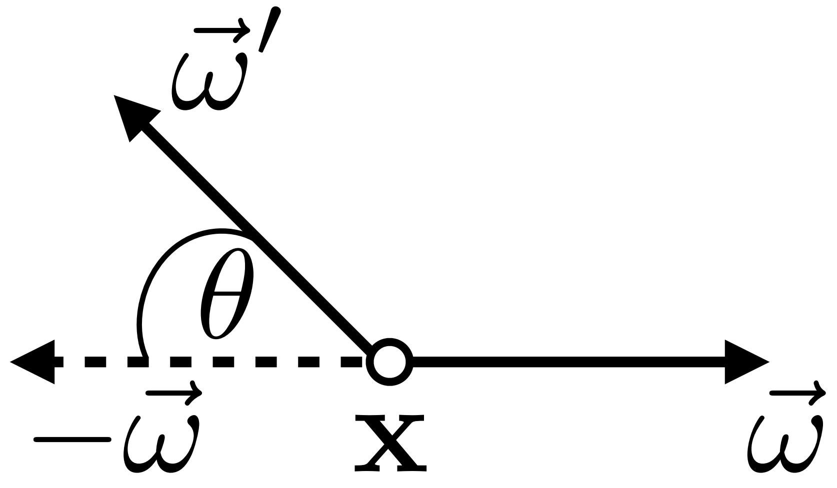

where

$$\cos\theta = -\vec{\omega} \cdot \vec{\omega}'$$

g is the asymmetry parameter

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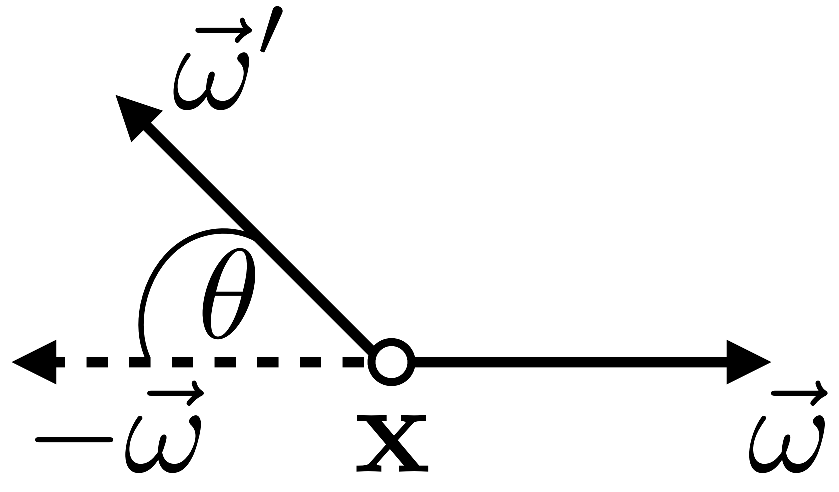

where

$$\cos\theta = -\vec{\omega} \cdot \vec{\omega}'$$

$g = 0$: isotropic scattering (on average)

$g > 0$: forward scattering

$g < 0$: backward scattering

g is the asymmetry parameter

Henyey-Greenstein Phase Function

$$f_{\mathrm{p}}(\theta) = \frac{1}{4\pi} \frac{1 - g^2}{(1 + g^2 + 2g(\cos\theta))^{3/2}} \qquad g \in [-1, 1]$$

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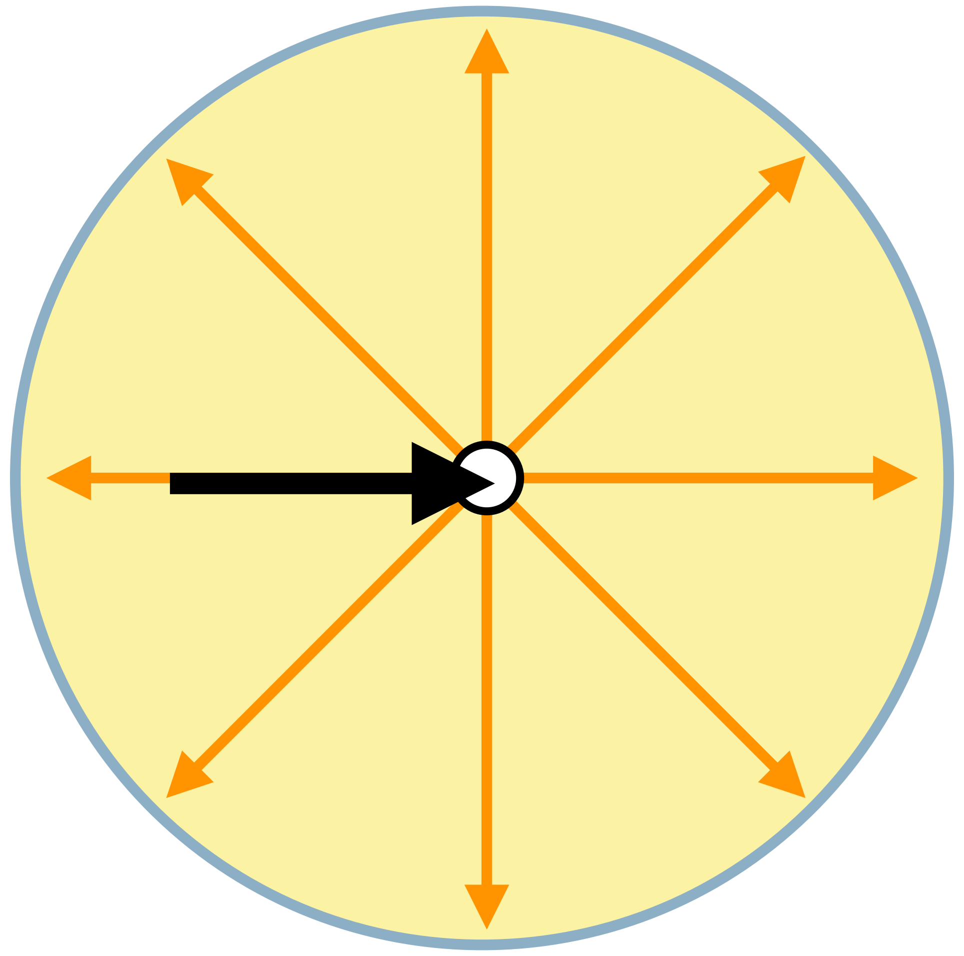
$$g = 0$$



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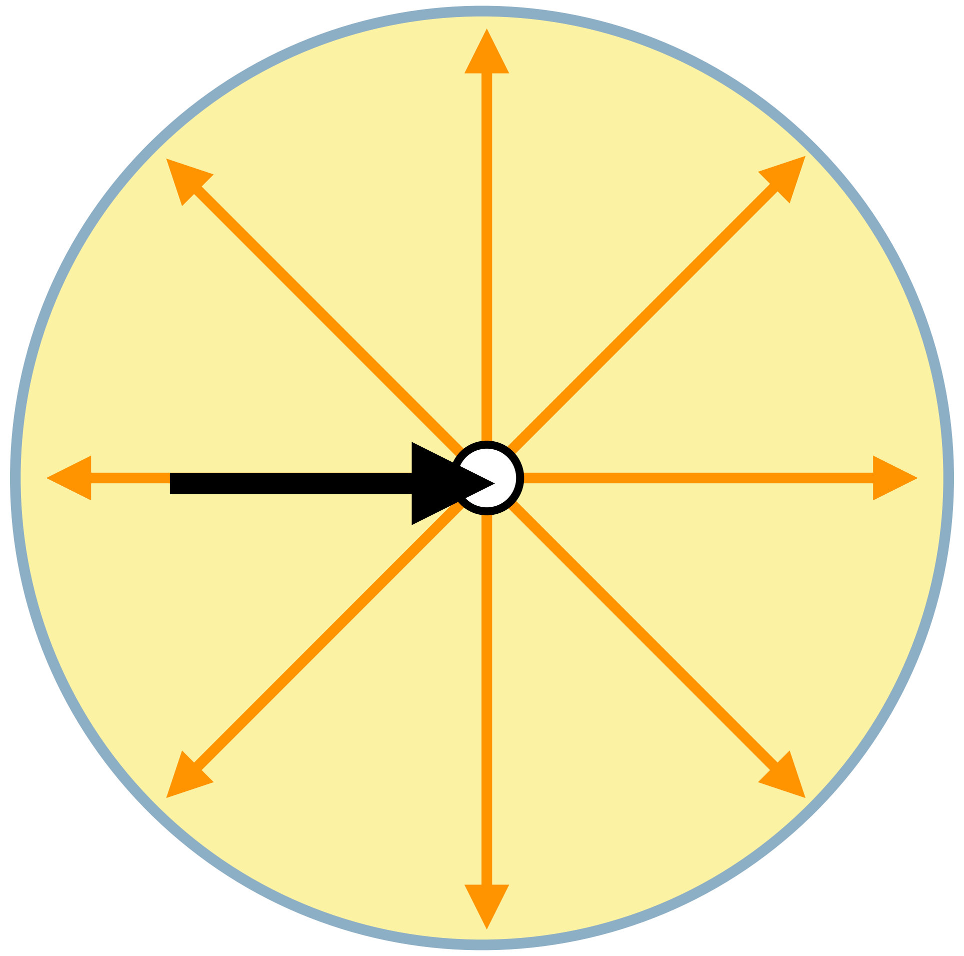
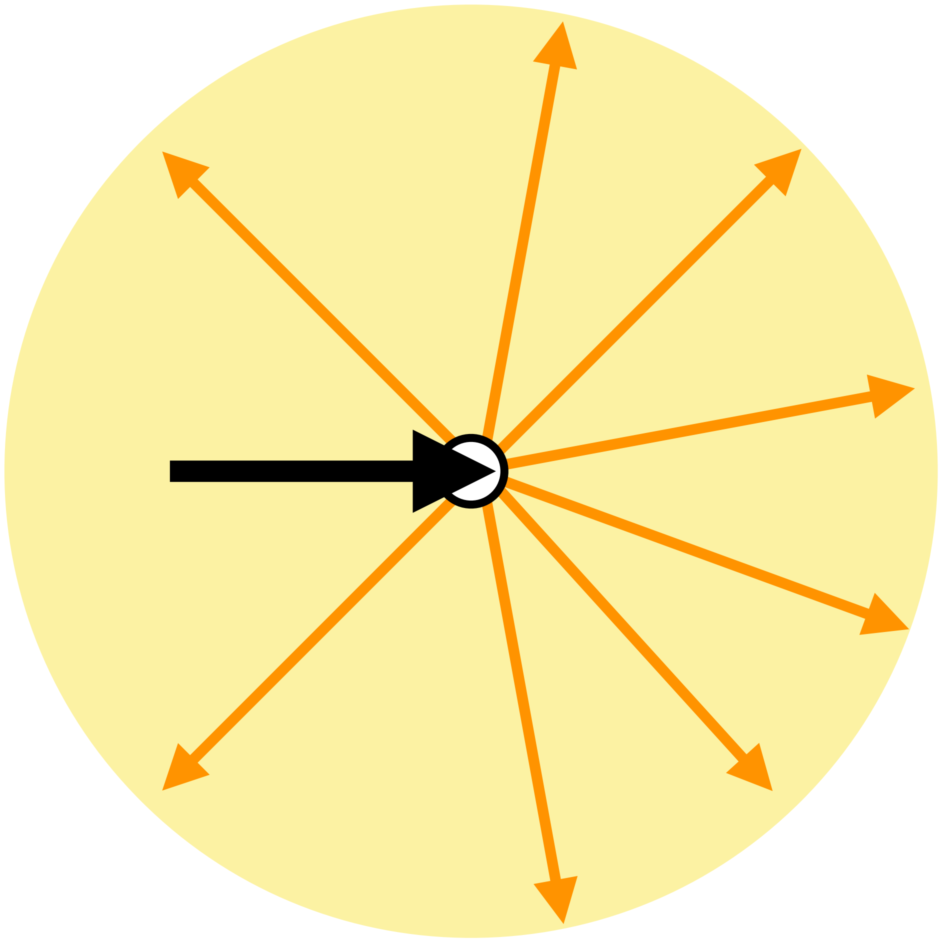


$g > 0$



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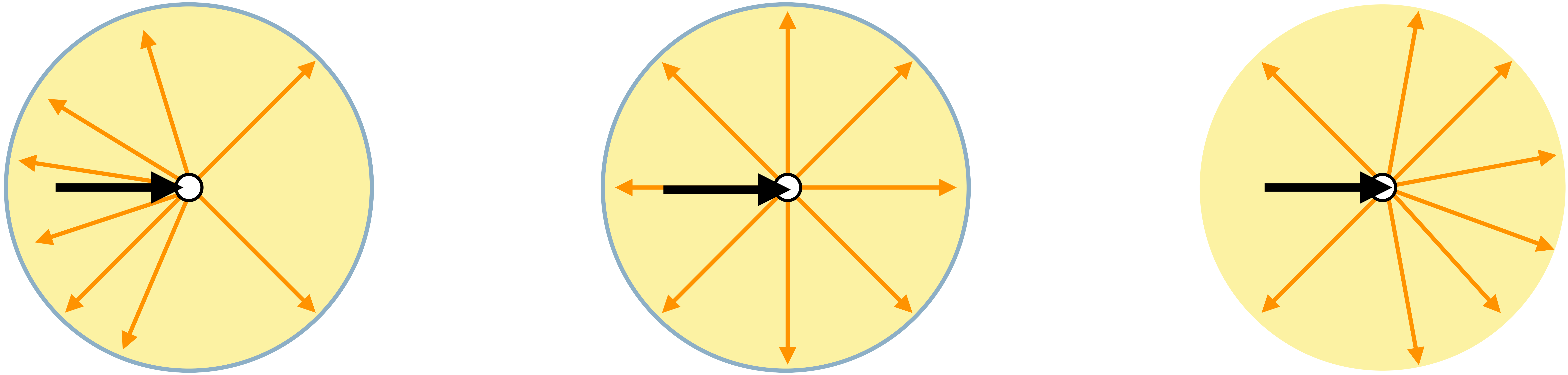

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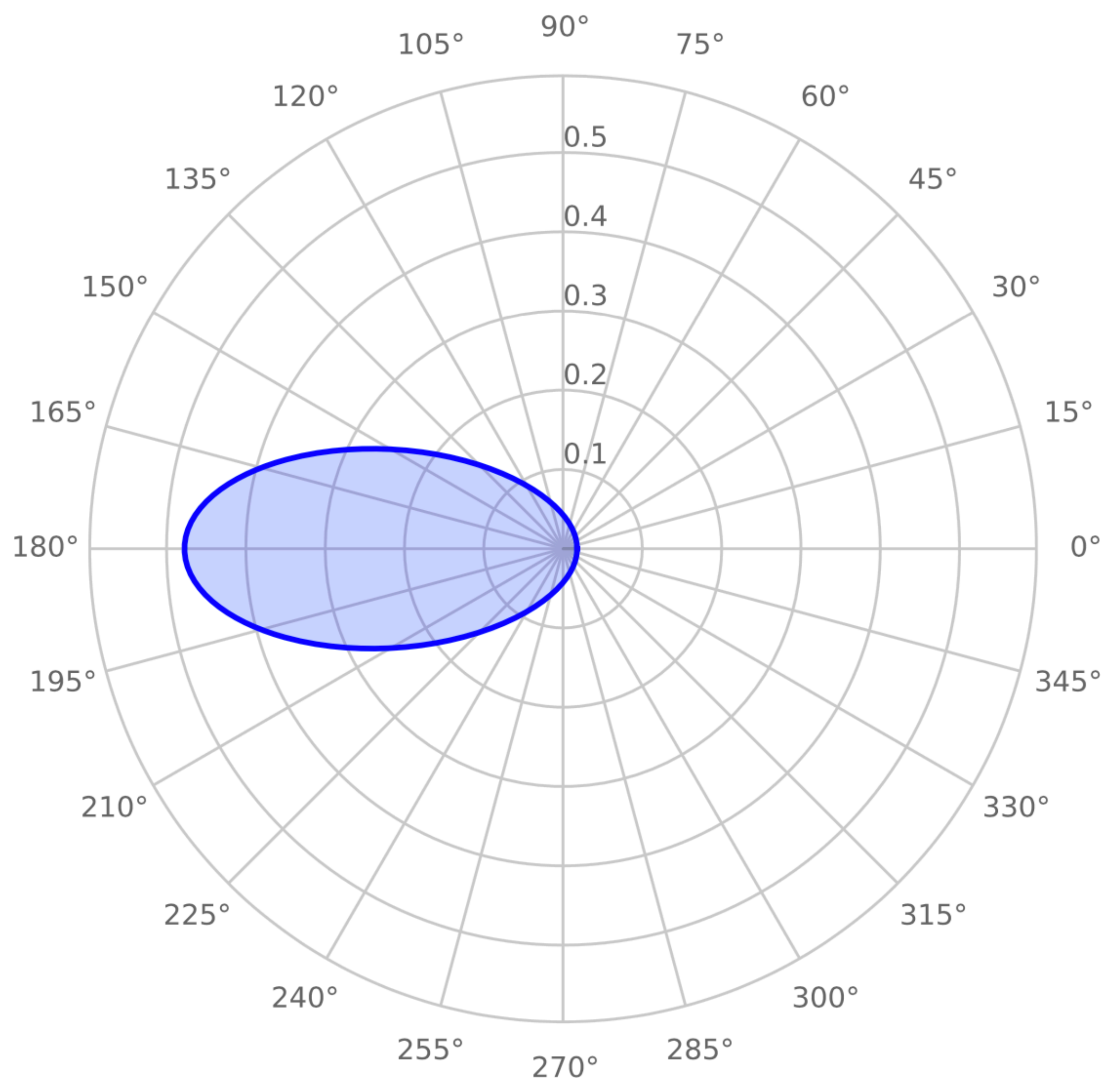
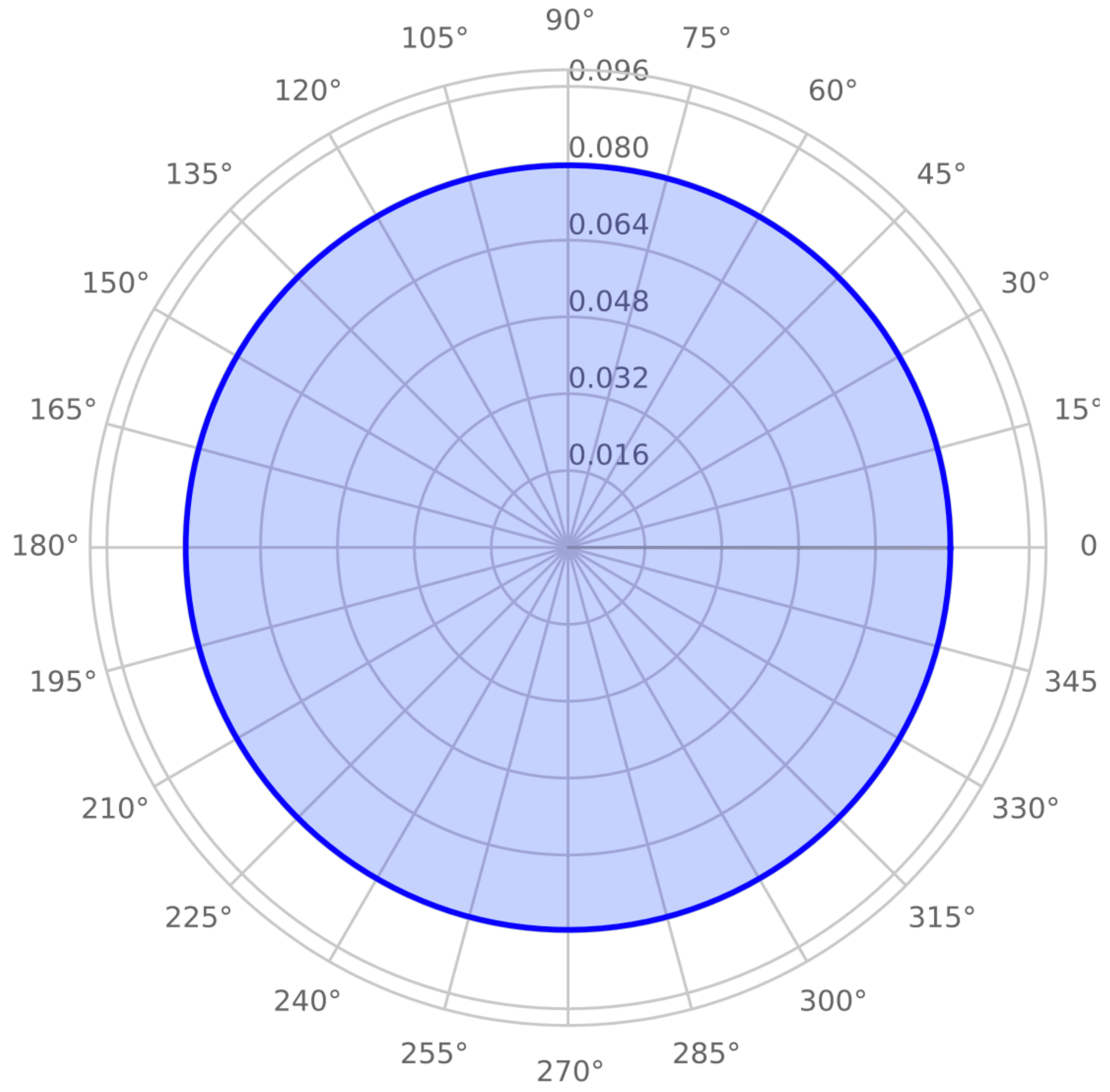
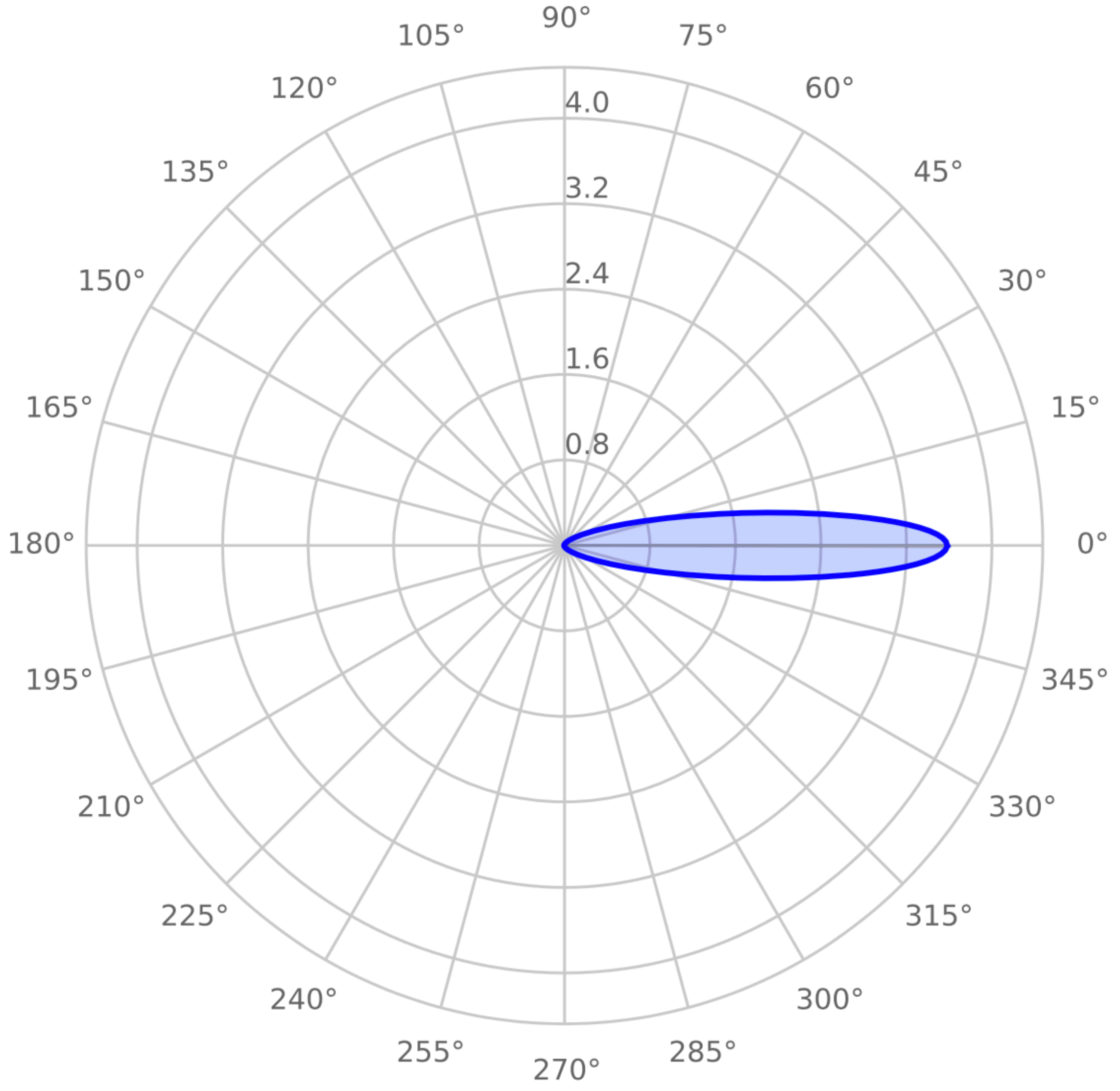
$g = -0.5$



$g = 0$



$g = 0.8$



Henyey-Greenstein Phase Function

$g = -0.7$

$g = 0.7$

Strong forward scattering

Strong backward scattering

PBRTv3 [2016]

Schlick's Phase Function

Empirical Phase Function

Faster approximation to HG

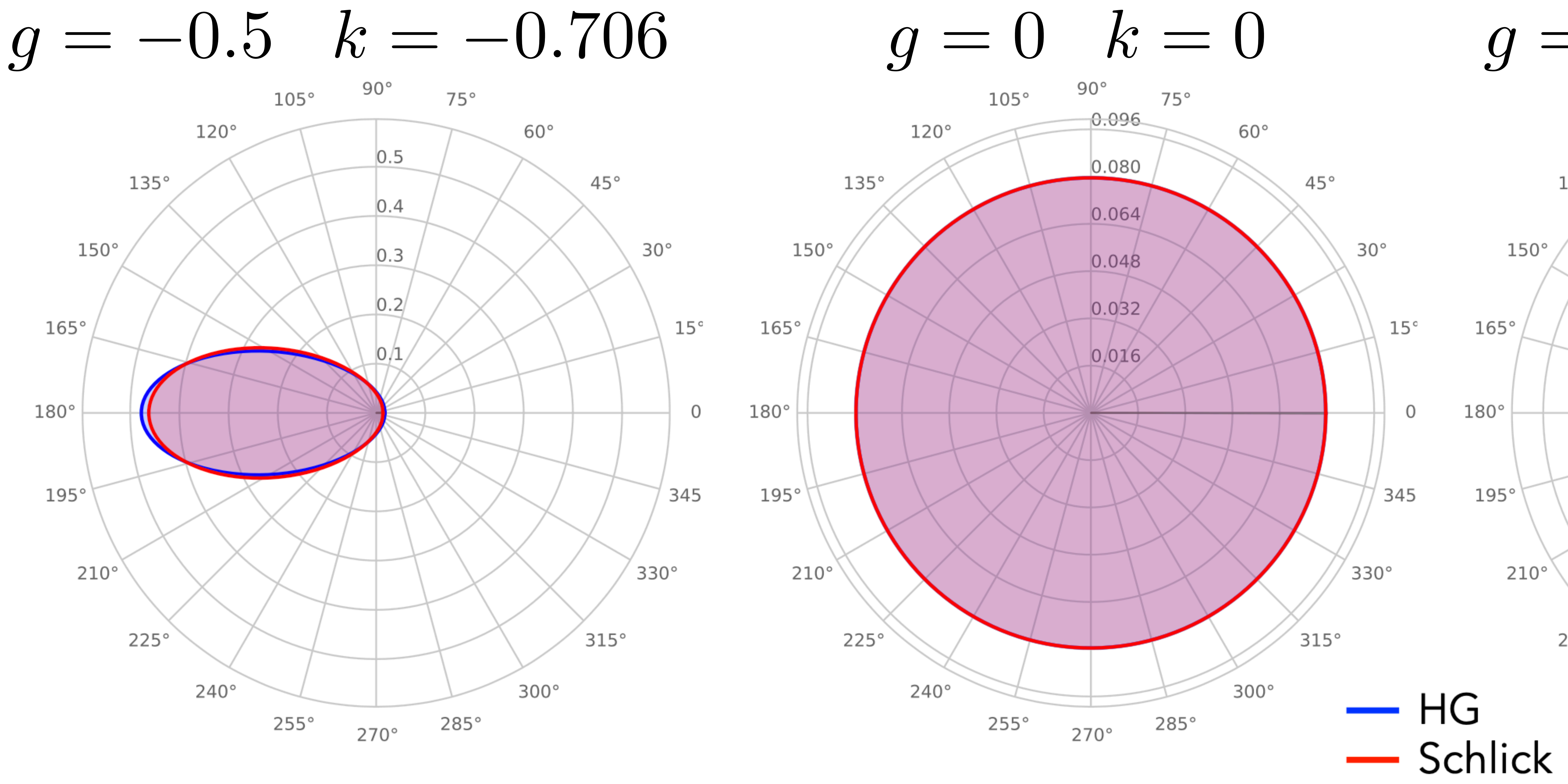
$$f_{\mathrm{p}}(\theta) = \frac{1}{4\pi} \frac{1 - k^2}{(1 - k\cos\theta)^2}$$

$$k = 1.55g - 0.55g^3$$

Schlick's Phase Function

Empirical Phase Function

Faster approximation to HG



$g = -0.5 \quad k = -0.706$

$g = 0 \quad k = 0$

$g = 0.8 \quad k = 0.96$

Rainbows

Lorenz-Mie Scattering

For large-size particles (scatterers), we cannot ignore the wave nature of light

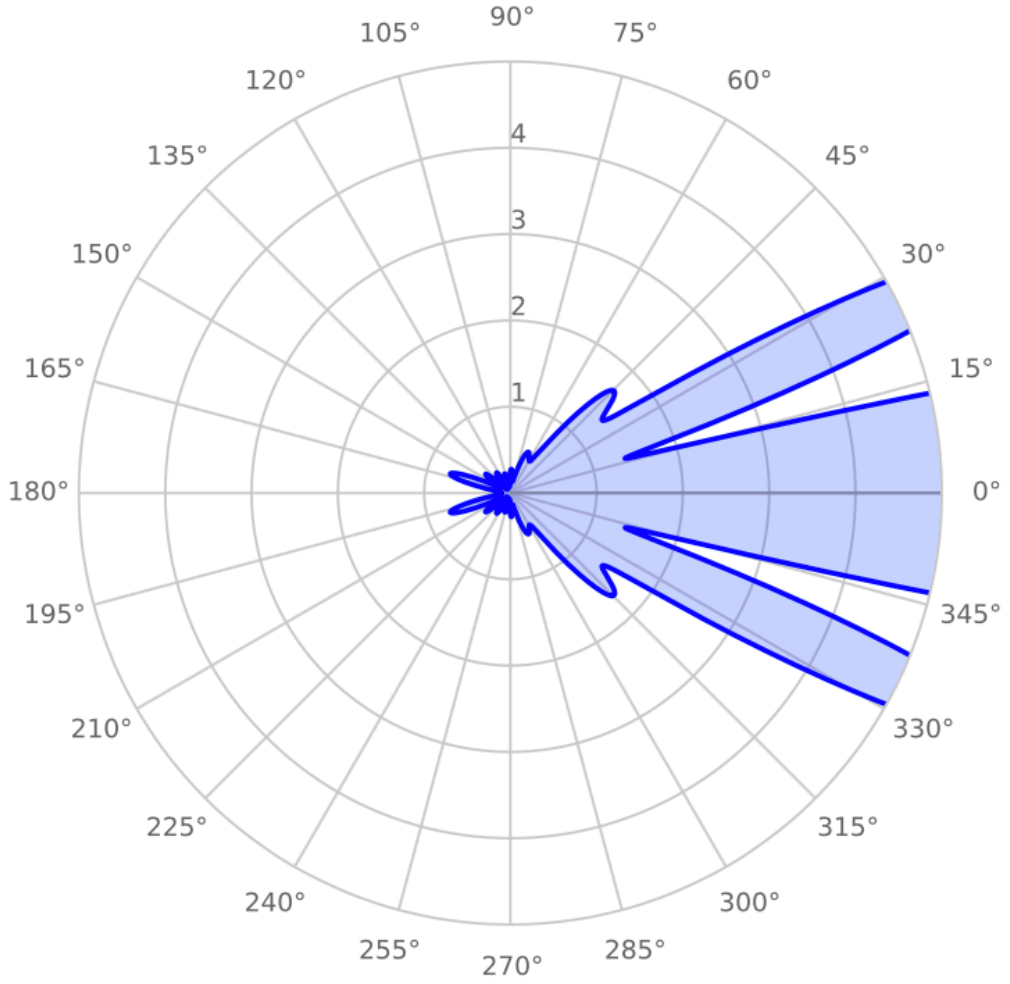
Solution to Maxwell's equations for scattering from many spherical dielectric particles

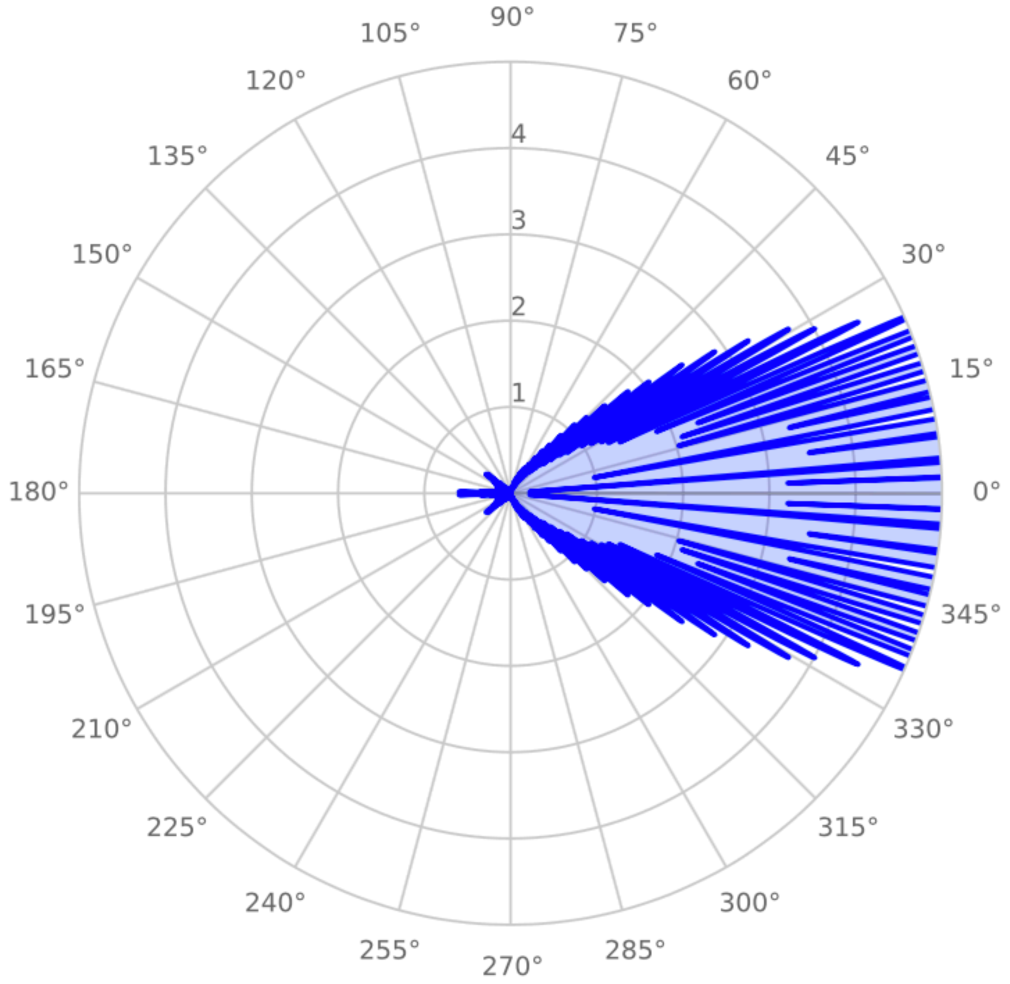
Explains many phenomena

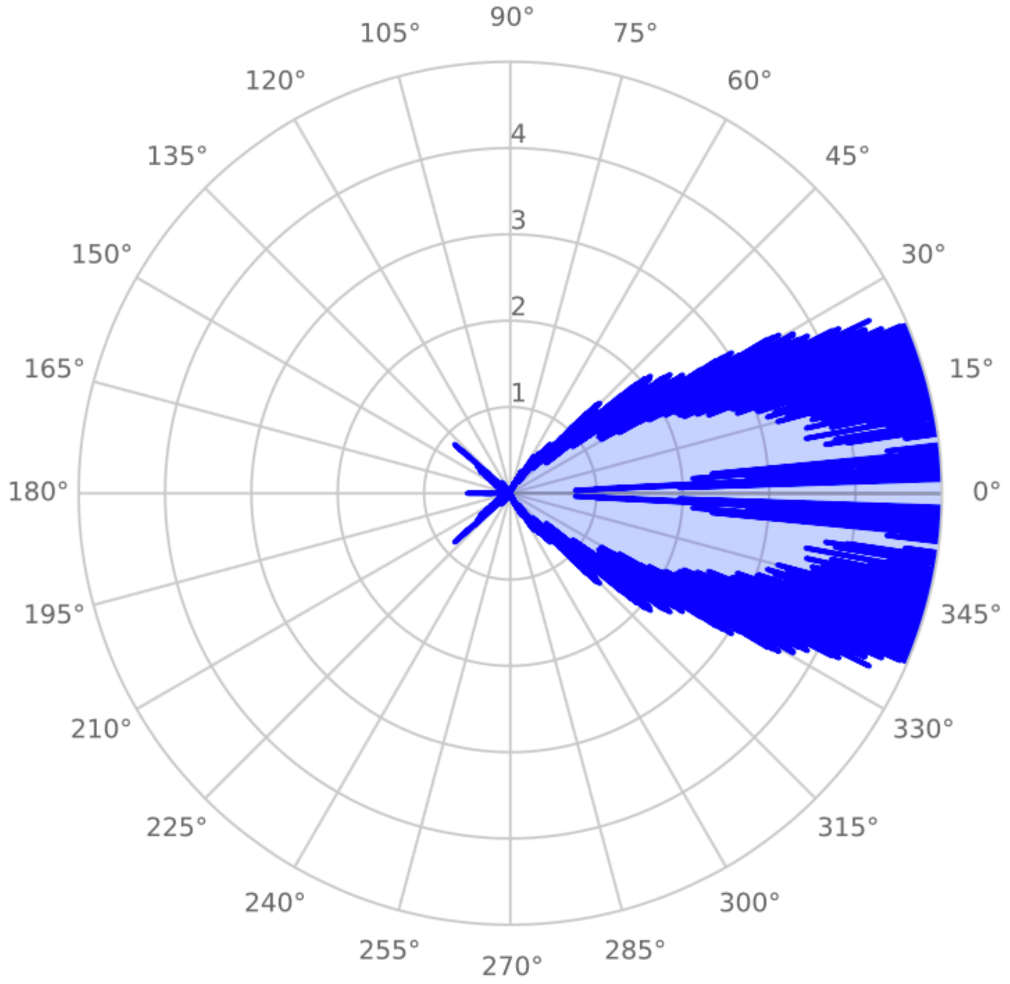
Complicated: solution is an infinite analytic series

Lorenz-Mie Scattering

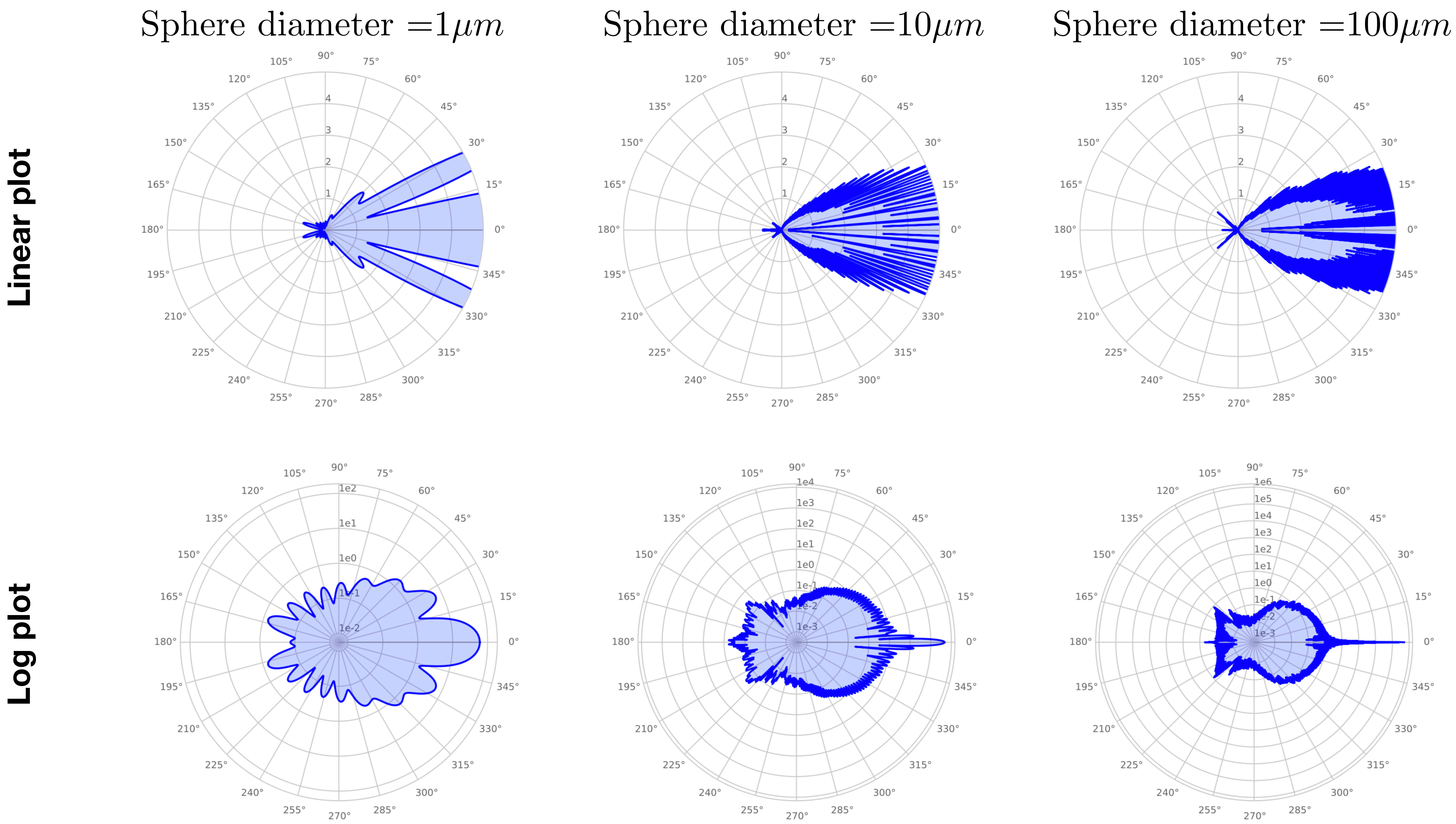
Linear plot


Sphere diameter $=1\mu m$


Sphere diameter $=10\mu m$


Sphere diameter $=100\mu m$

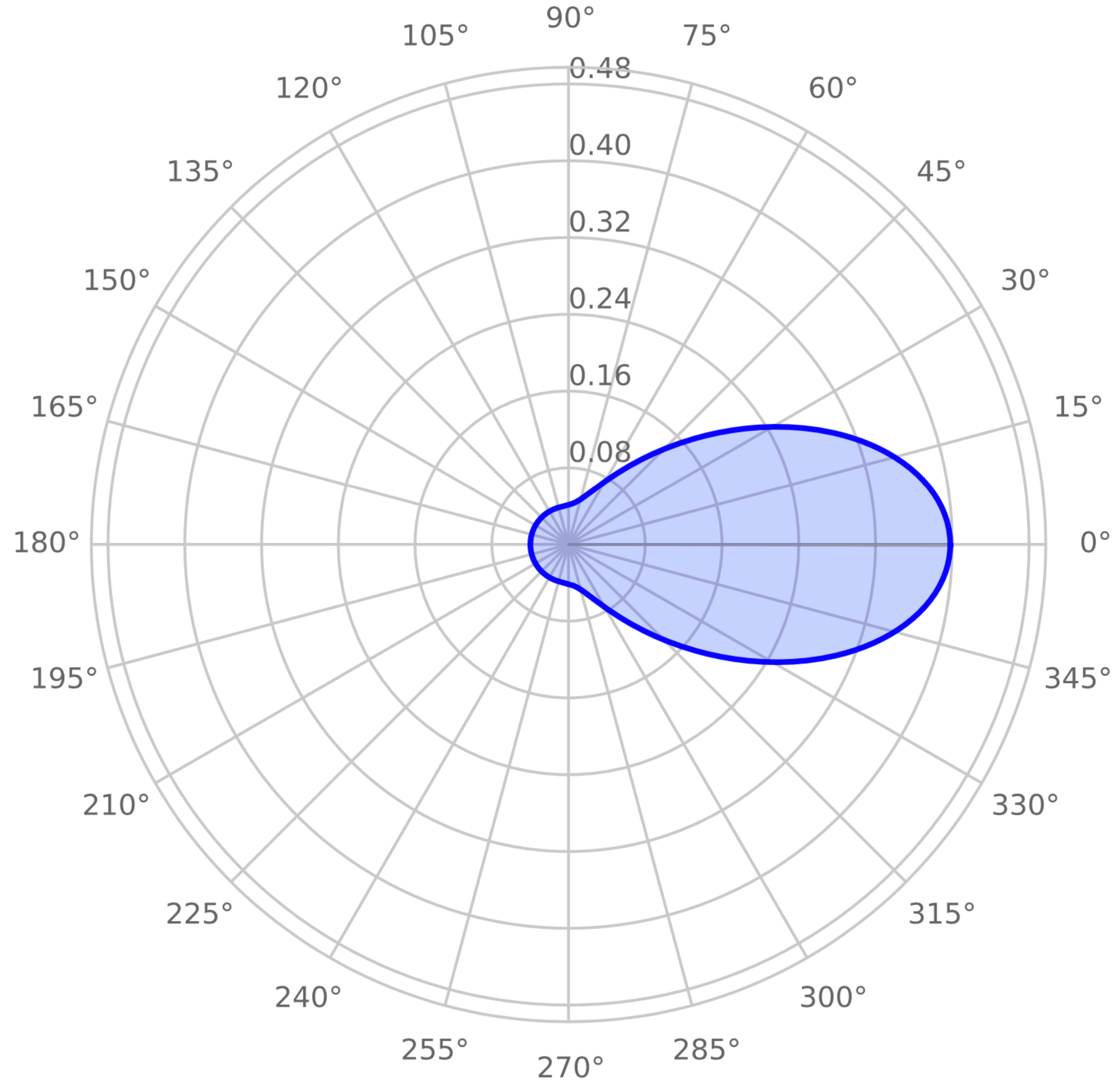
Lorenz-Mie Scattering



Lorenz-Mie Approximations

Hazy atmosphere

$$f_p^{\text{hazy}}(\theta) = \frac{1}{4\pi}\left(5 + \left(\frac{1+\cos\theta}{2}\right)^8\right)$$



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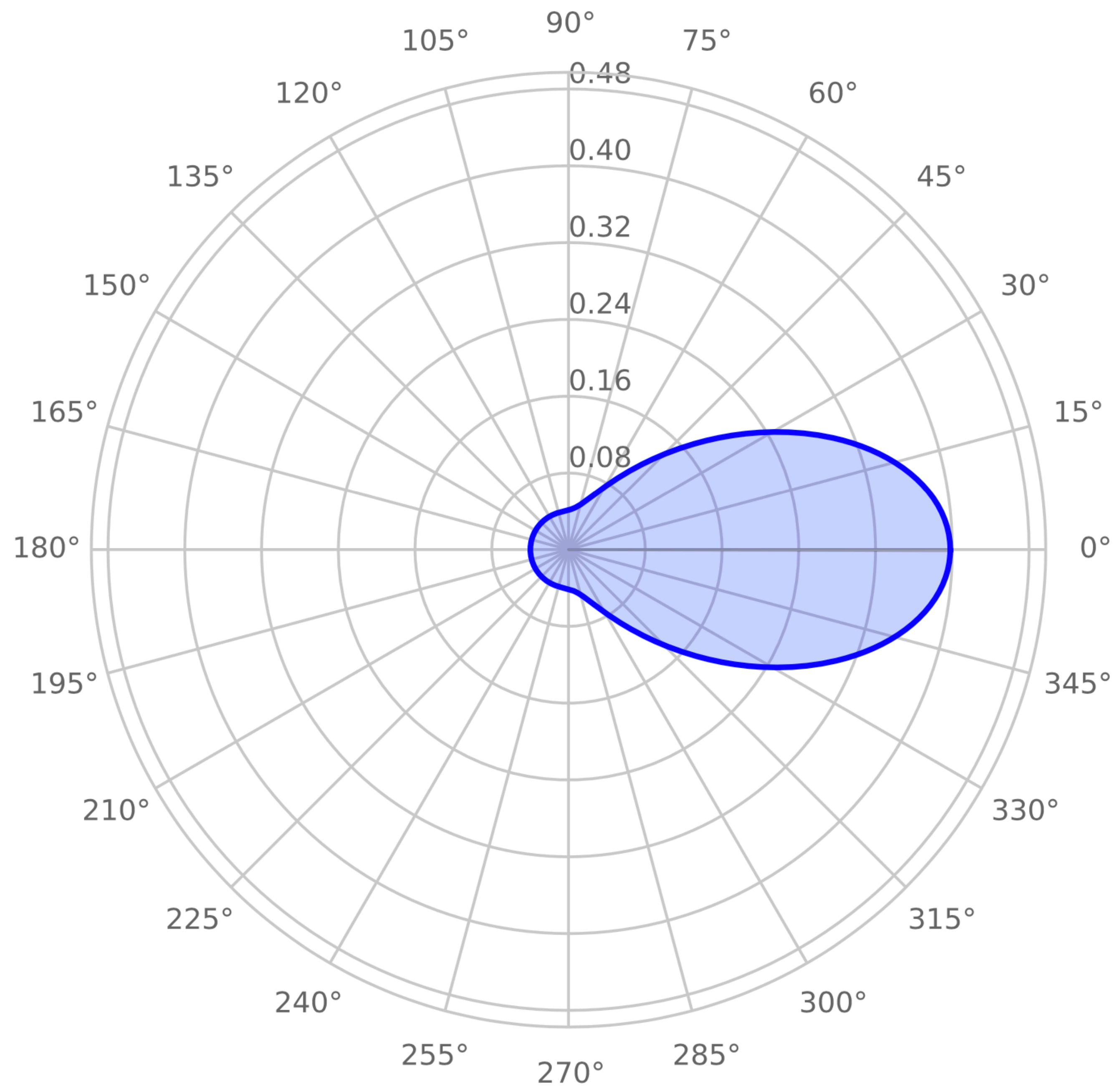
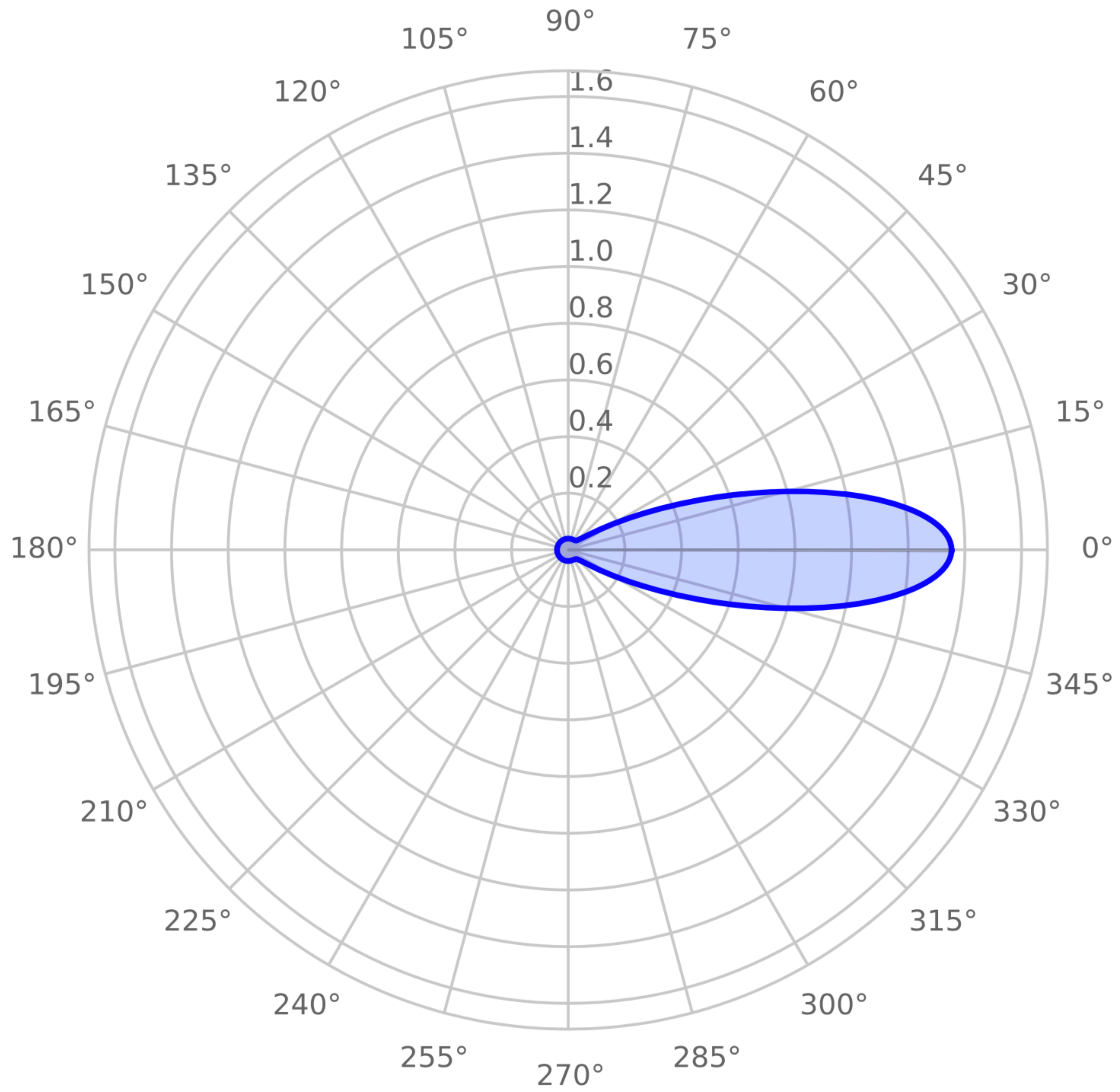
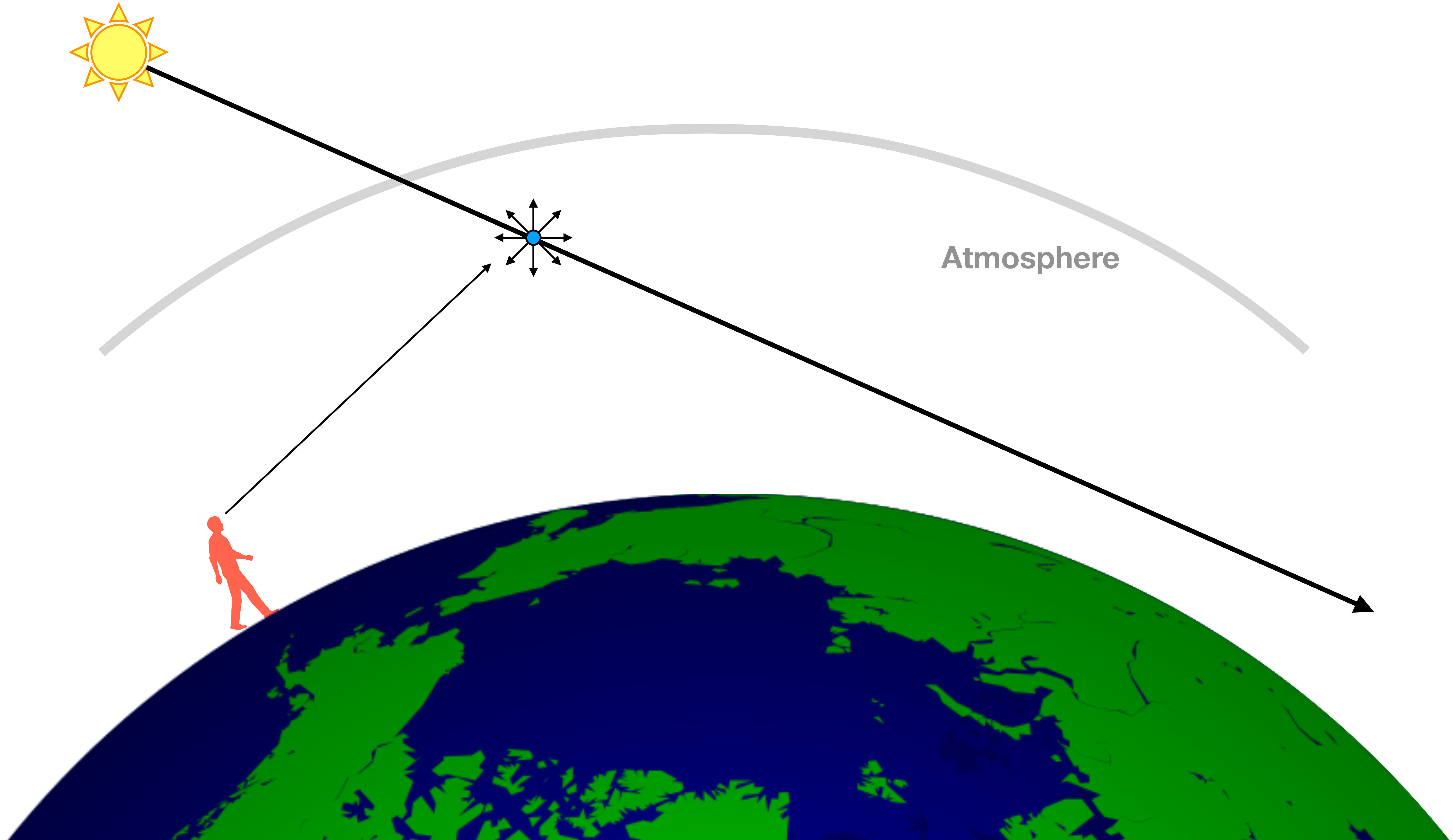


Murky atmosphere

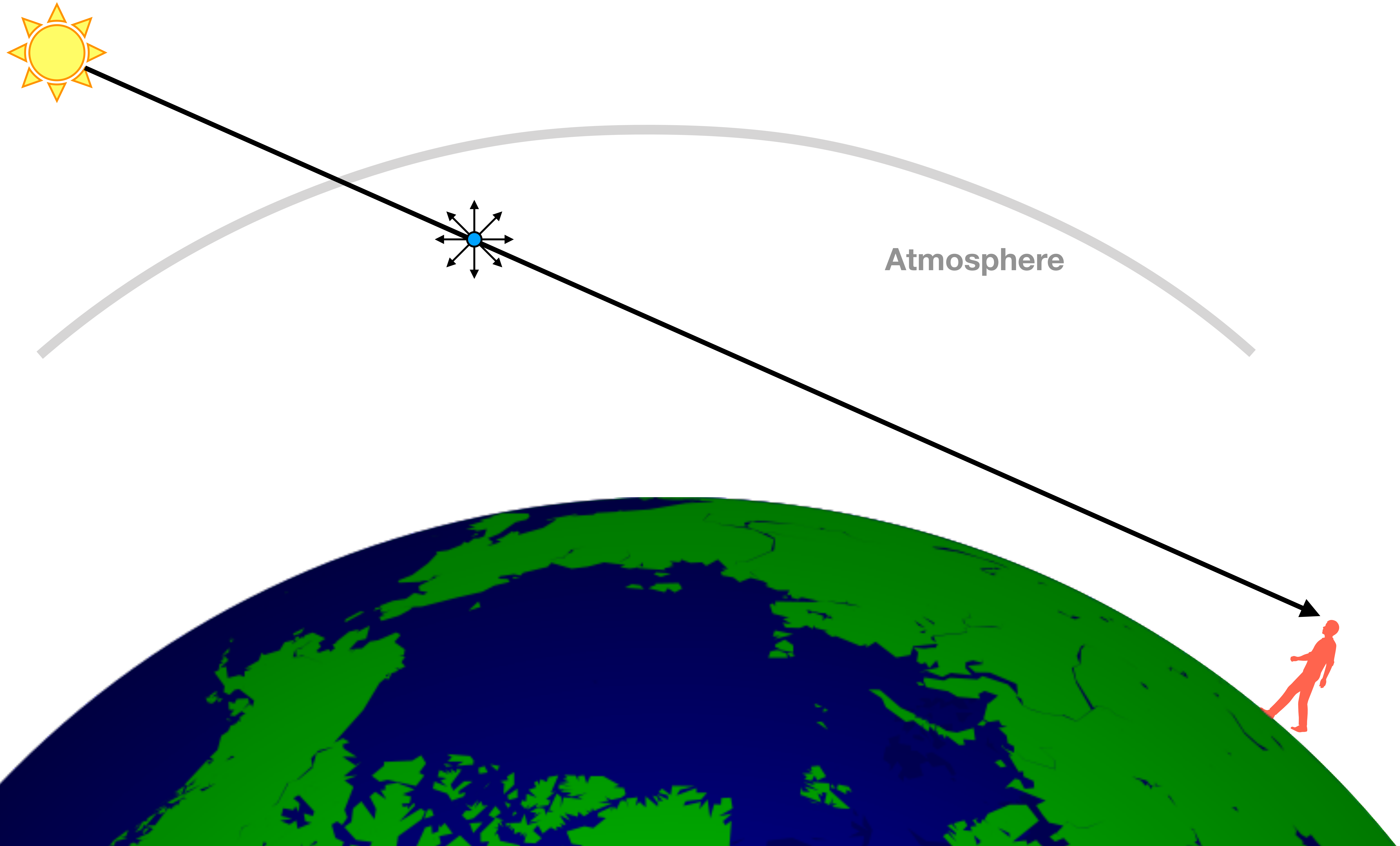
$$f_p^{\text{murky}}(\theta) = \frac{1}{4\pi}\left(17 + \left(\frac{1+\cos\theta}{2}\right)^{32}\right)$$



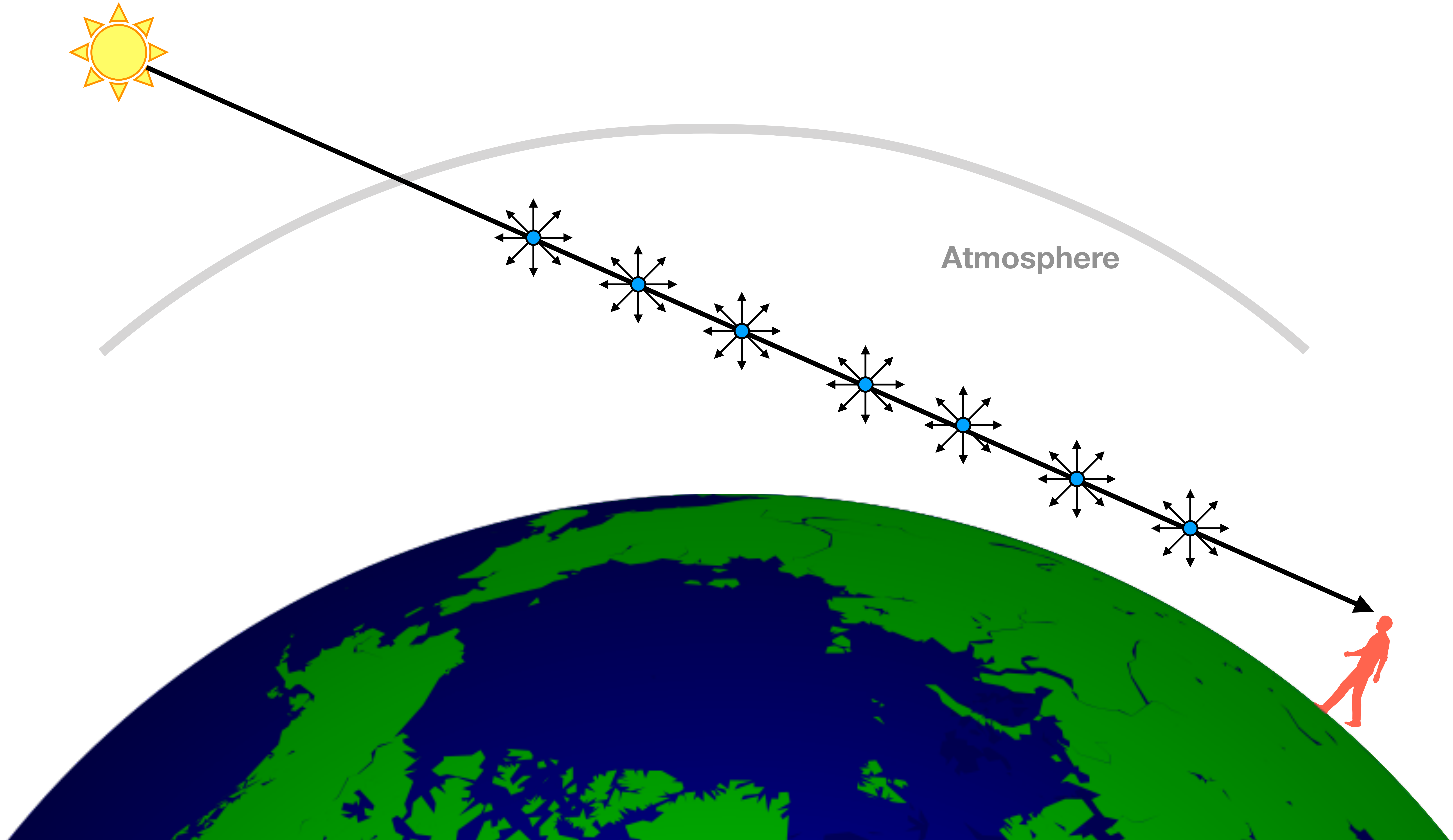
Why is the Sky Blue?



Why is the Sunset Red?



Why is the Sunset Red?



Rayleigh Scattering

forbes.com

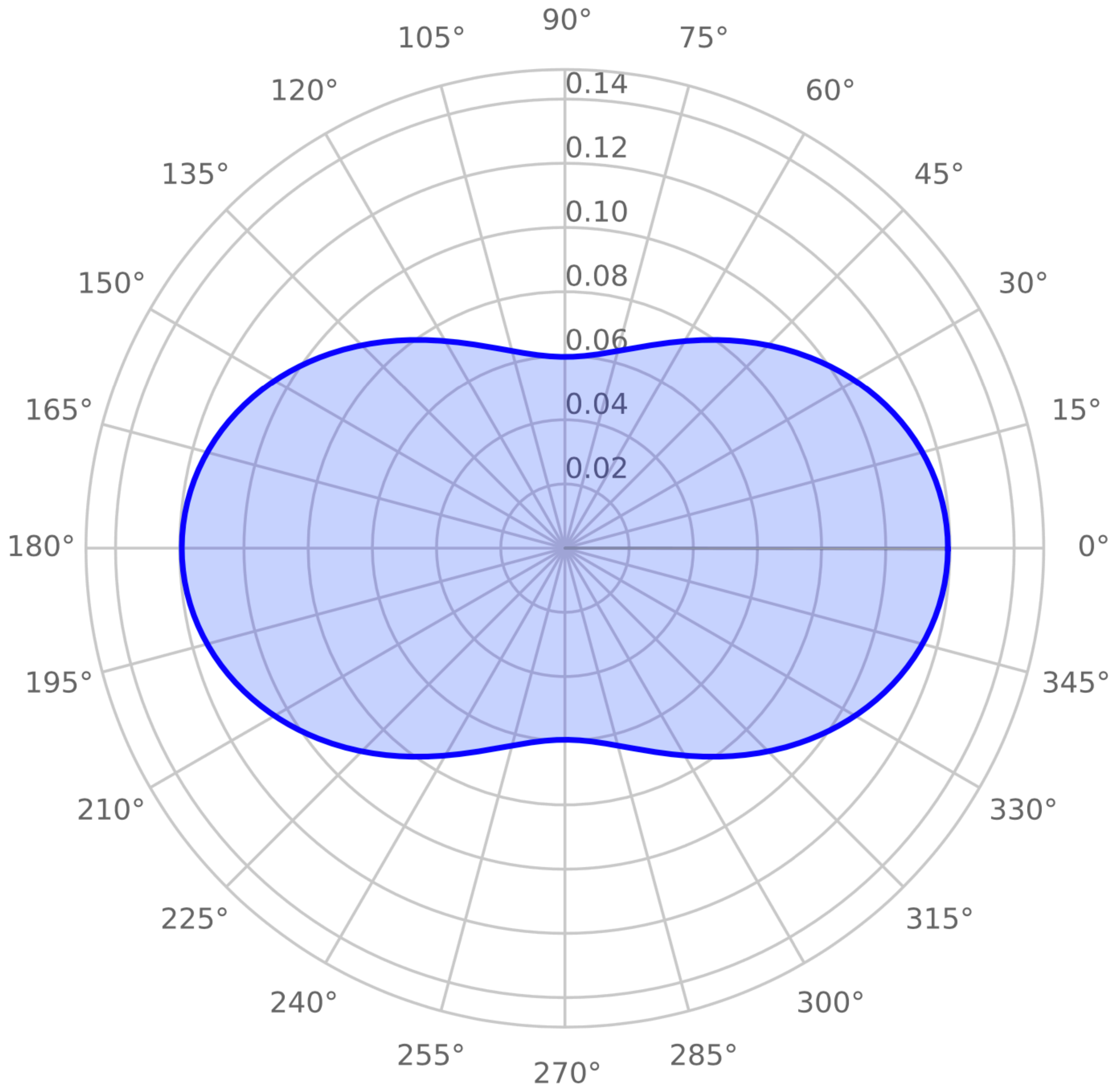
Rayleigh Scattering

Approximation of Lorenz-Mie for tiny particles (scatterers) that are typically smaller than 1/10th the wavelength of visible light

Used for atmospheric scattering, gasses, transparent solids

Highly wavelength dependent

Rayleigh Phase Function



$$f_p^{\mathrm{Rayleigh}}(\theta) = \frac{3}{16\pi}\left(1 + \cos^2\theta\right)$$

Scattering at right angles is half as likely as scattering forward or backward

Rayleigh Scattering

$$\beta_s^{\text{Rayleigh}}(\lambda, d, \eta, \rho) = \rho \frac{2\pi^5 d^6}{3\lambda^4} \left(\frac{\eta^2 - 1}{\eta^2 + 2} \right)^2$$

Rayleigh Scattering

Wavelength

$$\beta_s^{\text{Rayleigh}}(\lambda, d, \eta, \rho) = \rho \frac{2\pi^5 d^6}{3\lambda^4} \left(\frac{\eta^2 - 1}{\eta^2 + 2} \right)^2$$

Rayleigh Scattering

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Diameter of particles

Rayleigh Scattering

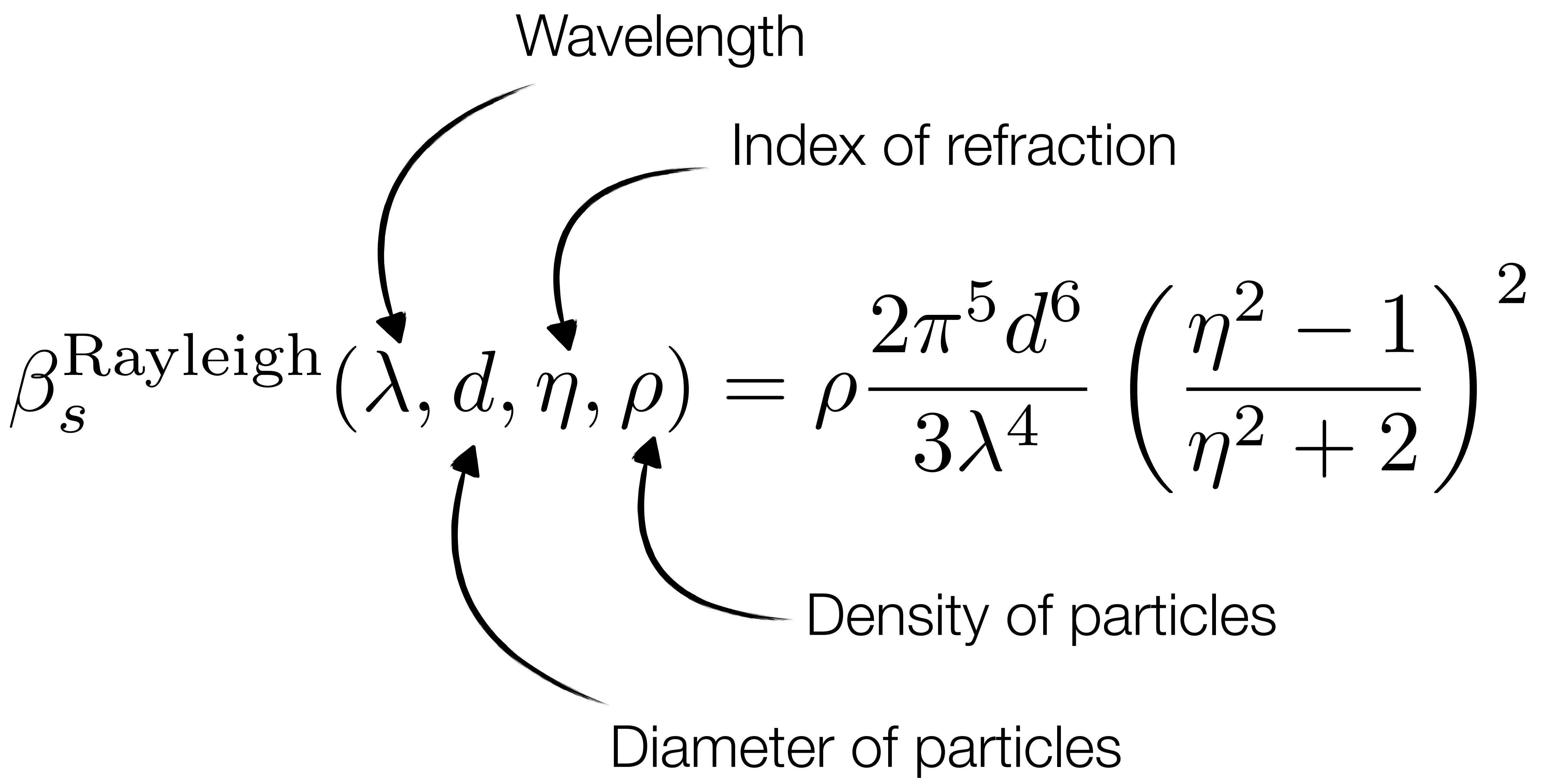
Wavelength

Index of refraction

$$\beta_s^{\text{Rayleigh}}(\lambda, d, \eta, \rho) = \rho \frac{2\pi^5 d^6}{3\lambda^4} \left(\frac{\eta^2 - 1}{\eta^2 + 2} \right)^2$$

Diameter of particles

Rayleigh Scattering



Wavelength

Index of refraction

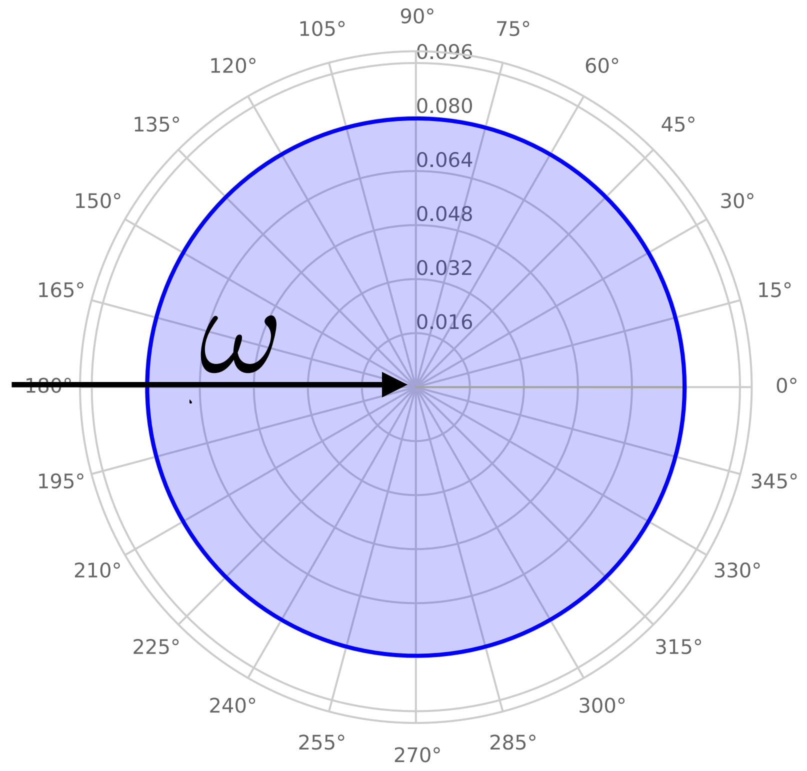
$$\beta_s^{\mathrm{Rayleigh}}(\lambda, d, \eta, \rho) = \rho \frac{2\pi^5 d^6}{3\lambda^4} \left(\frac{\eta^2 - 1}{\eta^2 + 2} \right)^2$$

Density of particles

Diameter of particles

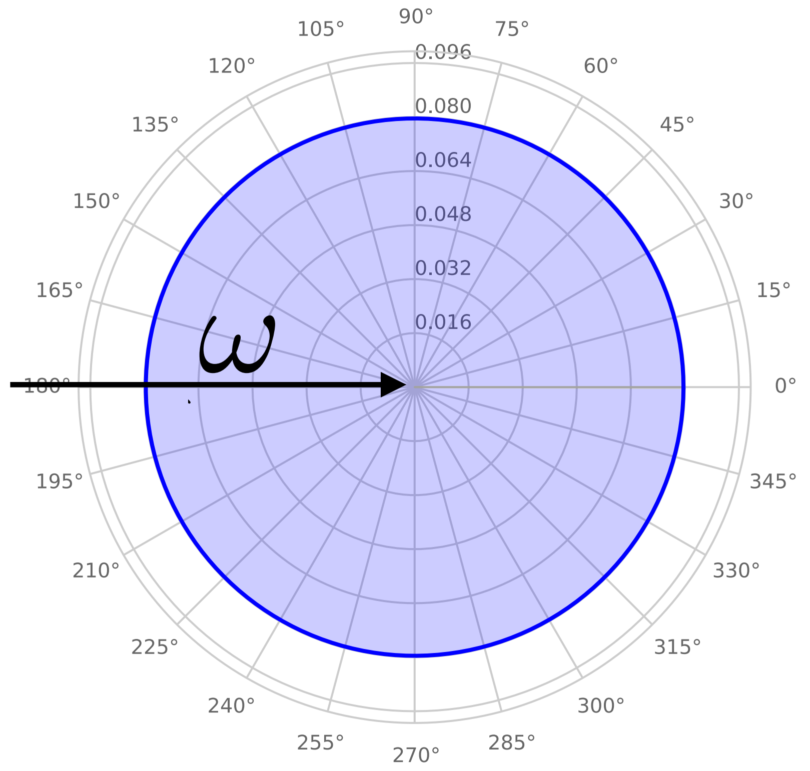
Recap: Phase Functions

Recap: Phase Functions

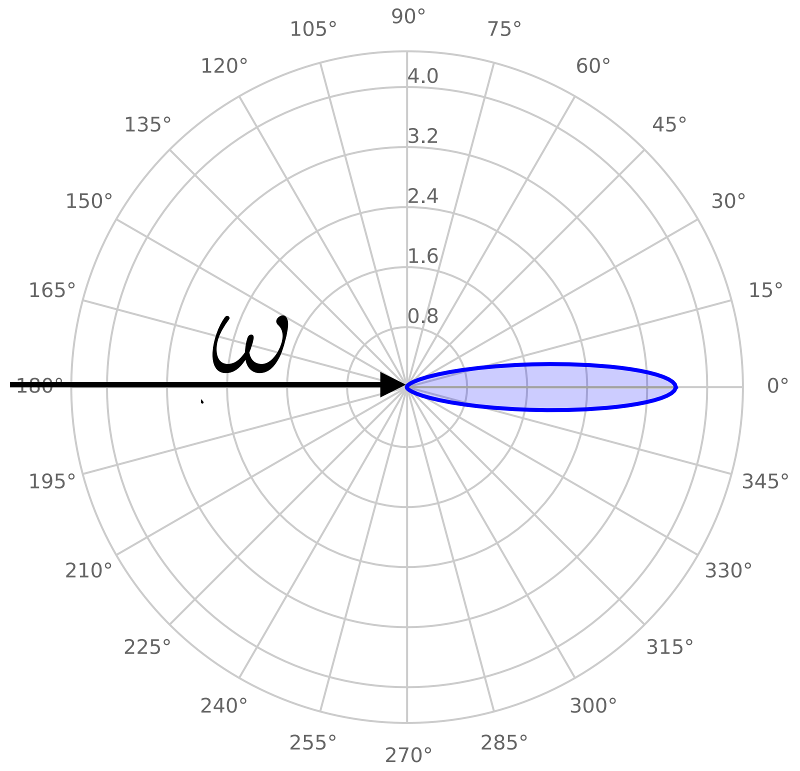


Isotropic

Recap: Phase Functions

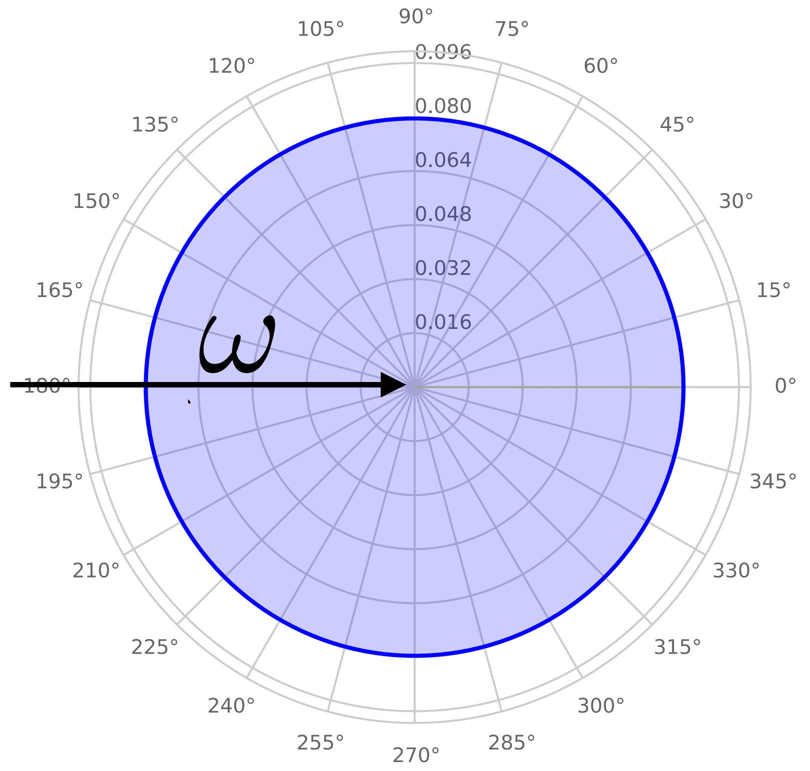


Isotropic

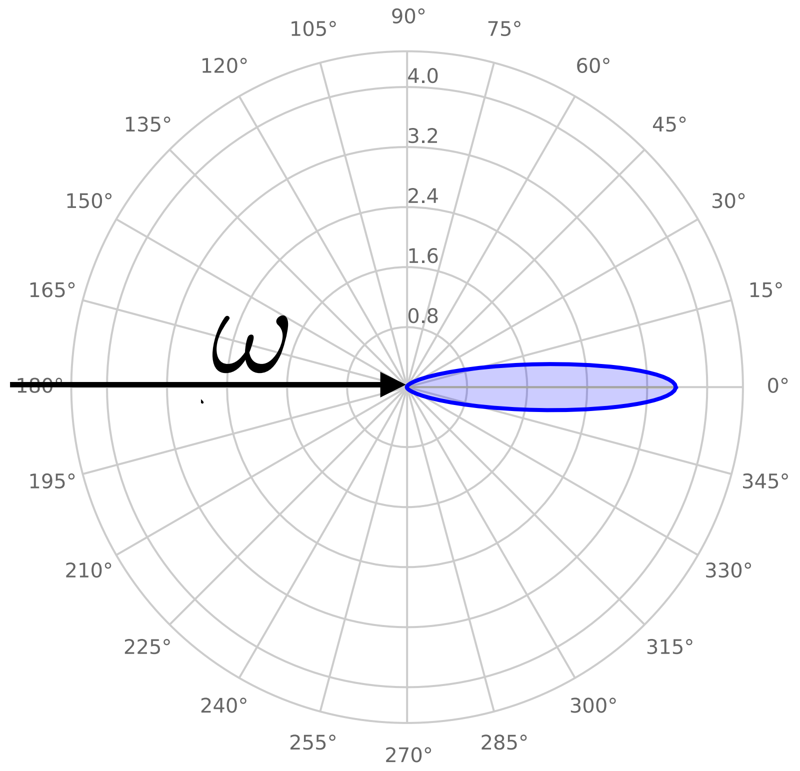


Henyey-Greenstein

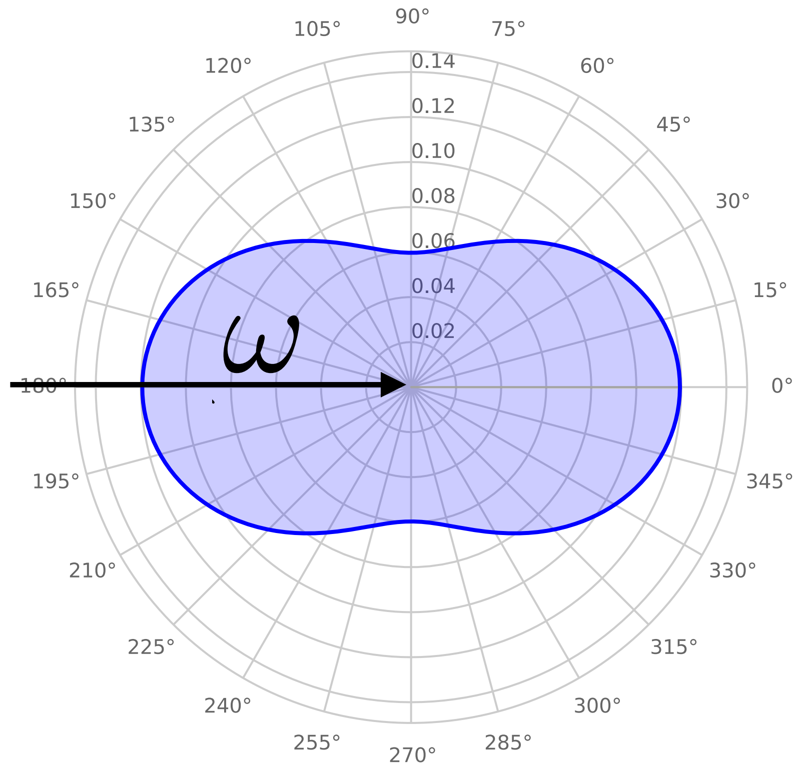
Recap: Phase Functions



Isotropic

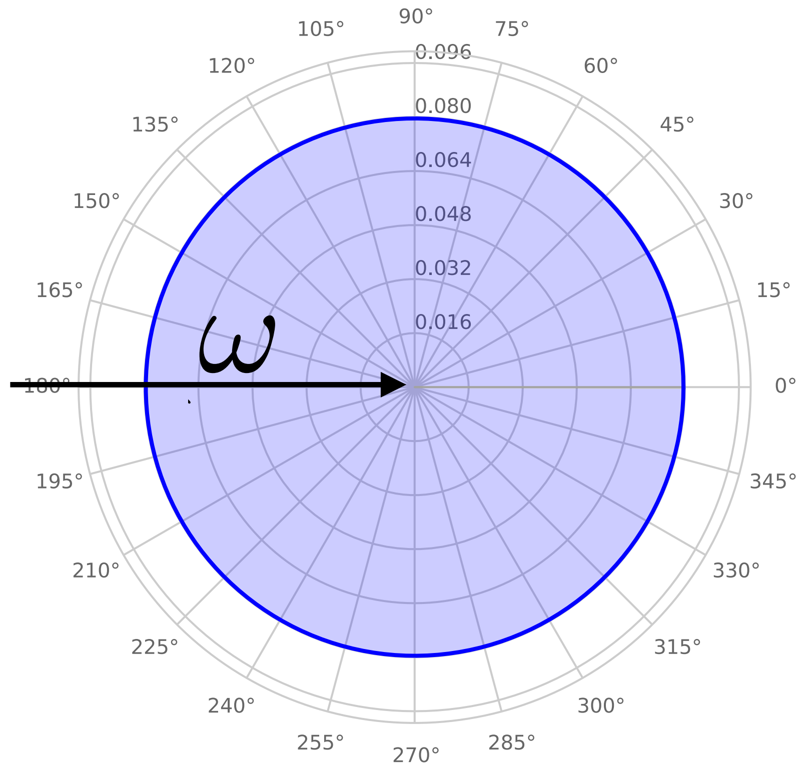


Henyey-Greenstein

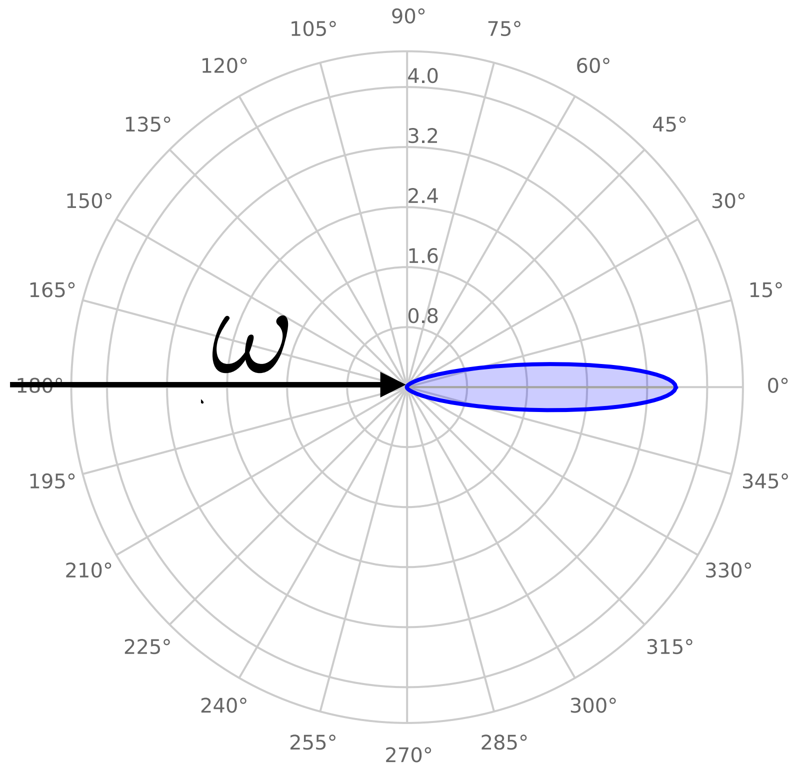


Rayleigh

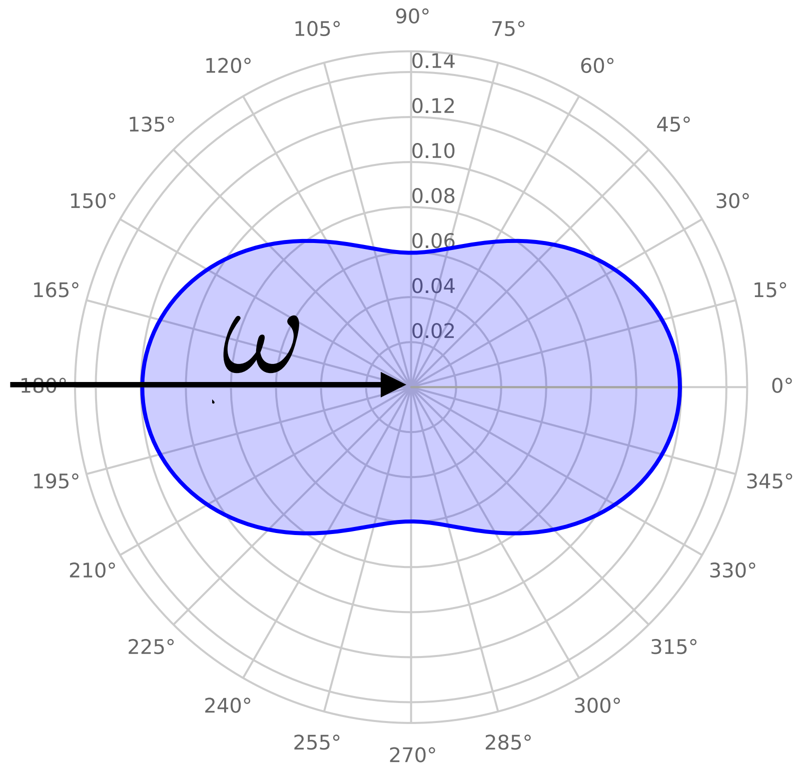
Recap: Phase Functions



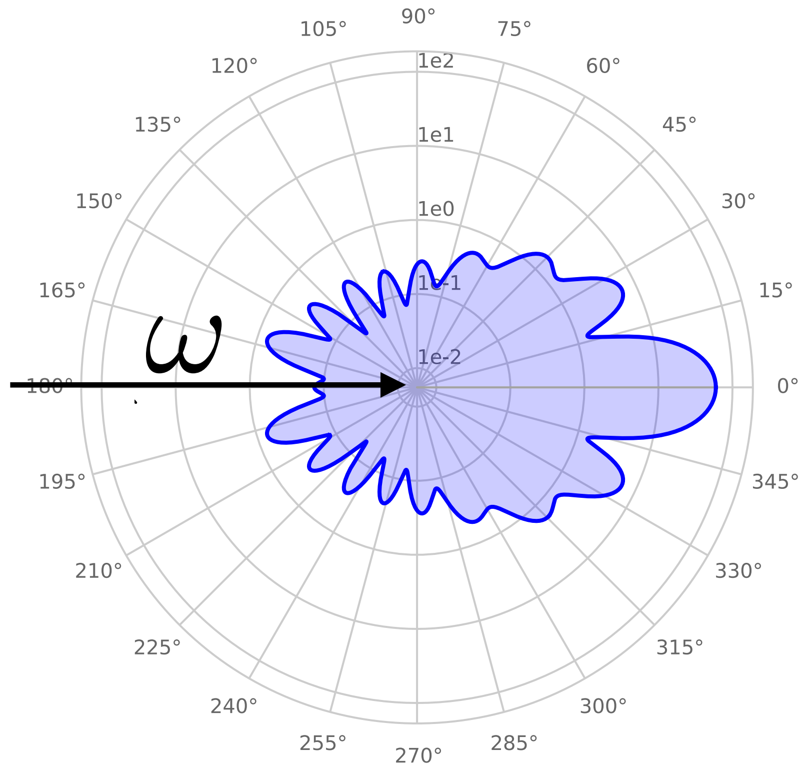
Isotropic



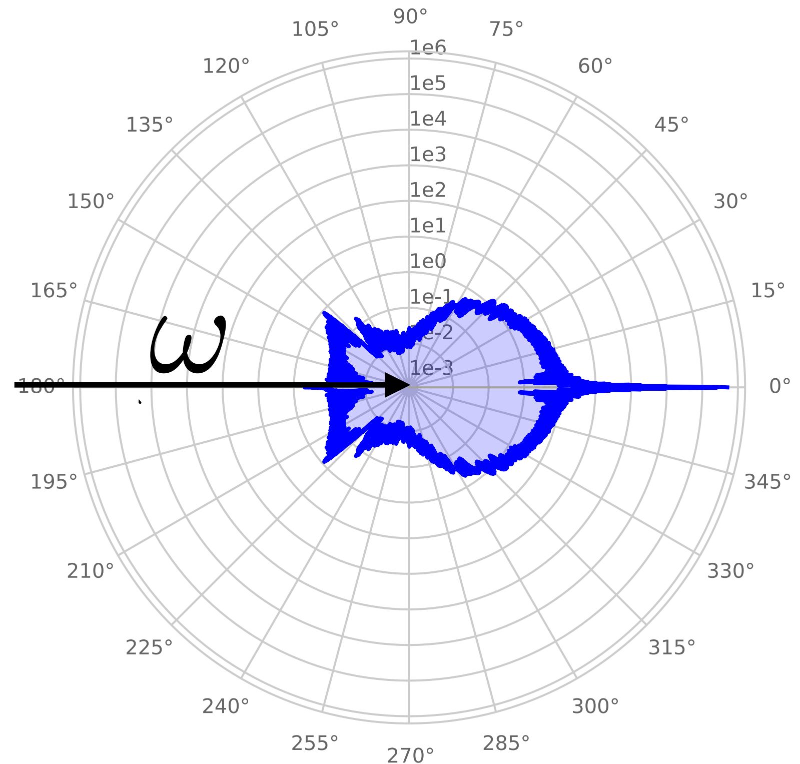
Henyey-Greenstein



Rayleigh



Lorenz-Mie small particles



Lorenz-Mie large particles

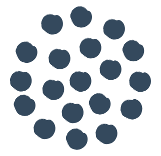
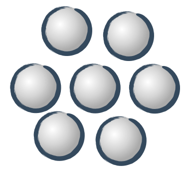
Anisotropy: Phase Function vs. Medium

Isotropic Medium

Slide after Jan Novak

Anisotropy: Phase Function vs. Medium

Isotropic Medium

Isotropic phase function	Anisotropic phase function
	

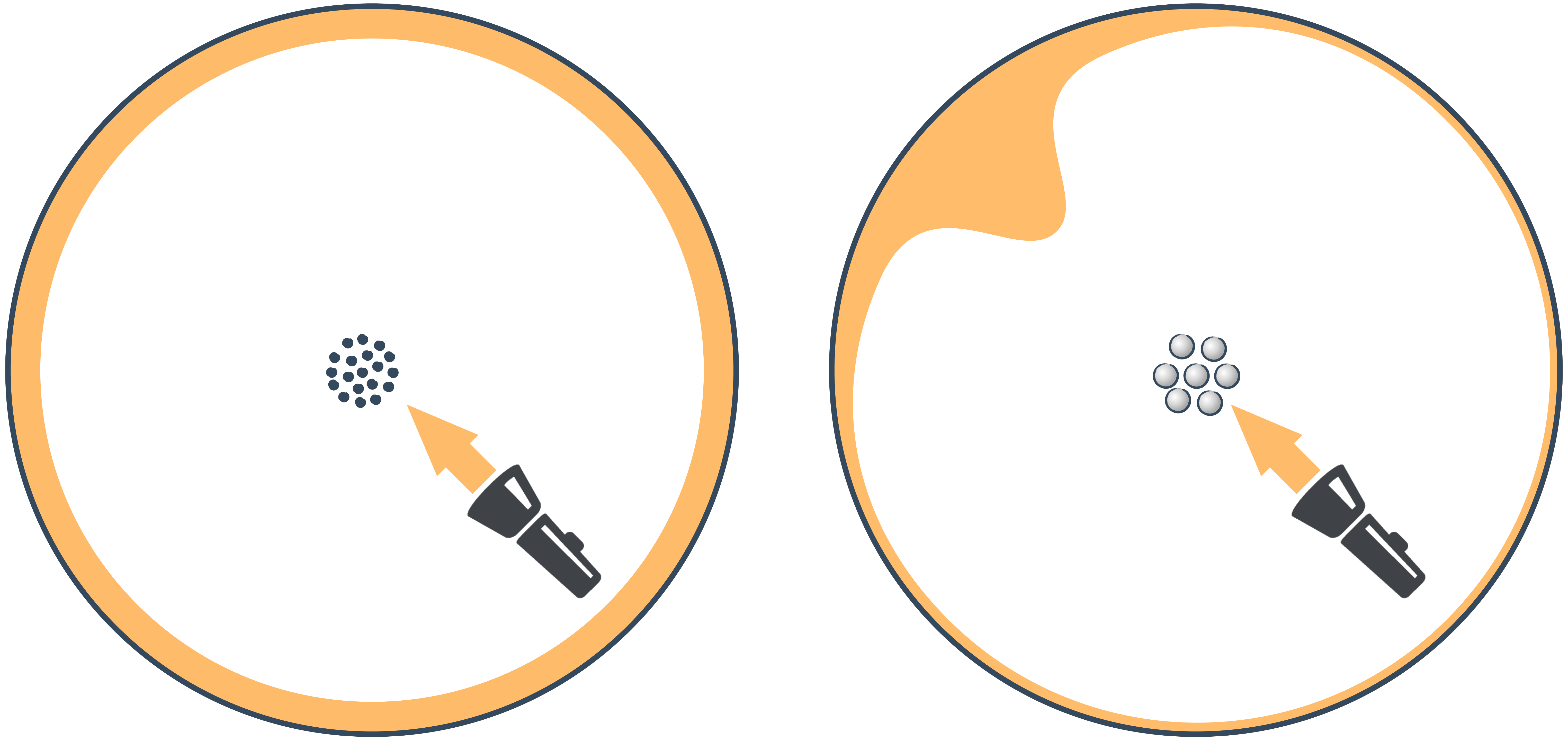
Slide after Jan Novak

Anisotropy: Phase Function vs. Medium

Isotropic Medium

Isotropic phase function

Anisotropic phase function

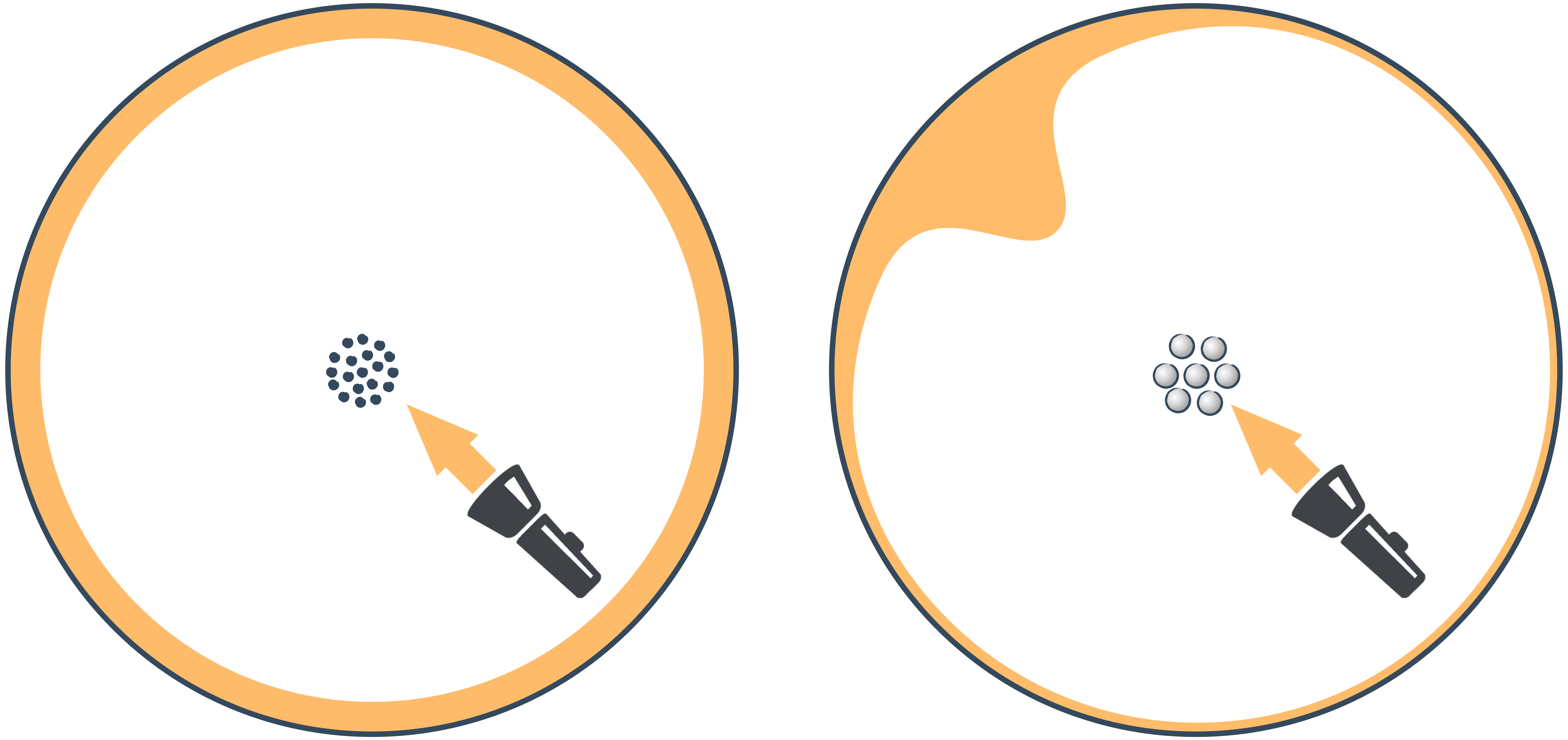


Slide after Jan Novak

Anisotropy: Phase Function vs. Medium

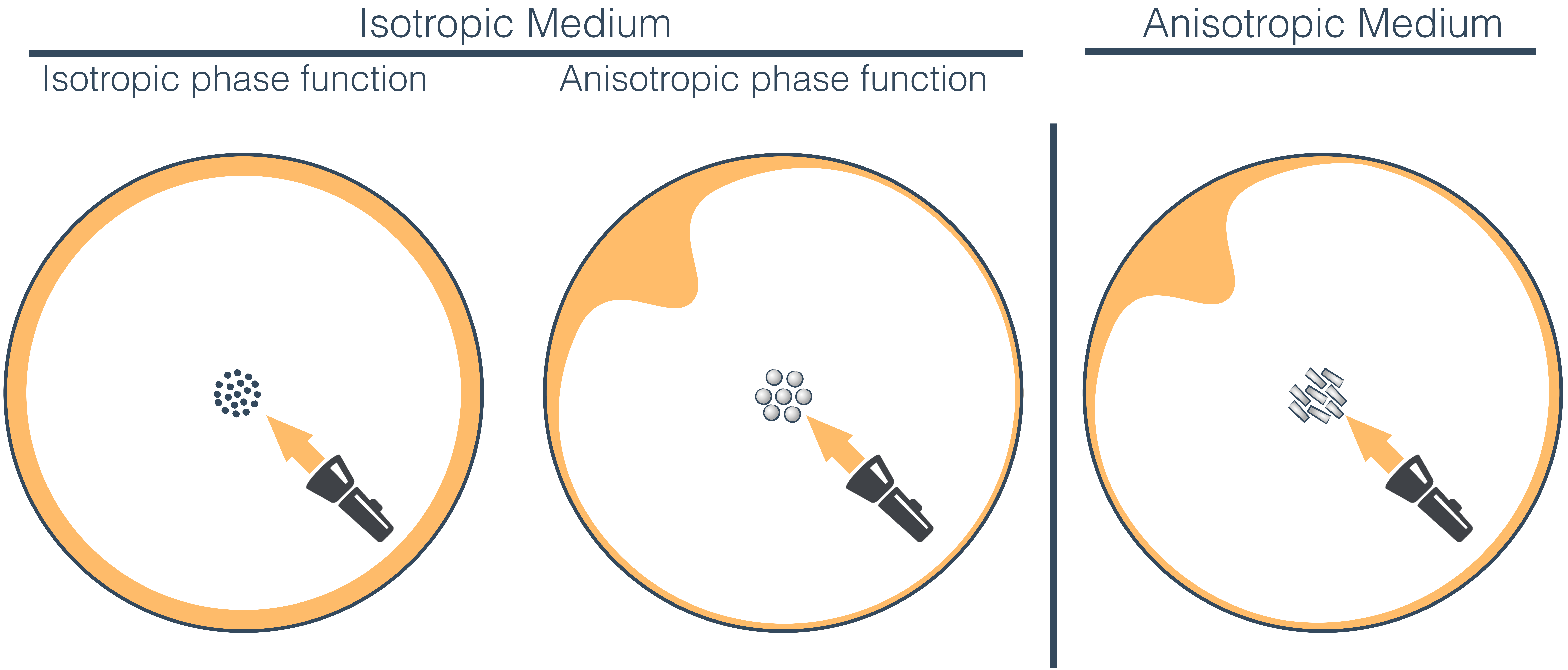
Isotropic Medium

Isotropic phase function	Anisotropic phase function



Slide after Jan Novak

Anisotropy: Phase Function vs. Medium



Isotropic Medium

Isotropic phase function

Anisotropic phase function

Anisotropic Medium

Slide after Jan Novak

Recap: Media Properties

Given:

Absorption coefficient	$\sigma_a(\mathbf{x})$	$[m^{-1}]$
Scattering coefficient	$\sigma_s(\mathbf{x})$	$[m^{-1}]$
Phase function	$f_p(\mathbf{x}, \vec{\omega}, \vec{\omega}')$	$[sr^{-1}]$

Recap: Media Properties

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Absorption coefficient	$\sigma_a(\mathbf{x})$	$[m^{-1}]$
Scattering coefficient	$\sigma_s(\mathbf{x})$	$[m^{-1}]$
Phase function	$f_p(\mathbf{x}, \vec{\omega}, \vec{\omega}')$	$[sr^{-1}]$

Derived:

Extinction coefficient	$\sigma_t(\mathbf{x}) = \sigma_a(\mathbf{x}) + \sigma_s(\mathbf{x})$	$[m^{-1}]$
Albedo	$\alpha(\mathbf{x}) = \sigma_s(\mathbf{x}) / \sigma_t(\mathbf{x})$	$[\text{None}]$
Mean-free path	$1/\sigma_t(\mathbf{x})$	$[m]$
Transmittance	$T_r(\mathbf{x}, \mathbf{y}) = e^{-\int_0^{\vert\vert\mathbf{x}-\mathbf{y}\vert\vert} \sigma_t(t)dt}$	$[\text{None}]$

For Homogeneous Isotropic Medium

Given:

Absorption coefficient	σ_a	$[m^{-1}]$
Scattering coefficient	σ_s	$[m^{-1}]$
Phase function	$\frac{1}{4\pi}$	$[sr^{-1}]$

Derived:

Extinction coefficient	$\sigma_t = \sigma_a + \sigma_s$	$[m^{-1}]$
Albedo	$\alpha = \sigma_s / \sigma_t$	$[\text{None}]$
Mean-free path	$1/\sigma_t$	$[m]$
Transmittance	$T_r(\mathbf{x}, \mathbf{y}) = e^{-\sigma_t \|\|\mathbf{x}-\mathbf{y}\|\|}$	$[\text{None}]$

Solving the Volumetric Rendering Equation

Complexity

Homogeneous vs. Heterogeneous

Scattering

- none
- fake
- single scattering
- multiple scattering

Volumetric Rendering Equation

$$
\begin{aligned}
L(\mathbf{x},\vec{\omega}) &= T_r(\mathbf{x},\mathbf{x}_z)L(\mathbf{x}_z,\vec{\omega}) \\
&+ \int_0^z T_r(\mathbf{x},\mathbf{x}_t)\sigma_a(\mathbf{x}_t)L_e(\mathbf{x}_t,\vec{\omega})dt \\
&+ \int_0^z T_r(\mathbf{x},\mathbf{x}_t)\sigma_s(\mathbf{x}_t)\int_{S^2} f_p(\mathbf{x}_t,\vec{\omega}',\vec{\omega})L_i(\mathbf{x}_t,\vec{\omega}')d\omega' dt
\end{aligned}
$$

Volumetric Rendering Equation

$$L(\mathbf{x},\vec{\omega}) = T_r(\mathbf{x},\mathbf{x}_z)L(\mathbf{x}_z,\vec{\omega})$$

Attenuated background radiance

$$+\int_0^z T_r(\mathbf{x},\mathbf{x}_t)\sigma_a(\mathbf{x}_t)L_e(\mathbf{x}_t,\vec{\omega})dt$$

$$+\int_0^z T_r(\mathbf{x},\mathbf{x}_t)\sigma_s(\mathbf{x}_t)\int_{S^2} f_p(\mathbf{x}_t,\vec{\omega}',\vec{\omega})L_i(\mathbf{x}_t,\vec{\omega}')d\omega' dt$$

Volumetric Rendering Equation

$$L(\mathbf{x},\vec{\omega}) = T_r(\mathbf{x},\mathbf{x}_z)L(\mathbf{x}_z,\vec{\omega})$$

Attenuated background radiance

$$+\int_0^z T_r(\mathbf{x},\mathbf{x}_t)\sigma_a(\mathbf{x}_t)L_e(\mathbf{x}_t,\vec{\omega})dt$$

Accumulated emitted radiance

$$+\int_0^z T_r(\mathbf{x},\mathbf{x}_t)\sigma_s(\mathbf{x}_t)\int_{S^2} f_p(\mathbf{x}_t,\vec{\omega}',\vec{\omega})L_i(\mathbf{x}_t,\vec{\omega}')d\omega' dt$$

Volumetric Rendering Equation

$$L(\mathbf{x},\vec{\omega}) = T_r(\mathbf{x},\mathbf{x}_z)L(\mathbf{x}_z,\vec{\omega})$$

Attenuated background radiance

$$+\int_0^z T_r(\mathbf{x},\mathbf{x}_t)\sigma_a(\mathbf{x}_t)L_e(\mathbf{x}_t,\vec{\omega})dt$$

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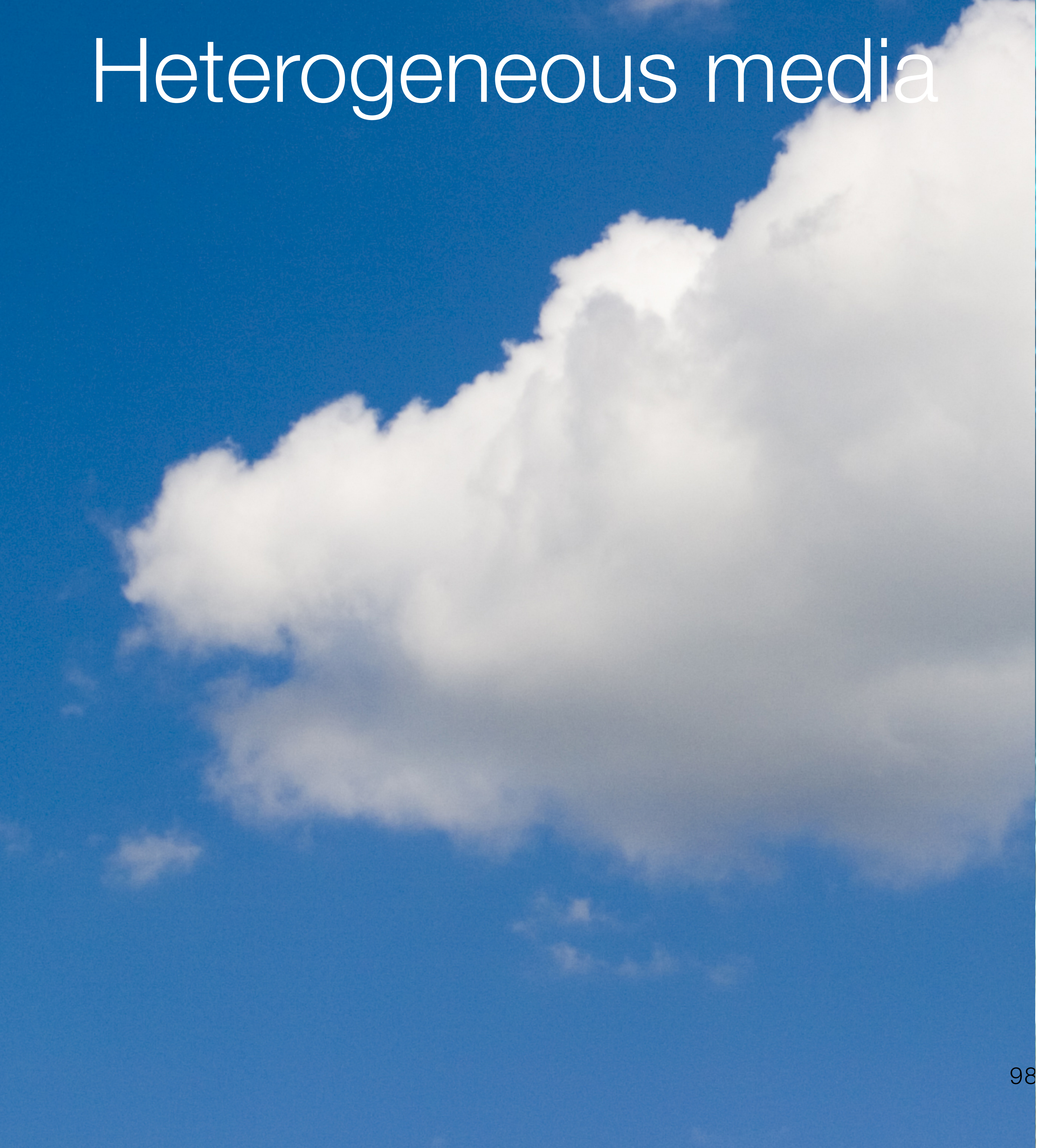
Accumulated in-scattered radiance

Heterogeneous/Homogeneous media

Homogeneous media

Heterogeneous media



Homogeneous media

Participating Media: Heterogeneous

Participating Media: Heterogeneous

Participating Media: Heterogeneous

$$L(\mathbf{x},\vec{\omega}) = \int_0^s T_r(\mathbf{x} \leftrightarrow \mathbf{x}_t)\sigma_s(\mathbf{x}_t)L_i(\mathbf{x}_t,\vec{\omega})dt + T_r(\mathbf{x} \leftrightarrow \mathbf{x}_s)L(\mathbf{x}_s,\vec{\omega})$$

Participating Media: Heterogeneous

$$L(\mathbf{x},\vec{\omega}) = \int_0^s T_r(\mathbf{x} \leftrightarrow \mathbf{x}_t)\sigma_s(\mathbf{x}_t)L_i(\mathbf{x}_t,\vec{\omega})dt + T_r(\mathbf{x} \leftrightarrow \mathbf{x}_s)L(\mathbf{x}_s,\vec{\omega})$$

Participating Media: Homogeneous

$$L(\mathbf{x},\vec{\omega}) = \int_0^s T_r(\mathbf{x} \leftrightarrow \mathbf{x}_t)\sigma_s(\mathbf{x}_t)L_i(\mathbf{x}_t,\vec{\omega})dt + T_r(\mathbf{x} \leftrightarrow \mathbf{x}_s)L(\mathbf{x}_s,\vec{\omega})$$

Participating Media: Homogeneous

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$$L(\mathbf{x},\vec{\omega}) = \sigma_s \int_0^s T_r(\mathbf{x} \leftrightarrow \mathbf{x}_t)L_i(\mathbf{x}_t,\vec{\omega})dt + T_r(\mathbf{x} \leftrightarrow \mathbf{x}_s)L(\mathbf{x}_s,\vec{\omega})$$

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Participating Media: Homogeneous

$$L(\mathbf{x},\vec{\omega}) = \int_0^s T_r(\mathbf{x} \leftrightarrow \mathbf{x}_t)\sigma_s(\mathbf{x}_t)L_i(\mathbf{x}_t,\vec{\omega})dt + T_r(\mathbf{x} \leftrightarrow \mathbf{x}_s)L(\mathbf{x}_s,\vec{\omega})$$

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$$L(\mathbf{x},\vec{\omega}) = \sigma_s \int_0^s e^{-t\sigma_t}L_i(\mathbf{x}_t,\vec{\omega})dt + e^{-s\sigma_t}L(\mathbf{x}_s,\vec{\omega})$$

Participating Media: Homogeneous

$$L(\mathbf{x},\vec{\omega}) = \int_0^s T_r(\mathbf{x} \leftrightarrow \mathbf{x}_t)\sigma_s(\mathbf{x}_t)L_i(\mathbf{x}_t,\vec{\omega})dt + T_r(\mathbf{x} \leftrightarrow \mathbf{x}_s)L(\mathbf{x}_s,\vec{\omega})$$

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Participating Media: Homogeneous

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Homogeneous Ambient Media

$$L(\mathbf{x}, \vec{\omega}) = \sigma_s \int_0^s e^{-t\sigma_t} L_i(\mathbf{x}_t, \vec{\omega}) dt + e^{-s\sigma_t} L(\mathbf{x}_s, \vec{\omega})$$

Homogeneous Ambient Media

Assume in-scattered radiance is an ambient constant

$$L(\mathbf{x},\vec{\omega}) = \sigma_s \int_0^s e^{-t\sigma_t} L_i(\mathbf{x}_t,\vec{\omega})dt + e^{-s\sigma_t} L(\mathbf{x}_s,\vec{\omega})$$

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Assume in-scattered radiance is an ambient constant

$$L(\mathbf{x}, \vec{\omega}) = \sigma_s \int_0^s e^{-t\sigma_t} L_i(\mathbf{x}_t, \vec{\omega}) dt + e^{-s\sigma_t} L(\mathbf{x}_s, \vec{\omega})$$

$$L(\mathbf{x}, \vec{\omega}) = \sigma_s \boxed{L_i} \int_0^s e^{-t\sigma_t} dt + e^{-s\sigma_t} L(\mathbf{x}_s, \vec{\omega})$$

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$$L(\mathbf{x},\vec{\omega}) = \sigma_s L_i \boxed{\int_0^s e^{-t\sigma_t} dt} + e^{-s\sigma_t} L(\mathbf{x}_s,\vec{\omega})$$

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$$L(\mathbf{x},\vec{\omega}) = \sigma_s \int_0^s e^{-t\sigma_t} L_i(\mathbf{x}_t,\vec{\omega})dt + e^{-s\sigma_t} L(\mathbf{x}_s,\vec{\omega})$$

$$L(\mathbf{x},\vec{\omega}) = \sigma_s L_i \int_0^s e^{-t\sigma_t} dt + e^{-s\sigma_t} L(\mathbf{x}_s,\vec{\omega})$$

$$L(\mathbf{x},\vec{\omega}) = \sigma_s L_i \boxed{\frac{1-e^{-s\sigma_t}}{\sigma_t}} + e^{-s\sigma_t} L(\mathbf{x}_s,\vec{\omega})$$

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Assume in-scattered radiance is an ambient constant

$$L(\mathbf{x},\vec{\omega}) = \sigma_s \int_0^s e^{-t\sigma_t} L_i(\mathbf{x}_t,\vec{\omega})dt + e^{-s\sigma_t} L(\mathbf{x}_s,\vec{\omega})$$

$$L(\mathbf{x},\vec{\omega}) = \sigma_s L_i \int_0^s e^{-t\sigma_t} dt + e^{-s\sigma_t} L(\mathbf{x}_s,\vec{\omega})$$

$$L(\mathbf{x},\vec{\omega}) = \sigma_s L_i \frac{1-e^{-s\sigma_t}}{\sigma_t} + e^{-s\sigma_t} L(\mathbf{x}_s,\vec{\omega})$$

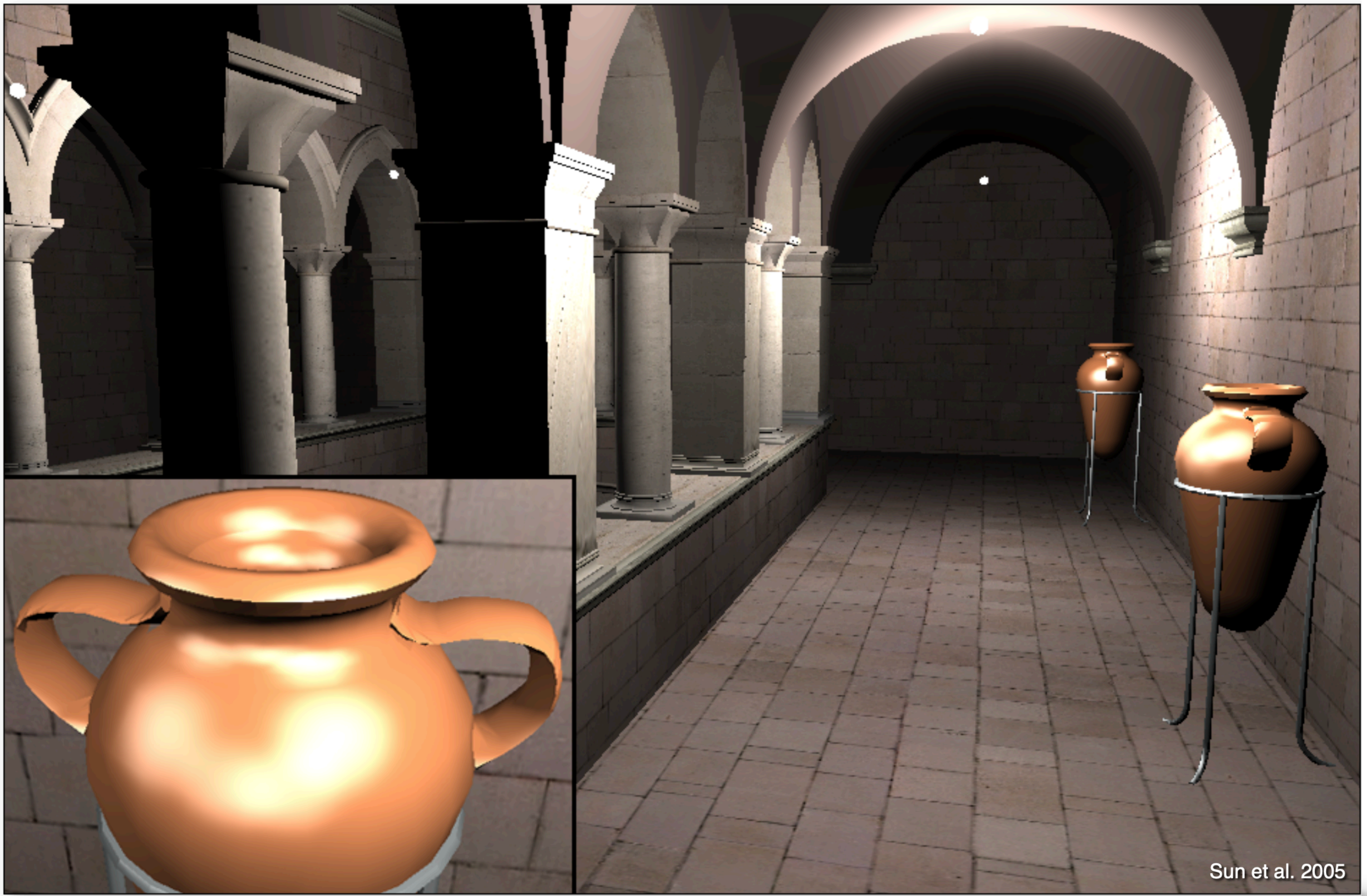
$$L(\mathbf{x},\vec{\omega}) = \mathrm{lerp}\left(\frac{\sigma_s}{\sigma_t} L_i, L(\mathbf{x}_s,\vec{\omega}), e^{-s\sigma_t}\right)$$

Homogeneous Ambient Media

Fog

Sun et al. 2005

Clear Day



Sun et al. 2005

Fog

Volumetric Rendering Equation

$$
\begin{aligned}
L(\mathbf{x},\vec{\omega}) &= T_r(\mathbf{x},\mathbf{x}_z)L(\mathbf{x}_z,\vec{\omega}) \\
&\quad + \int_0^z T_r(\mathbf{x},\mathbf{x}_t)\sigma_a(\mathbf{x}_t)L_e(\mathbf{x}_t,\vec{\omega})dt \\
&\quad + \int_0^z T_r(\mathbf{x},\mathbf{x}_t)\sigma_s(\mathbf{x}_t)\int_{S^2} f_p(\mathbf{x}_t,\vec{\omega}',\vec{\omega})L_i(\mathbf{x}_t,\vec{\omega}')d\omega' dt
\end{aligned}
$$

Accumulated in-scattered radiance

In-scattered Radiance

$$L(\mathbf{x},\omega) = \int_0^z T_r(\mathbf{x},\mathbf{x}_t)\sigma_s(\mathbf{x}_t)\int_{S^2} f_p(\mathbf{x}_t,\vec{\omega}',\vec{\omega})L_i(\mathbf{x}_t,\vec{\omega}')d\omega' dt$$

In-scattered Radiance

$$L(\mathbf{x},\omega) = \int_0^z T_r(\mathbf{x},\mathbf{x}_t)\sigma_s(\mathbf{x}_t) \int_{S^2} f_p(\mathbf{x}_t,\vec{\omega}',\vec{\omega})L_i(\mathbf{x}_t,\vec{\omega}')d\omega' dt$$

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$$L_s(\mathbf{x},\omega) = \int_{S^2} f_p(\mathbf{x}_t,\vec{\omega}',\vec{\omega})L_i(\mathbf{x}_t,\vec{\omega}')d\omega' dt$$

In-scattered Radiance

$$L(\mathbf{x},\omega) = \int_0^z T_r(\mathbf{x},\mathbf{x}_t)\sigma_s(\mathbf{x}_t)\int_{S^2} f_p(\mathbf{x}_t,\vec{\omega}',\vec{\omega})L_i(\mathbf{x}_t,\vec{\omega}')d\omega' dt$$

$$L_s(\mathbf{x},\omega) = \int_{S^2} f_p(\mathbf{x}_t,\vec{\omega}',\vec{\omega})L_i(\mathbf{x}_t,\vec{\omega}')d\omega' dt$$

Single scattering L_i arrives directly from a light source (direct illumination)

In-scattered Radiance

$$L(\mathbf{x},\omega) = \int_0^z T_r(\mathbf{x},\mathbf{x}_t)\sigma_s(\mathbf{x}_t)\int_{S^2} f_p(\mathbf{x}_t,\vec{\omega}',\vec{\omega})L_i(\mathbf{x}_t,\vec{\omega}')d\omega' dt$$

$$L_s(\mathbf{x},\omega) = \int_{S^2} f_p(\mathbf{x}_t,\vec{\omega}',\vec{\omega})L_i(\mathbf{x}_t,\vec{\omega}')d\omega' dt$$

Single scattering

L_i arrives directly from a light source (direct illumination)

$$L_i(\mathbf{x},\vec{\omega}) = T_r(\mathbf{x},r(\mathbf{x},\vec{\omega}))L_e(r(\mathbf{x},\vec{\omega}),-\vec{\omega})$$

In-scattered Radiance

$$L(\mathbf{x},\omega) = \int_0^z T_r(\mathbf{x},\mathbf{x}_t)\sigma_s(\mathbf{x}_t)\int_{S^2} f_p(\mathbf{x}_t,\vec{\omega}',\vec{\omega})L_i(\mathbf{x}_t,\vec{\omega}')d\omega' dt$$

$$L_s(\mathbf{x},\omega) = \int_{S^2} f_p(\mathbf{x}_t,\vec{\omega}',\vec{\omega})L_i(\mathbf{x}_t,\vec{\omega}')d\omega' dt$$

Single scattering

L_i arrives directly from a light source (direct illumination)

$$L_i(\mathbf{x},\vec{\omega}) = T_r(\mathbf{x},r(\mathbf{x},\vec{\omega}))L_e(r(\mathbf{x},\vec{\omega}),-\vec{\omega})$$

Multiple scattering

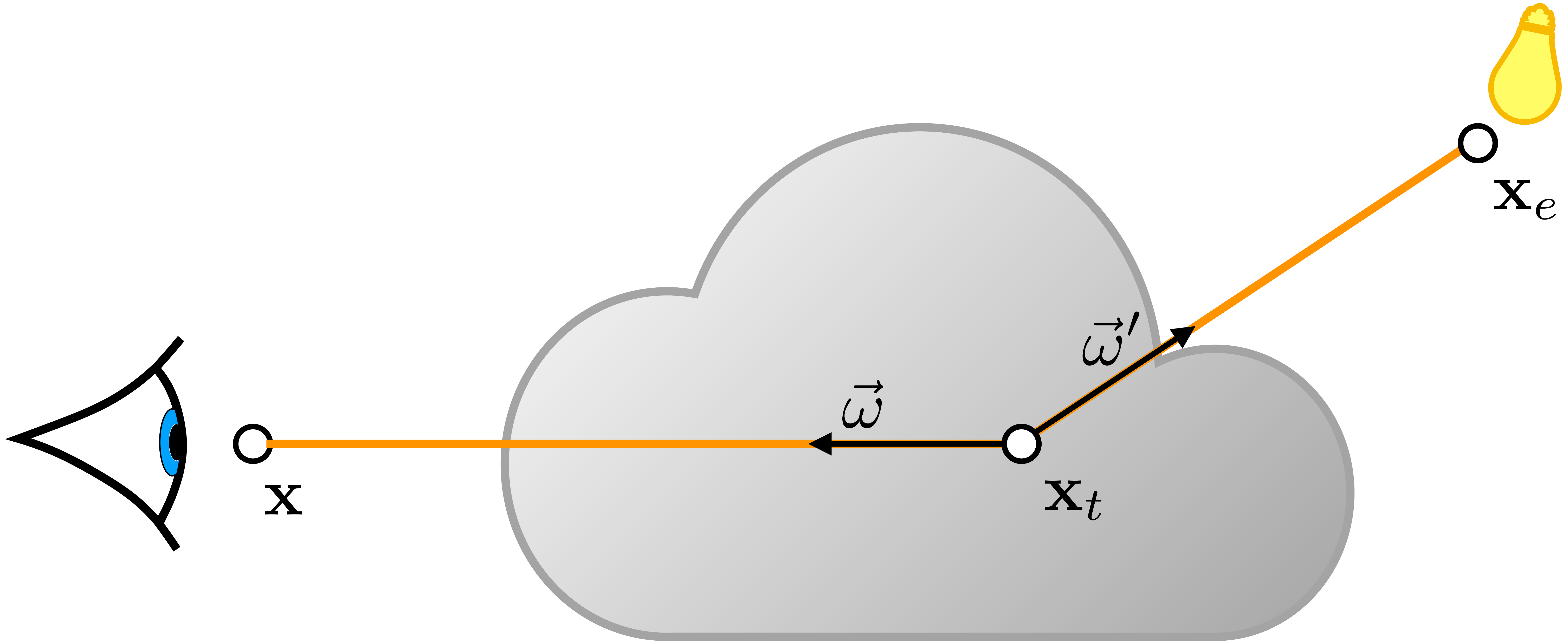
arrives through multiple bounces (indirect illumination)

Single Scattering

$$L(\mathbf{x}, \vec{\omega}) = \int_0^z T_r(\mathbf{x}, \mathbf{x}_t)\sigma_s(\mathbf{x}_t) \int_{S^2} f_p(\mathbf{x}, \vec{\omega}, \vec{\omega}')T_r(\mathbf{x}_t, \mathbf{x}_e)L_e(\mathbf{x}_e, -\vec{\omega})d\vec{\omega}'dt$$

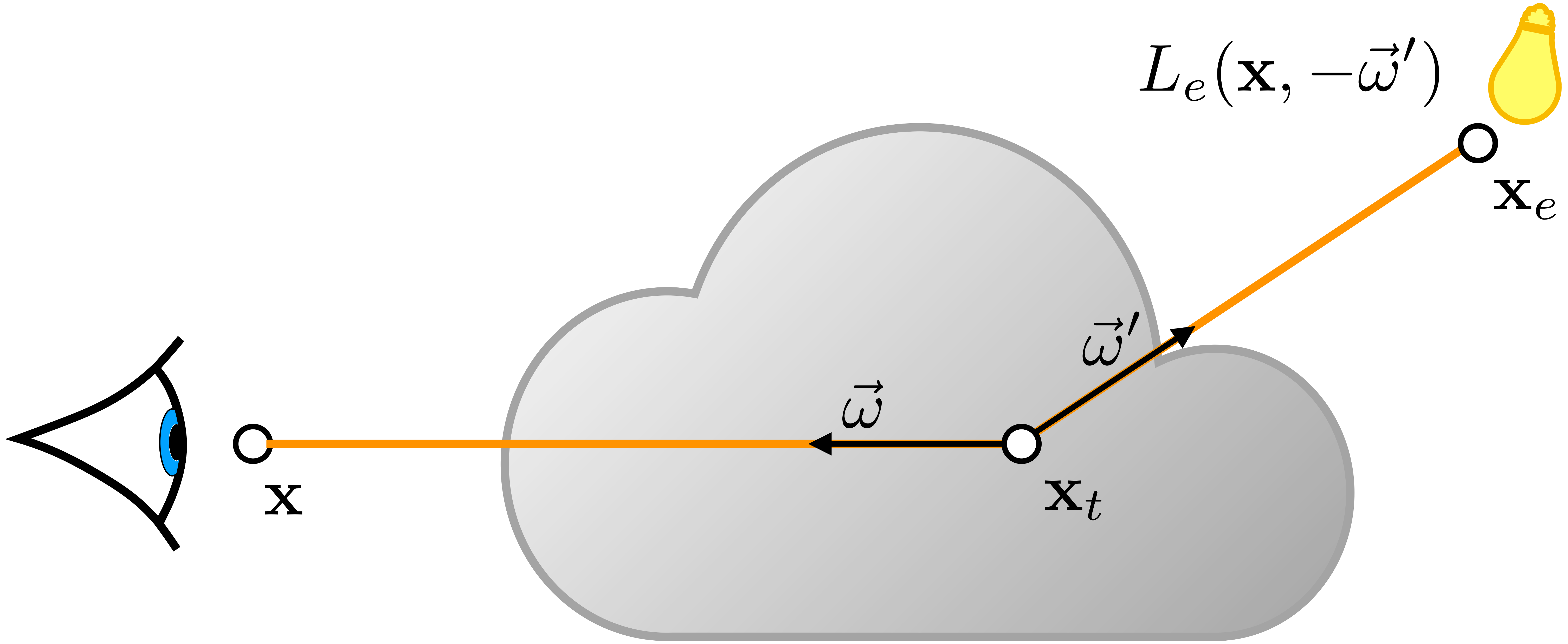
Single Scattering

$$L(\mathbf{x},\vec{\omega}) = \int_0^z T_r(\mathbf{x},\mathbf{x}_t)\sigma_s(\mathbf{x}_t)\int_{S^2} f_p(\mathbf{x},\vec{\omega},\vec{\omega}')T_r(\mathbf{x}_t,\mathbf{x}_e)L_e(\mathbf{x}_e,-\vec{\omega})d\vec{\omega}'dt$$



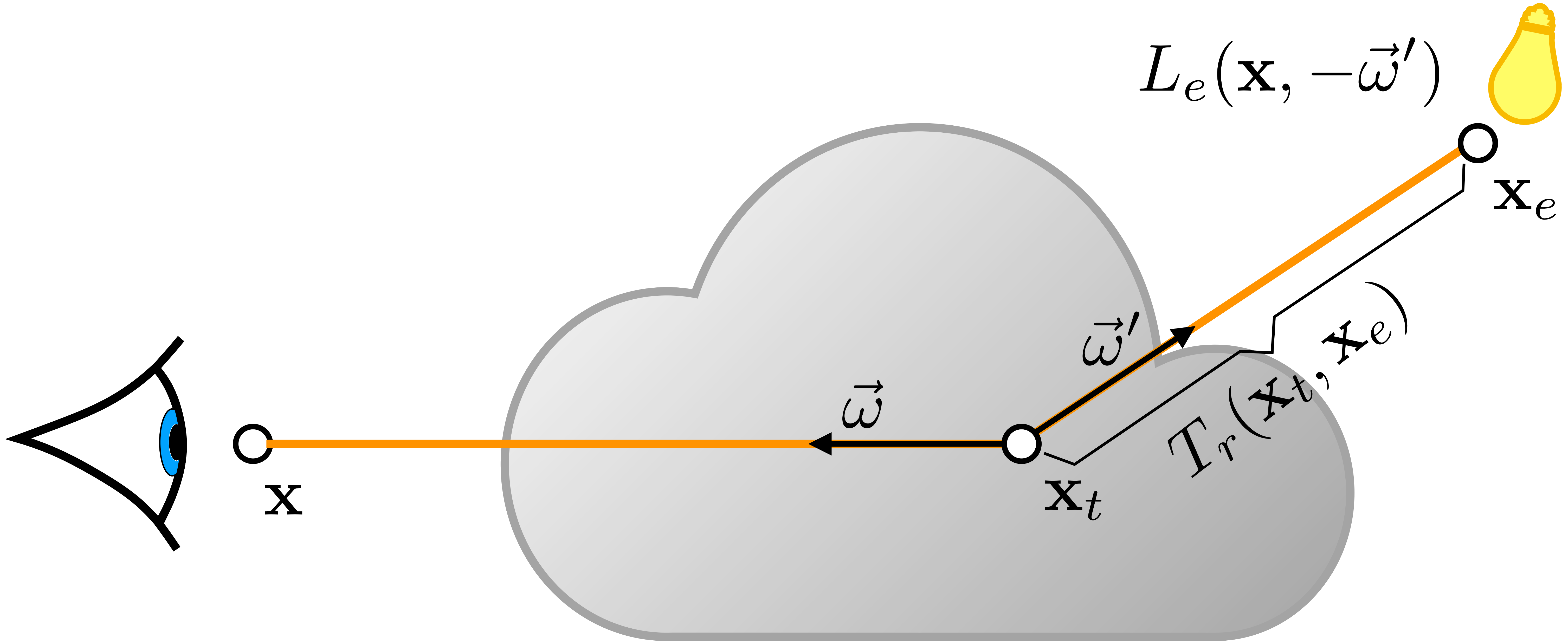
Single Scattering

$$L(\mathbf{x},\vec{\omega}) = \int_0^z T_r(\mathbf{x},\mathbf{x}_t)\sigma_s(\mathbf{x}_t)\int_{S^2} f_p(\mathbf{x},\vec{\omega},\vec{\omega}')T_r(\mathbf{x}_t,\mathbf{x}_e)L_e(\mathbf{x}_e,-\vec{\omega})d\vec{\omega}'dt$$



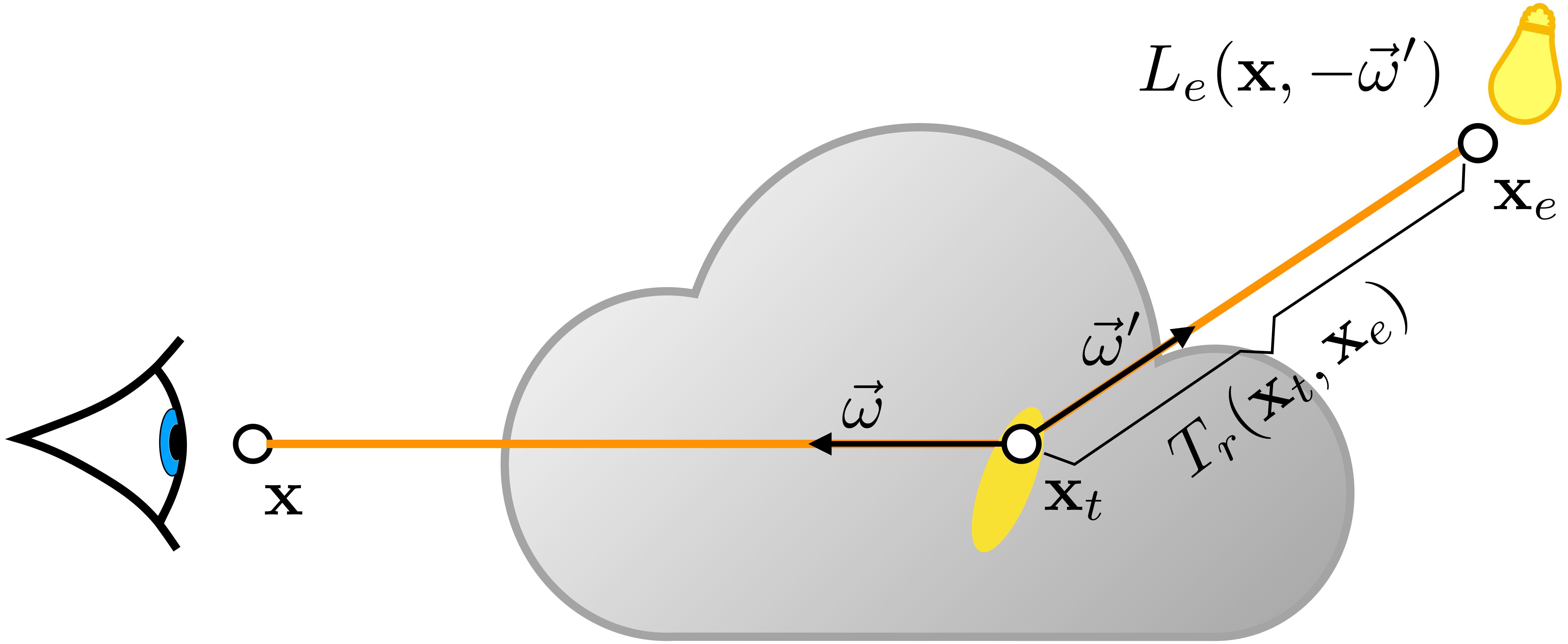
Single Scattering

$$L(\mathbf{x},\vec{\omega}) = \int_0^z T_r(\mathbf{x},\mathbf{x}_t)\sigma_s(\mathbf{x}_t)\int_{S^2} f_p(\mathbf{x},\vec{\omega},\vec{\omega}')T_r(\mathbf{x}_t,\mathbf{x}_e)L_e(\mathbf{x}_e,-\vec{\omega})d\vec{\omega}'dt$$



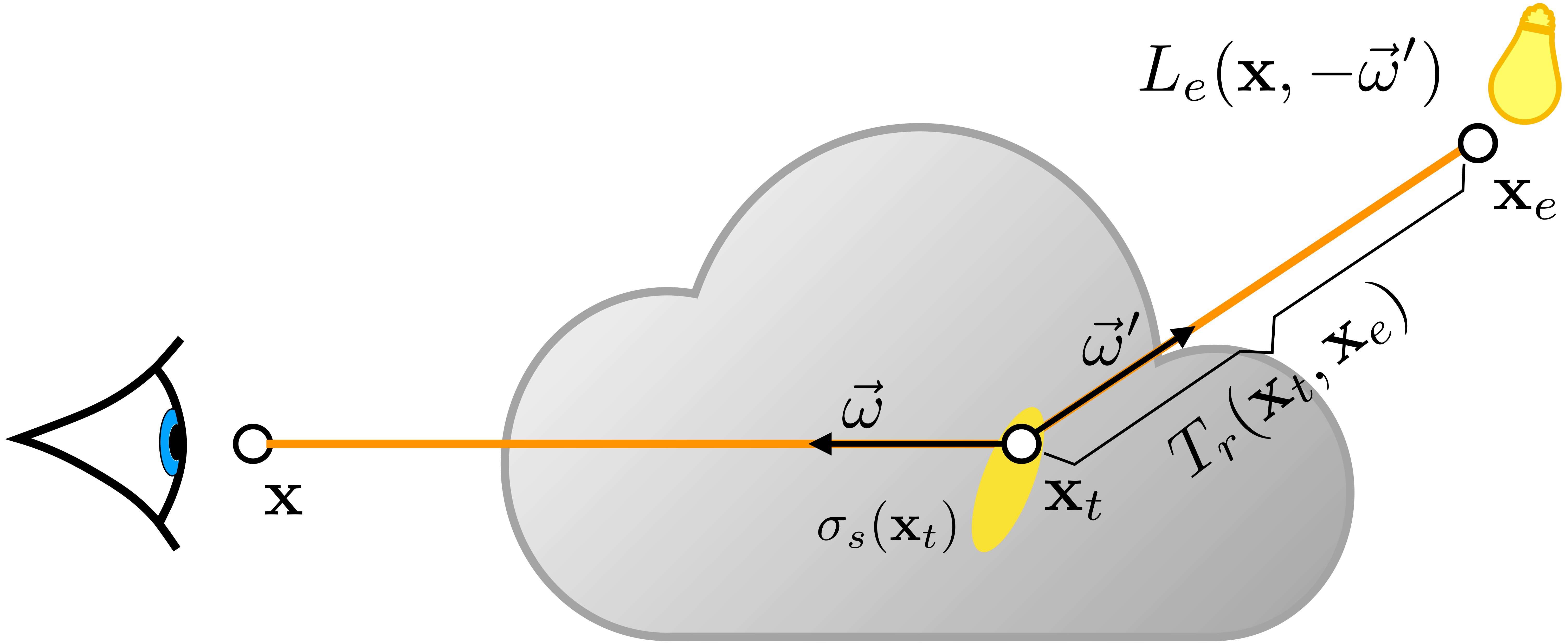
Single Scattering

$$L(\mathbf{x},\vec{\omega}) = \int_0^z T_r(\mathbf{x},\mathbf{x}_t)\sigma_s(\mathbf{x}_t)\int_{S^2} f_p(\mathbf{x},\vec{\omega},\vec{\omega}')T_r(\mathbf{x}_t,\mathbf{x}_e)L_e(\mathbf{x}_e,-\vec{\omega})d\vec{\omega}'dt$$



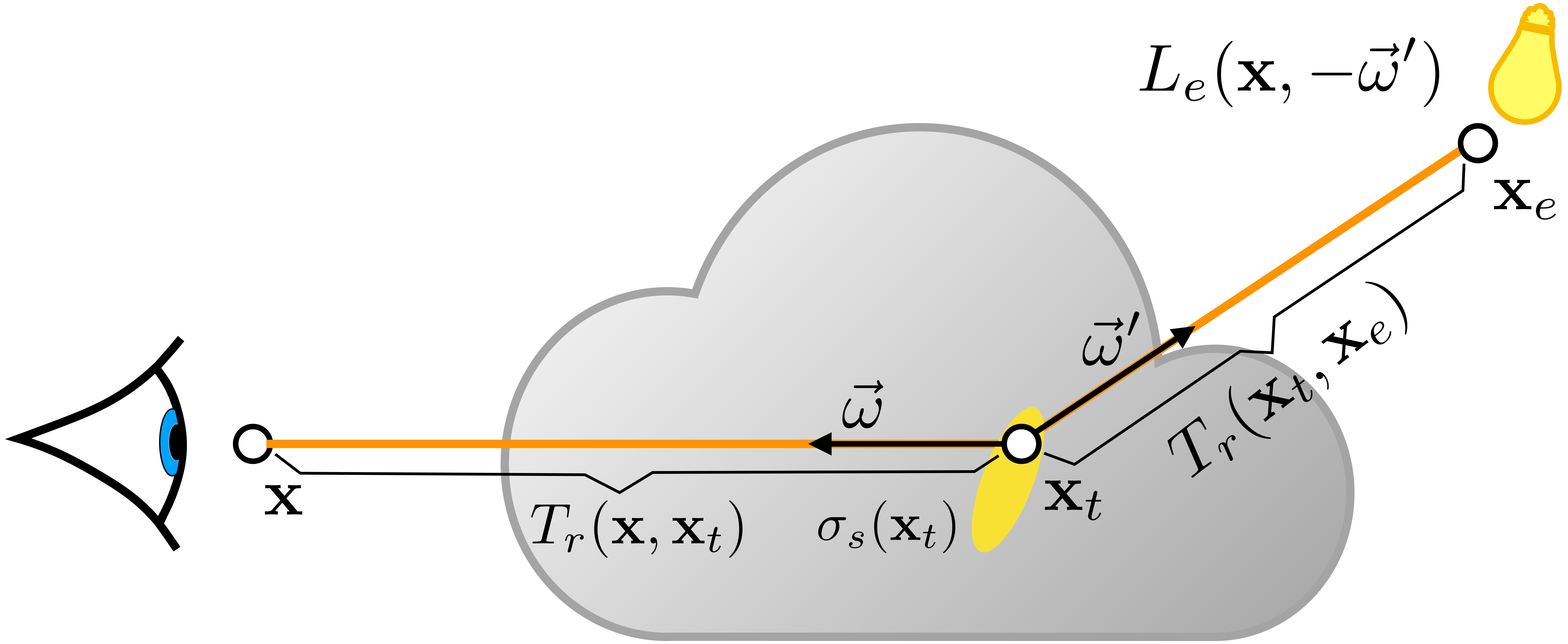
Single Scattering

$$L(\mathbf{x},\vec{\omega}) = \int_0^z T_r(\mathbf{x},\mathbf{x}_t)\sigma_s(\mathbf{x}_t)\int_{S^2} f_p(\mathbf{x},\vec{\omega},\vec{\omega}')T_r(\mathbf{x}_t,\mathbf{x}_e)L_e(\mathbf{x}_e,-\vec{\omega})d\vec{\omega}'dt$$



Single Scattering

$$L(\mathbf{x},\vec{\omega}) = \int_0^z T_r(\mathbf{x},\mathbf{x}_t)\sigma_s(\mathbf{x}_t)\int_{S^2} f_p(\mathbf{x},\vec{\omega},\vec{\omega}')T_r(\mathbf{x}_t,\mathbf{x}_e)L_e(\mathbf{x}_e,-\vec{\omega})d\vec{\omega}'dt$$



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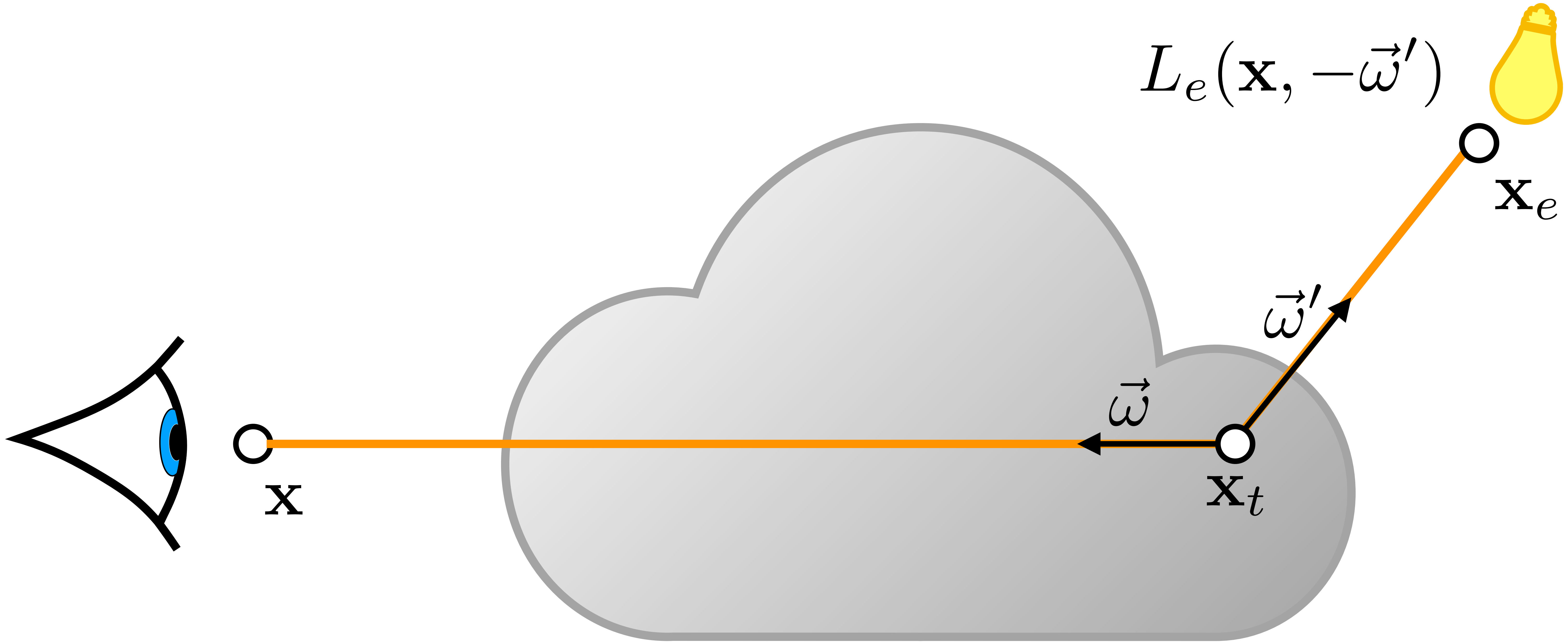
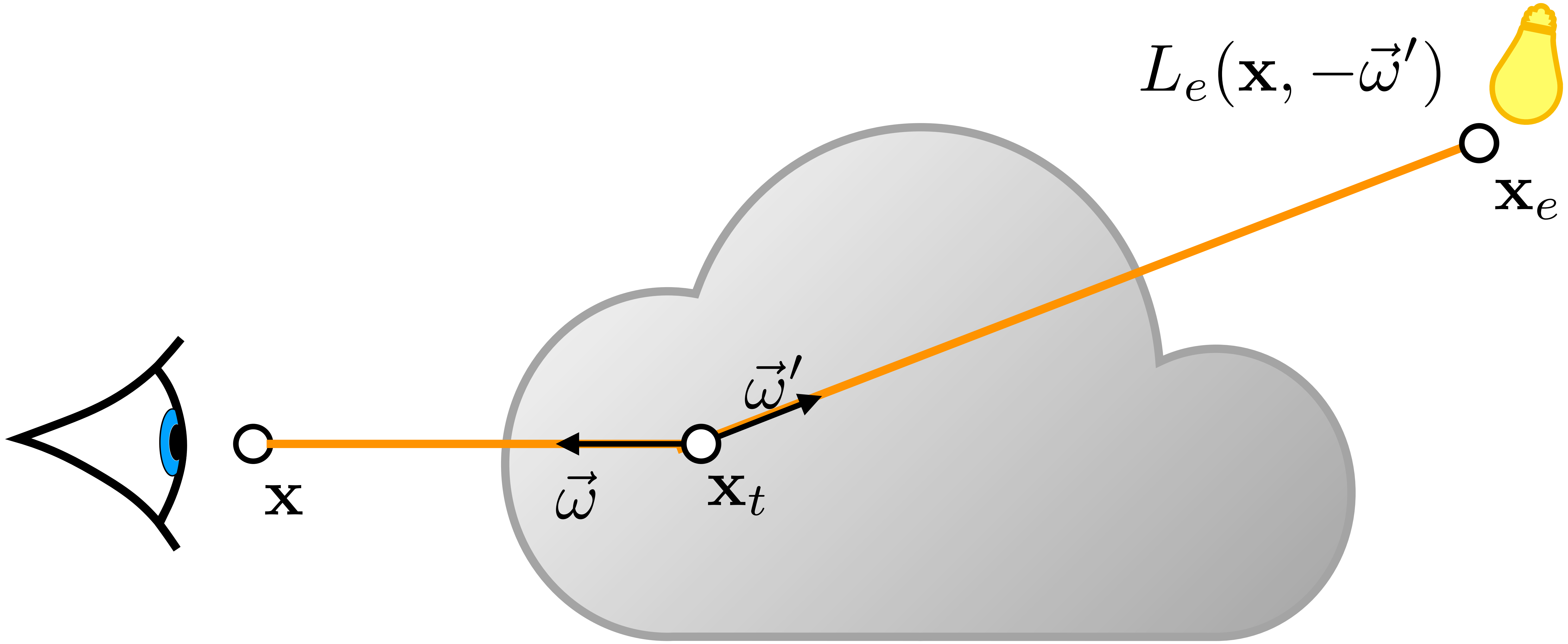


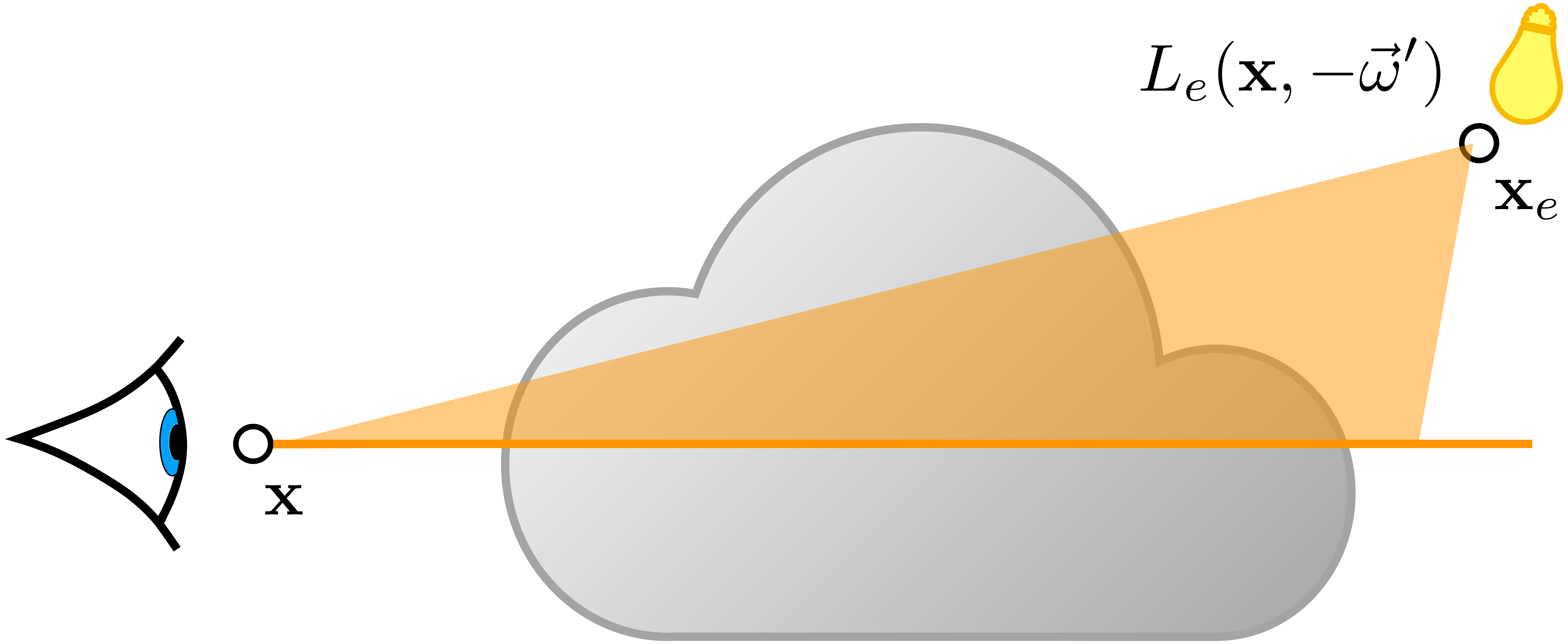
Single Scattering

$$L(\mathbf{x},\vec{\omega}) = \int_0^z T_r(\mathbf{x},\mathbf{x}_t)\sigma_s(\mathbf{x}_t)\int_{S^2} f_p(\mathbf{x},\vec{\omega},\vec{\omega}')T_r(\mathbf{x}_t,\mathbf{x}_e)L_e(\mathbf{x}_e,-\vec{\omega})d\vec{\omega}'dt$$



Single Scattering

$$L(\mathbf{x},\vec{\omega}) = \int_0^z T_r(\mathbf{x},\mathbf{x}_t)\sigma_s(\mathbf{x}_t)\int_{S^2} f_p(\mathbf{x},\vec{\omega},\vec{\omega}')T_r(\mathbf{x}_t,\mathbf{x}_e)L_e(\mathbf{x}_e,-\vec{\omega})d\vec{\omega}'dt$$



Single Scattering

$$L(\mathbf{x},\vec{\omega}) = \int_0^z T_r(\mathbf{x},\mathbf{x}_t)\sigma_s(\mathbf{x}_t)\int_{S^2} f_p(\mathbf{x},\vec{\omega},\vec{\omega}')T_r(\mathbf{x}_t,\mathbf{x}_e)L_e(\mathbf{x}_e,-\vec{\omega})d\vec{\omega}'dt$$

Single Scattering

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Semi-analytic solutions

Sun et al. [2005]

Pegoraro et al. [2009, 2010]

Single Scattering

$$L(\mathbf{x},\vec{\omega}) = \int_0^z T_r(\mathbf{x},\mathbf{x}_t)\sigma_s(\mathbf{x}_t)\int_{S^2} f_p(\mathbf{x},\vec{\omega},\vec{\omega}')T_r(\mathbf{x}_t,\mathbf{x}_e)L_e(\mathbf{x}_e,-\vec{\omega})d\vec{\omega}'dt$$

Semi-analytic solutions

Sun et al. [2005]

Pegoraro et al. [2009, 2010]

Numerical solutions

Ray marching

Equiangular sampling

Analytic Single Scattering

$$L(\mathbf{x}, \vec{\omega}) = \int_0^z T_r(\mathbf{x}, \mathbf{x}_t)\sigma_s(\mathbf{x}_t) \int_{S^2} f_p(\mathbf{x}, \vec{\omega}, \vec{\omega}')T_r(\mathbf{x}_t, \mathbf{x}_e)L_e(\mathbf{x}_e, -\vec{\omega})d\vec{\omega}'dt$$

Assumptions:

Analytic Single Scattering

$$L(\mathbf{x},\vec{\omega}) = \int_0^z T_r(\mathbf{x},\mathbf{x}_t)\sigma_s(\mathbf{x}_t)\int_{S^2} f_p(\mathbf{x},\vec{\omega},\vec{\omega}')T_r(\mathbf{x}_t,\mathbf{x}_e)L_e(\mathbf{x}_e,-\vec{\omega})d\vec{\omega}'dt$$

Assumptions:

Homogeneous

Analytic Single Scattering

$$L(\mathbf{x},\vec{\omega}) = \int_0^z T_r(\mathbf{x},\mathbf{x}_t)\sigma_s(\mathbf{x}_t)\int_{S^2} f_p(\mathbf{x},\vec{\omega},\vec{\omega}')T_r(\mathbf{x}_t,\mathbf{x}_e)L_e(\mathbf{x}_e,-\vec{\omega})d\vec{\omega}'dt$$

Assumptions:

Homogeneous

Point or spot light

Analytic Single Scattering

$$L(\mathbf{x},\vec{\omega}) = \int_0^z T_r(\mathbf{x},\mathbf{x}_t)\sigma_s(\mathbf{x}_t)\int_{S^2} f_p(\mathbf{x},\vec{\omega},\vec{\omega}')T_r(\mathbf{x}_t,\mathbf{x}_e)L_e(\mathbf{x}_e,-\vec{\omega})d\vec{\omega}'dt$$

Assumptions:

Homogeneous

Point or spot light

Relatively simple phase function

Analytic Single Scattering

$$L(\mathbf{x},\vec{\omega}) = \int_0^z T_r(\mathbf{x},\mathbf{x}_t)\sigma_s(\mathbf{x}_t)\int_{S^2} f_p(\mathbf{x},\vec{\omega},\vec{\omega}')T_r(\mathbf{x}_t,\mathbf{x}_e)L_e(\mathbf{x}_e,-\vec{\omega})d\vec{\omega}'dt$$

Assumptions:

Homogeneous

Point or spot light

Relatively simple phase function

No occlusion

Analytic Single Scattering

$$L(\mathbf{x}, \vec{\omega}) = \int_0^z T_r(\mathbf{x}, \mathbf{x}_t)\sigma_s(\mathbf{x}_t) \int_{S^2} f_p(\mathbf{x}, \vec{\omega}, \vec{\omega}')T_r(\mathbf{x}_t, \mathbf{x}_e)L_e(\mathbf{x}_e, -\vec{\omega})d\vec{\omega}'dt$$

Assumptions:

- Homogeneous
- Point or spot light
- Relatively simple phase function
- No occlusion

Analytic Single Scattering

$$L(\mathbf{x}, \vec{\omega}) = \int_0^z T_r(\mathbf{x}, \mathbf{x}_t)\sigma_s(\mathbf{x}_t) \int_{S^2} f_p(\mathbf{x}, \vec{\omega}, \vec{\omega}')T_r(\mathbf{x}_t, \mathbf{x}_e)L_e(\mathbf{x}_e, -\vec{\omega})d\vec{\omega}' dt$$

Assumptions:

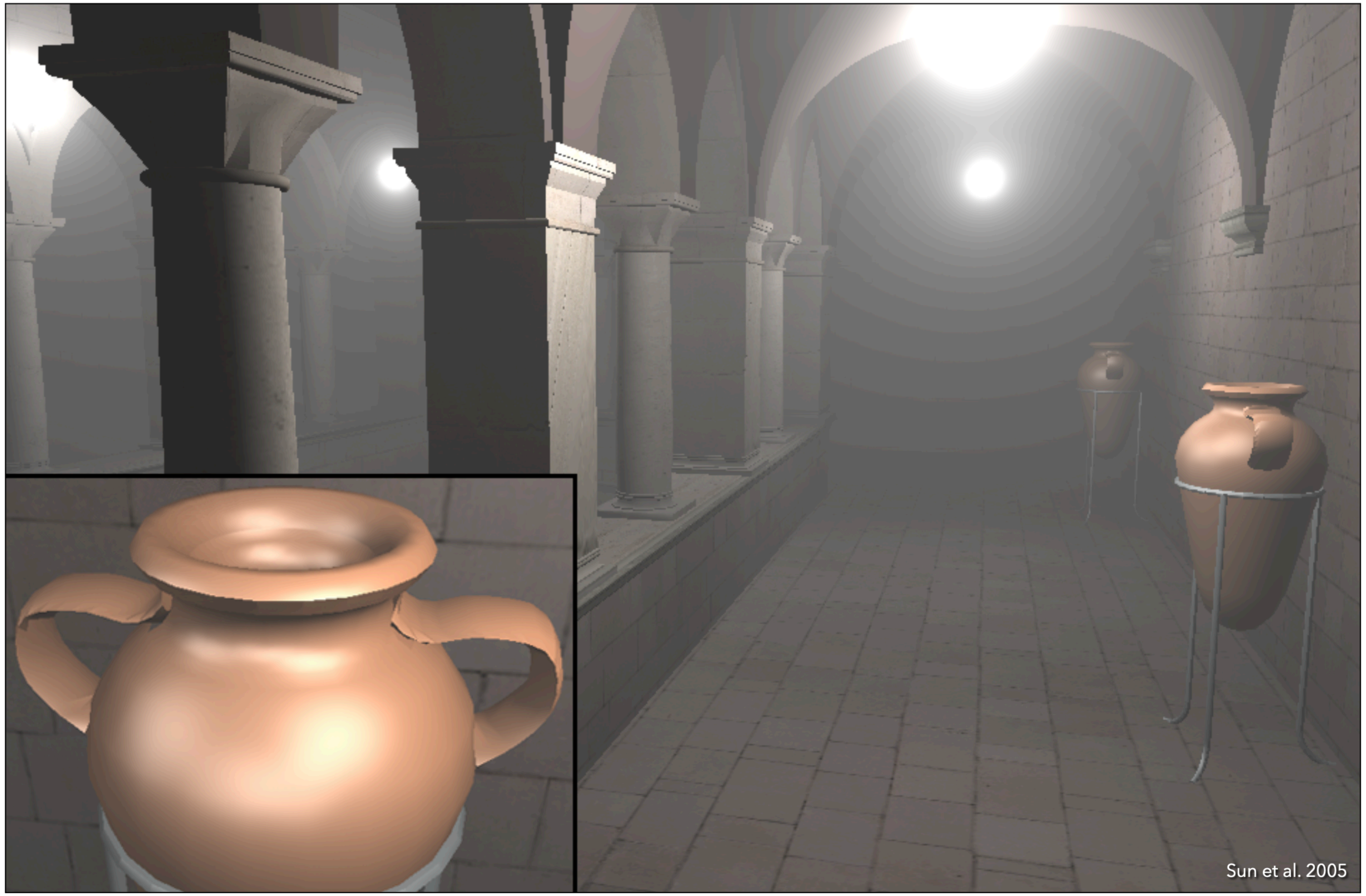
- Homogeneous
- Point or spot light
- Relatively simple phase function
- No occlusion

$$L_{(}\mathbf{x}, \vec{\omega}) = \frac{\Phi}{4\pi}\frac{1}{4\pi}\int_0^z e^{-\sigma_t||\mathbf{x},\mathbf{x}_t||}\frac{e^{-\sigma_t||\mathbf{x}_t,\mathbf{x}_p||}}{e^{-\sigma_t||\mathbf{x}_t,\mathbf{x}_p||^2}}dt$$

OpenGL Fog

Sun et al. 2005

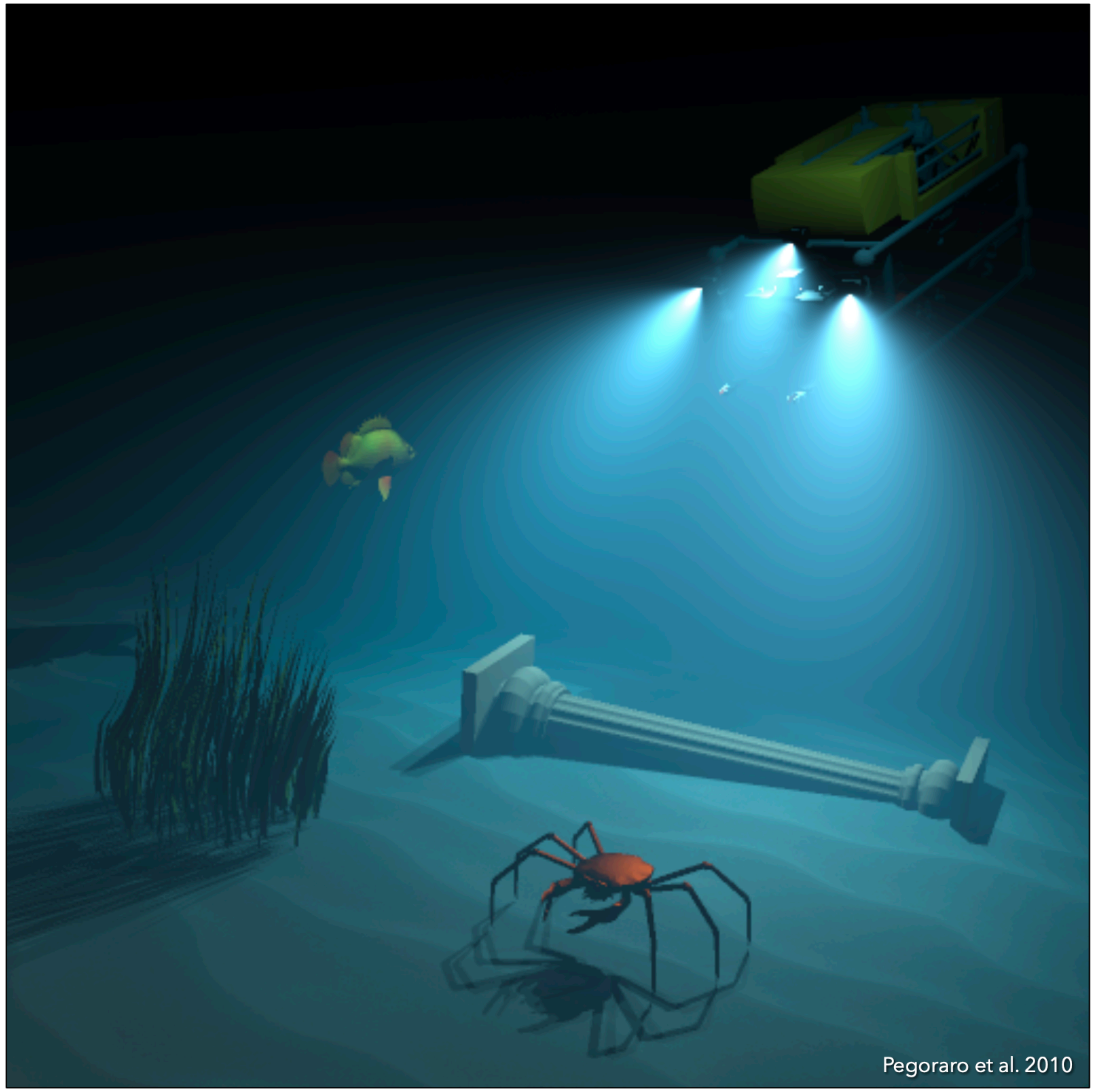
Analytic Single Scattering



Sun et al. 2005

Analytic Single Scattering


Pegoraro et al. 2009


Pegoraro et al. 2010

Analytic Single Scattering

$$L_m(x_a, x_b, \vec{\omega}) = \frac{\kappa_s}{h} e^{\kappa_t (x_a - x_h)} 2 \sum_{n=0}^{N-1} c(n) \sum_{k=0}^{2n} d(n,k) \int_{v_a}^{v_b} \frac{e^{-Hv}}{(v^2+1)^{n+1}} v^k \mathrm{d}v$$

$$\begin{aligned}
\int \frac{e^{av}}{(v^2+1)^m} v^n \mathrm{d}v = \frac{1}{2^{m-1}} \sum_{l=0}^{m-1} \frac{1}{2^l} \binom{m-1+l}{m-1} \Bigg(\sum_{k=0}^{\min\{m-1-l,n\}} \binom{n}{k} \Bigg(& \frac{a^{m-1-l-k}}{(m-1-l-k)!} E(a, v, m-n-l+k) \\
& - e^{av} \sum_{j=1}^{m-1-l-k} \frac{(j-1)!}{(m-1-l-k)!} \frac{a^{m-1-l-k-j}}{(v^2+1)^j} \sum_{\substack{i=(m-n-l+k-j) \bmod 2 \\ i+=2}}^{\leq j} (-1)^{\frac{m-n-l+k-j+i}{2}} \binom{j}{i} v^i \Bigg) \\
+ \frac{e^{av}}{a} \sum_{k=0}^{\leq n-m+l} \binom{n}{k} \sum_{j=0}^{n-m+l-k} \frac{(n-m+l-k)!}{j!} & \frac{1}{(-a)^{n-m+l-k-j}} \sum_{\substack{i=(-m+l+k-j) \bmod 2 \\ i+=2}}^{\leq j} (-1)^{\frac{-m+l+k-j+i}{2}} \binom{j}{i} v^i \Bigg)
\end{aligned}$$

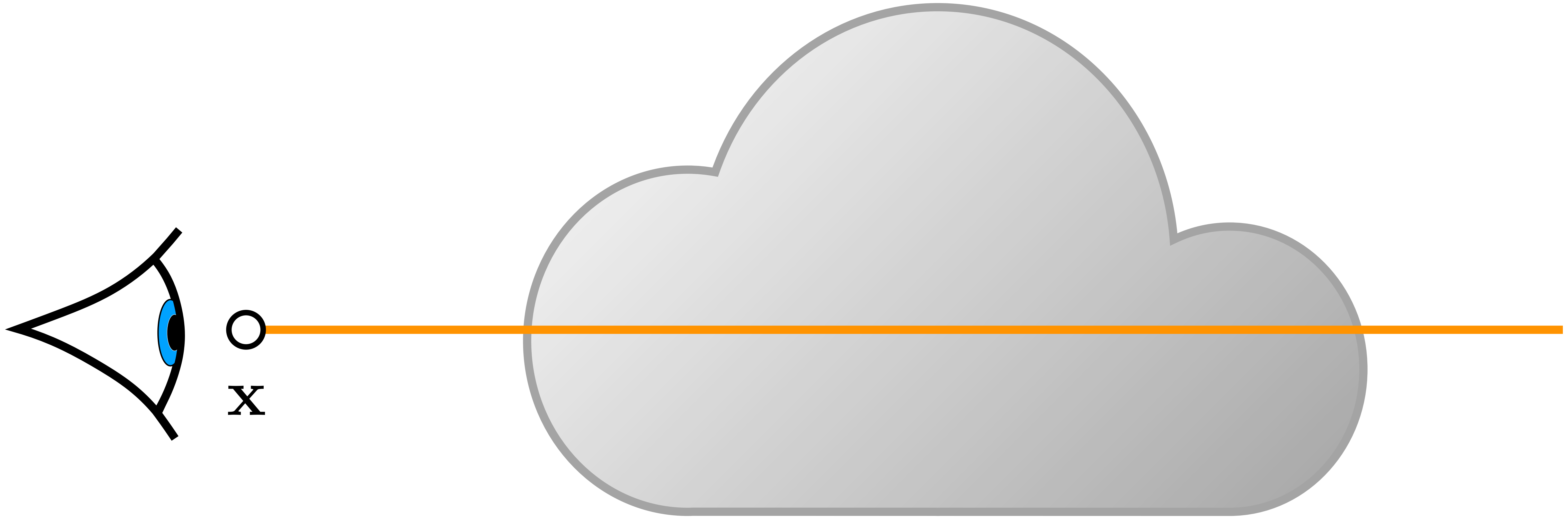
No shadows, implementation nightmare, computationally intensive,...

Let's try brute force!

Ray Marching

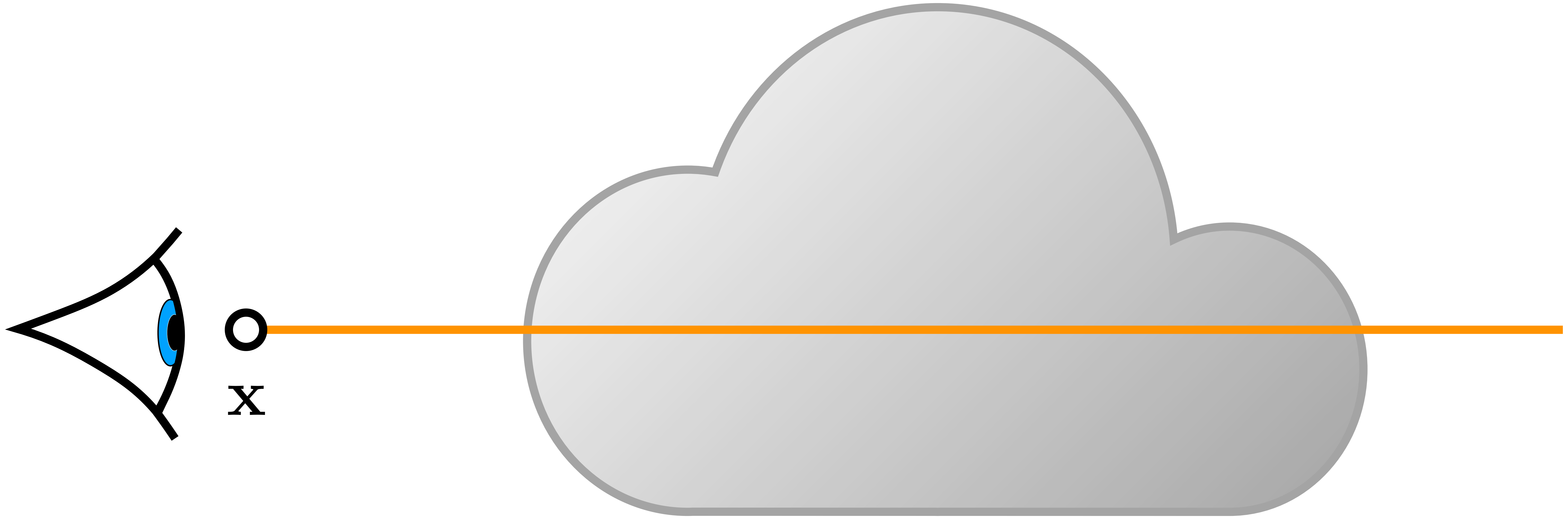
$$L(\mathbf{x},\vec{\omega}) = \int_0^z T_r(\mathbf{x},\mathbf{x}_t)\sigma_s(\mathbf{x}_t)L_s(\mathbf{x}_t,\vec{\omega})dt$$

Approximate with Riemann summation



Ray Marching

$$L_(\mathbf{x}, \vec{\omega}) \approx \sum_{k=0}^{N} T_r(\mathbf{x}, \mathbf{x}_{t,k})\sigma_s(\mathbf{x}_{t,k})L_s(\mathbf{x}_{t,k}, \vec{\omega})\Delta t$$



Ray Marching

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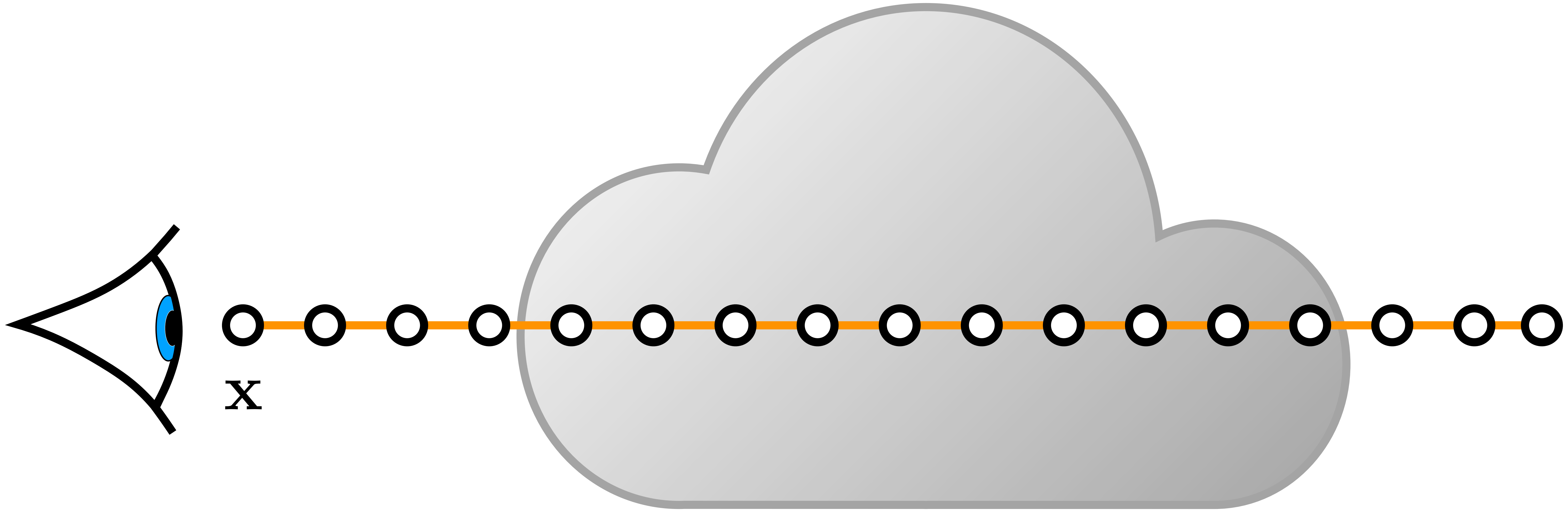
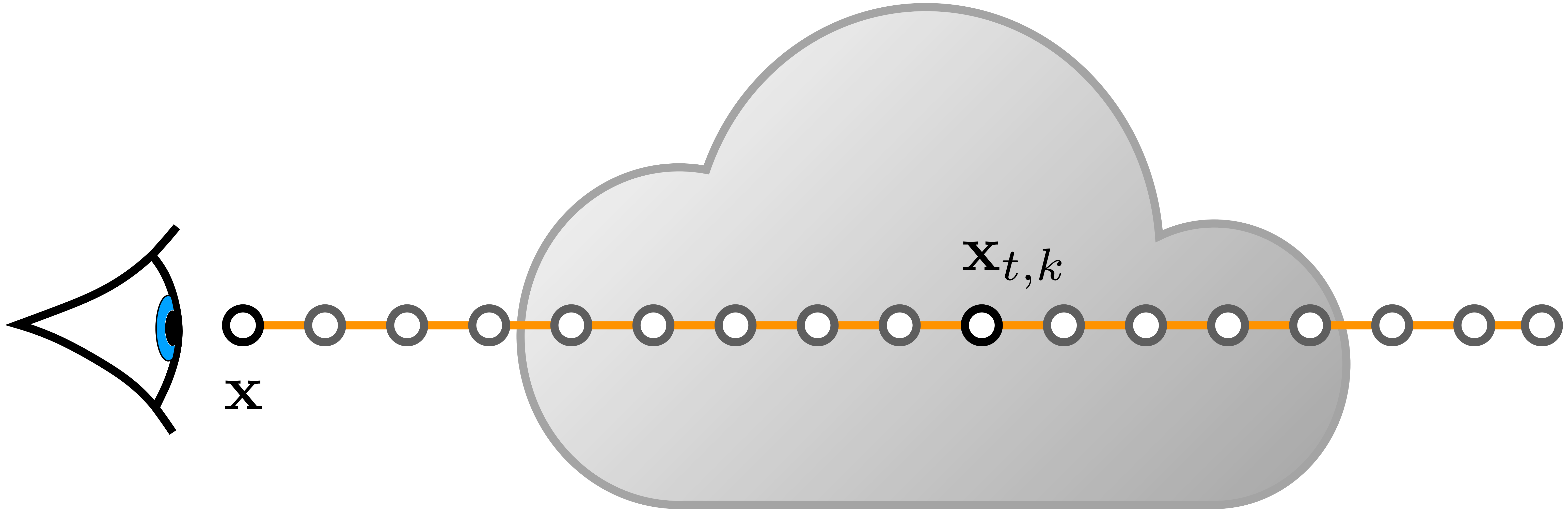


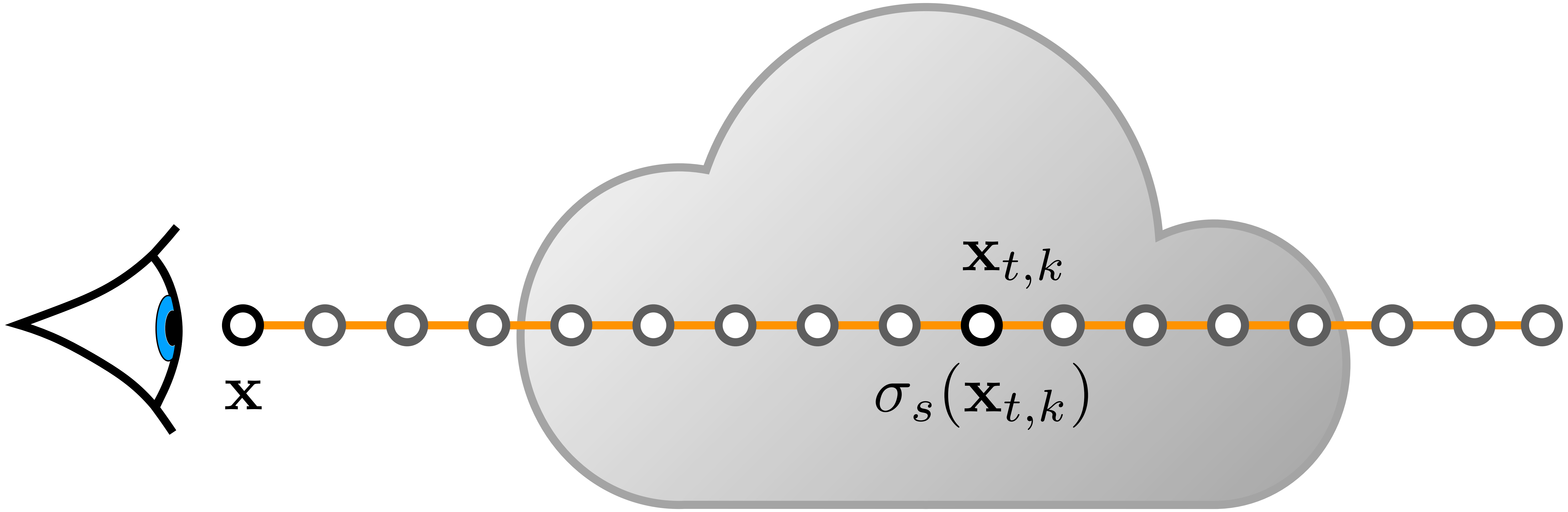
Ray Marching

$$L_{(}\mathbf{x},\vec{\omega}) \approx \sum_{k=0}^{N} T_r(\mathbf{x},\mathbf{x}_{t,k})\sigma_s(\mathbf{x}_{t,k})L_s(\mathbf{x}_{t,k},\vec{\omega})\Delta t$$



Ray Marching

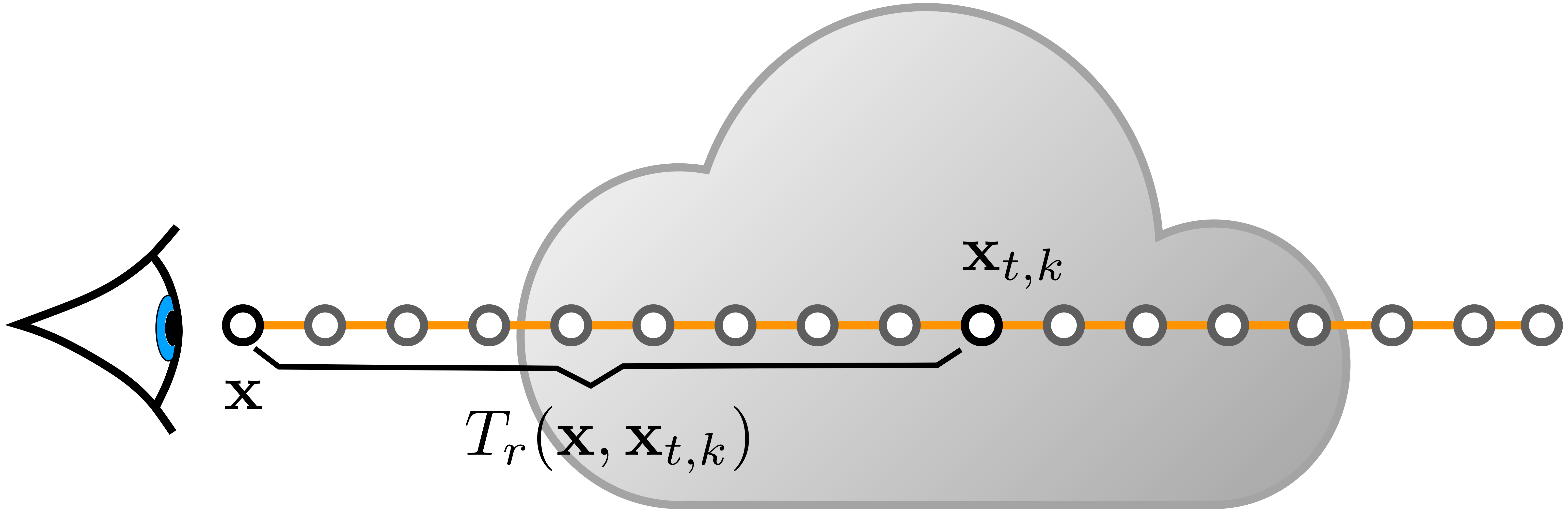
$$L_(\mathbf{x}, \vec{\omega}) \approx \sum_{k=0}^{N} T_r(\mathbf{x}, \mathbf{x}_{t,k}) \sigma_s(\mathbf{x}_{t,k}) L_s(\mathbf{x}_{t,k}, \vec{\omega}) \Delta t$$



Ray Marching

$$L_(\mathbf{x},\vec{\omega}) \approx \sum_{k=0}^{N} T_r(\mathbf{x},\mathbf{x}_{t,k})\sigma_s(\mathbf{x}_{t,k})L_s(\mathbf{x}_{t,k},\vec{\omega})\Delta t$$

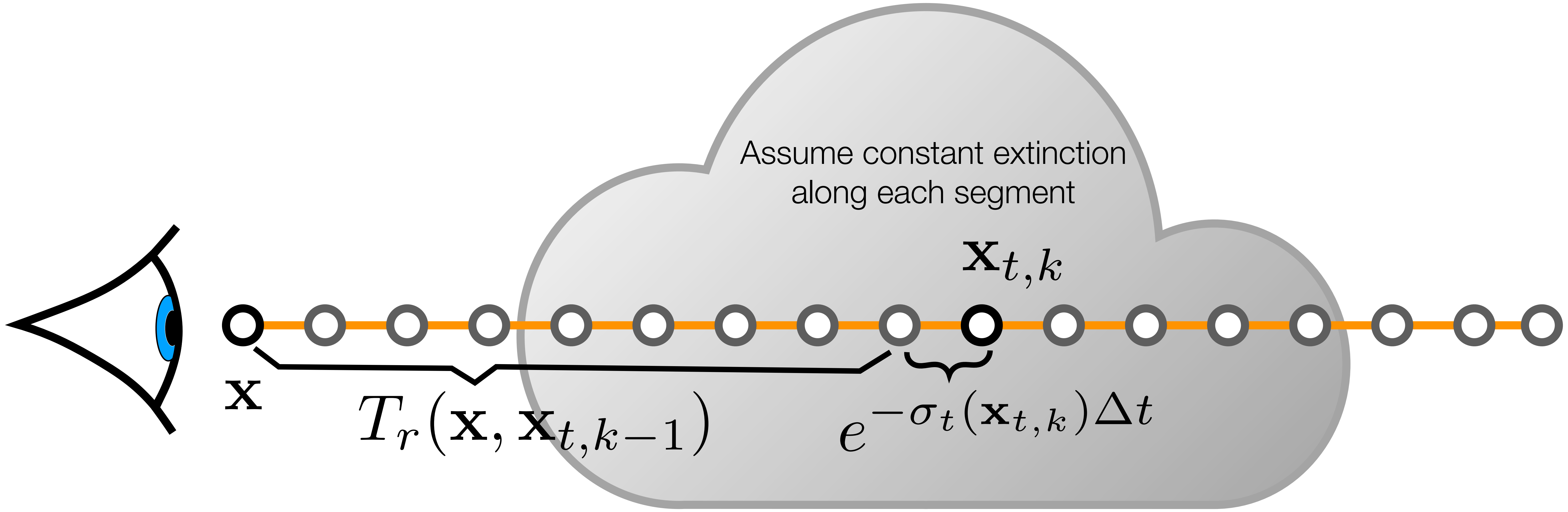
Homogeneous volume: $T_r(\mathbf{x},\mathbf{x}_{t,k}) = e^{-\sigma_t||\mathbf{x},\mathbf{x}_{t,k}||}$



Ray Marching

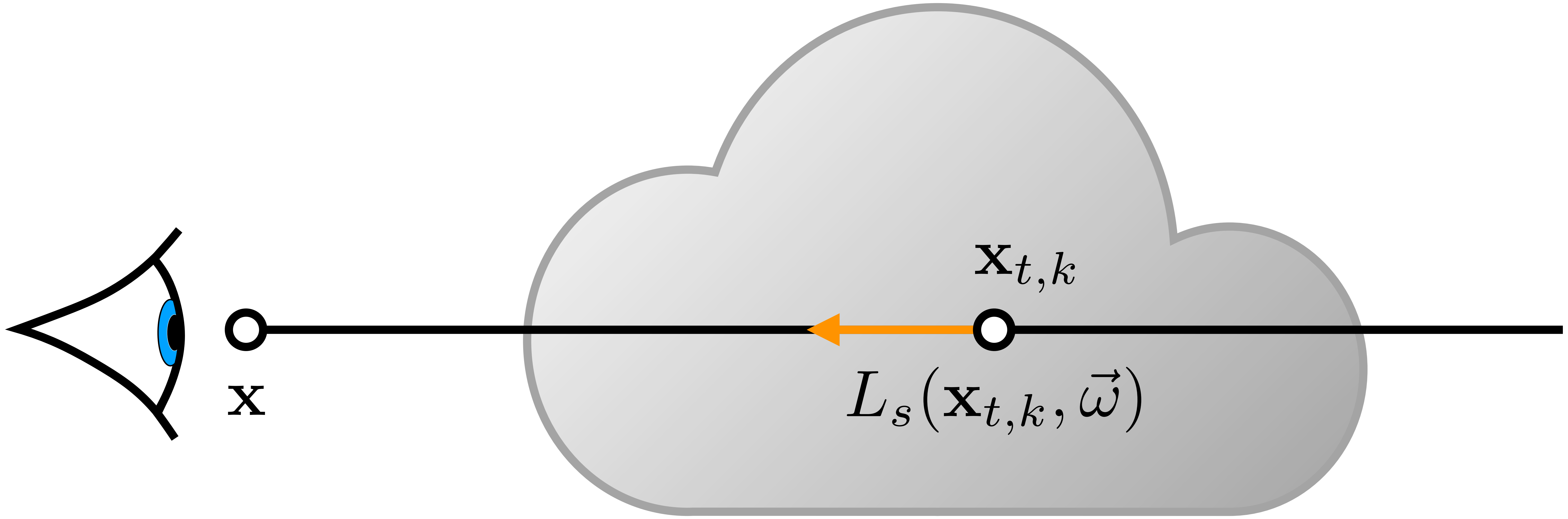
$$L_(\mathbf{x},\vec{\omega}) \approx \sum_{k=0}^{N} T_r(\mathbf{x},\mathbf{x}_{t,k})\sigma_s(\mathbf{x}_{t,k})L_s(\mathbf{x}_{t,k},\vec{\omega})\Delta t$$

Heterogeneous volume: $T_r(\mathbf{x},\mathbf{x}_{t,k}) = T_r(\mathbf{x},\mathbf{x}_{t,k-1})e^{-\sigma_t(\mathbf{x}_{t,k})\Delta t}$


Assume constant extinction along each segment

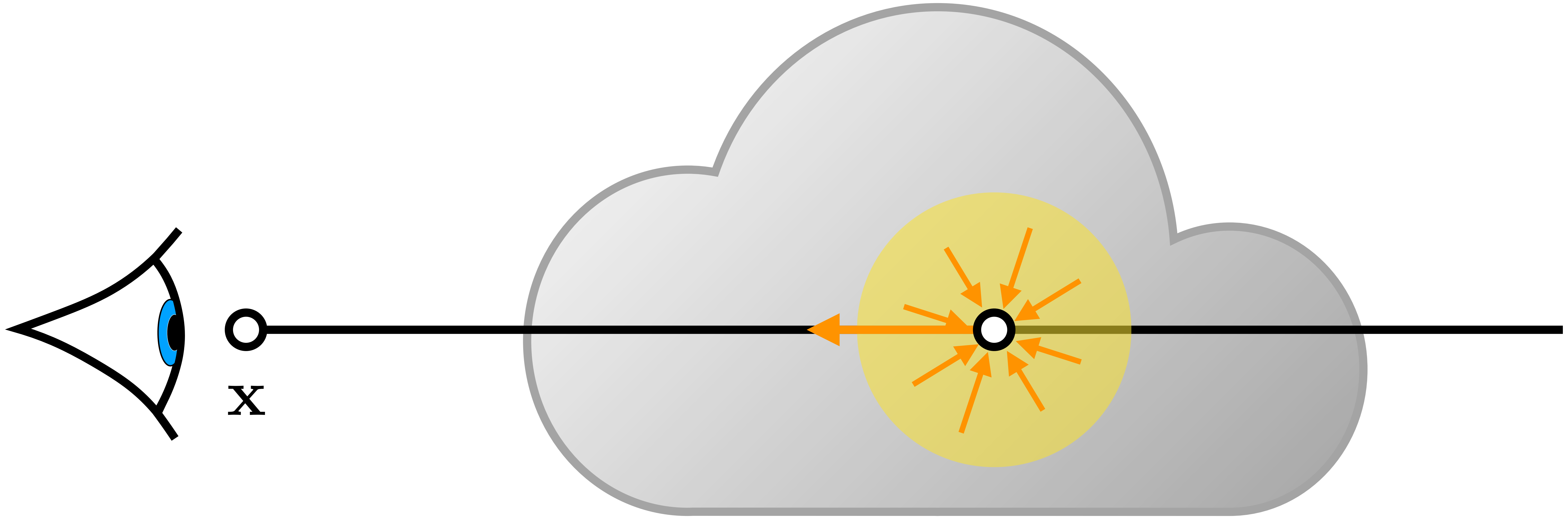
Ray Marching

$$L_{(}\mathbf{x}, \vec{\omega}) \approx \sum_{k=0}^{N} T_r(\mathbf{x}, \mathbf{x}_{t,k})\sigma_s(\mathbf{x}_{t,k})L_s(\mathbf{x}_{t,k}, \vec{\omega})\Delta t$$



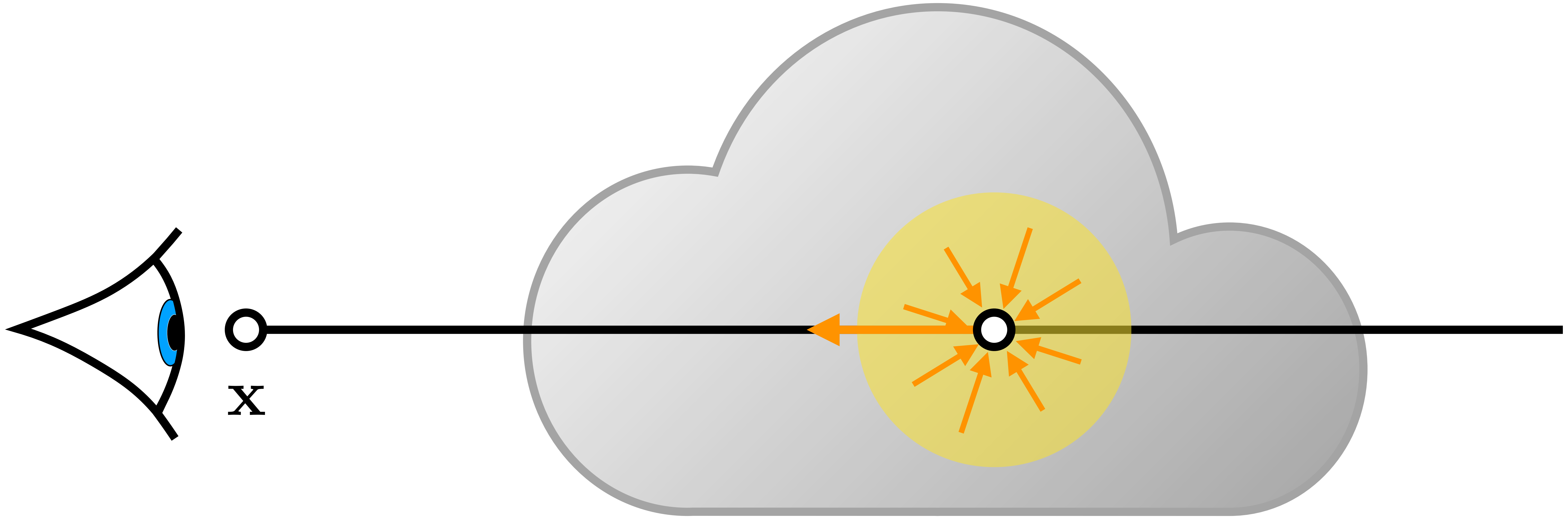
Ray Marching

$$L_s(\mathbf{x}_t, \vec{\omega}) = \int_{S^2} f_p(\mathbf{x}_t, \vec{\omega}, \vec{\omega}')L_i(\mathbf{x}_t, \vec{\omega}')d\vec{\omega}'$$



Ray Marching

$$L_s(\mathbf{x}_t, \vec{\omega}) \approx \frac{1}{M} \sum_{j=1}^{M} \frac{f_p(\mathbf{x}_t, \vec{\omega}, \vec{\omega}'_j) L_i(\mathbf{x}_t, \vec{\omega}'_j)}{p(\vec{\omega}'_j)}$$



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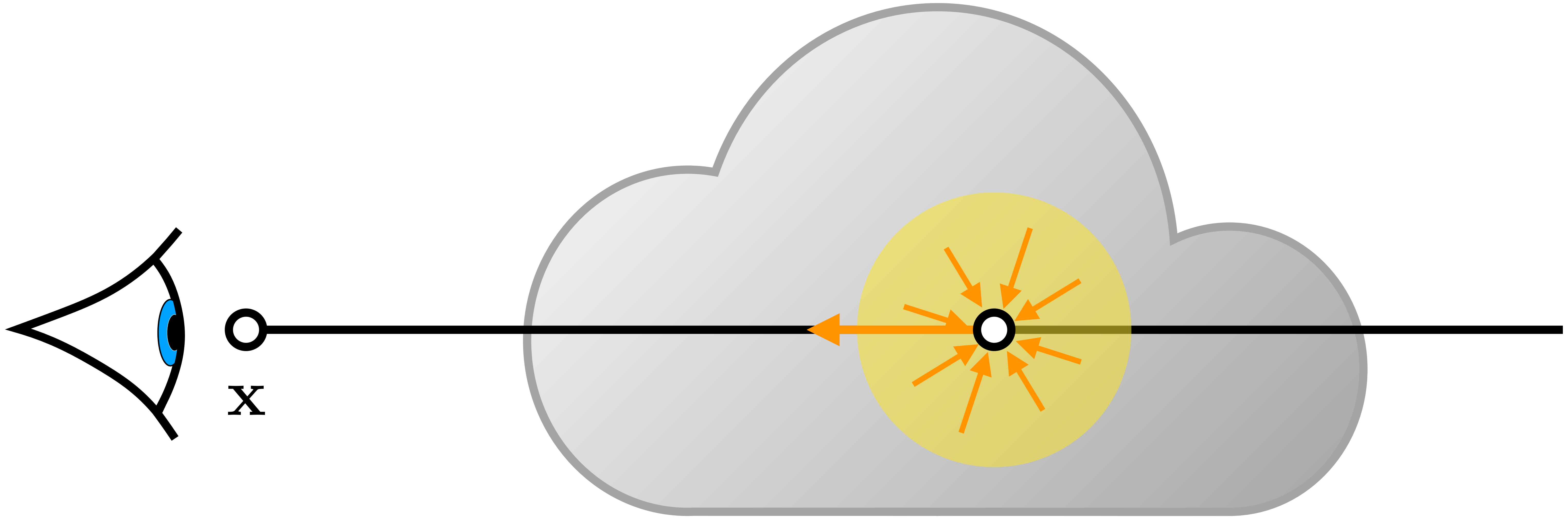

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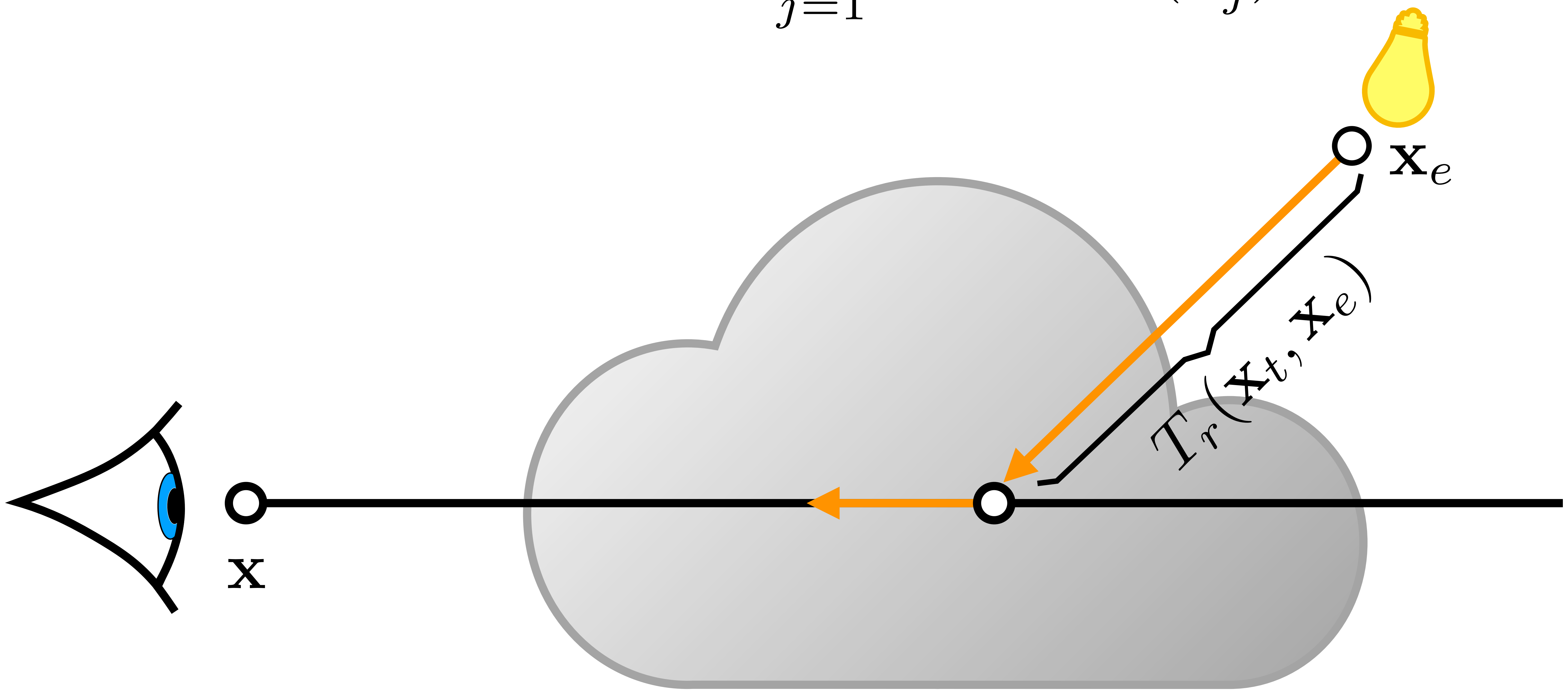

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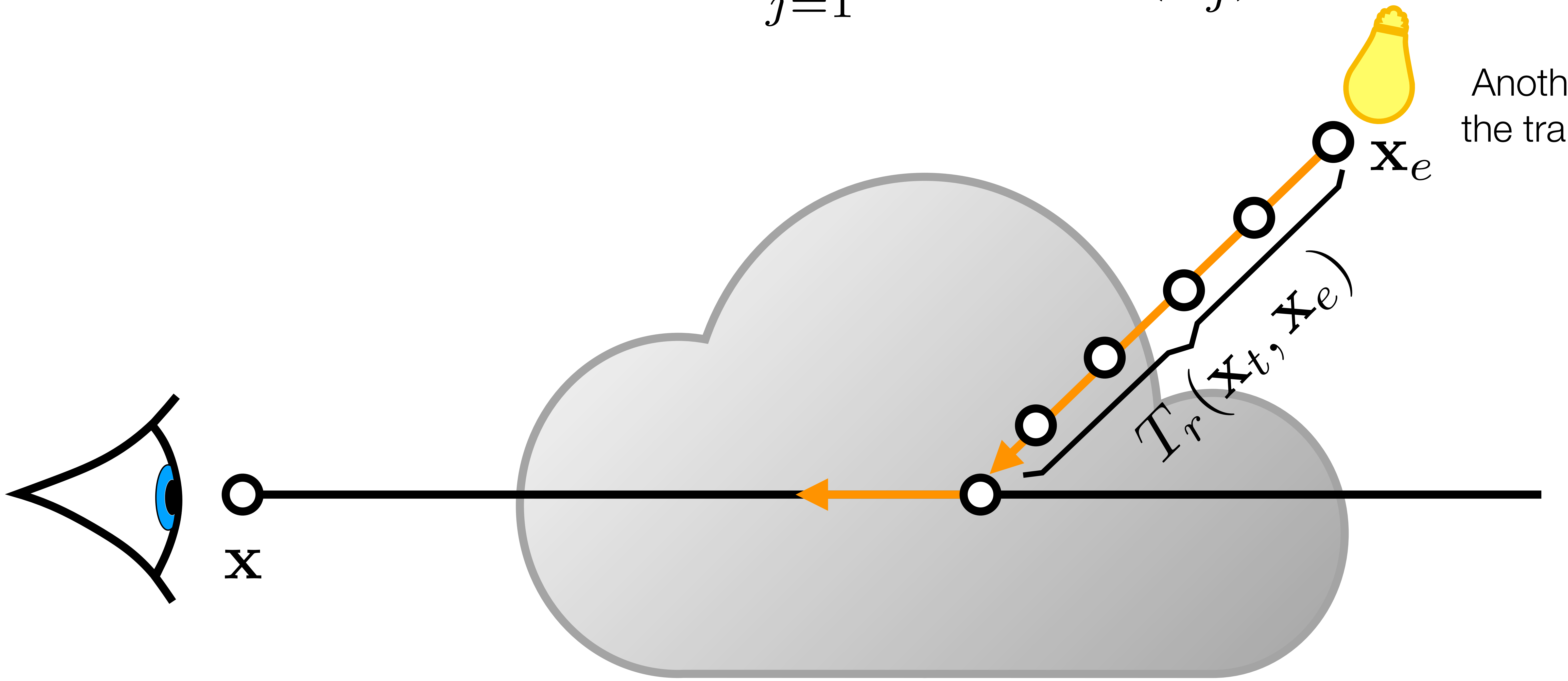
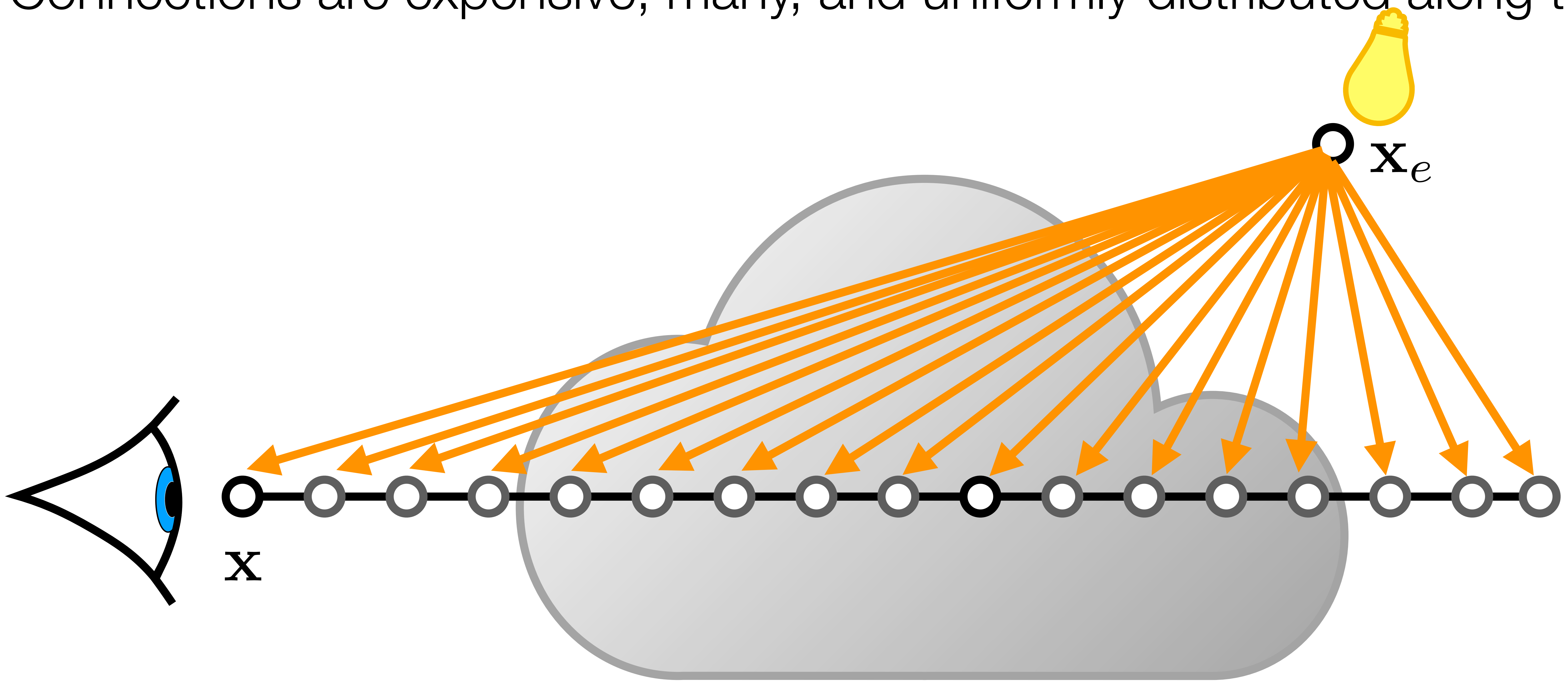


Another ray marching needed to estimate the transmittance along the connection ray (in the heterogeneous media)

Ray Marching in Heterogeneous Media

Marching towards the light source

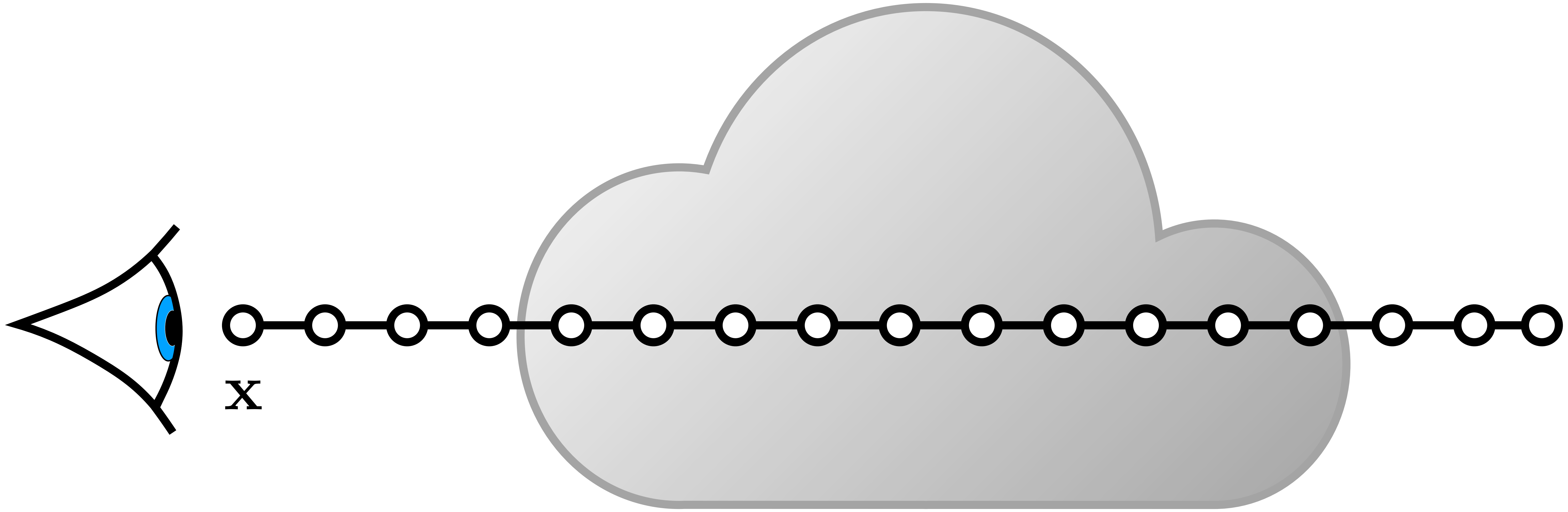
- Connections are expensive, many, and uniformly distributed along the primary ray



Decoupled Transmittance and in-scattering

1. Ray march and cache transmittance

   - Choose step-size w.r.t. frequency content to accurately capture variations



Decoupled Transmittance and in-scattering

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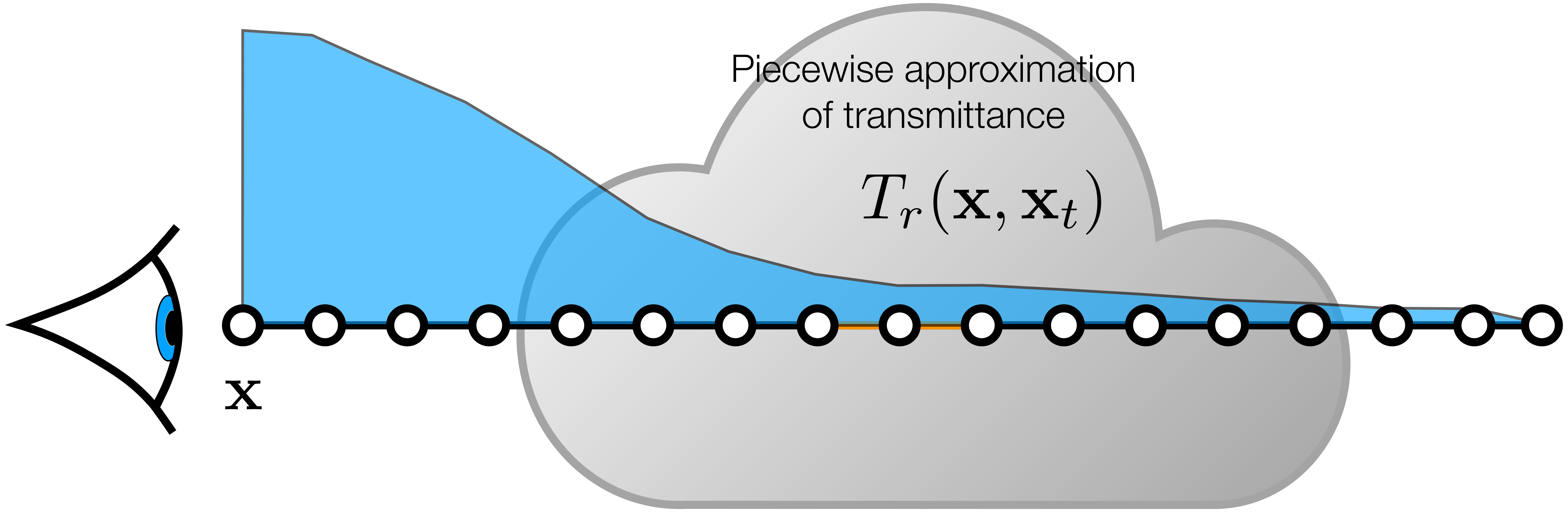
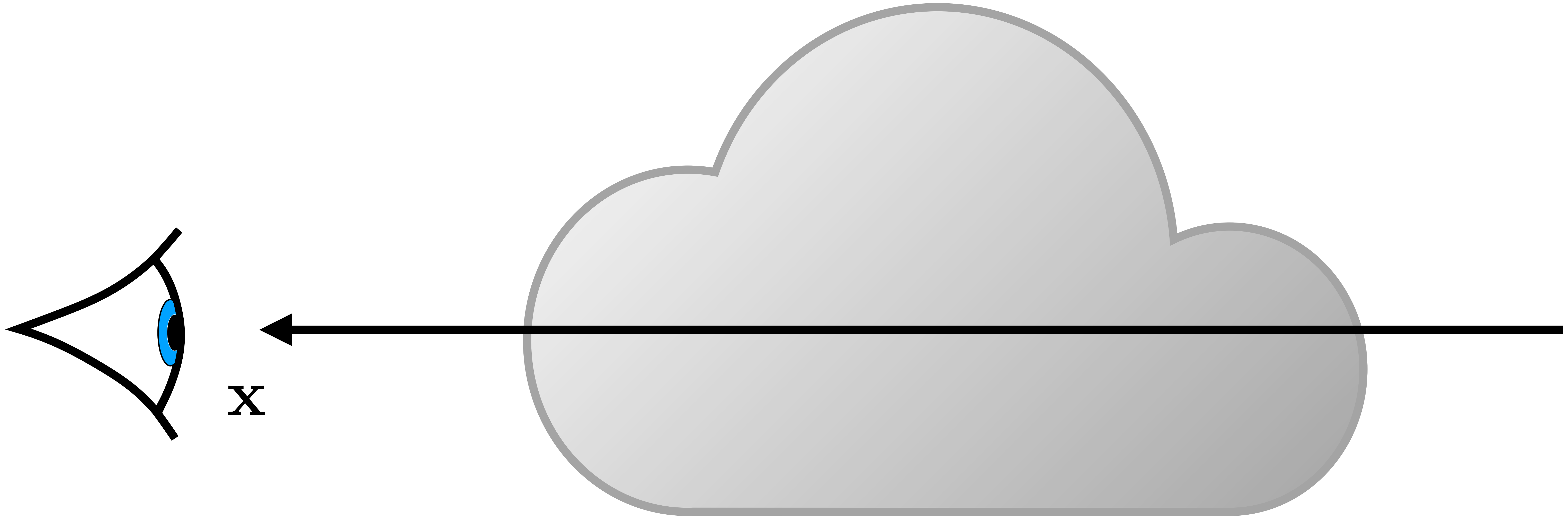


Decoupled Transmittance and in-scattering

2. Estimate in-scattering using MC integration

    - Distribute samples proportional to (part of) the integrand



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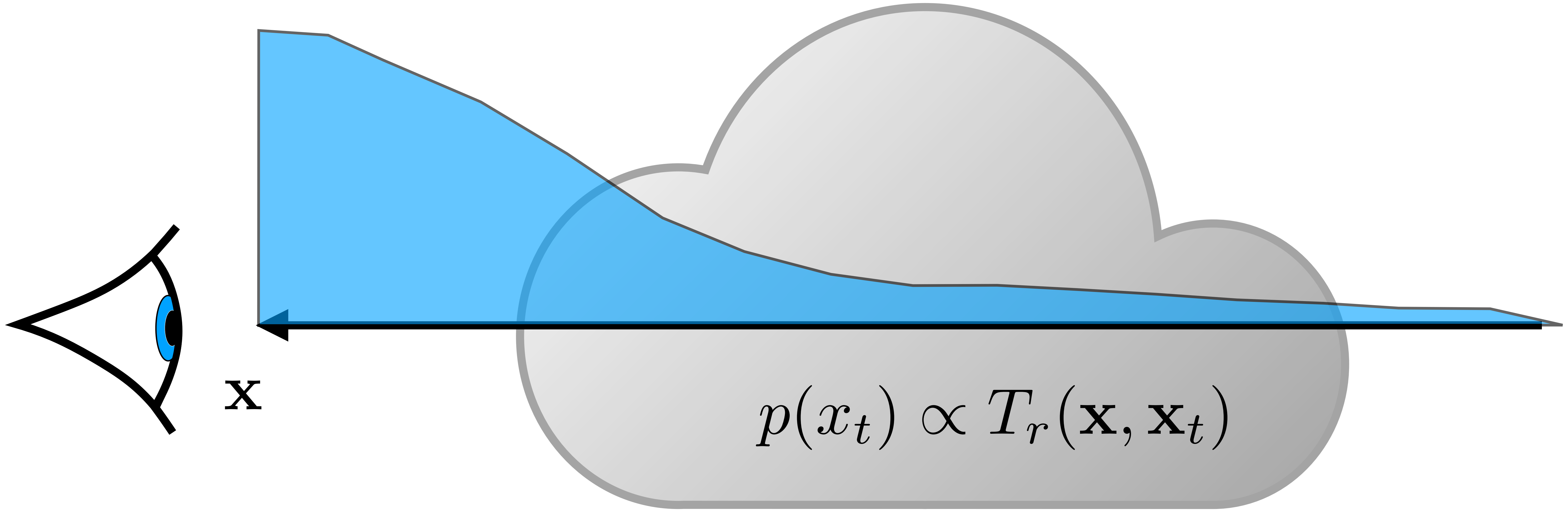

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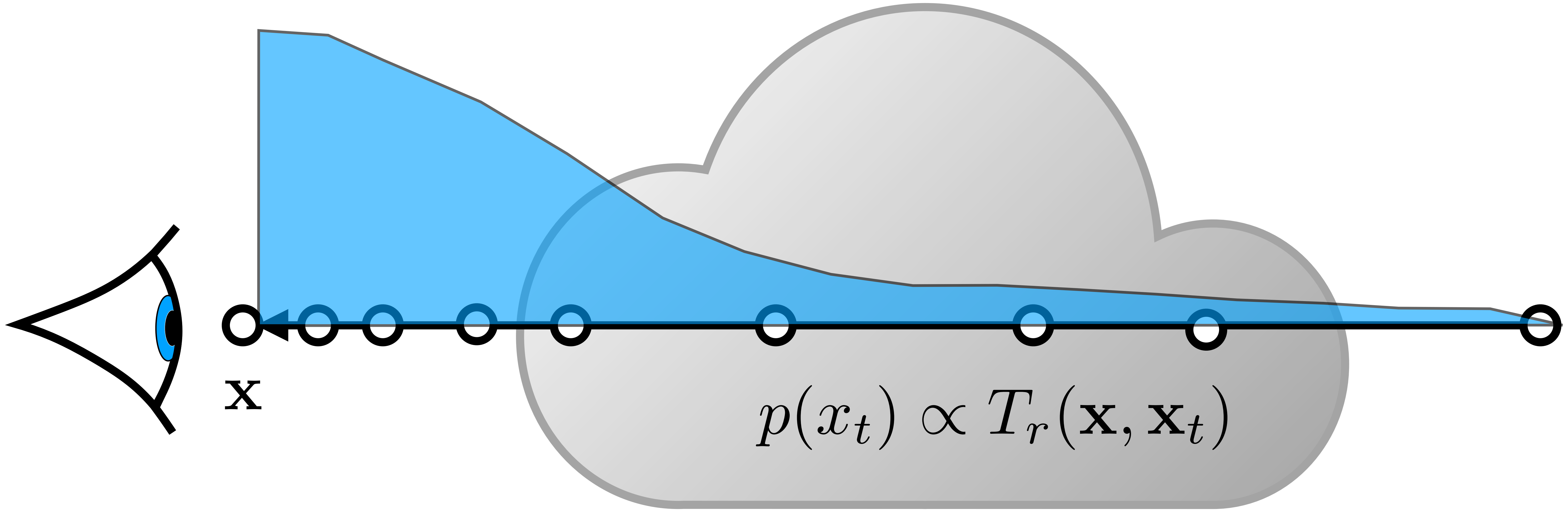

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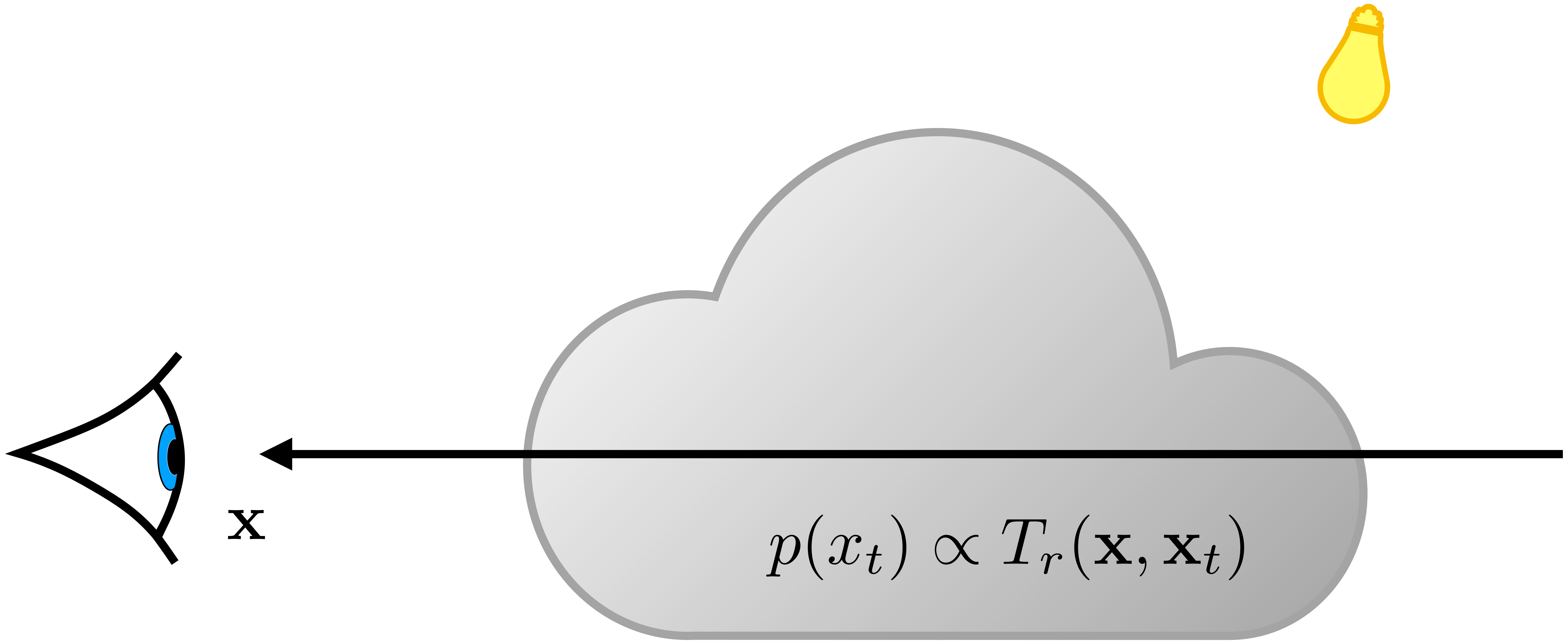

$p(x_t) \propto T_r(\mathbf{x}, \mathbf{x}_t)$

Decoupled Transmittance and in-scattering

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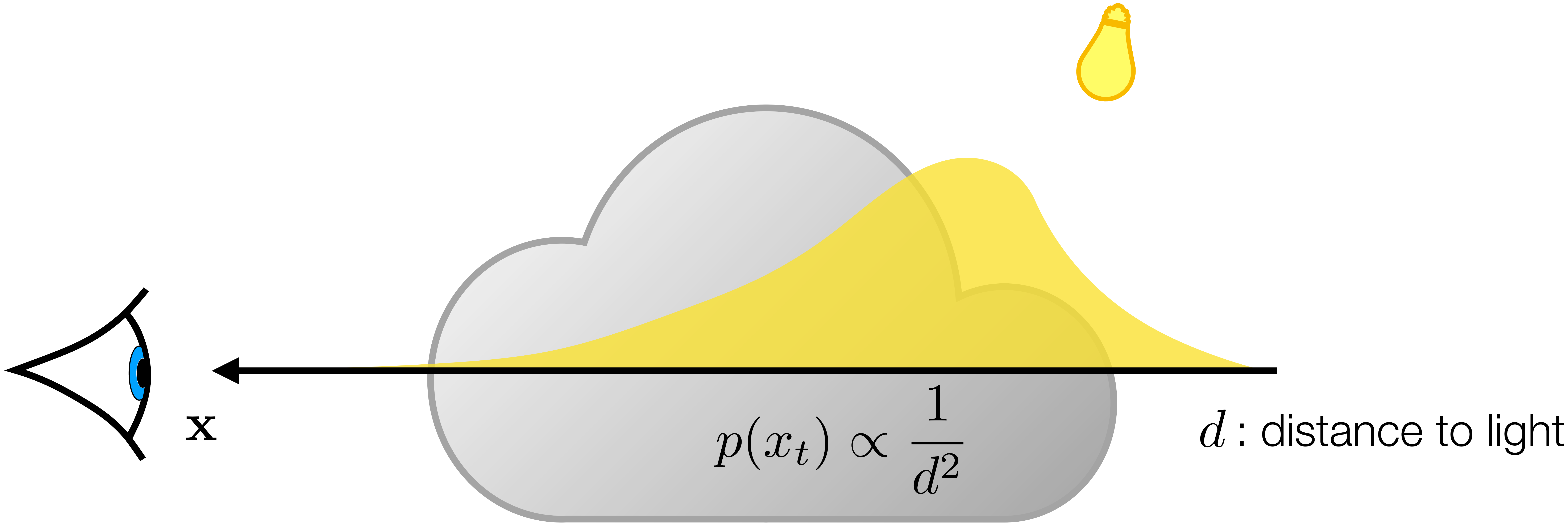
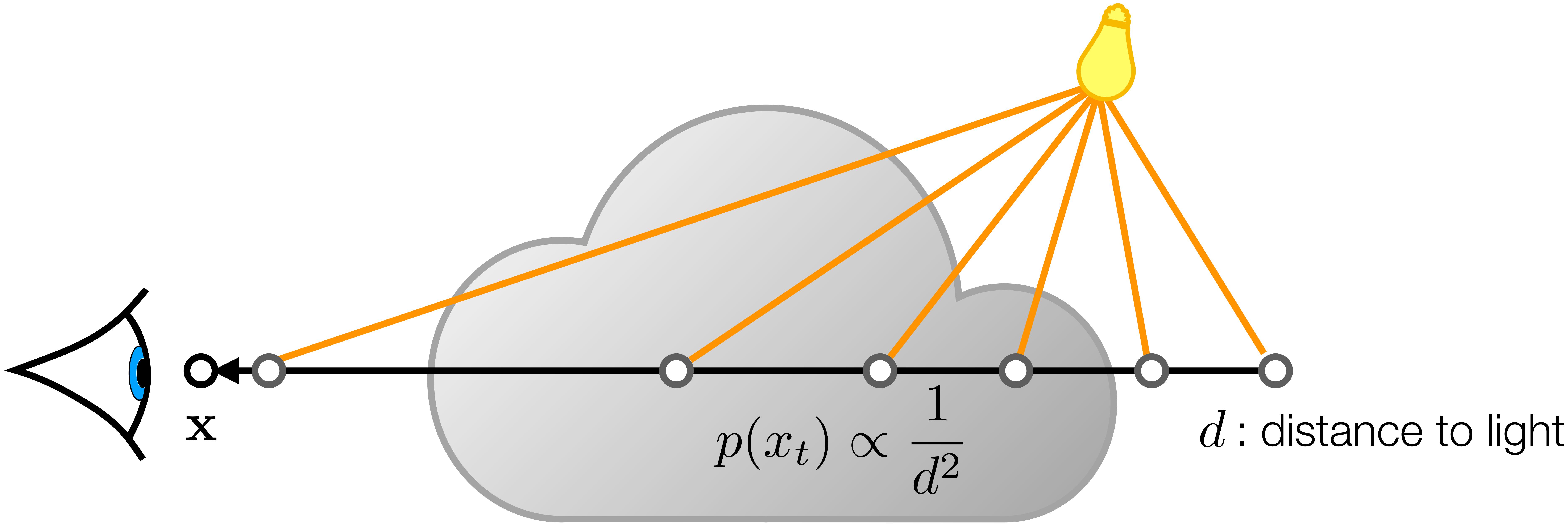


Decoupled Transmittance and in-scattering

2. Estimate in-scattering using MC integration

   - Distribute samples proportional to (part of) the integrand



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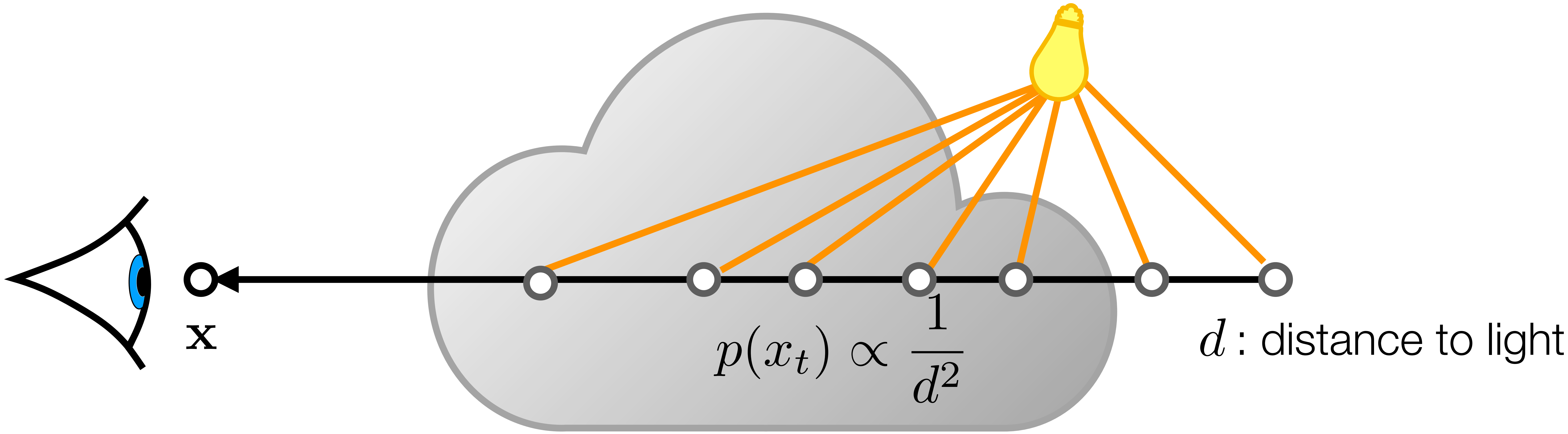

Decoupled Transmittance and in-scattering

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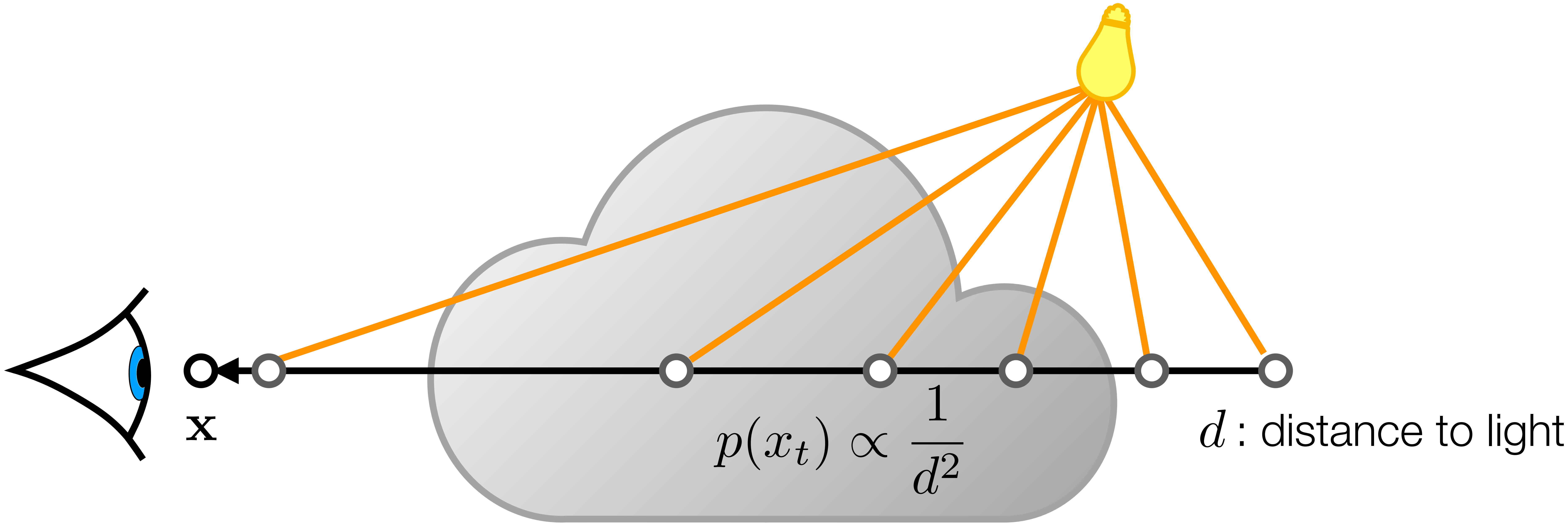

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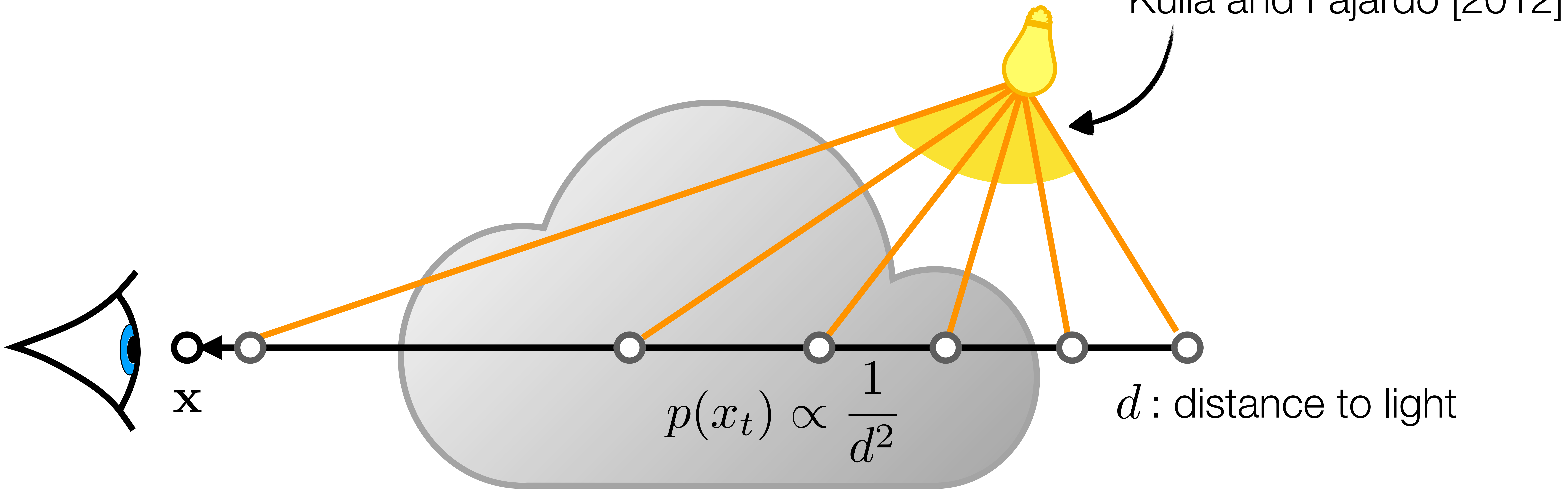
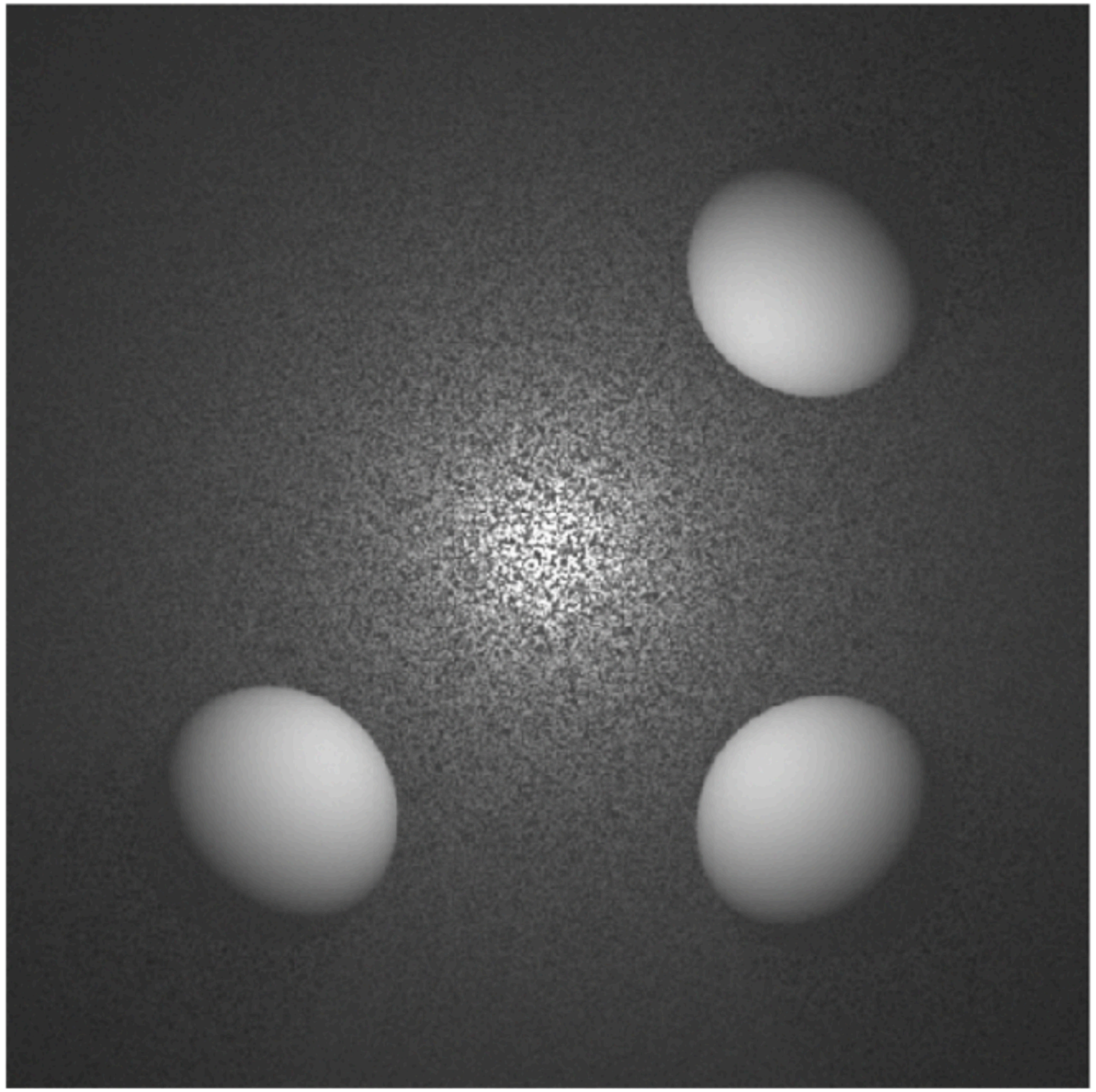
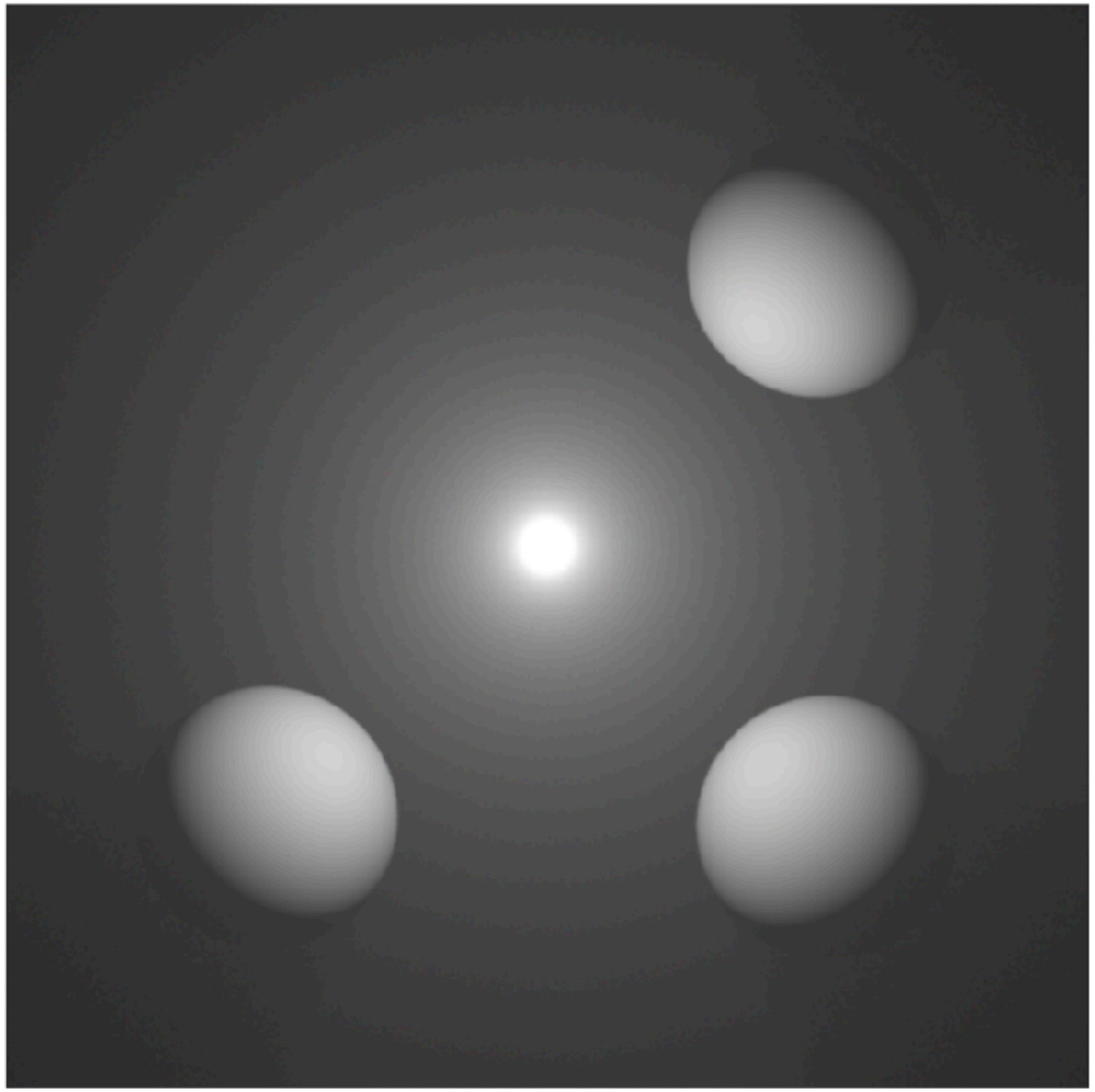


Decoupled Transmittance and in-scattering

Ray Marching



Equi-angular sampling



Single scattering

Multiple scattering

Volumetric Path Tracing

Volumetric Path Tracing

Motivation

Same as with path tracing: avoid the exponential growth

Paths can:

Reflect / Refract off surfaces

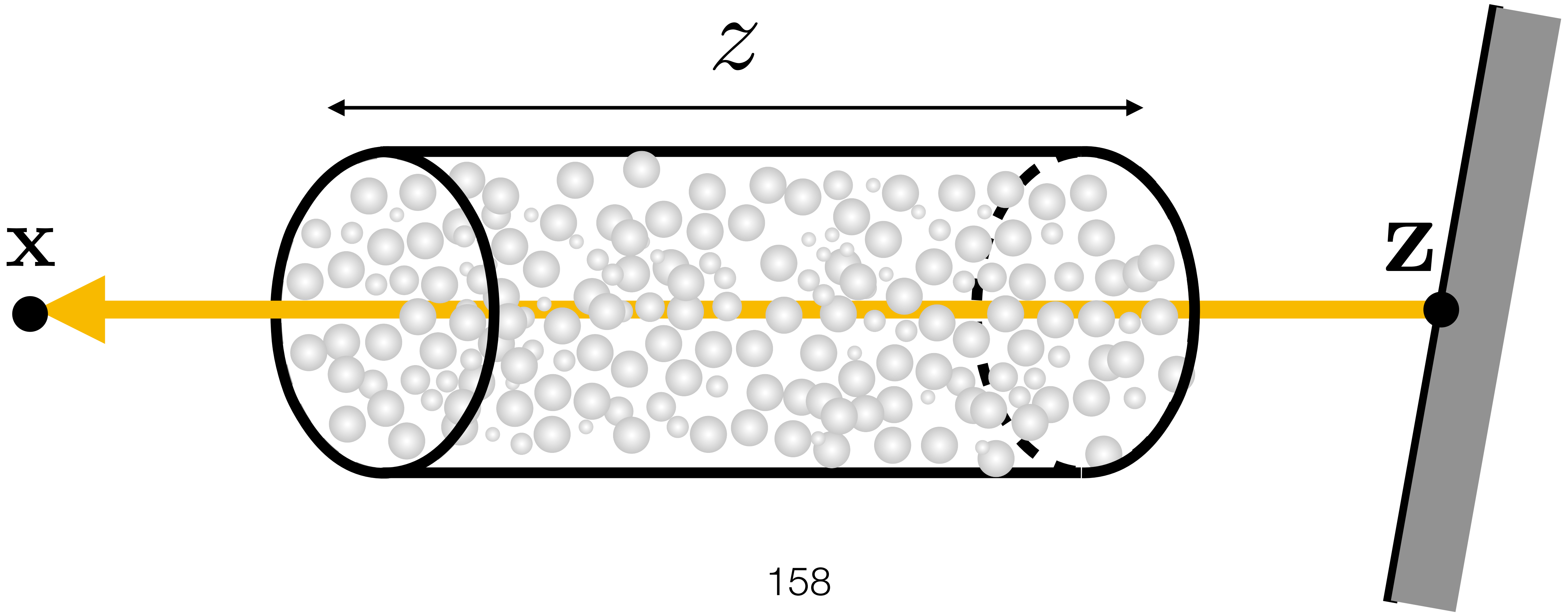
Scatter inside a volume

Volumetric Rendering Equation

$$L(\mathbf{x},\vec{\omega}) = \int_0^z T_r(\mathbf{x},\mathbf{x}_t)\sigma_a(\mathbf{x}_t)L_e(\mathbf{x}_t,\vec{\omega})dt$$

$$+ \int_0^z T_r(\mathbf{x},\mathbf{x}_t)\sigma_s(\mathbf{x}_t)L_s(\mathbf{x}_t,\omega)dt$$

$$+ \; T_r(\mathbf{x},\mathbf{x}_z)L(\mathbf{x}_z,\vec{\omega})$$



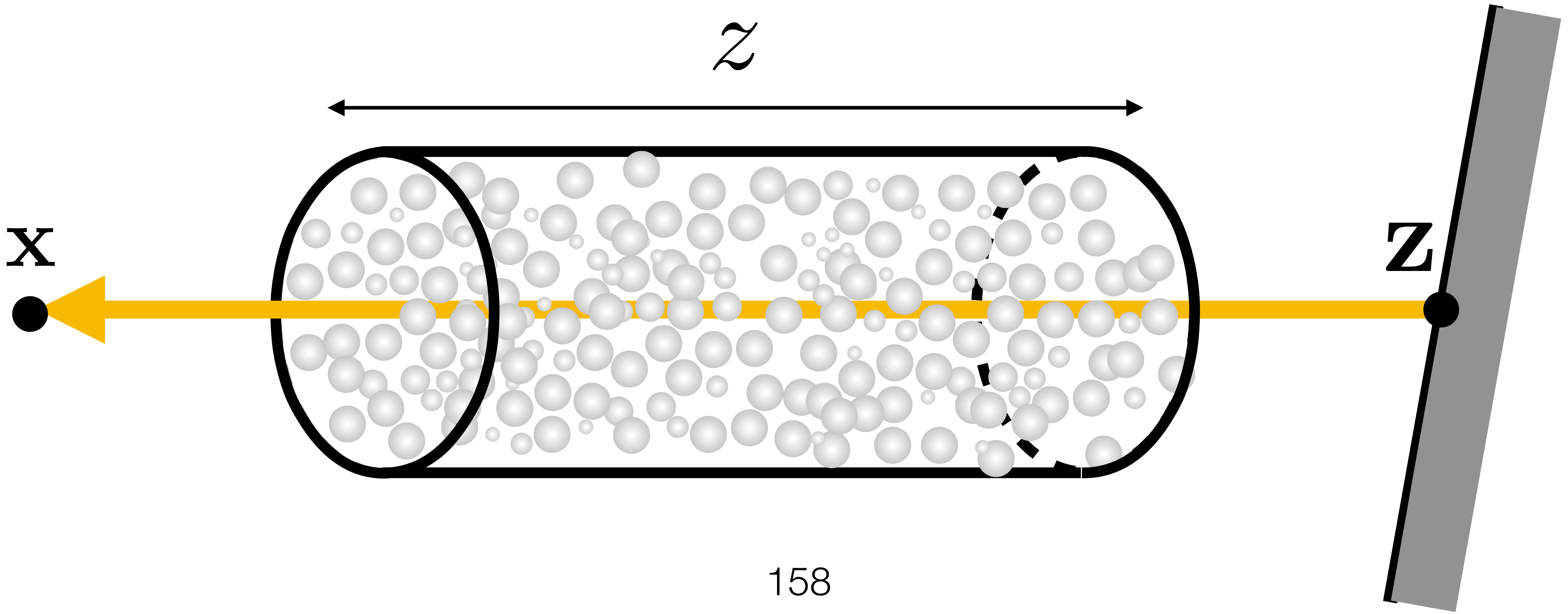
Volumetric Rendering Equation

$$L(\mathbf{x},\vec{\omega}) = \int_0^z T_r(\mathbf{x},\mathbf{x}_t)\sigma_a(\mathbf{x}_t)L_e(\mathbf{x}_t,\vec{\omega})dt$$

Accumulated emitted radiance

$$+\int_0^z T_r(\mathbf{x},\mathbf{x}_t)\sigma_s(\mathbf{x}_t)L_s(\mathbf{x}_t,\omega)dt$$

$$+\ T_r(\mathbf{x},\mathbf{x}_z)L(\mathbf{x}_z,\vec{\omega})$$



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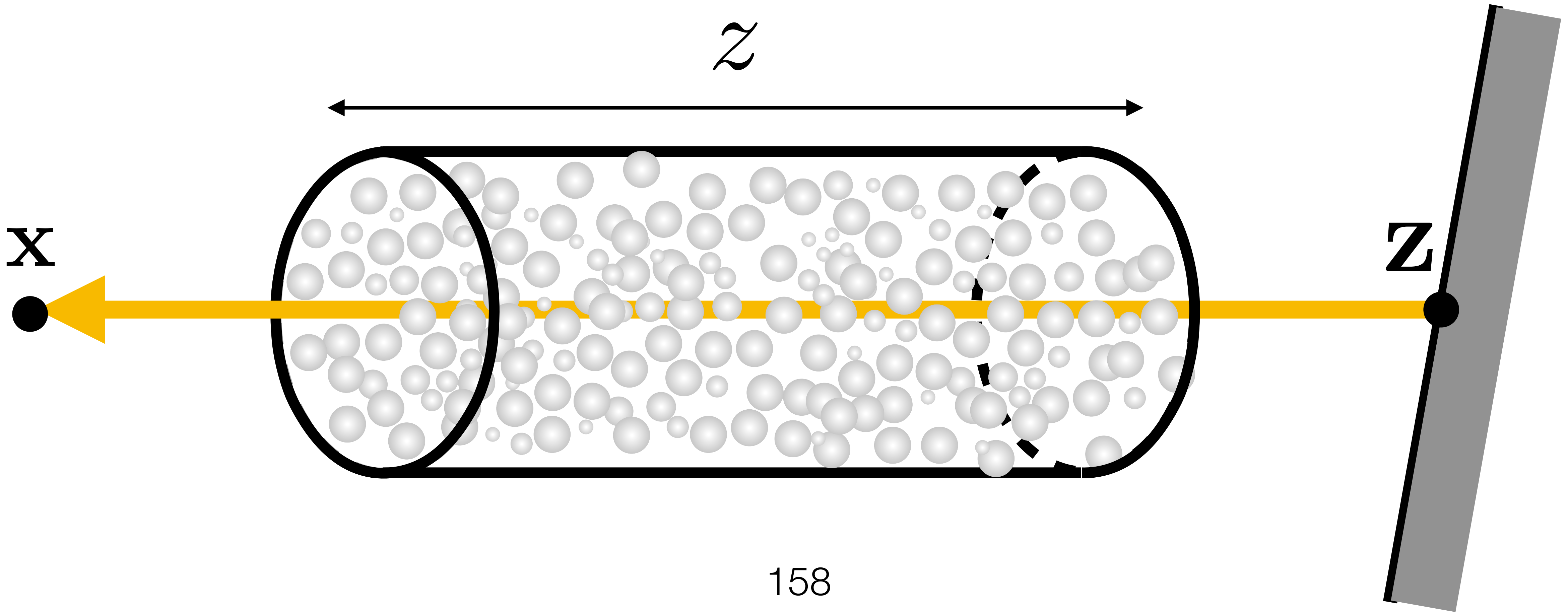
$$+\boxed{\int_0^z T_r(\mathbf{x},\mathbf{x}_t)\sigma_s(\mathbf{x}_t)L_s(\mathbf{x}_t,\omega)dt}$$

Accumulated in-scattered radiance

$$+\; T_r(\mathbf{x},\mathbf{x}_z)L(\mathbf{x}_z,\vec{\omega})$$



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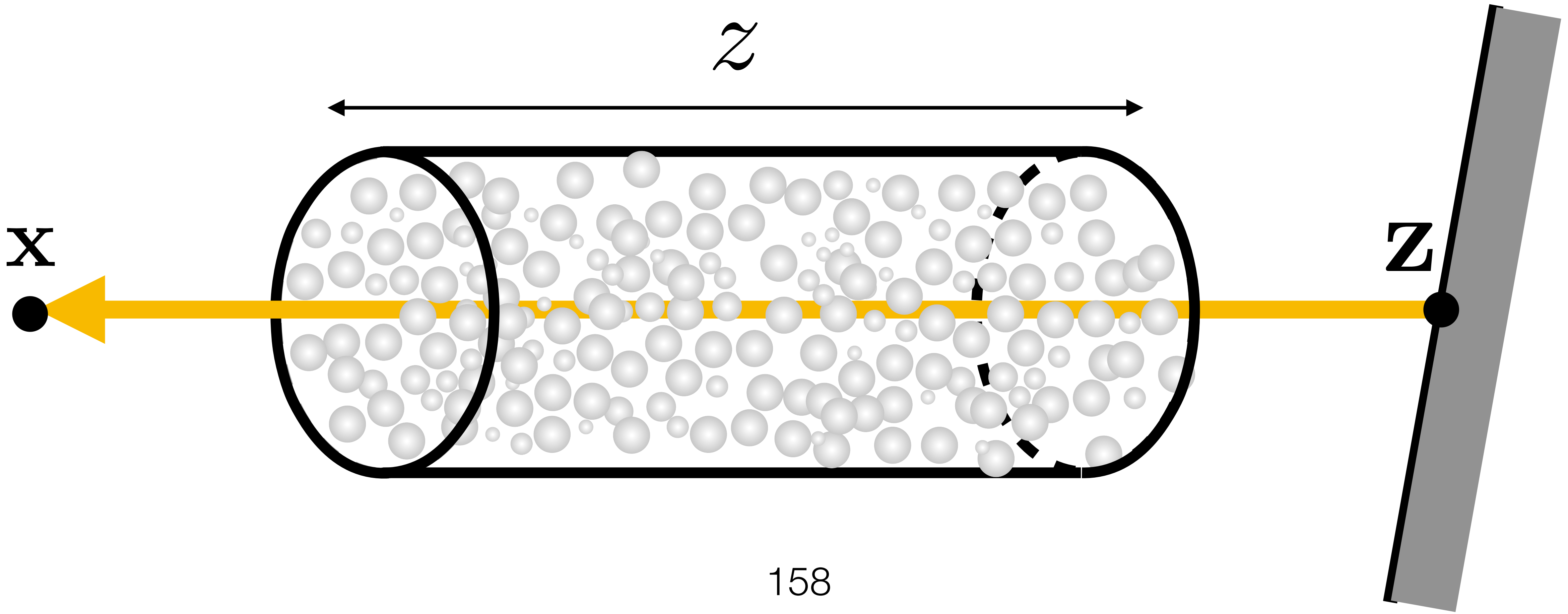
Accumulated in-scattered radiance

$$+ T_r(\mathbf{x}, \mathbf{x}_z)L(\mathbf{x}_z, \vec{\omega})$$

Attenuated background radiance



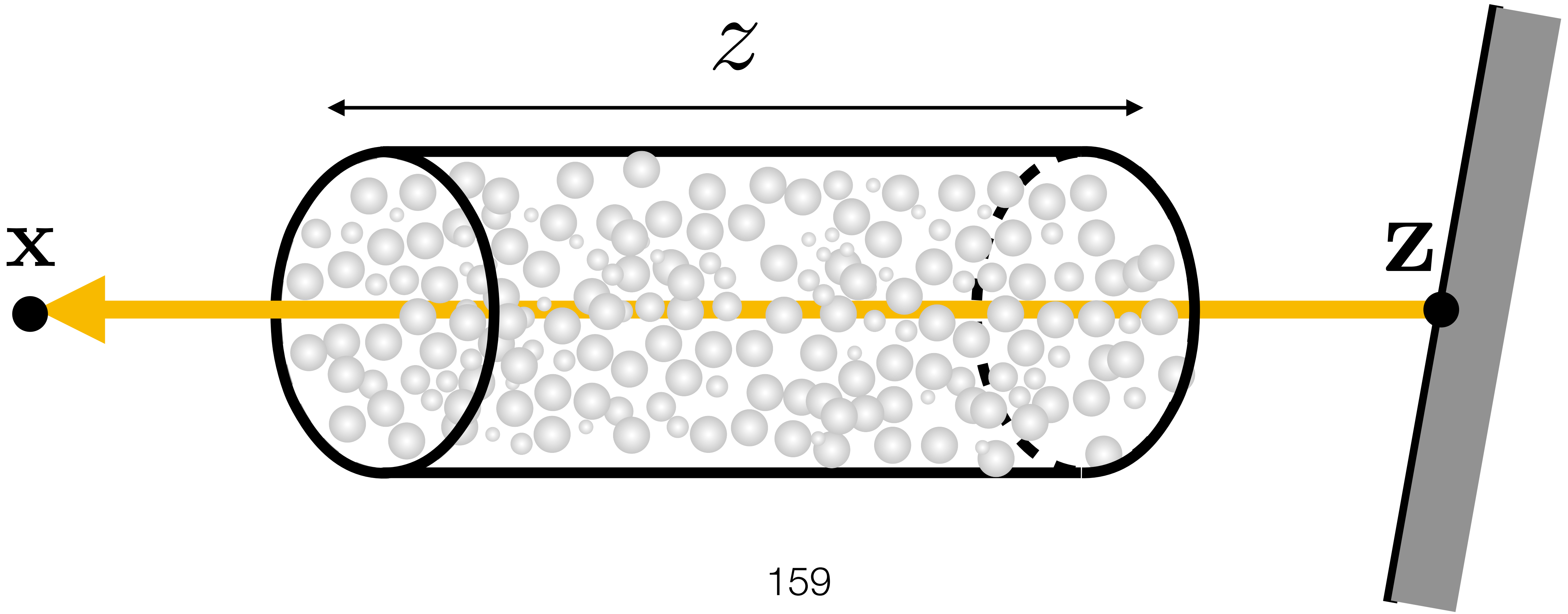
Volumetric Rendering Equation

Accumulated emitted + in-scattered radiance

$$L(\mathbf{x},\vec{\omega}) = \int_0^z T_r(\mathbf{x},\mathbf{x}_t)\Big[\sigma_a(\mathbf{x}_t)L_e(\mathbf{x}_t,\vec{\omega}) + \sigma_s(\mathbf{x}_t)L_s(\mathbf{x}_t,\vec{\omega})\Big]dt$$

$$+T_r(\mathbf{x},\mathbf{x}_z)L(\mathbf{x}_z,\vec{\omega})$$

Attenuated background radiance



Volumetric Rendering Equation

$$L(\mathbf{x}, \vec{\omega}) = \int_0^z T_r(\mathbf{x}, \mathbf{x}_t) \Big[\sigma_a(\mathbf{x}_t) L_e(\mathbf{x}_t, \vec{\omega}) + \sigma_s(\mathbf{x}_t) L_s(\mathbf{x}_t, \vec{\omega}) \Big] dt$$

$$+ T_r(\mathbf{x}, \mathbf{x}_z) L(\mathbf{x}_z, \vec{\omega})$$

1-Sample Monte Carlo Estimator

$$\langle L(\mathbf{x},\vec{\omega})\rangle = \frac{T_r(\mathbf{x},\mathbf{x}_t)}{p(t)}\Big[\sigma_a(\mathbf{x}_t)L_e(\mathbf{x}_t,\vec{\omega}) + \sigma_s(\mathbf{x}_t)L_s(\mathbf{x}_t,\vec{\omega})\Big]$$

$$+\frac{T_r(\mathbf{x},\mathbf{x}_z)}{P(z)}L(\mathbf{x}_z,\vec{\omega})$$

1-Sample Monte Carlo Estimator

$$\langle L(\mathbf{x},\vec{\omega})\rangle = \frac{T_r(\mathbf{x},\mathbf{x}_t)}{p(t)}\Big[\sigma_a(\mathbf{x}_t)L_e(\mathbf{x}_t,\vec{\omega}) + \sigma_s(\mathbf{x}_t)L_s(\mathbf{x}_t,\vec{\omega})\Big]$$

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$p(t)$	Probability density of distance t
$P(z)$	Probability of exceeding distance z

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$$+\frac{T_r(\mathbf{x},\mathbf{x}_z)}{P(z)}L(\mathbf{x}_z,\vec{\omega})$$

$p(t)$	Probability density of distance t
$P(z)$	Probability of exceeding distance z
$p(\vec{\omega}_i)$	Probability density of direction $\vec{\omega}_i$

Volumetric Path Tracing

1. Sample distance to next interaction

2. Scatter in the volume or bounce off a surface

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Volumetric Path Tracing with NEE

Sampling the Phase Function

Isotropic:

Henyey-Greenstein:

Sampling the Phase Function

Isotropic: Uniform sphere sampling

Henyey-Greenstein:

Sampling the Phase Function

Isotropic: Uniform sphere sampling

Henyey-Greenstein: Using the inversion method we can derive

$$\cos\theta = \frac{1}{2g}\left(1 + g^2 - \left(\frac{1-g^2}{1-g+2g\xi_1}\right)^2\right)$$

$$\phi = 2\pi\xi_2$$

PDF is the value of the HG phase function

Free-path Sampling

Free-path or free-flight distance:

- Distance to the next interaction in the medium
- Dense media (e.g. milk): short mean-free path
- Thin media (e.g. atmosphere): long mean-free path

Free-path Sampling

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Ideally, we want to sample according to (part of) of the integrand:

$$p(\mathbf{x}_t|(\mathbf{x},\vec{\omega})) \propto T_r(\mathbf{x},\mathbf{x}_t)$$

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Ideally, we want to sample according to (part of) of the integrand:

$$p(\mathbf{x}_t|(\mathbf{x},\vec{\omega})) \propto T_r(\mathbf{x},\mathbf{x}_t)$$

simplified notation

$$p(t) \propto T_r(t)$$

Free-path Sampling

Homogeneous media: $T_r(t) = e^{-\sigma_t t}$

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$$p(t) = \frac{e^{-\sigma_t t}}{\int_0^\infty e^{-\sigma_t s} ds} = \sigma_t e^{-\sigma_t t}$$

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CDF: $P(t) = \int_0^t e^{-\sigma_t s} ds = 1 - e^{-\sigma_t t}$

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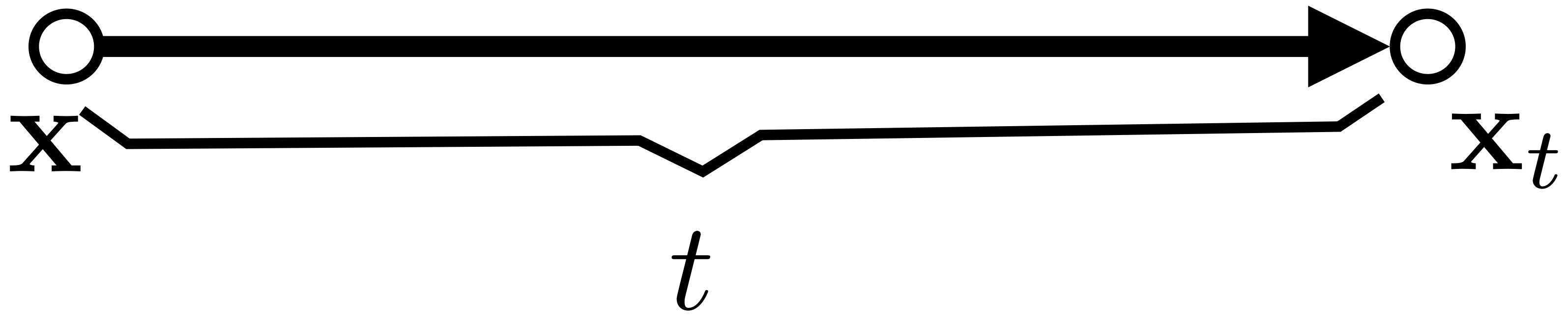
CDF: $$P(t) = \int_0^t e^{-\sigma_t s} ds = 1 - e^{-\sigma_t t}$$

Inverted CDF: $$P^{-1}(\xi) = -\frac{log_e(1-\xi)}{\sigma_t}$$

Free-path Sampling

Homogeneous media: $T_r(t) = e^{-\sigma_t t}$

Recipe:

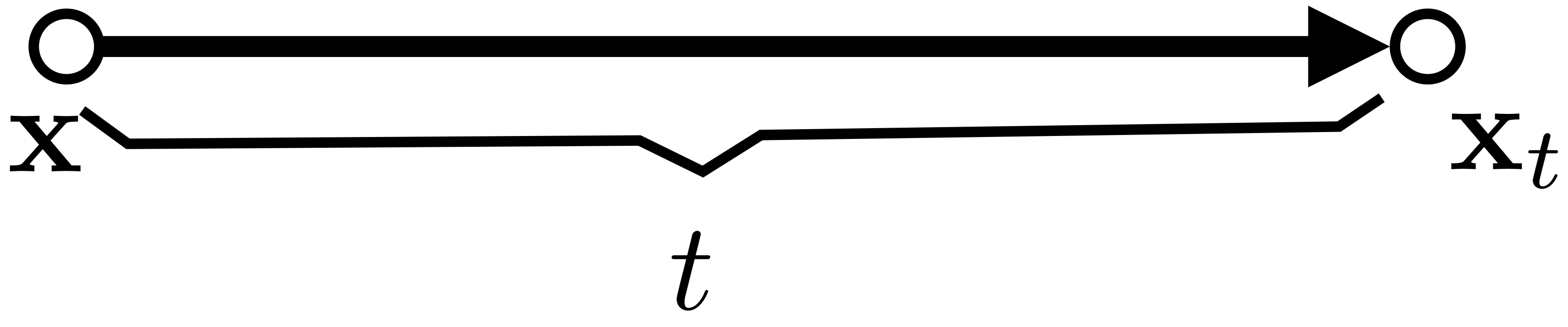


Free-path Sampling

Homogeneous media: $T_r(t) = e^{-\sigma_t t}$

Recipe:

Generate a random number ξ



Free-path Sampling

Homogeneous media: $T_r(t) = e^{-\sigma_t t}$

Recipe:

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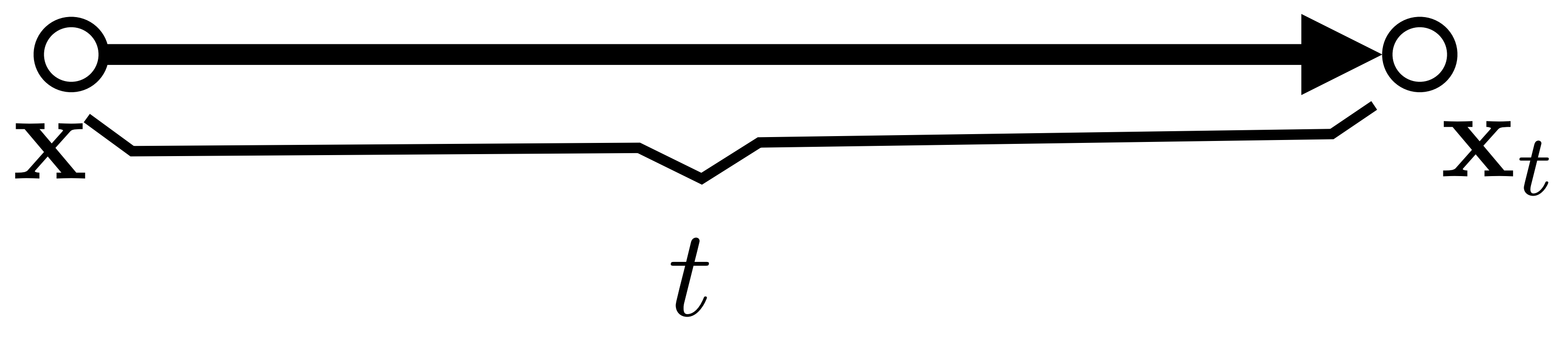
Sample distance



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Recipe:

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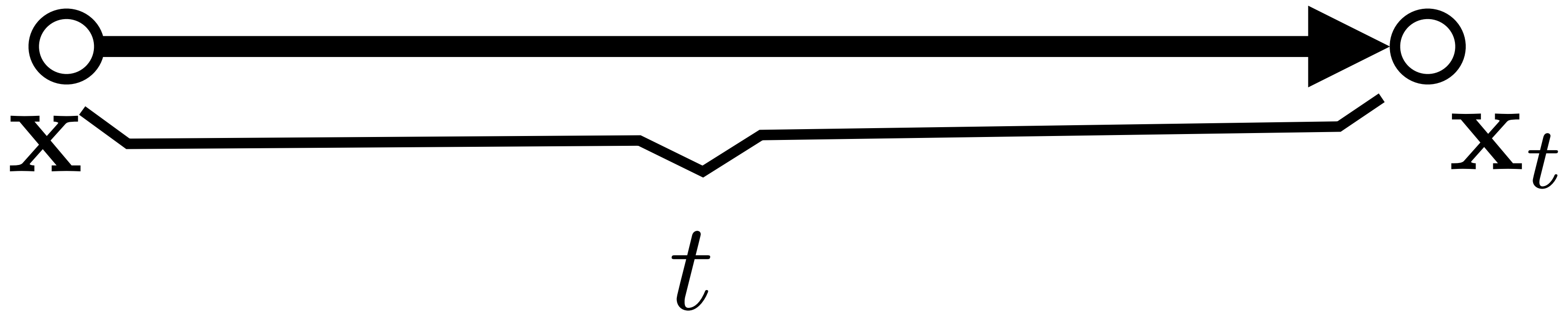
Sample distance $t = -\frac{log_e(1-\xi)}{\sigma_t}$



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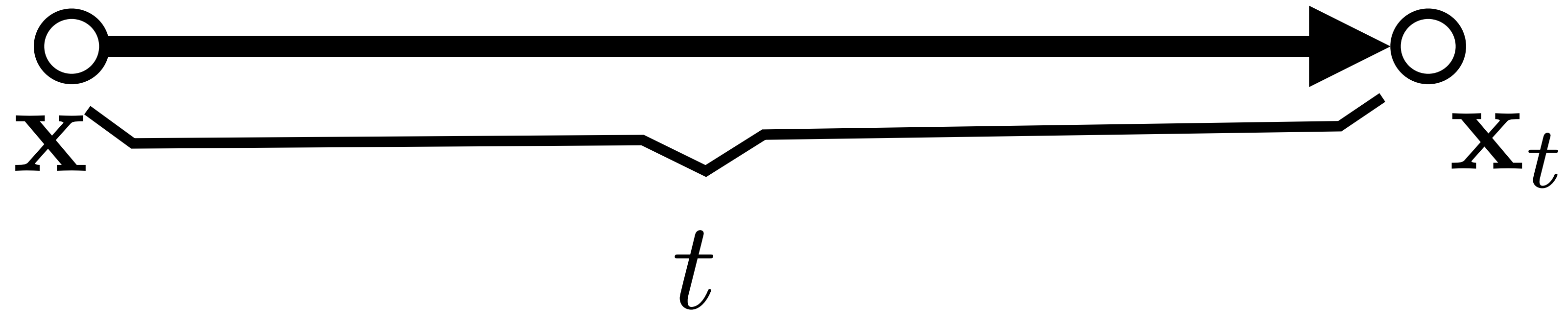
Compute PDF



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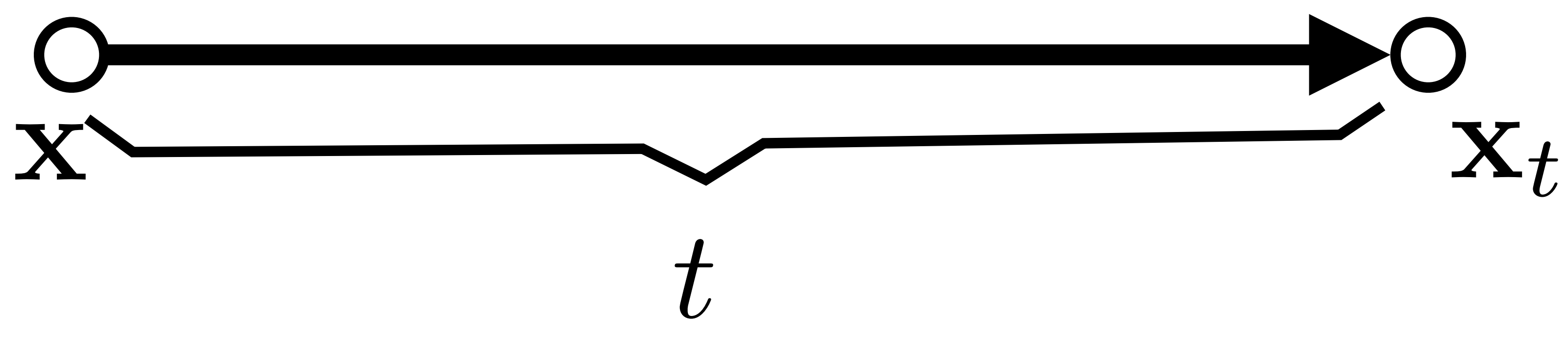
Sample distance $t = -\frac{log_e(1-\xi)}{\sigma_t}$

Compute PDF $p(t) = \sigma_t e^{-\sigma_t t}$



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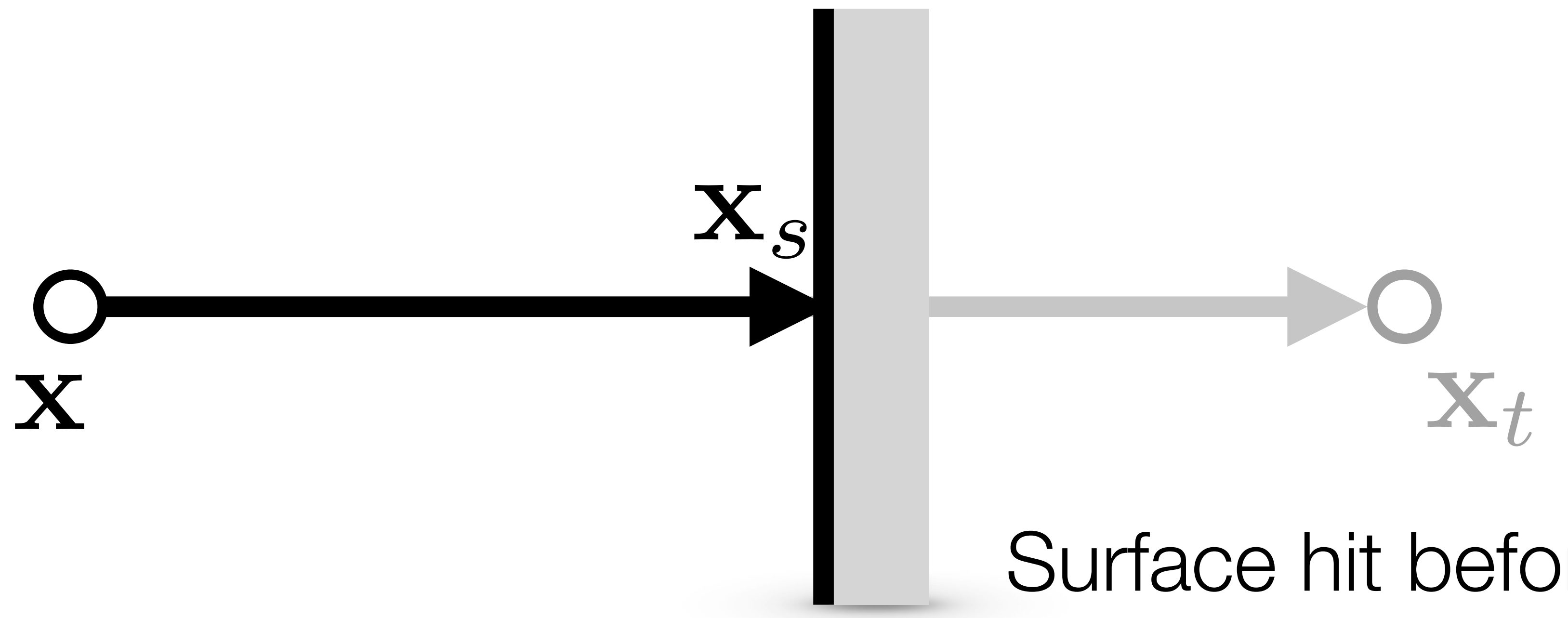

Surface hit before reaching t

Free-path Sampling

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Recipe:

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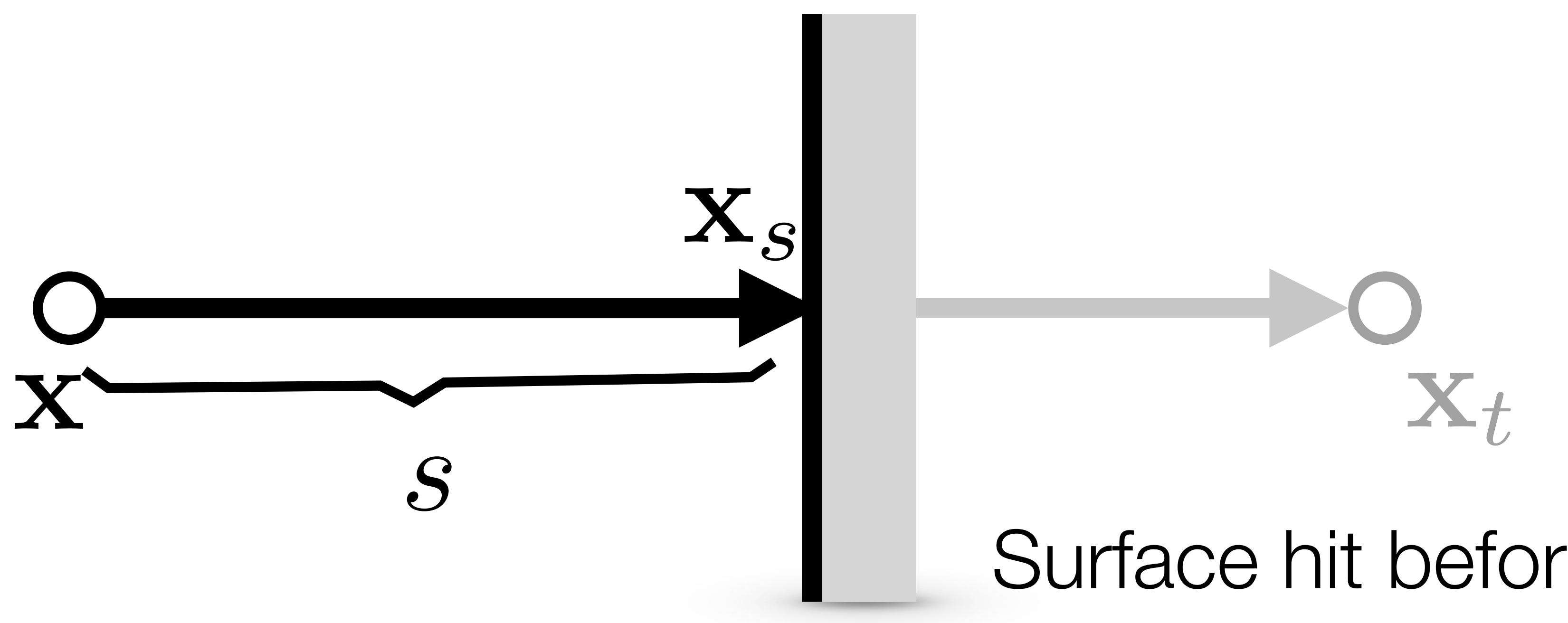

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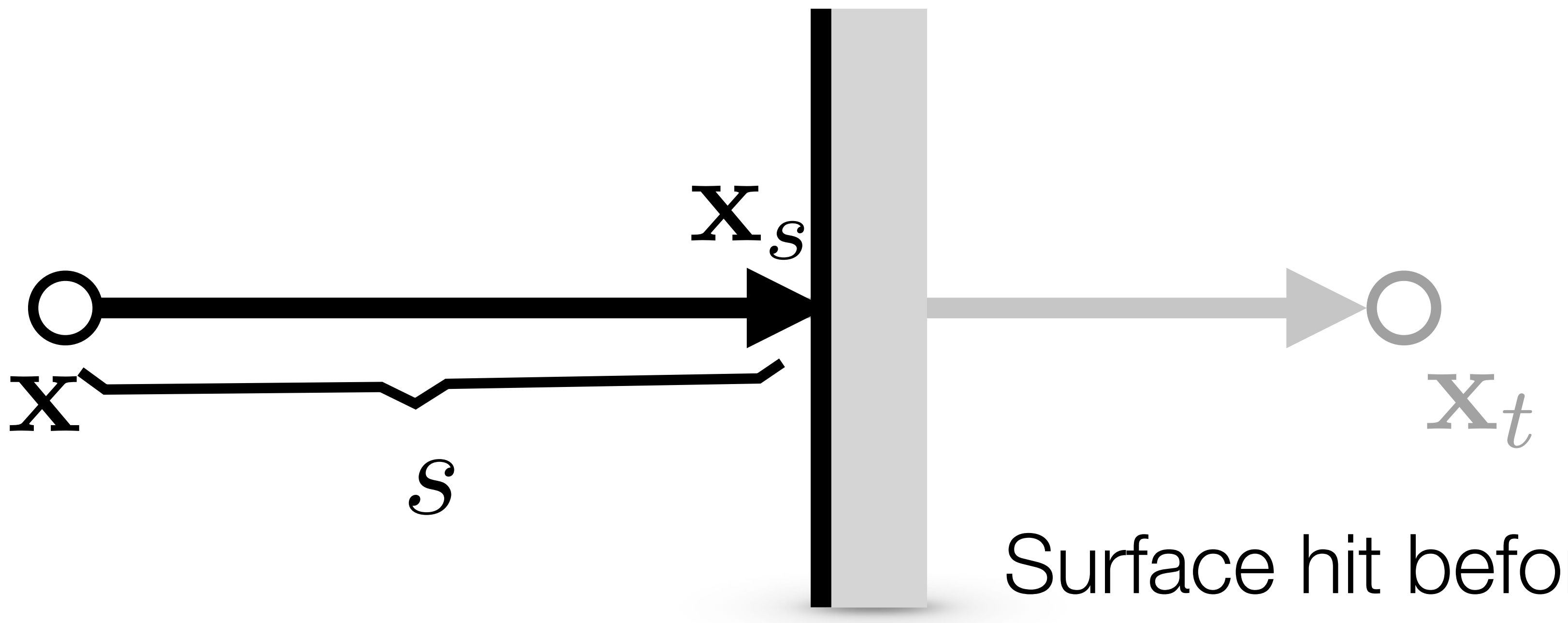


Surface hit before reaching t

Free-path Sampling

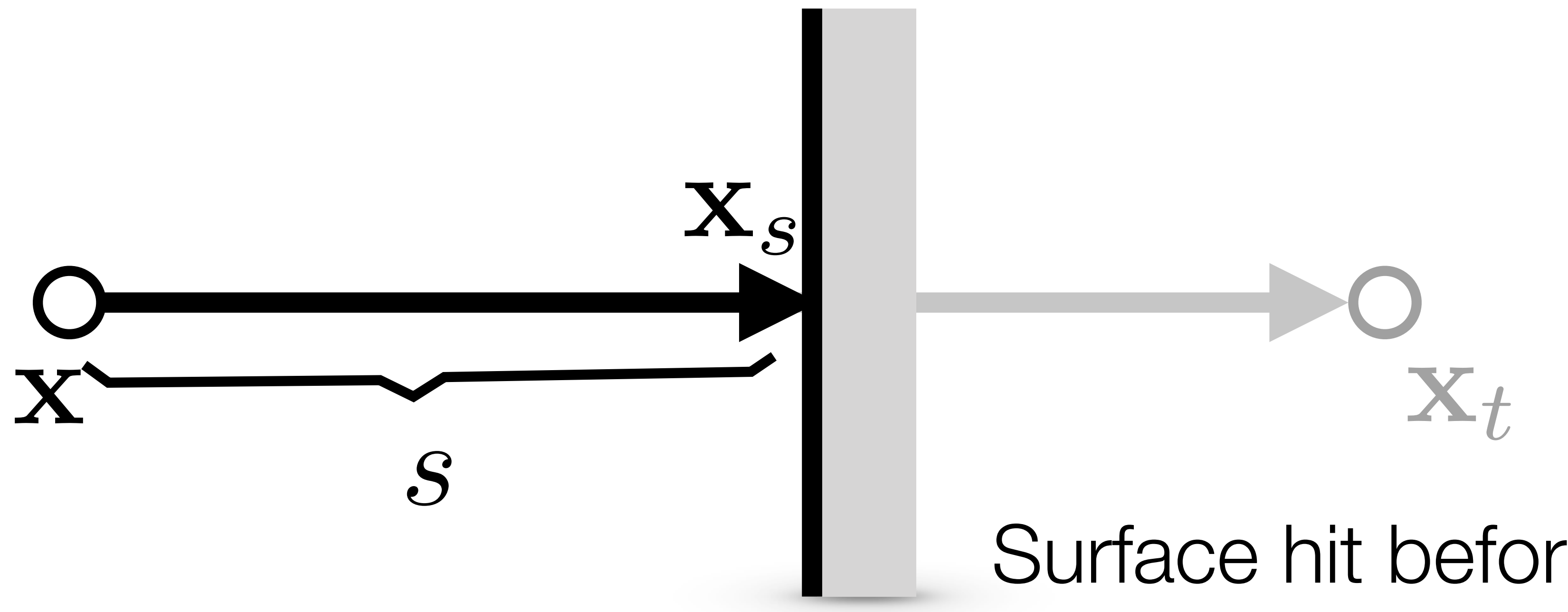
Homogeneous media: $T_r(t) = e^{-\sigma_t t}$

Recipe:

Generate a random number ξ

Sample distance $$t = -\cancel{\frac{log_e(1-\xi)}{\sigma_t}} = s$$

Compute PDF $p(t) = \sigma_t e^{-\sigma_t t}$



Surface hit before reaching t

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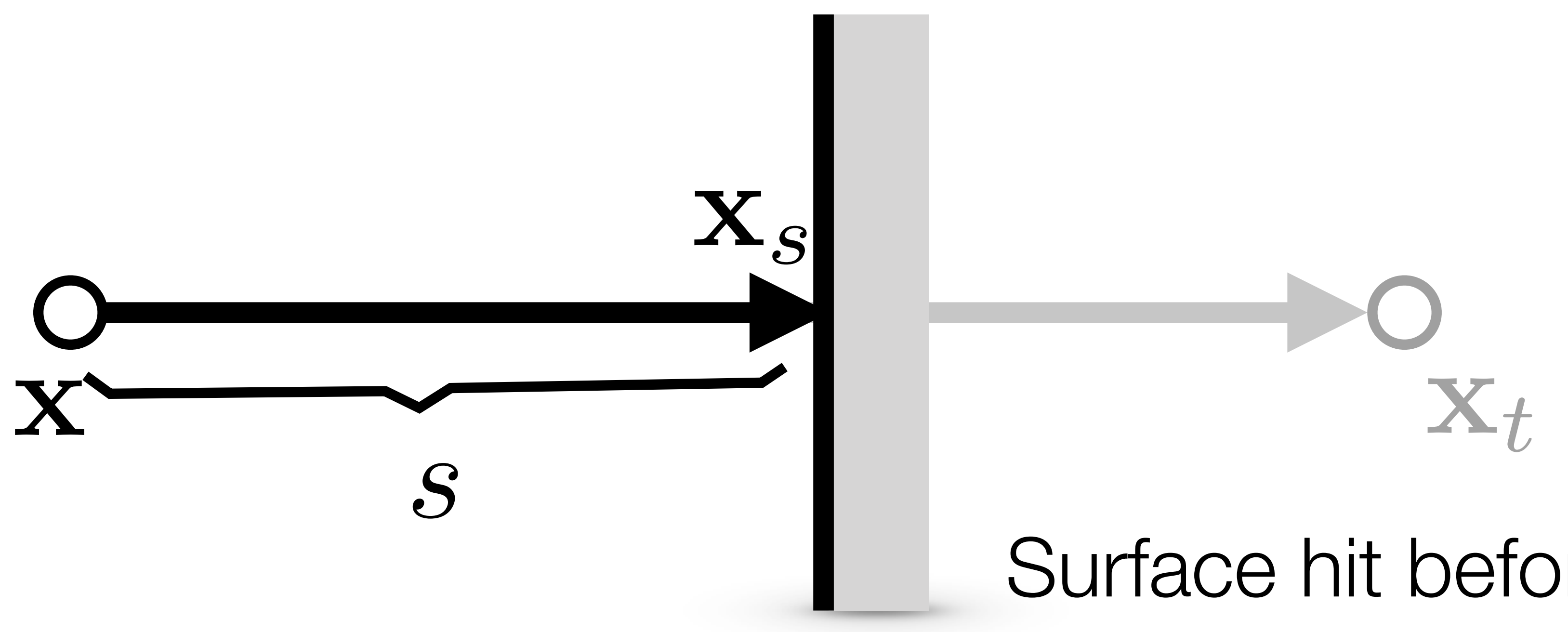
Compute PDF $$p(t) = \xcancel{\sigma_t e^{-\sigma_t t}}$$



Surface hit before reaching t

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Sample distance $t = -\cancel{\frac{log_e(1-\xi)}{\sigma_t}} = s$

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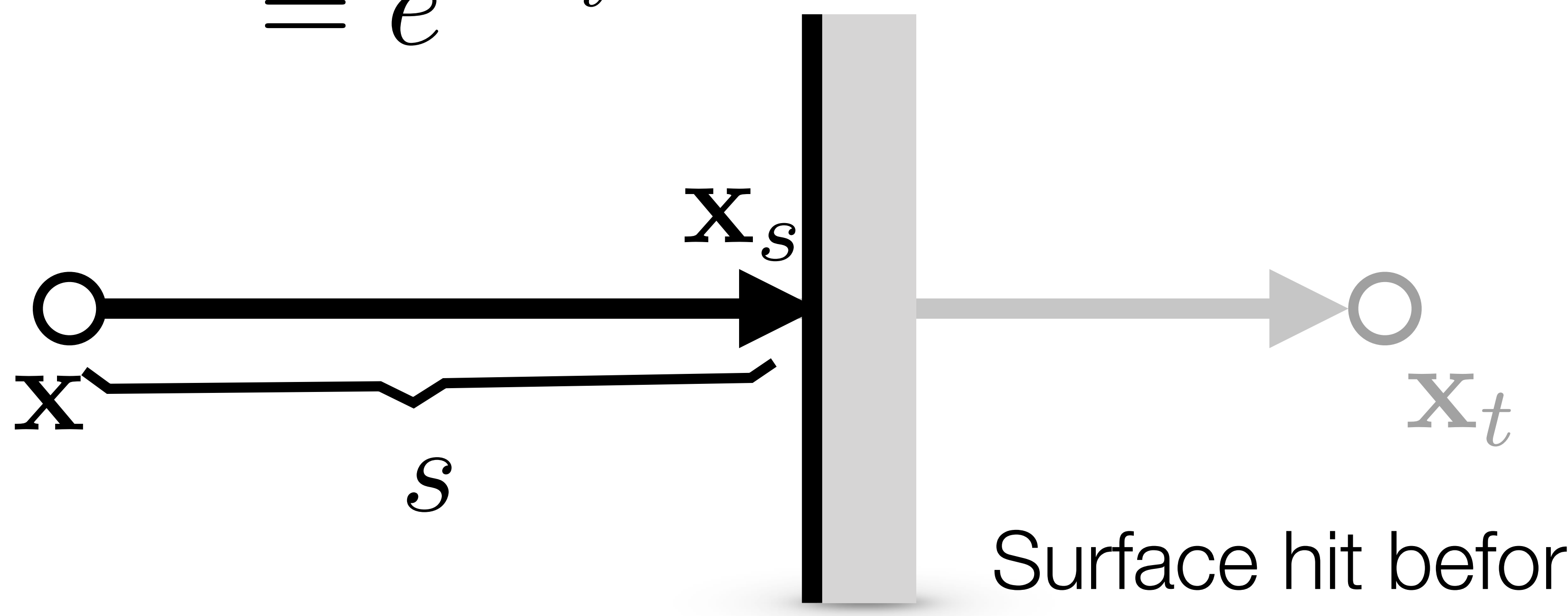


Surface hit before reaching t

Free-path Sampling

Homogeneous media: $T_r(t) = e^{-\sigma_t t}$

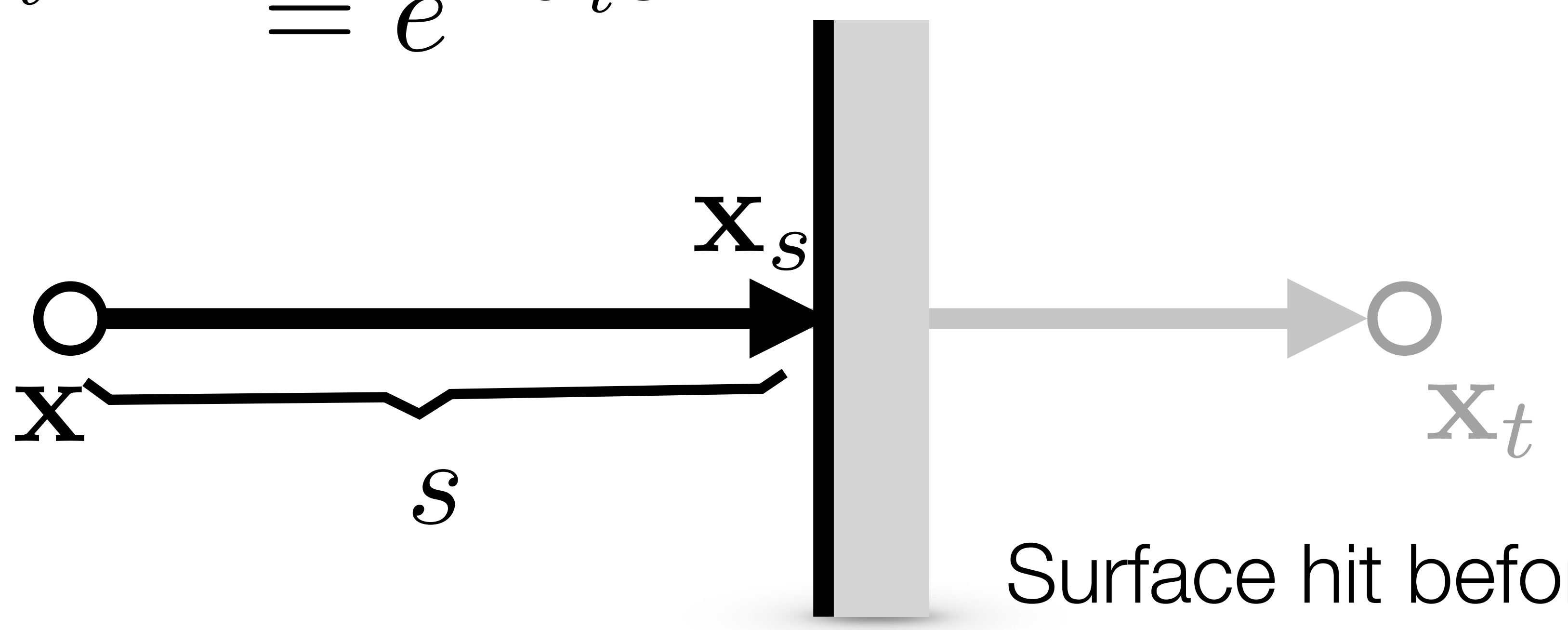
Recipe:

Generate a random number ξ

Sample distance $t = -\cancel{\frac{log_e(1-\xi)}{\sigma_t}} = s$

Compute PDF $p(t) = \cancel{\sigma_t} e^{-\sigma_t t} = e^{-\sigma_t s}$

Note: This is now a probability, not a probability density



Surface hit before reaching t

What about heterogeneous media?

Free-path Sampling

Heterogeneous medium: $T_r(t) = e^{\int_0^t -\sigma_t(s)ds}$

Free-path Sampling

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- Closed form solutions exist but for only simple media

e.g., linearly or exponentially varying extinction

Free-path Sampling

Heterogeneous medium: $T_r(t) = e^{\int_0^t -\sigma_t(s)ds}$

- Closed form solutions exist but for only simple media

  e.g., linearly or exponentially varying extinction

- Other solutions:
  - Regular tracking (3D DDA)
  - Ray marching
  - Delta tracking

Free-path Sampling

How to sample the flight distance to the next interaction?

Free-path Sampling

How to sample the flight distance to the next interaction?

Random variable representing flight distance

$$T(t) = e^{-\int_0^t \mu_{\mathrm{t}}(s)\mathrm{d}s} = P(X > t)$$

Free-path Sampling

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CDF

$$P(X \leq t) = F(t)$$

Free-path Sampling

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Partition of unity

Free-path Sampling

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Random variable representing flight distance

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CDF

$$P(X \leq t) = F(t)$$

Partition of unity

$$F(t) = 1 - T(t)$$

Recipe for generating samples

Free-path Sampling

Cumulative distribution function (**CDF**)

$$F(t) = 1 - T(t) = 1 - e^{-\tau(t)}$$

Free-path Sampling

Cumulative distribution function (**CDF**)

$$F(t) = 1 - T(t) = 1 - e^{-\tau(t)}$$

Probability density function (**PDF**)

$$p(t) = \frac{\mathrm{d}F(t)}{\mathrm{d}t} = \frac{\mathrm{d}}{\mathrm{d}t}\left(1 - e^{-\tau(t)}\right) = \mu_{\mathrm{t}}(t)e^{-\tau(t)}$$

Free-path Sampling

Cumulative distribution function (**CDF**)

$$F(t) = 1 - T(t) = 1 - e^{-\tau(t)}$$

Probability density function (**PDF**)

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Inverted cumulative distr. function (**CDF^{-1}**)

$$\xi = 1 - e^{-\tau(t)}$$

Solve for t

Free-path Sampling

Cumulative distribution function (**CDF**)

$$F(t) = 1 - T(t) = 1 - e^{-\tau(t)}$$

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Inverted cumulative distr. function (**CDF⁻¹**)

$$\xi = 1 - e^{-\tau(t)}$$

Solve for t

$$\tau(t) = -\ln(1-\xi)$$

Free-path Sampling

Cumulative distribution function (**CDF**)

$$F(t) = 1 - T(t) = 1 - e^{-\tau(t)}$$

Probability density function (**PDF**)

$$p(t) = \frac{\mathrm{d}F(t)}{\mathrm{d}t} = \frac{\mathrm{d}}{\mathrm{d}t}\left(1 - e^{-\tau(t)}\right) = \mu_{\mathrm{t}}(t)e^{-\tau(t)}$$

Inverted cumulative distr. function (**CDF^{-1}**)

$$\xi = 1 - e^{-\tau(t)}$$

Solve for t

$$\int_0^t \mu_{\mathrm{t}}(s)\mathrm{d}s = -\ln(1-\xi)$$

Free-path Sampling

Cumulative distribution function (**CDF**)

$$F(t) = 1 - T(t) = 1 - e^{-\tau(t)}$$

Probability density function (**PDF**)

$$p(t) = \frac{\mathrm{d}F(t)}{\mathrm{d}t} = \frac{\mathrm{d}}{\mathrm{d}t}\left(1 - e^{-\tau(t)}\right) = \mu_\mathrm{t}(t)e^{-\tau(t)}$$

Inverted cumulative distr. function (**CDF^{-1}**)

$$\xi = 1 - e^{-\tau(t)}$$

Solve for t

$$\int_0^t \mu_\mathrm{t}(s)\mathrm{d}s = -\ln(1-\xi)$$

Approaches for finding t:

1) **ANALYTIC (closed-form CDF^{-1})**
2) **SEMI-ANALYTIC (regular tracking)**
3) **APPROXIMATE (ray marching)**

Free-path Sampling

Inverted cumulative distr. function (**CDF^{-1}**)

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$$\int_0^t \mu_{\mathrm{t}}(s)\mathrm{d}s = t\mu_{\mathrm{t}}$$

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Opt. thickness

$$\int_0^t \mu_\mathrm{t}(s)\mathrm{d}s = t\mu_\mathrm{t}$$

$$\Rightarrow$$

Expression for t

$$t = -\frac{\ln(1-\xi)}{\mu_\mathrm{t}}$$

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Example: **homogeneous** medium ($\mu_{\mathrm{t}}(\mathbf{x}) = \mu_{\mathrm{t}}$)

Opt. thickness

Inverted CDF

$$\int_0^t \mu_{\mathrm{t}}(s)\mathrm{d}s = t\mu_{\mathrm{t}} \quad \Rightarrow \quad F^{-1}(\xi) = -\frac{\ln(1-\xi)}{\mu_{\mathrm{t}}}$$

Analytic Approach

Inverted cumulative distr. function (**CDF^{-1}**)

$$\int_0^t \mu_{\mathrm{t}}(s)\mathrm{d}s = -\ln(1-\xi)$$

Homogeneous volume

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Homogeneous volume

Sampling in homogeneous vol:

1) Draw a random number ξ
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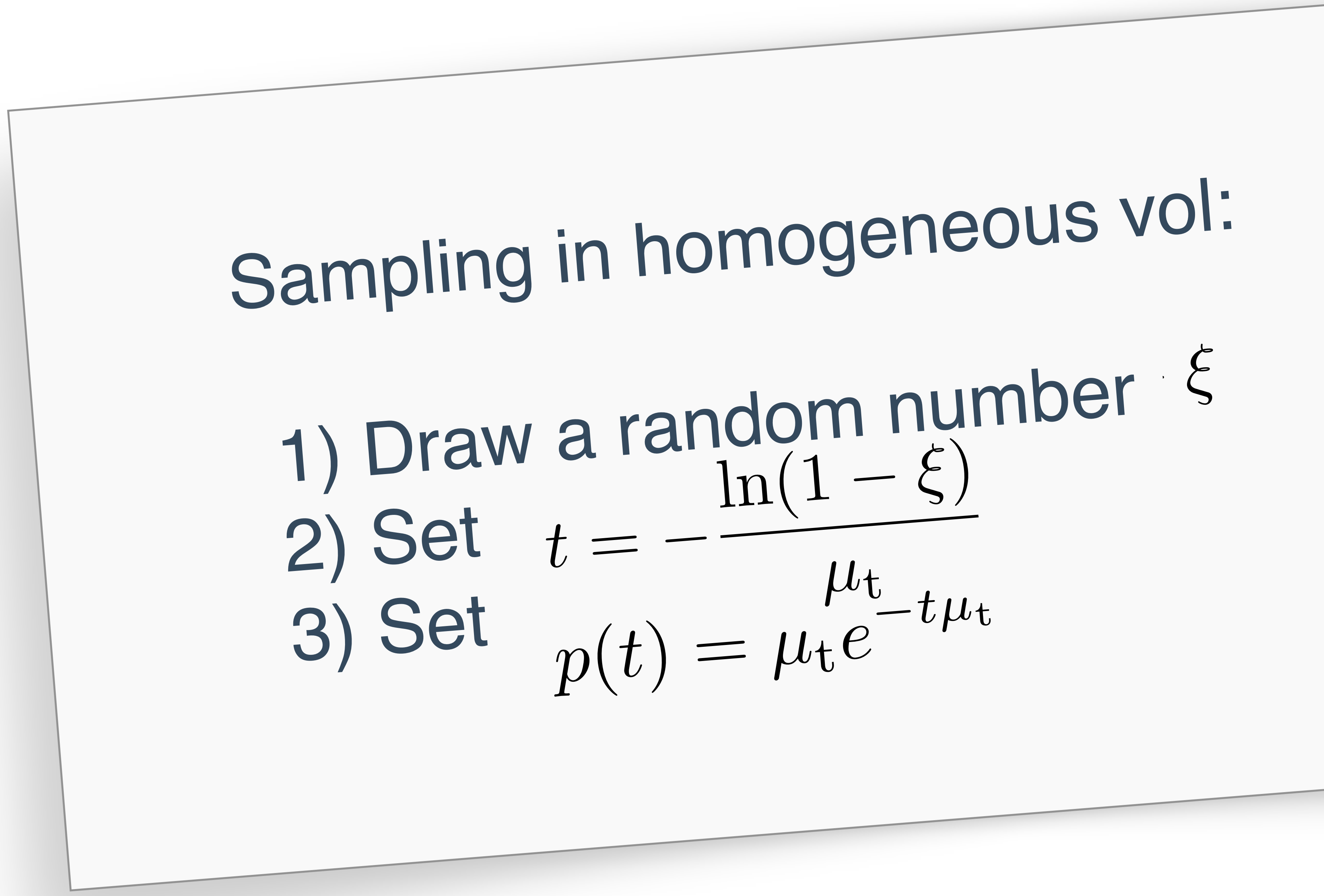

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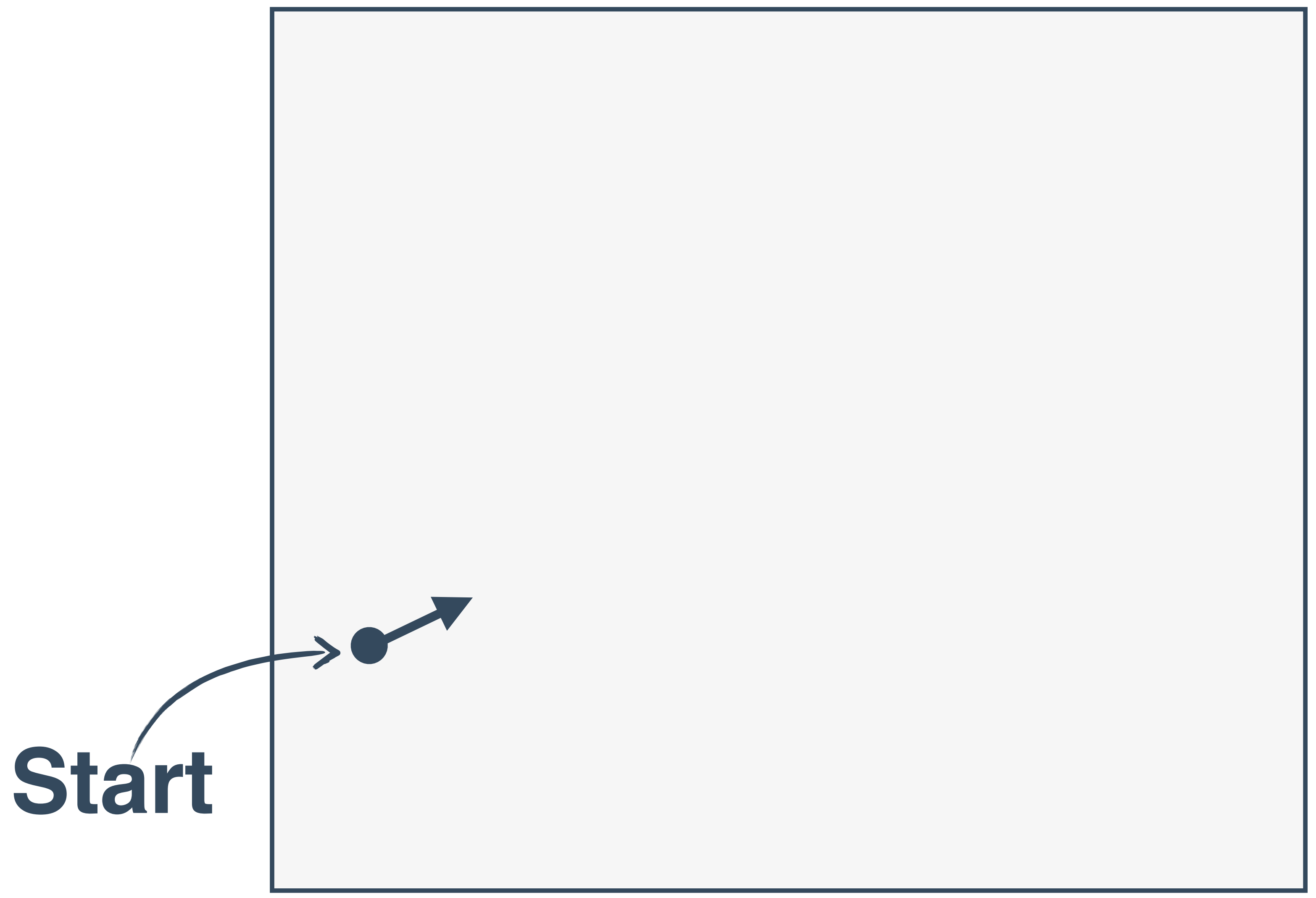

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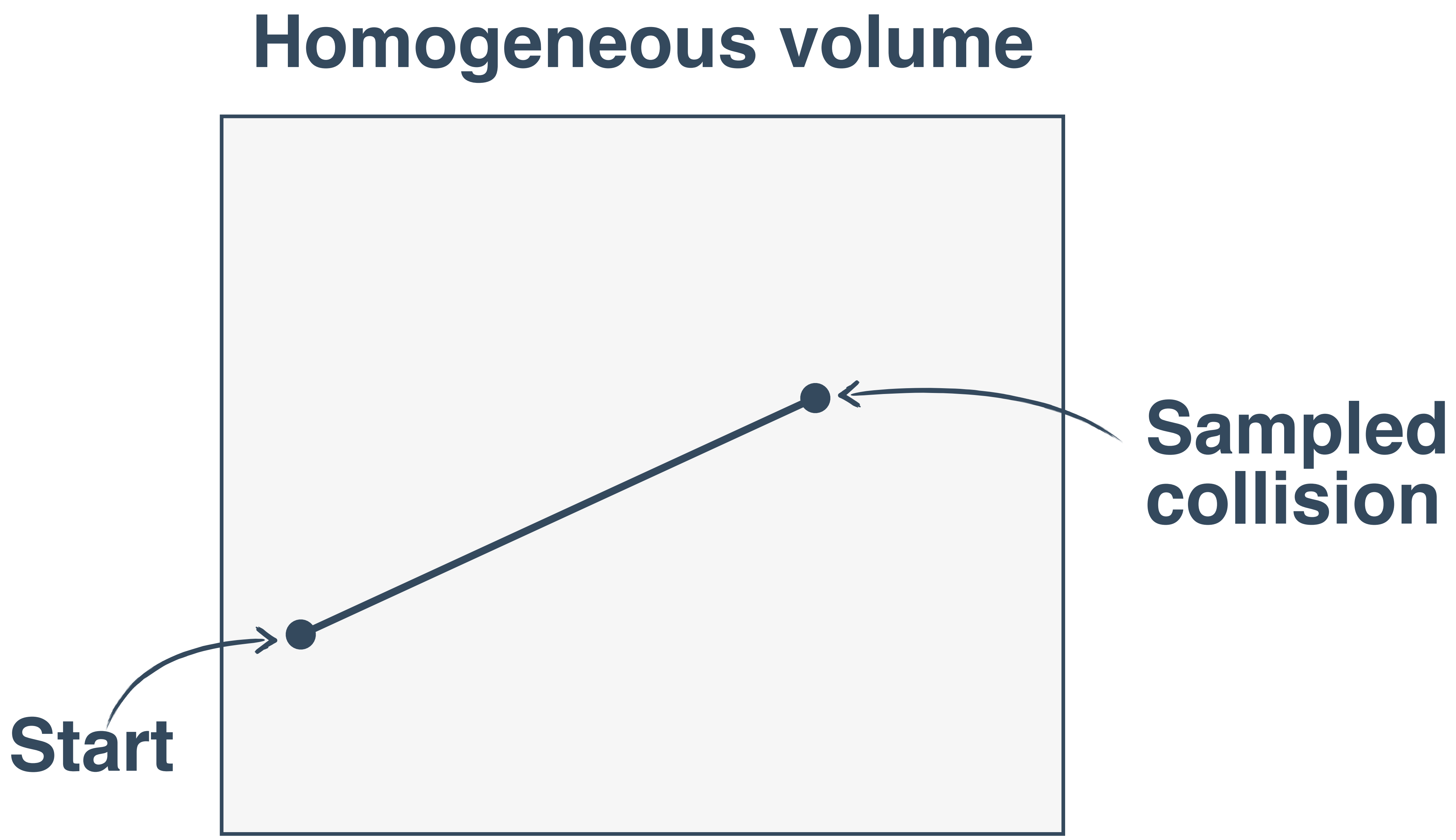

Homogeneous volume

Start

Sampled collision

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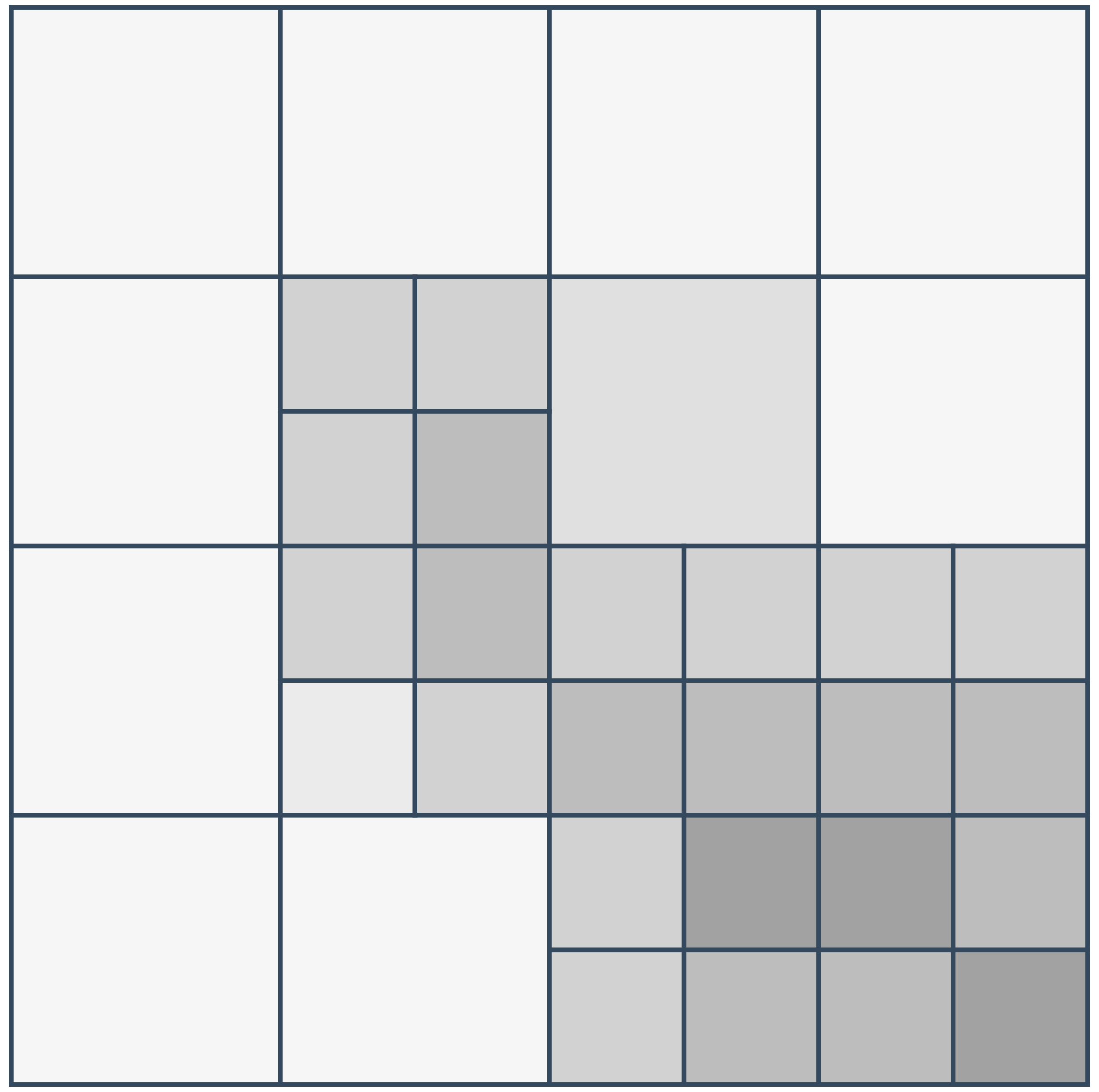
(Hierarchical) voxel grid



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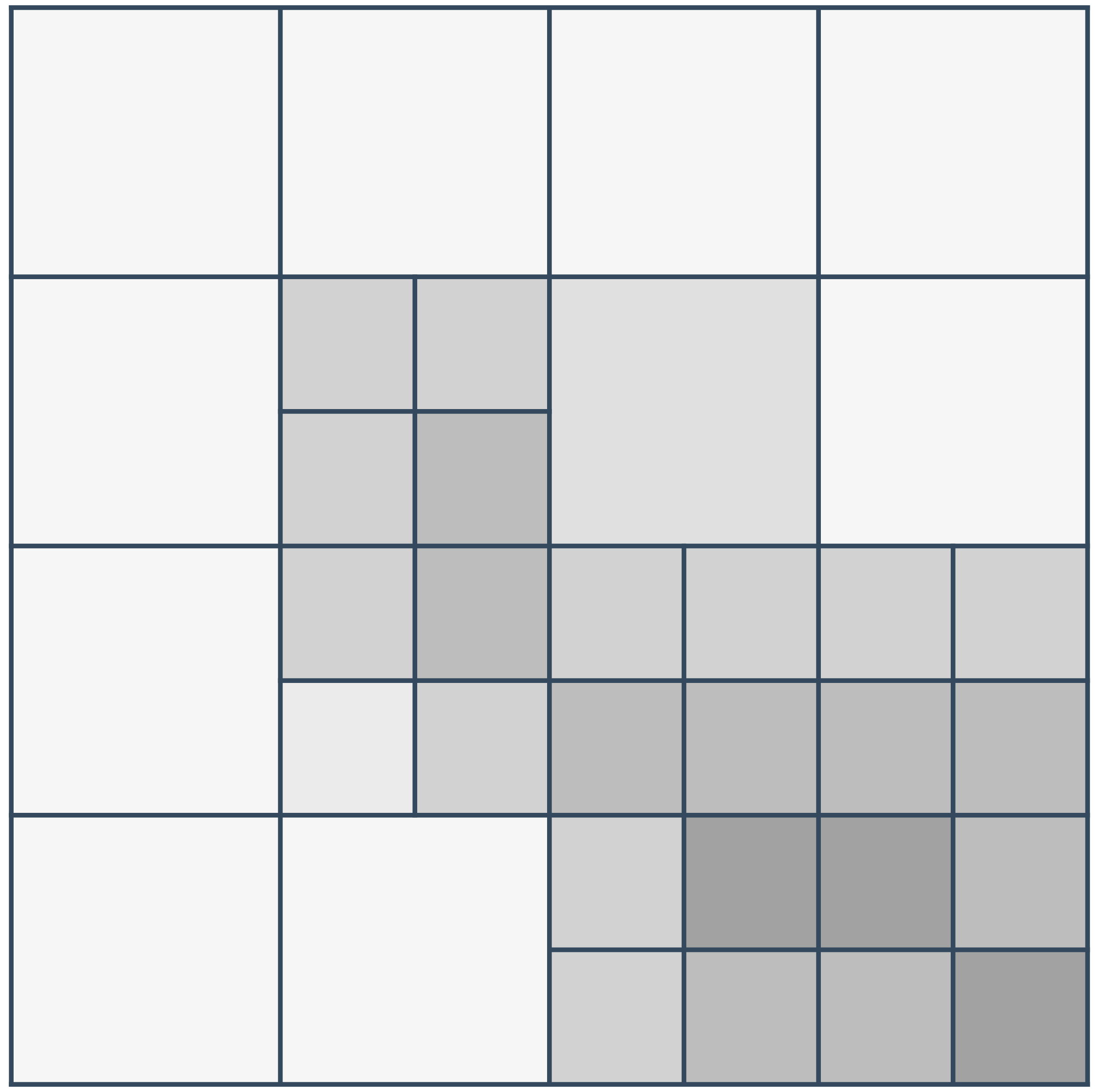

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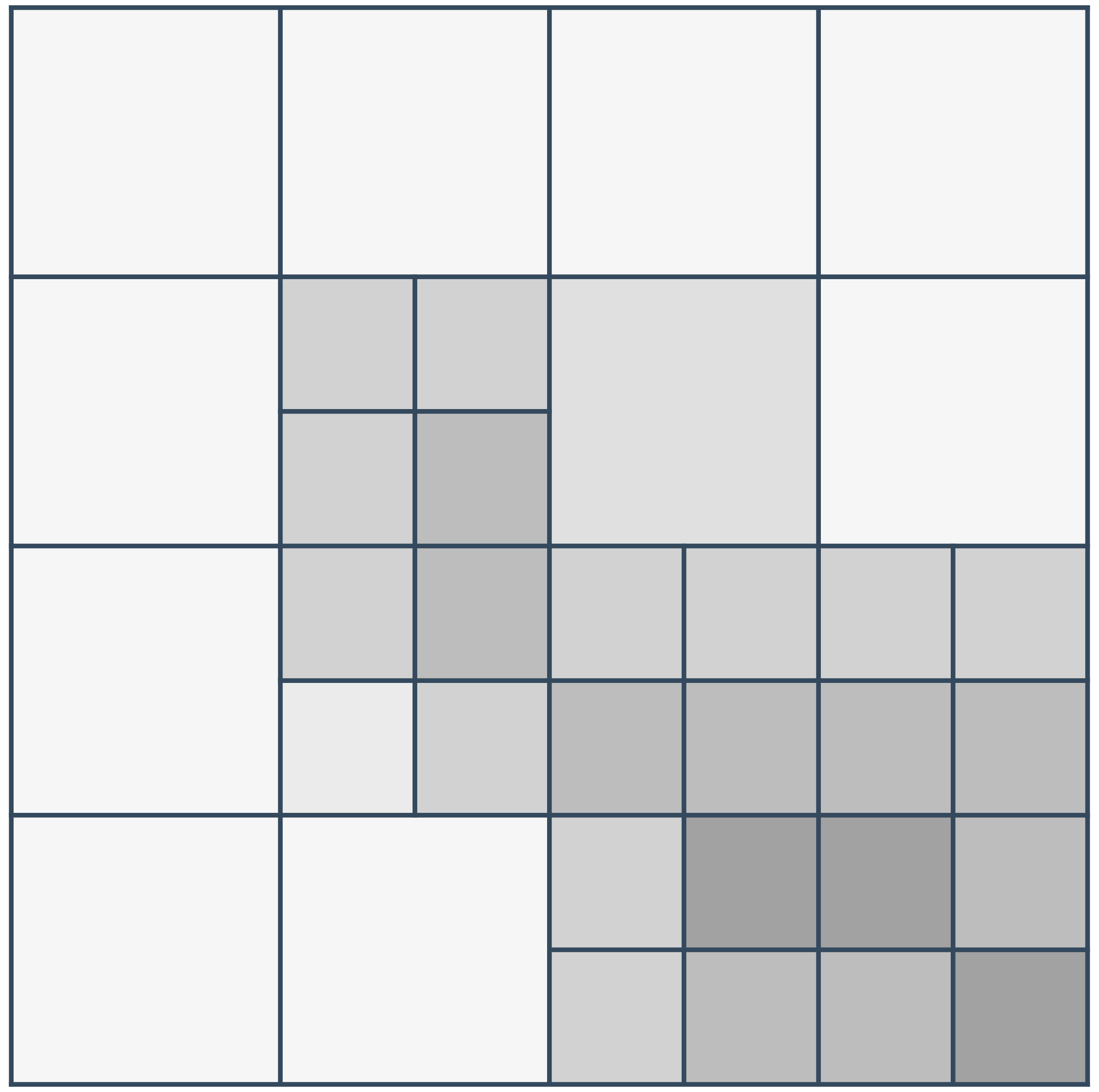

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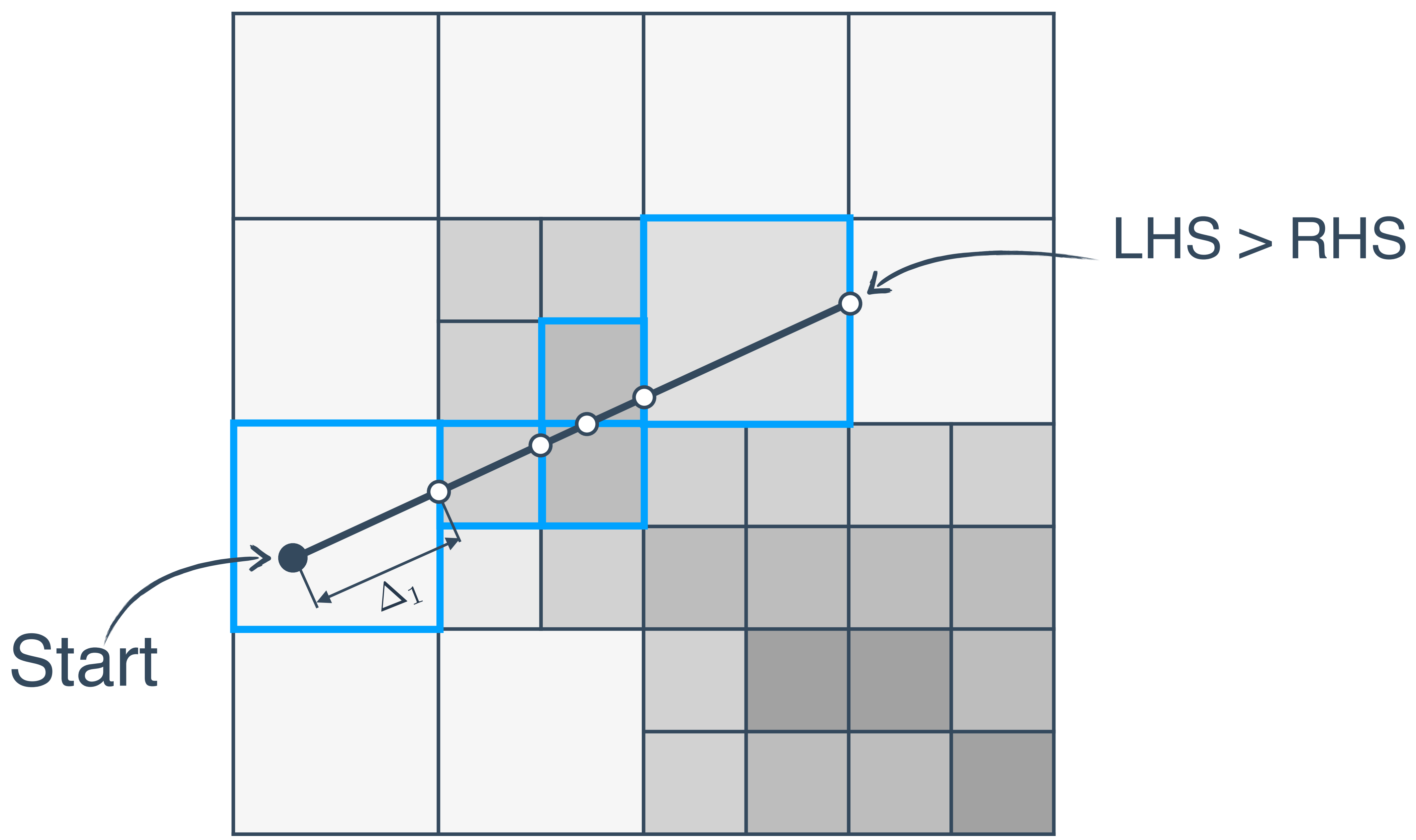

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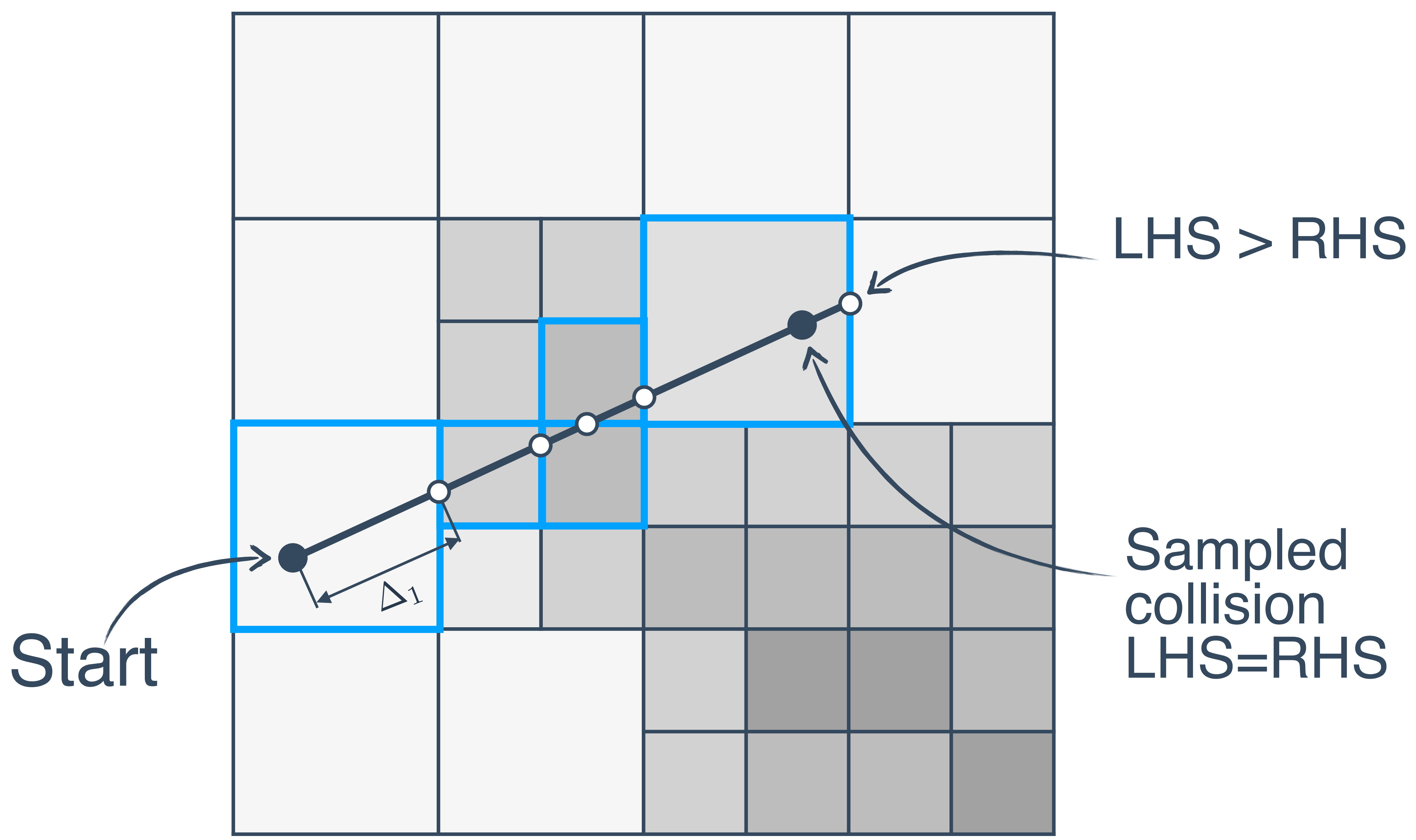


LHS > RHS

Start

Sampled collision LHS=RHS

Ray Marching

Find the collision distance approximately

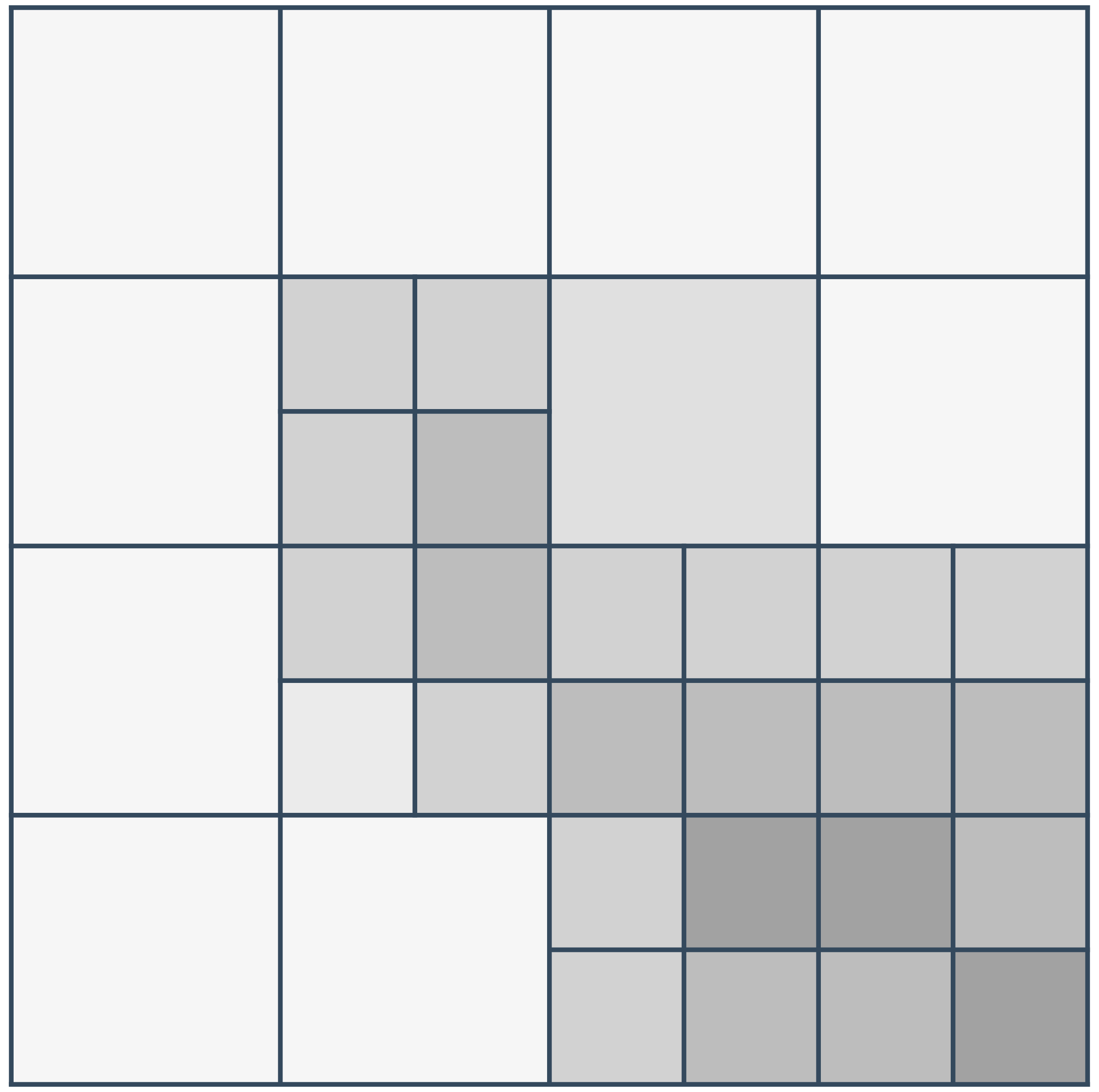
$$\int_0^t \mu_{\mathrm{t}}(s)\mathrm{d}s = -\ln(1-\xi)$$

$$\neq$$

$$\sum_{i=1}^{k} \mu_{\mathrm{t},i}\,\underset{\uparrow}{\Delta} = -\ln(1-\xi)$$

Constant step

(Hierarchical) voxel grid



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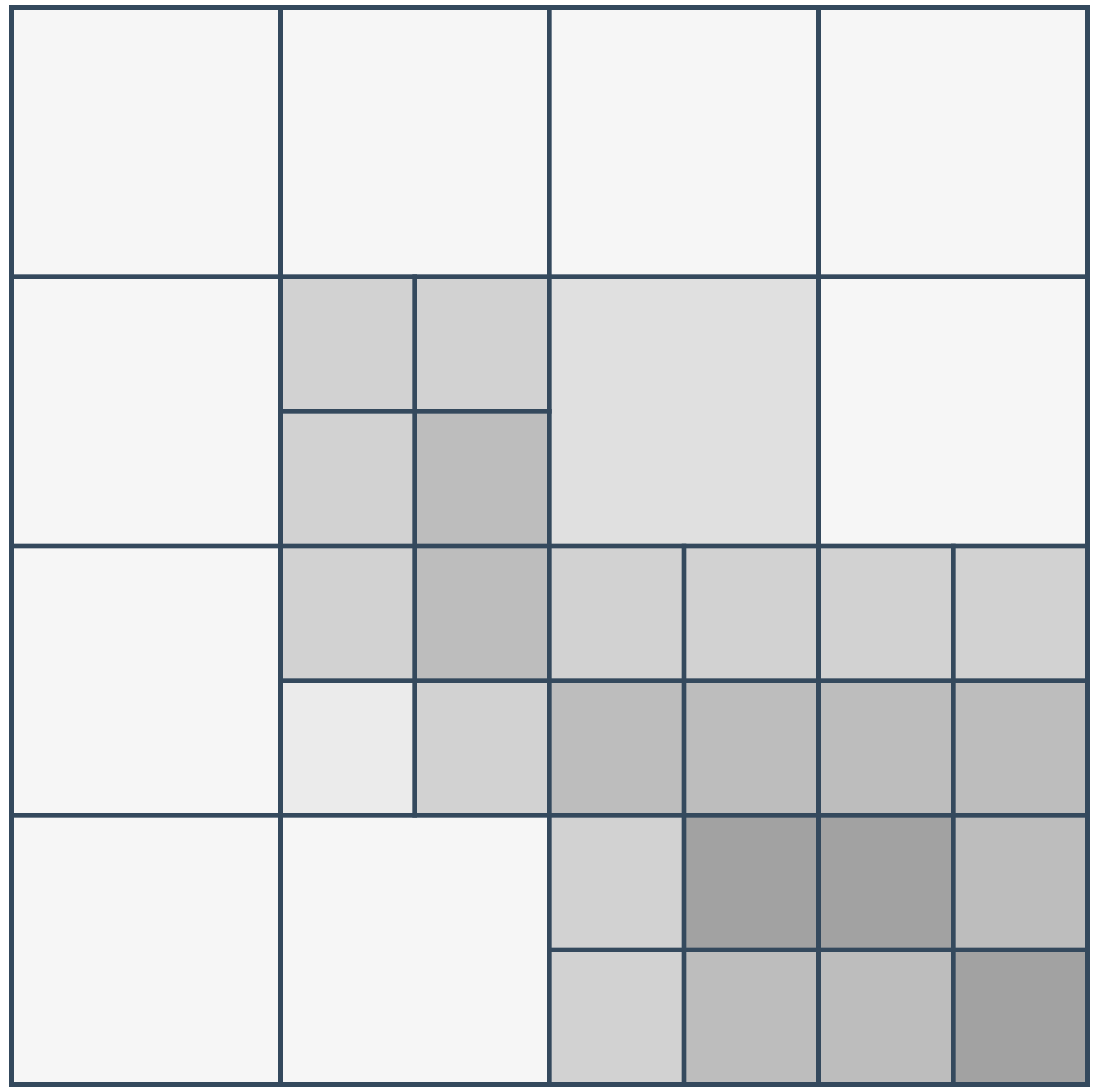

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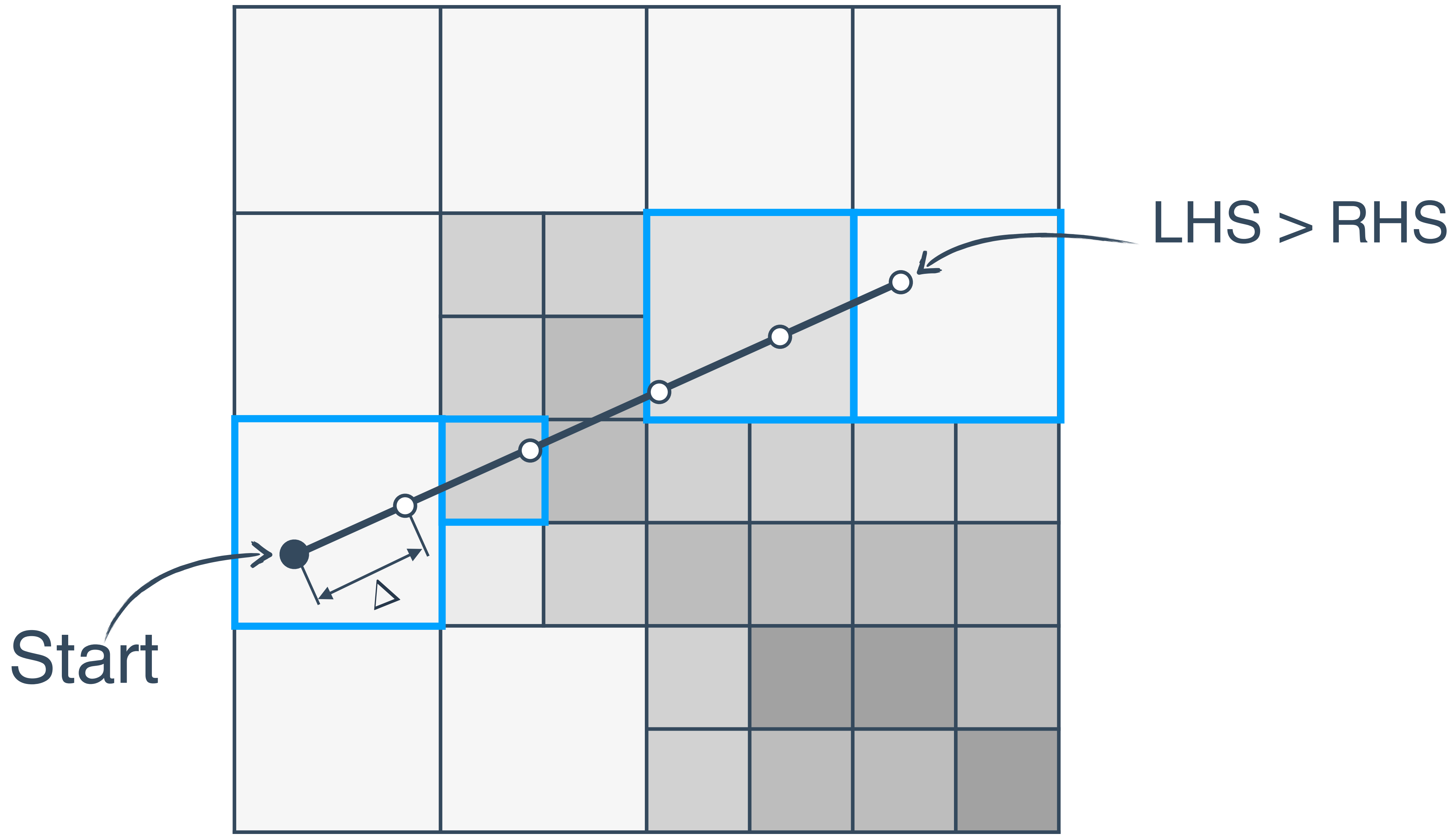

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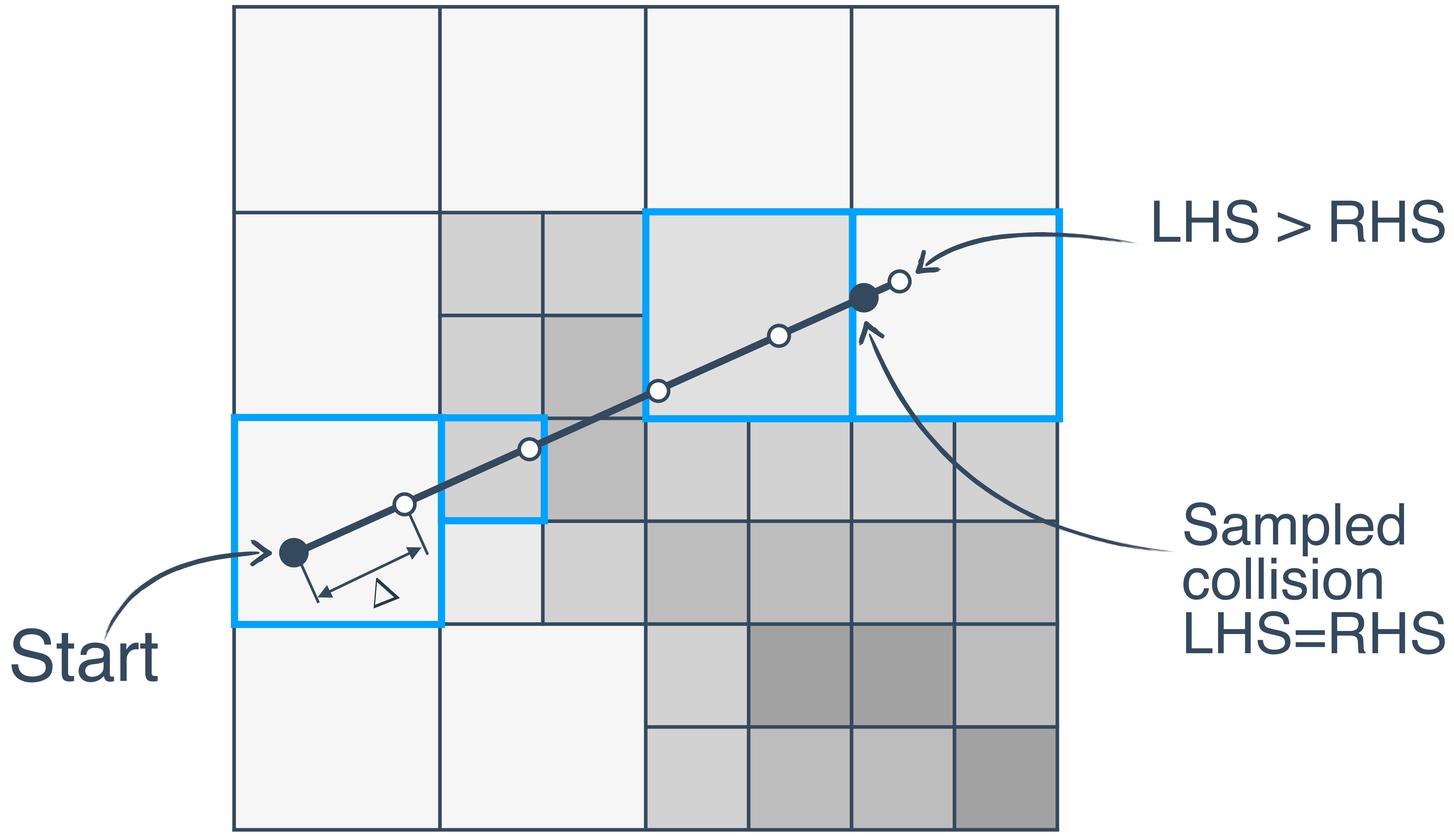

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$$\approx$$

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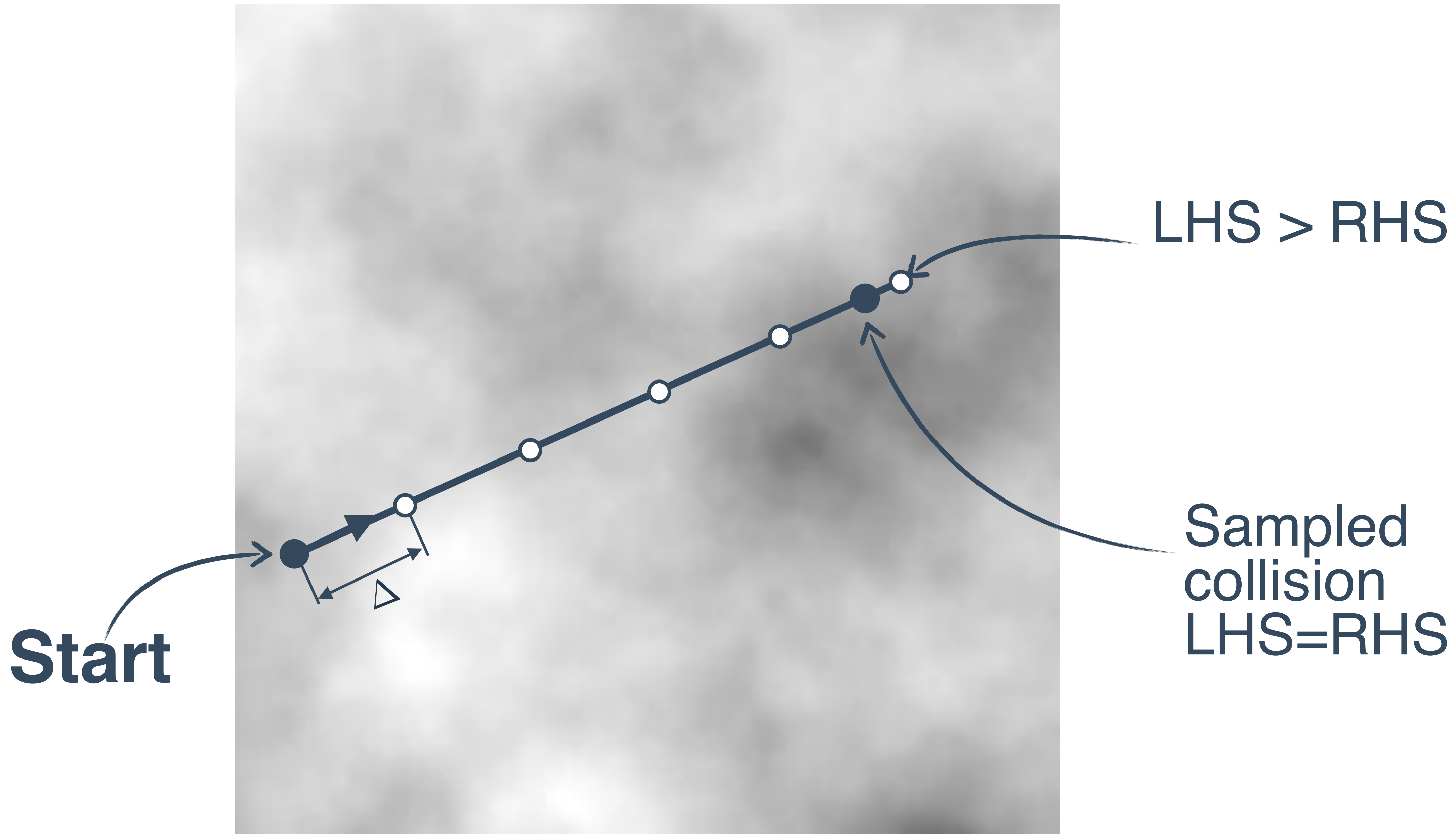
General volume



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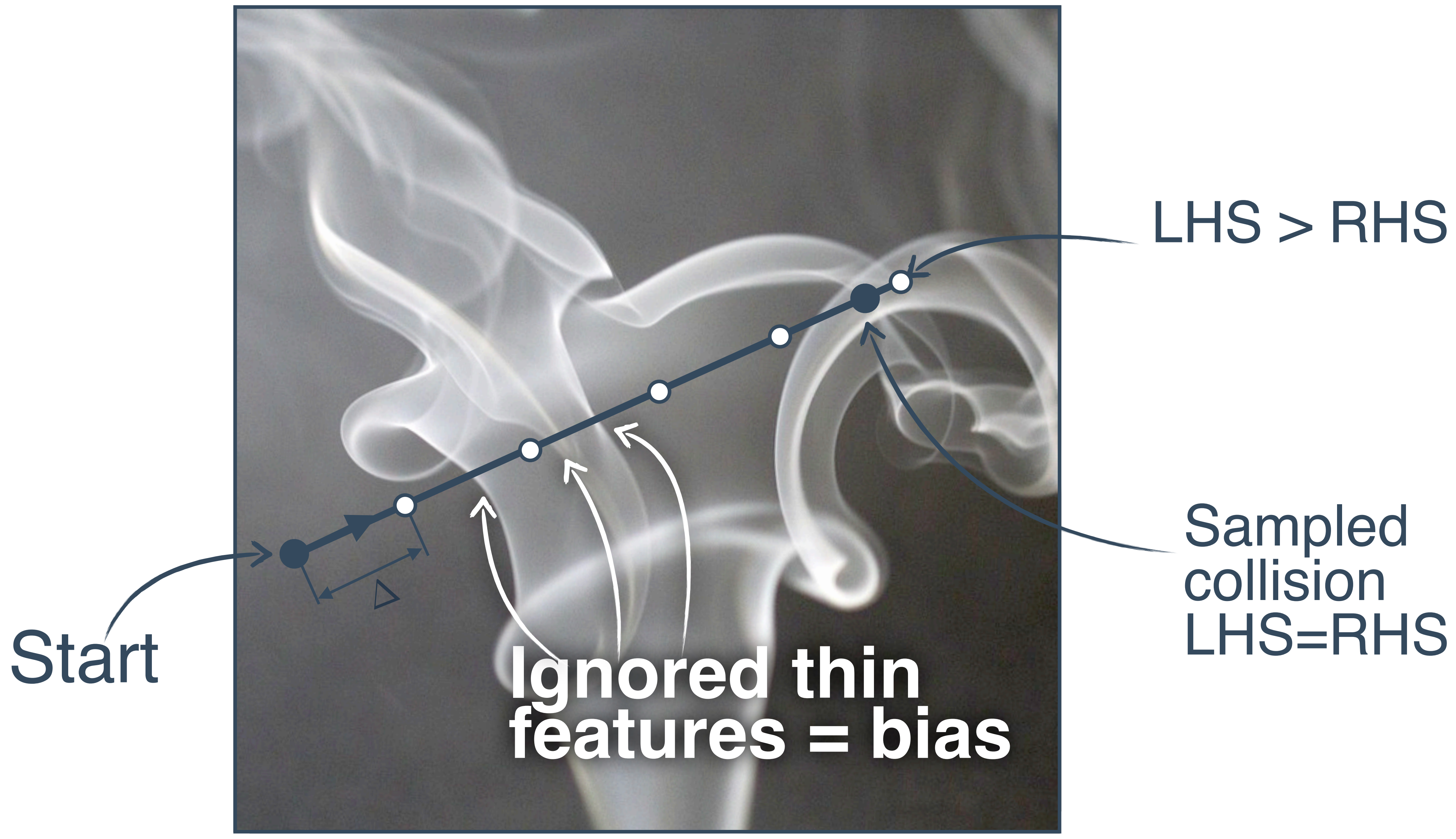

Free-path Sampling

ANALYTIC CDF^{-1}	REGULAR TRACKING	RAY MARCHING

Free-path Sampling

ANALYTIC CDF^{-1}	REGULAR TRACKING	RAY MARCHING
▸ Efficient & simple, limited to few volumes		
▸ Simple volumes (e.g. homogeneous)		
▸ Unbiased		

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▸ Efficient & simple, limited to few volumes	▸ Iterative, inefficient if free paths cross many boundaries	
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▸ Unbiased	▸ Unbiased	▸ Biased

Common approach: sample optical thickness, find corresponding distance

Delta Tracking

a.k.a. Woodcock tracking, pseudo scattering, hole tracking, null-collision method,…

Physical Interpretation

Add **FICTITIOUS MATTER** to homogenize heterogeneous extinction

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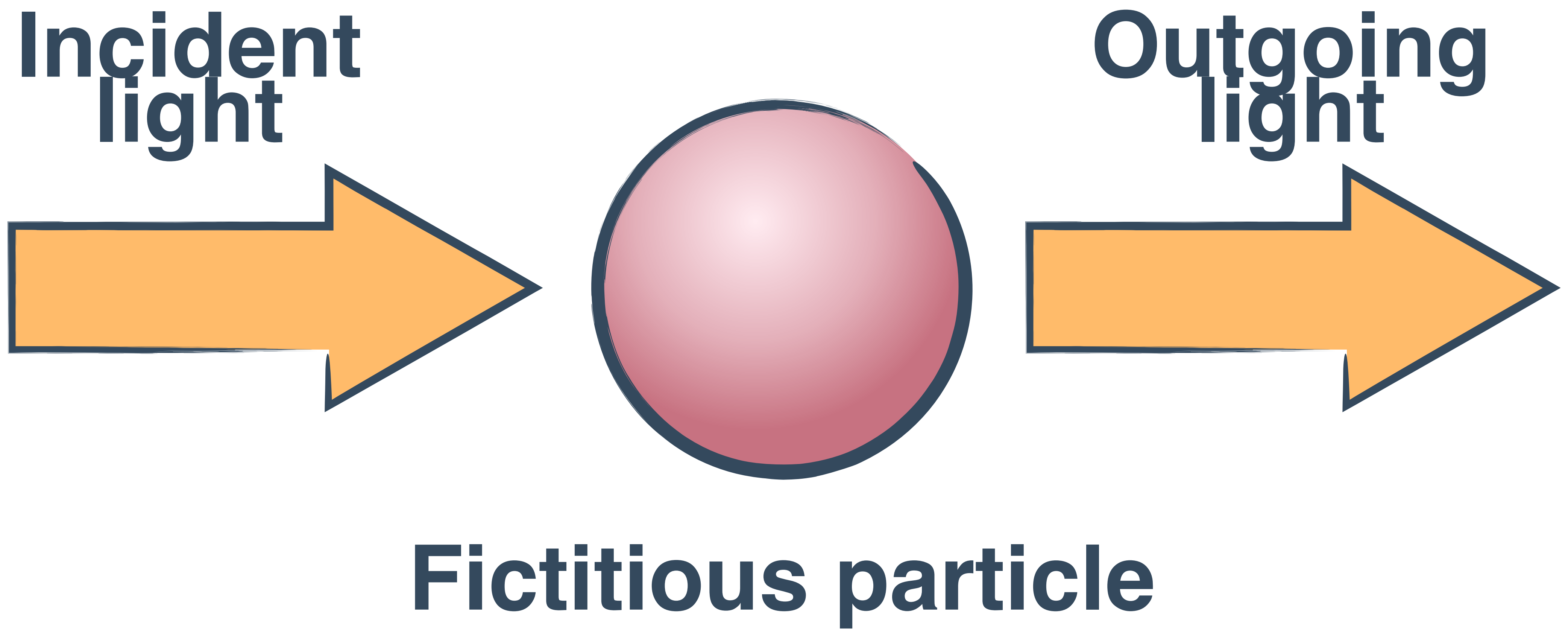

Presence of fictitious matter does not impact light transport

Physical Interpretation

HOMOGENIZATION

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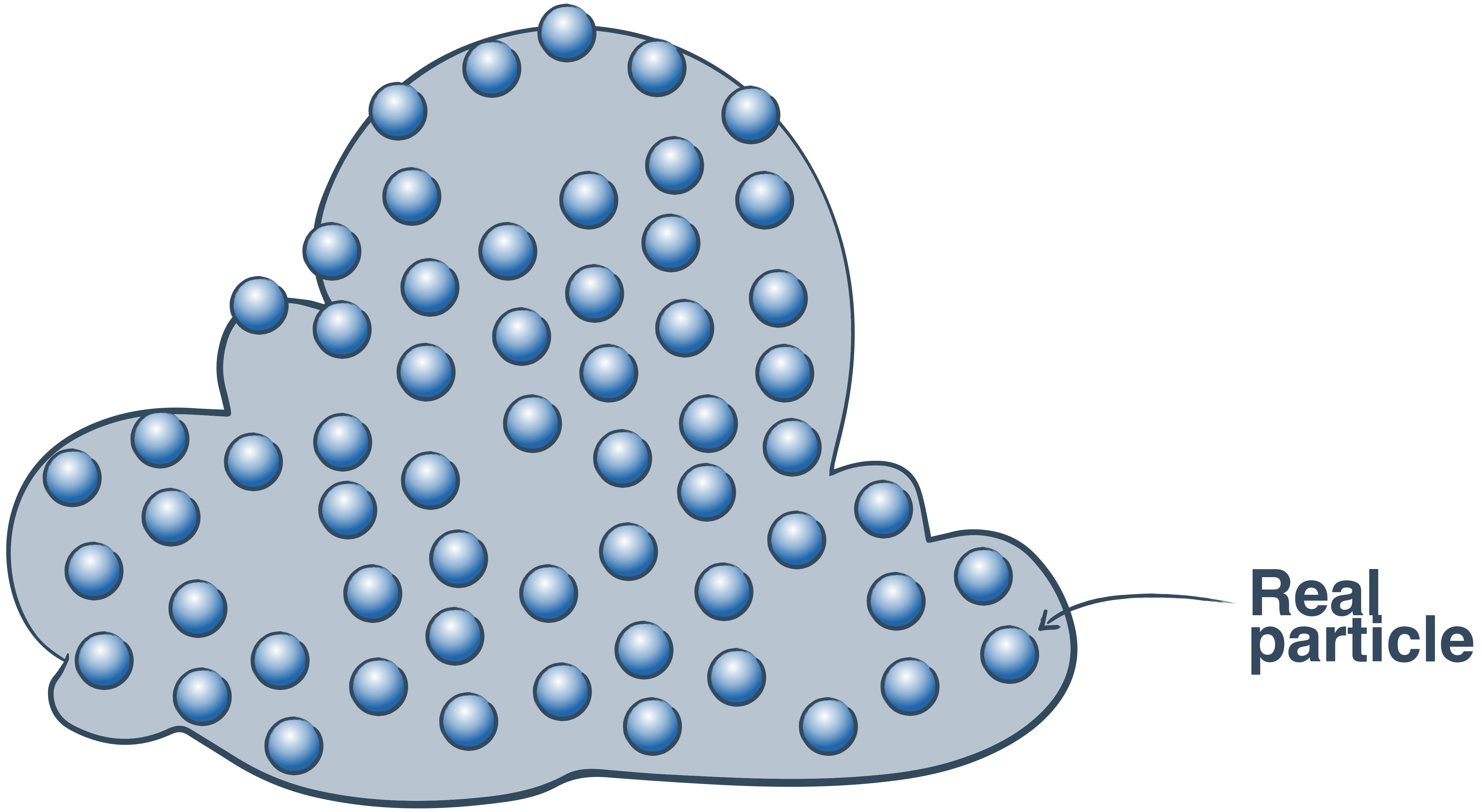

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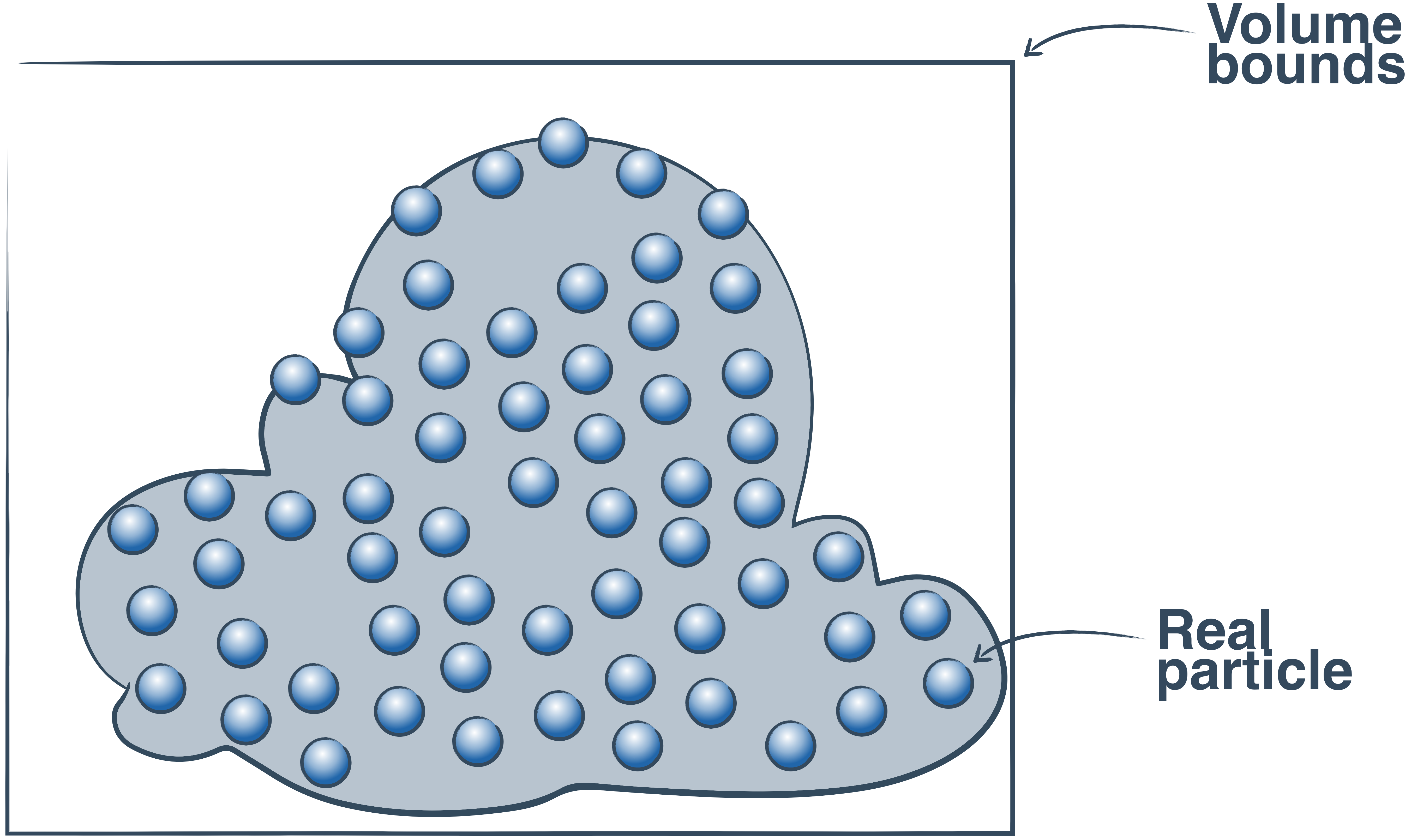

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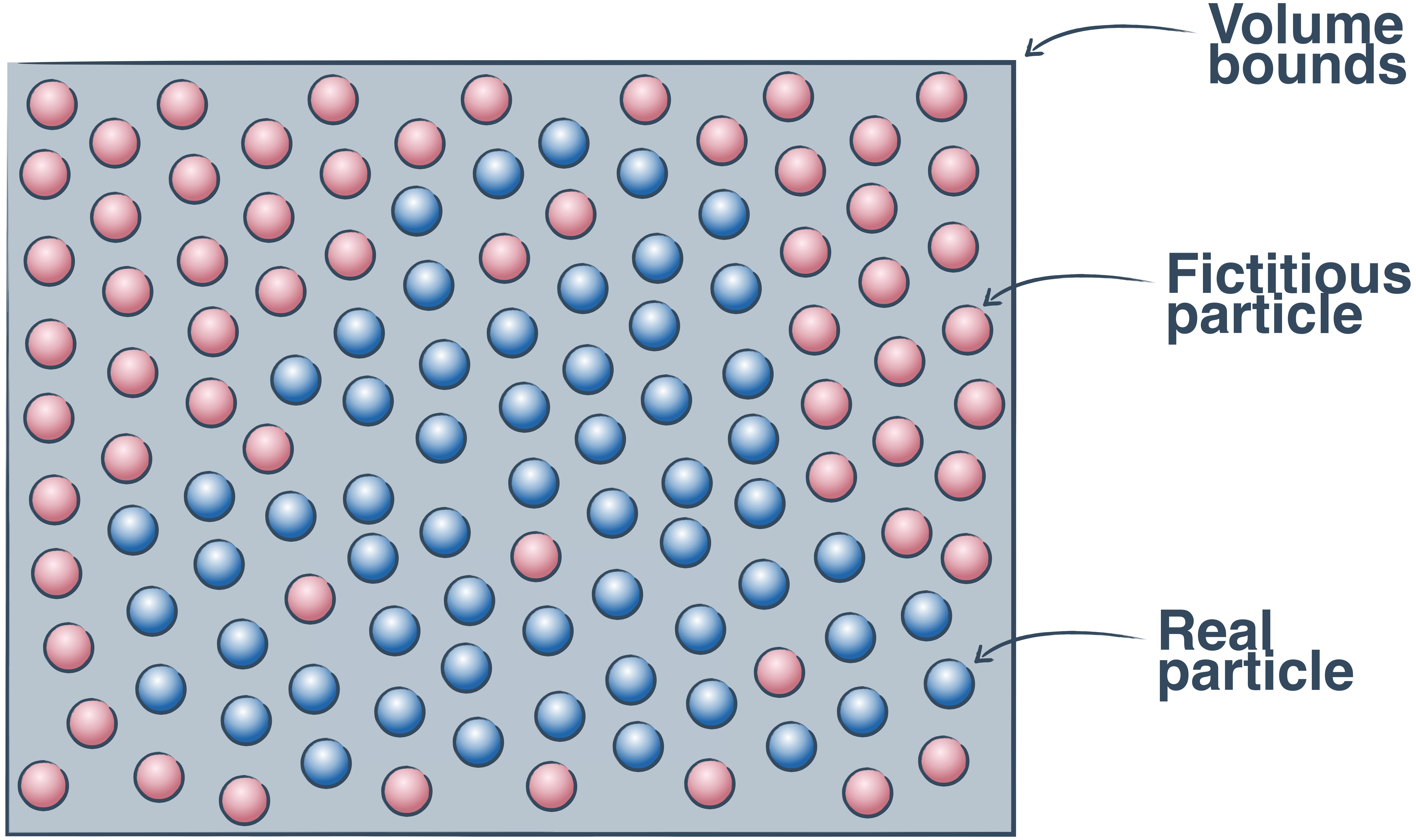

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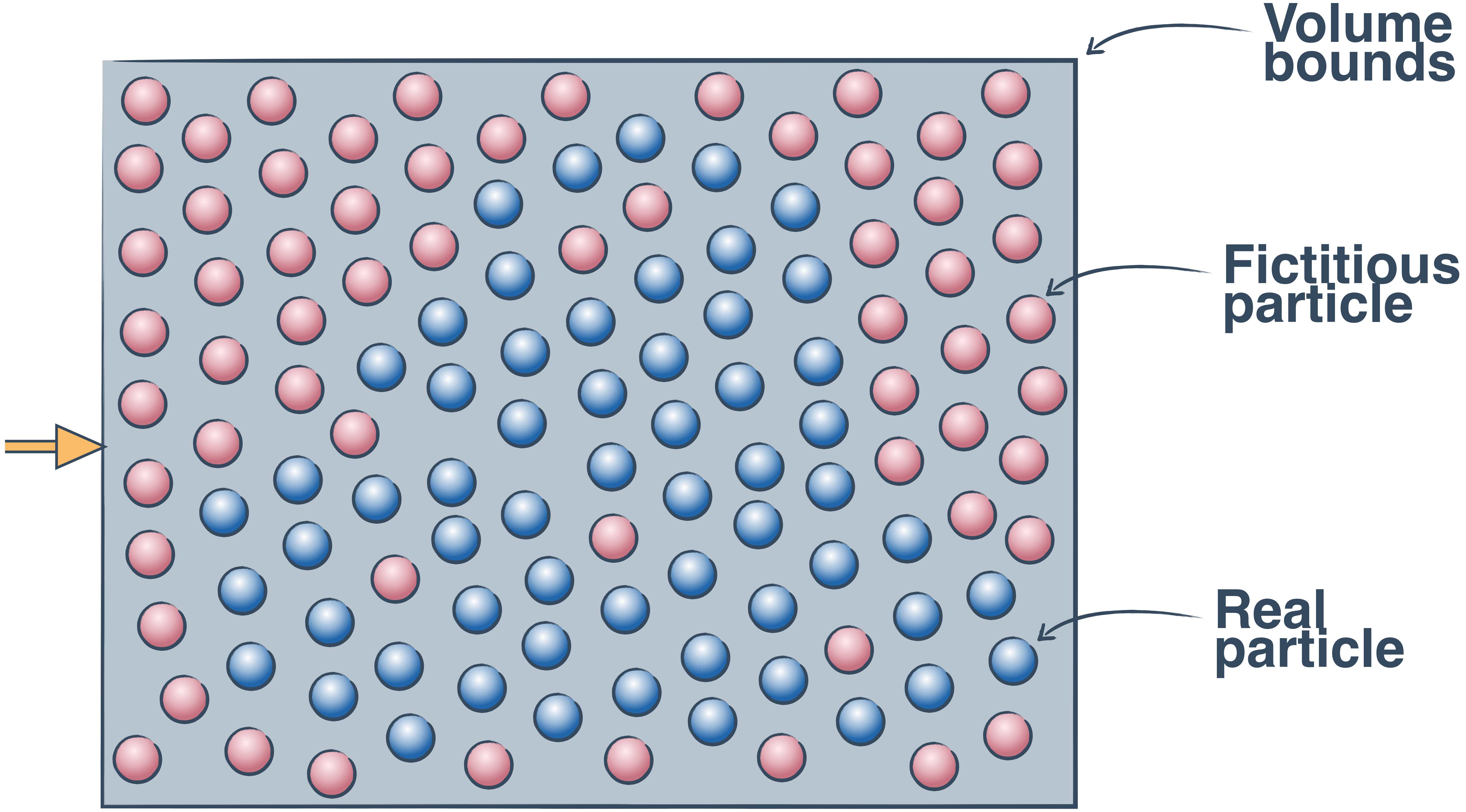

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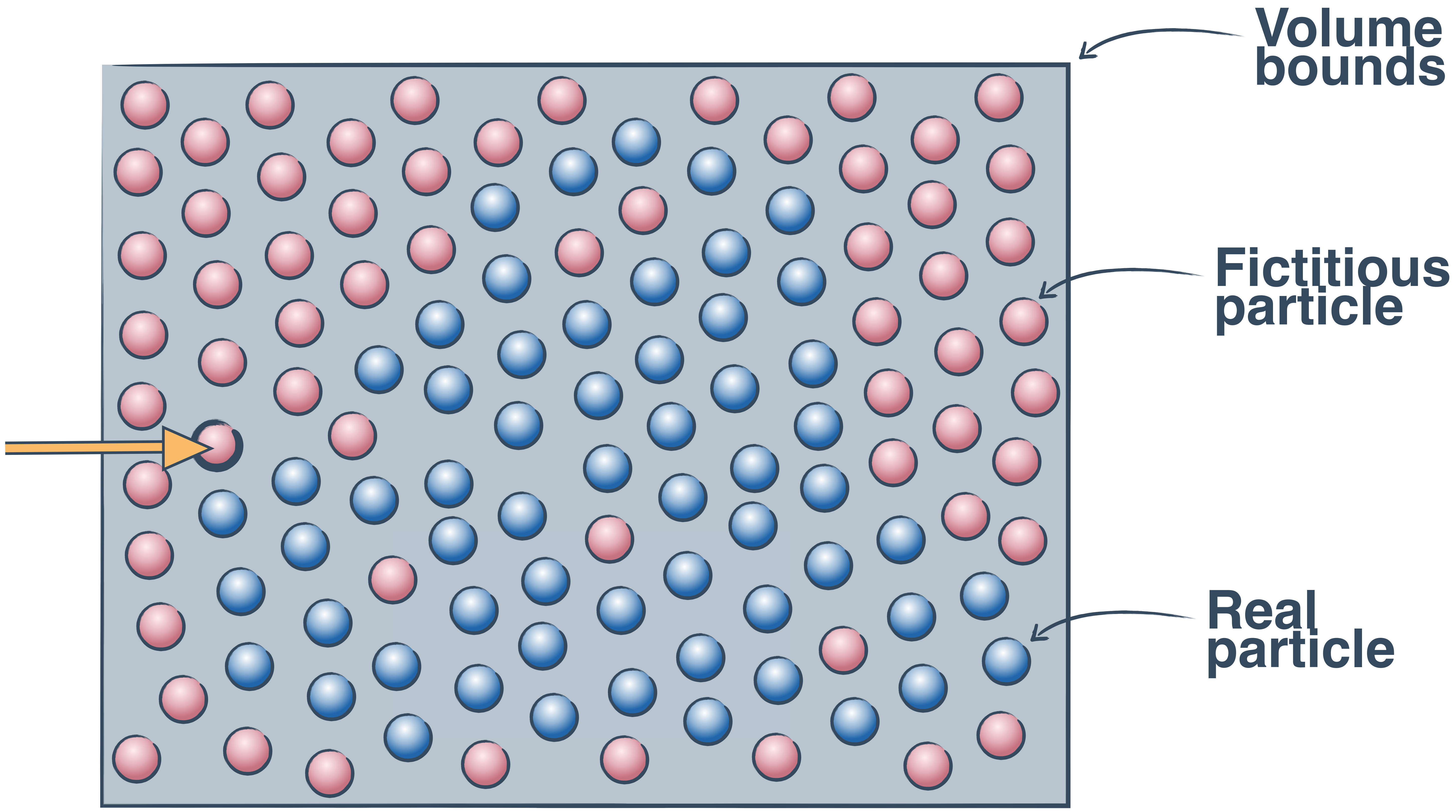

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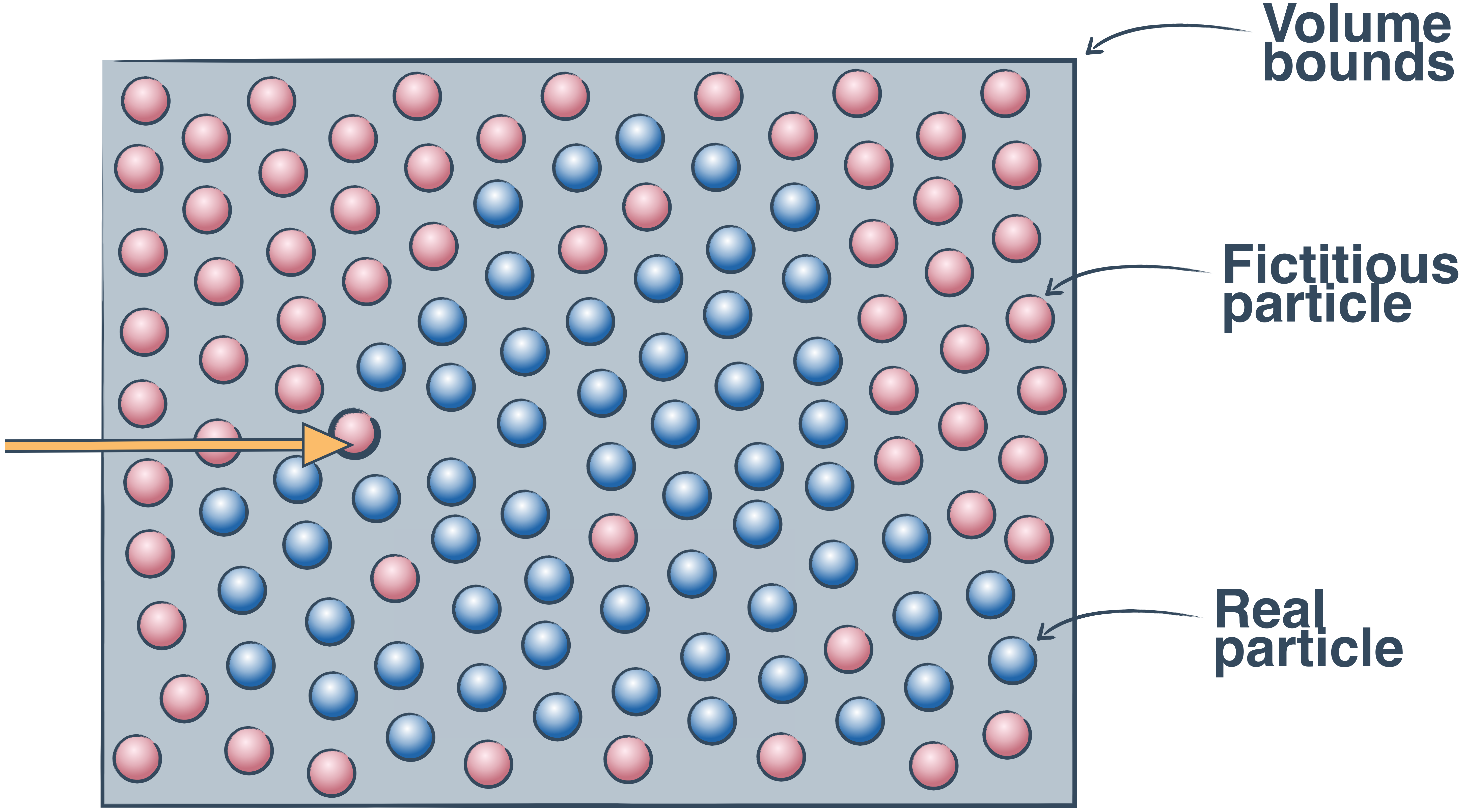

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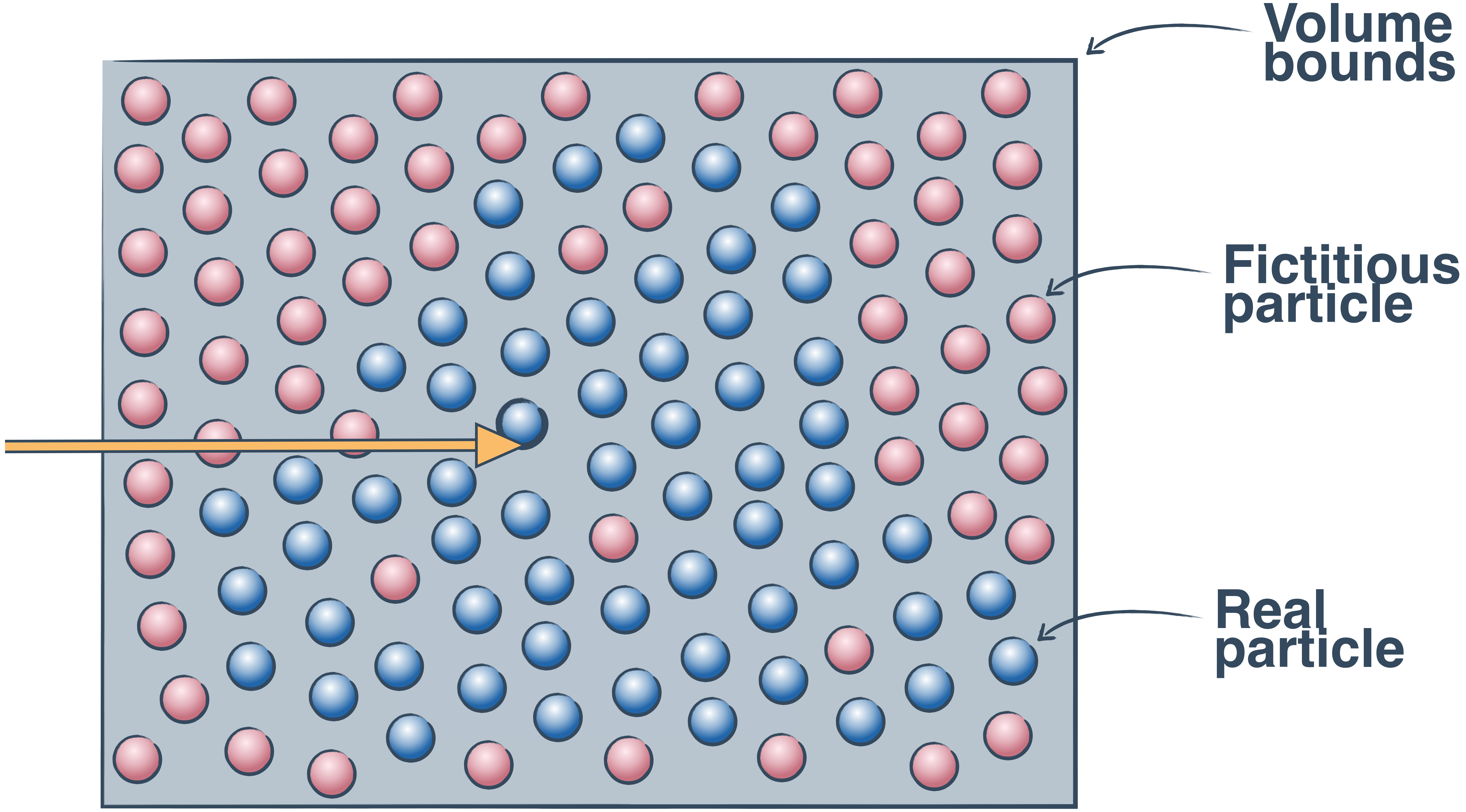

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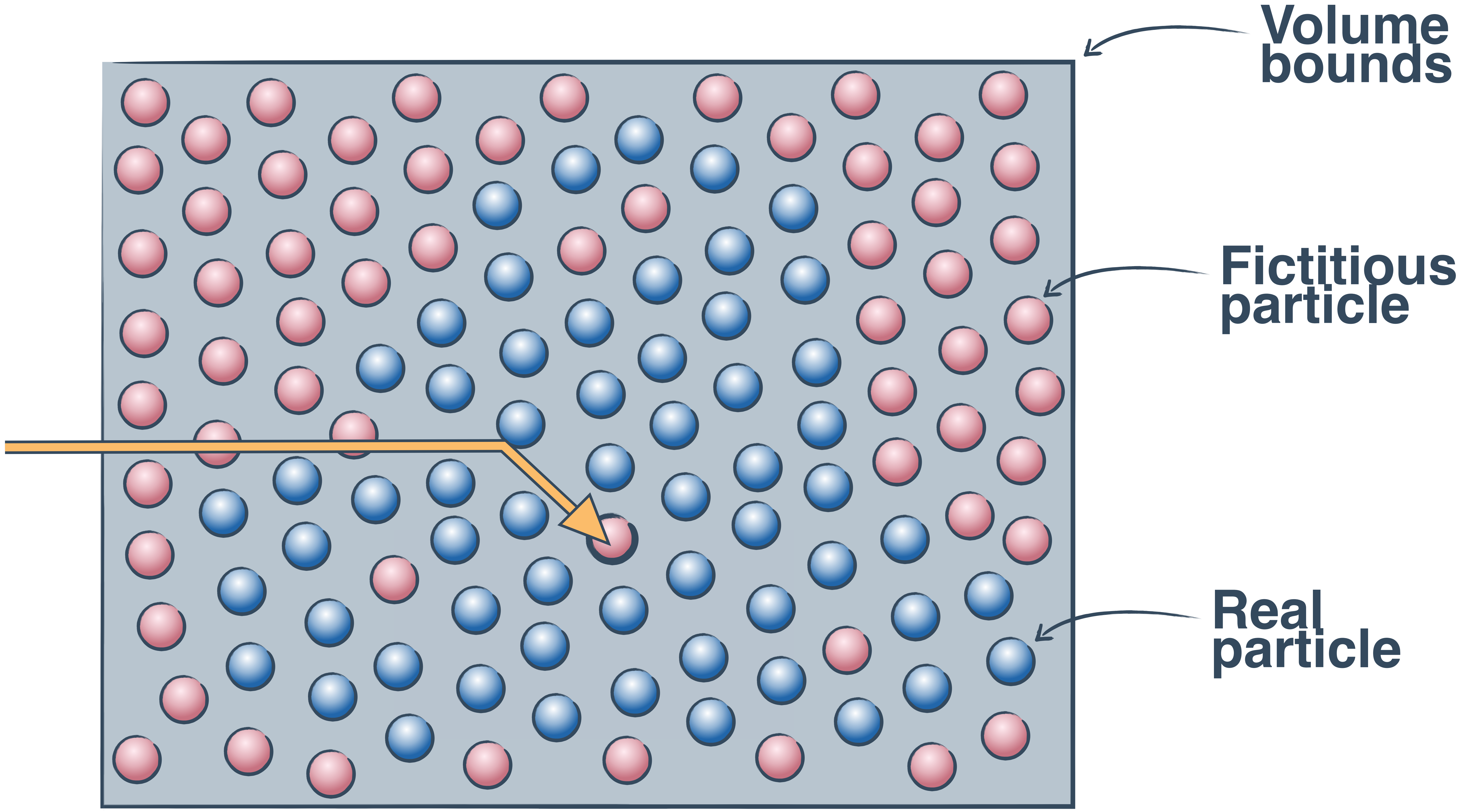

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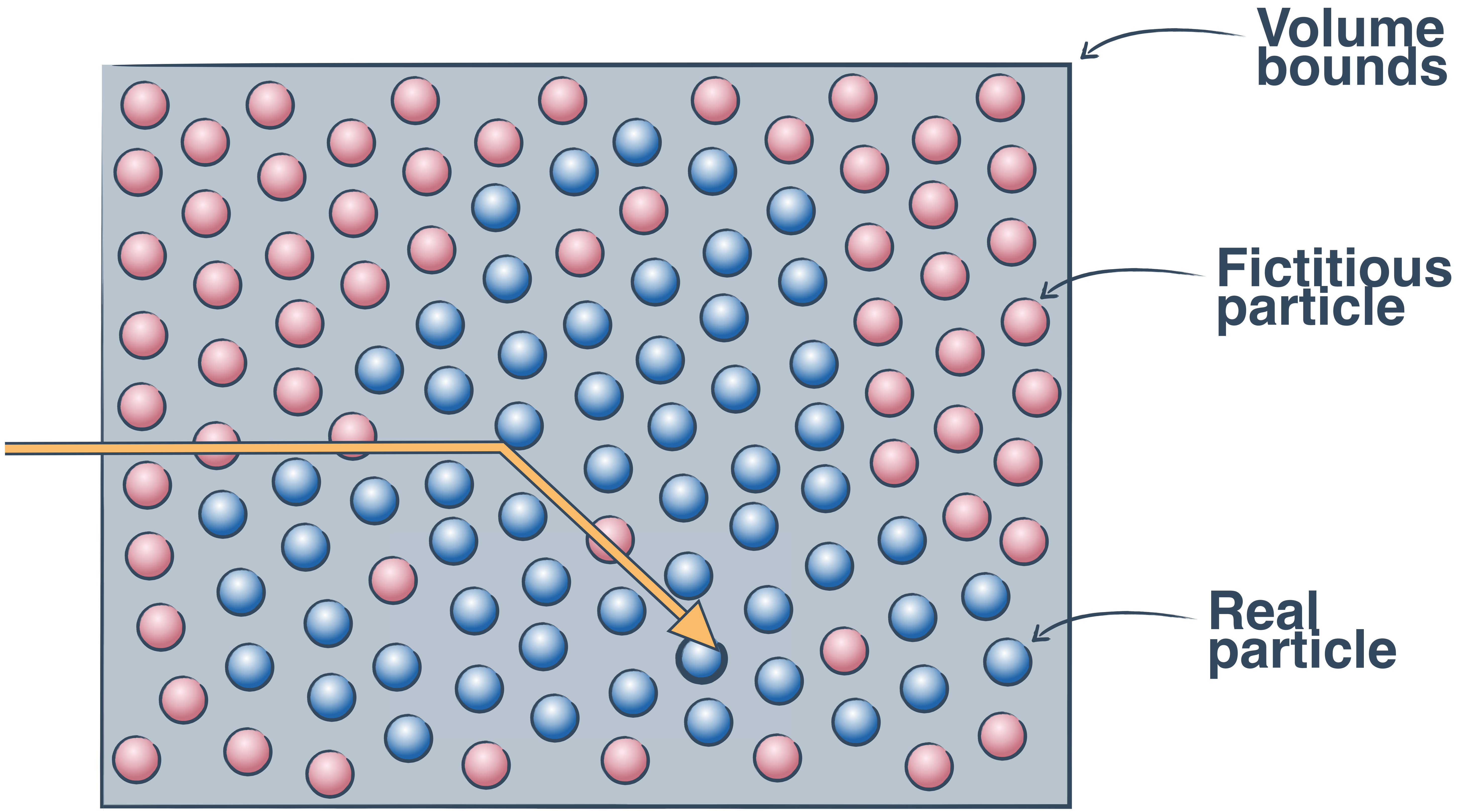

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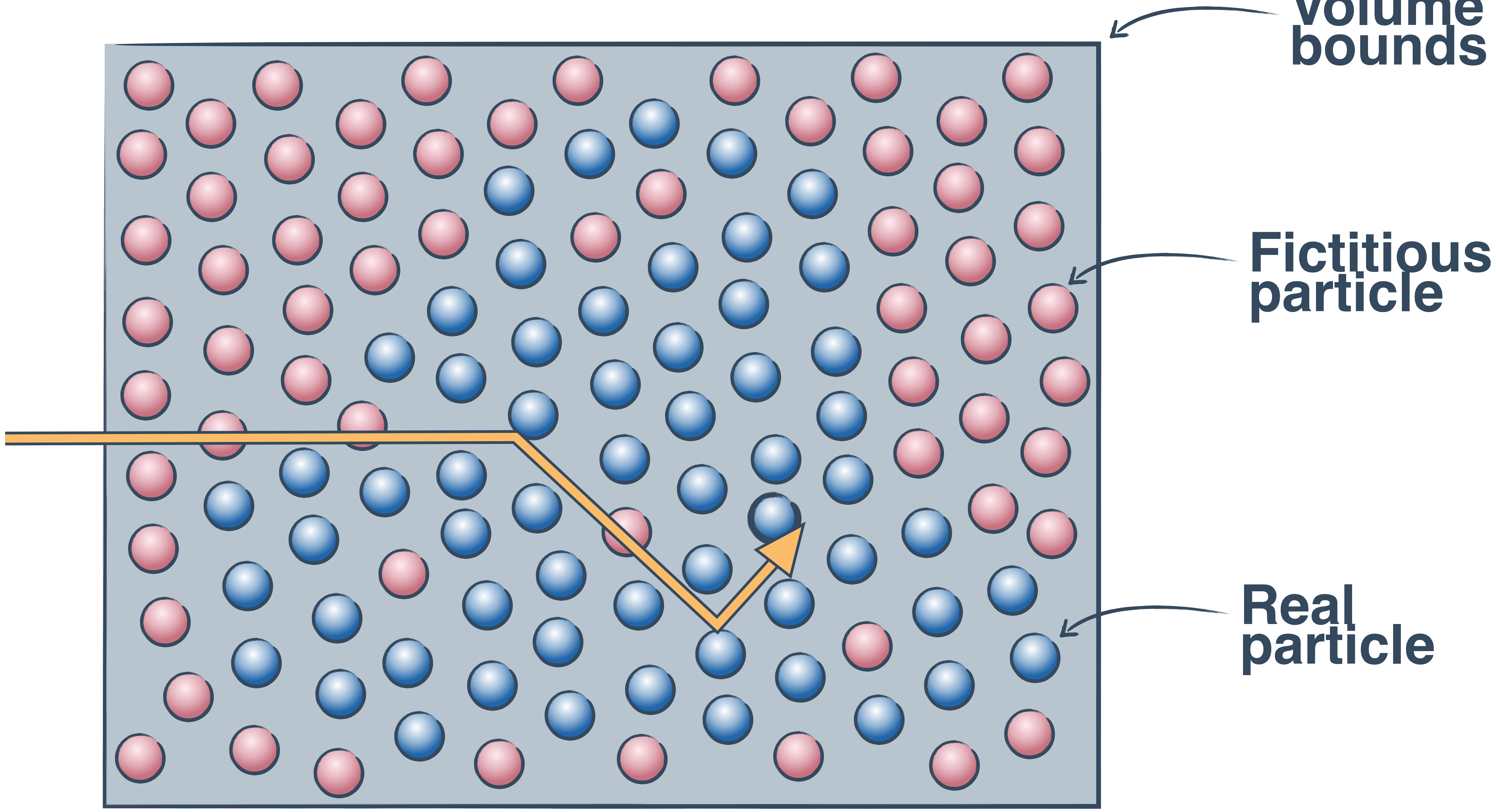

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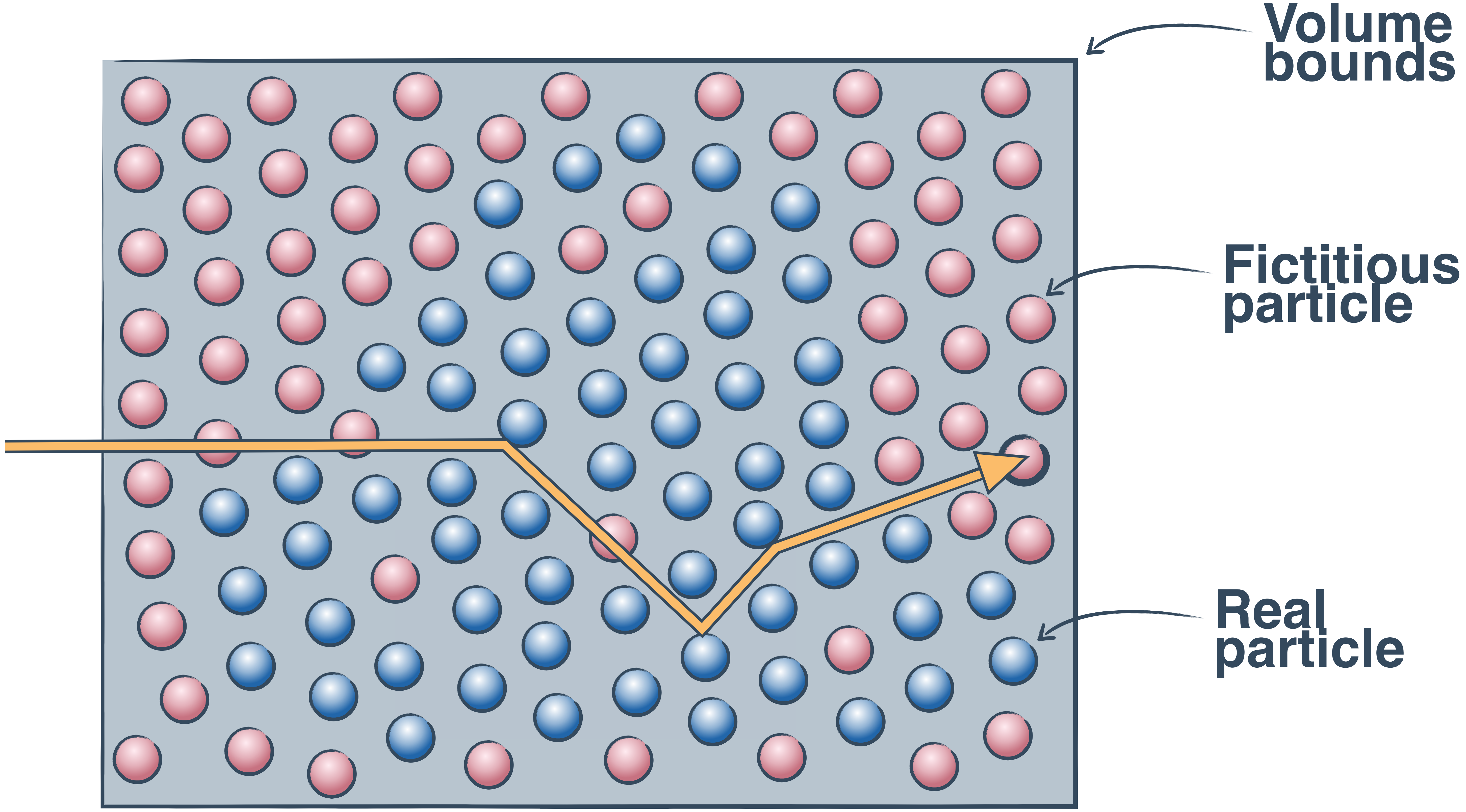

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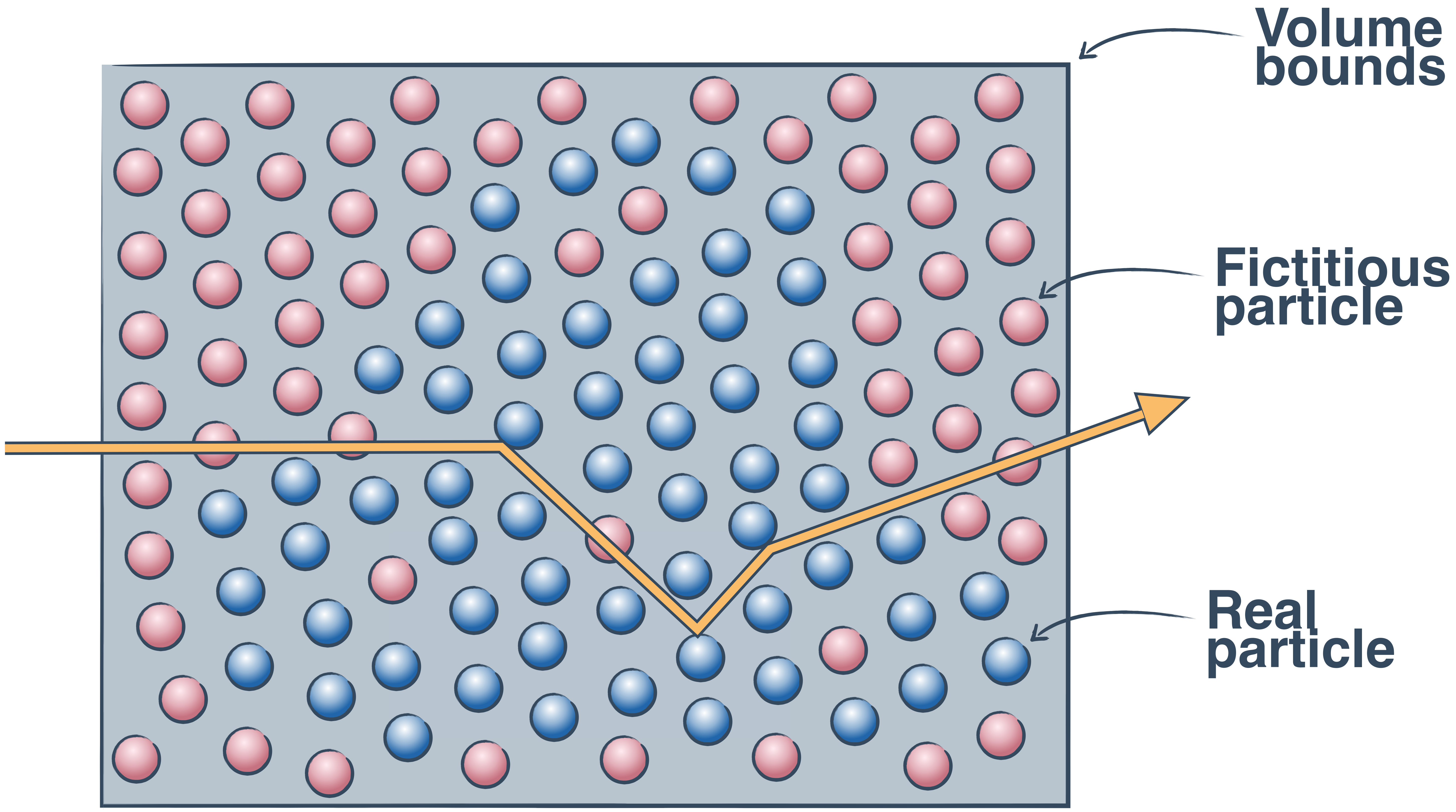

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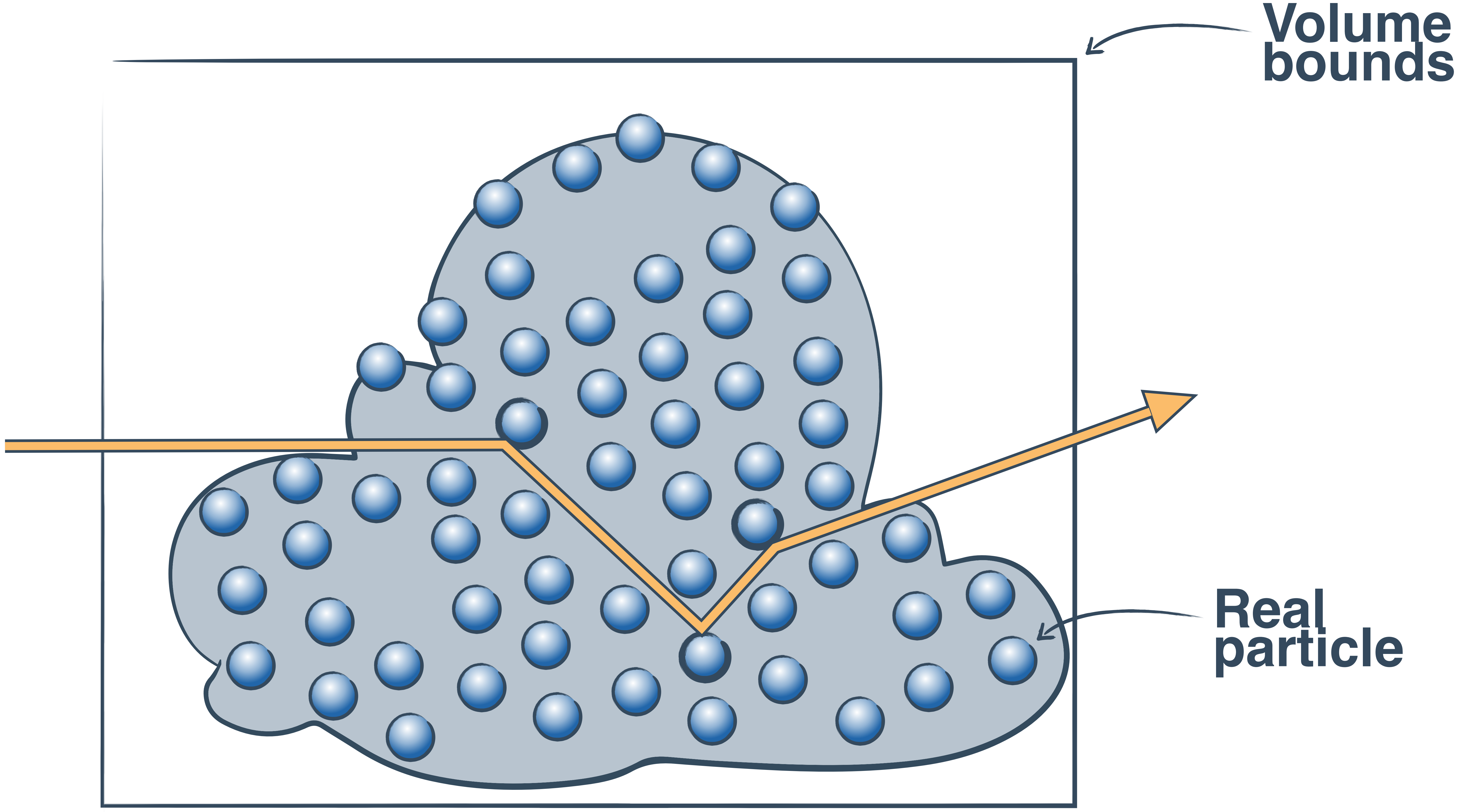
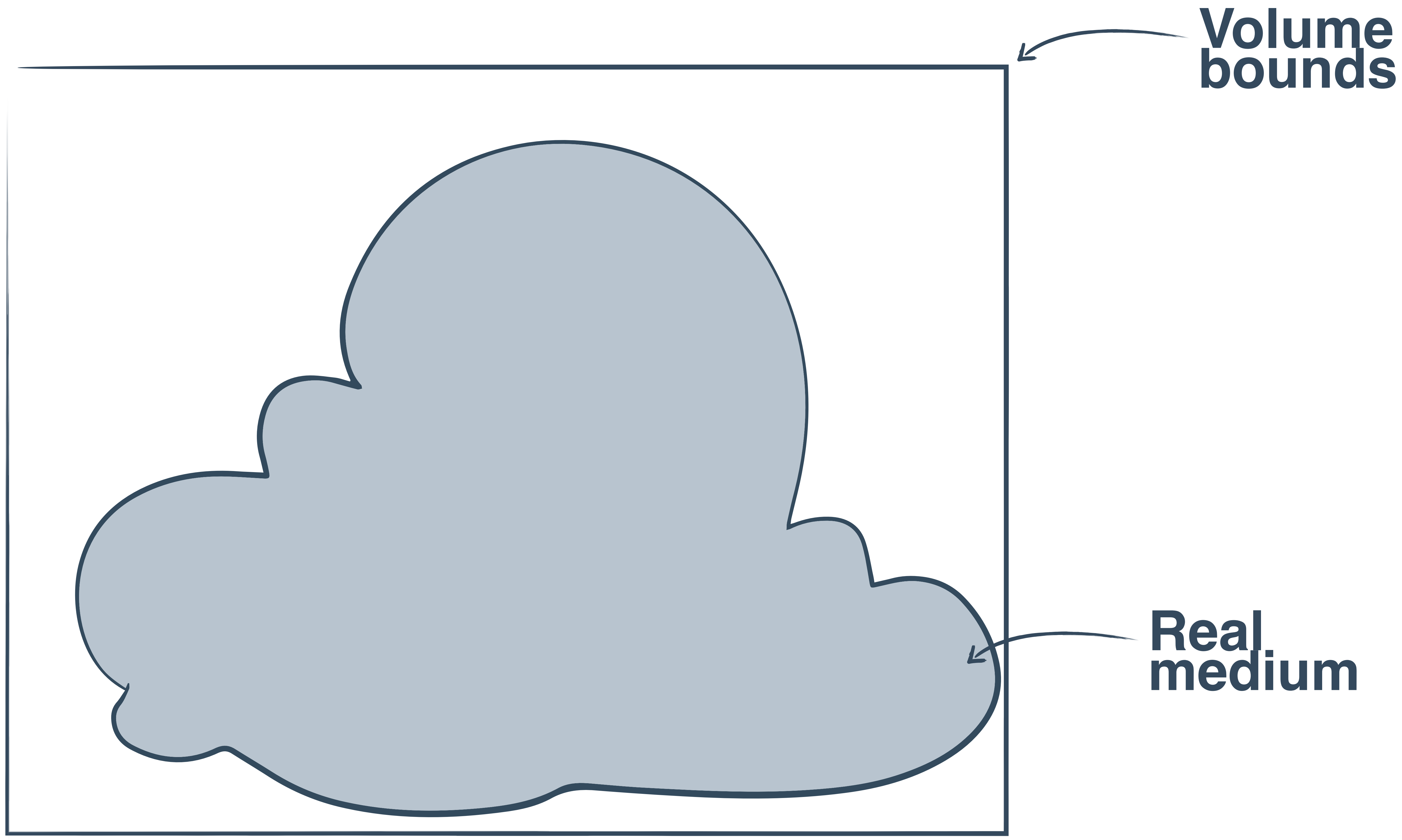


Stochastic Sampling



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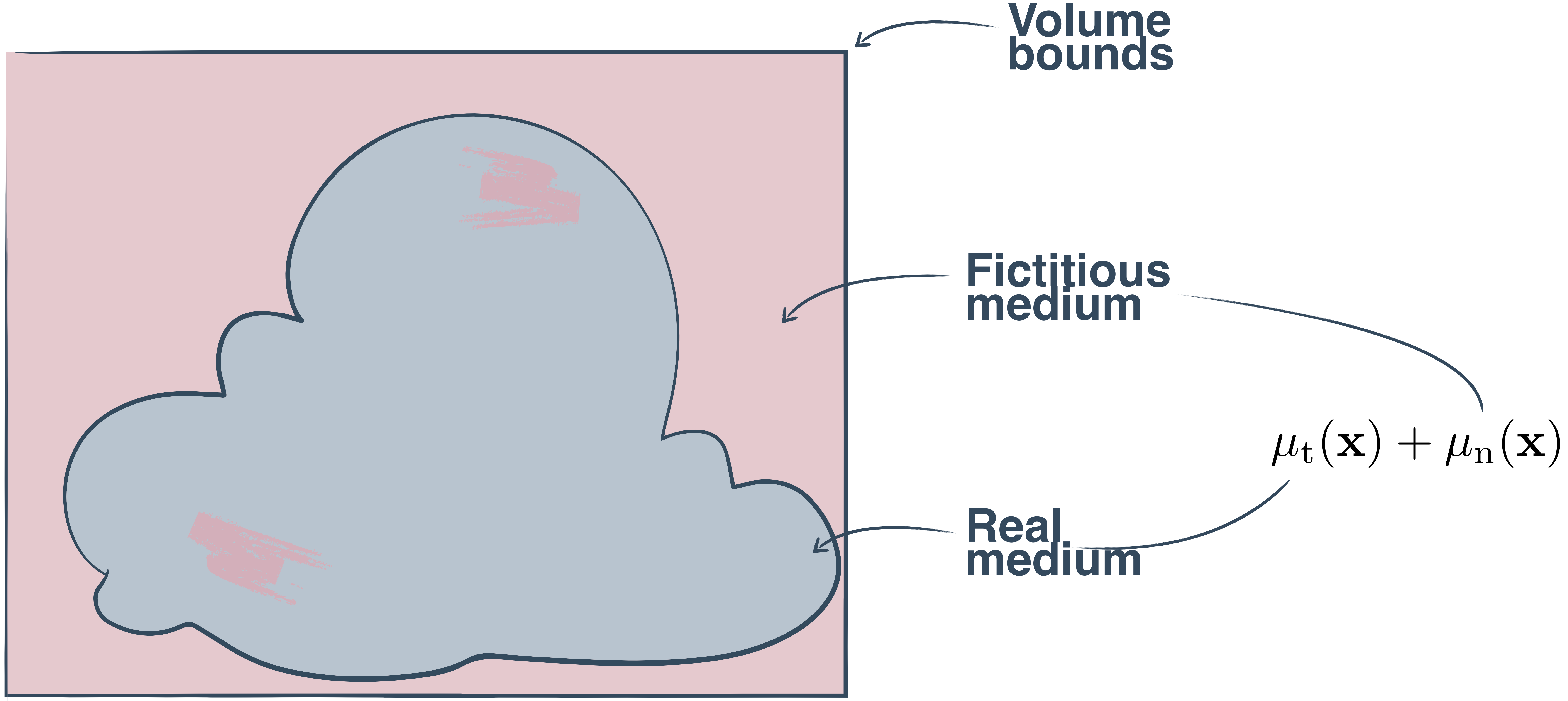

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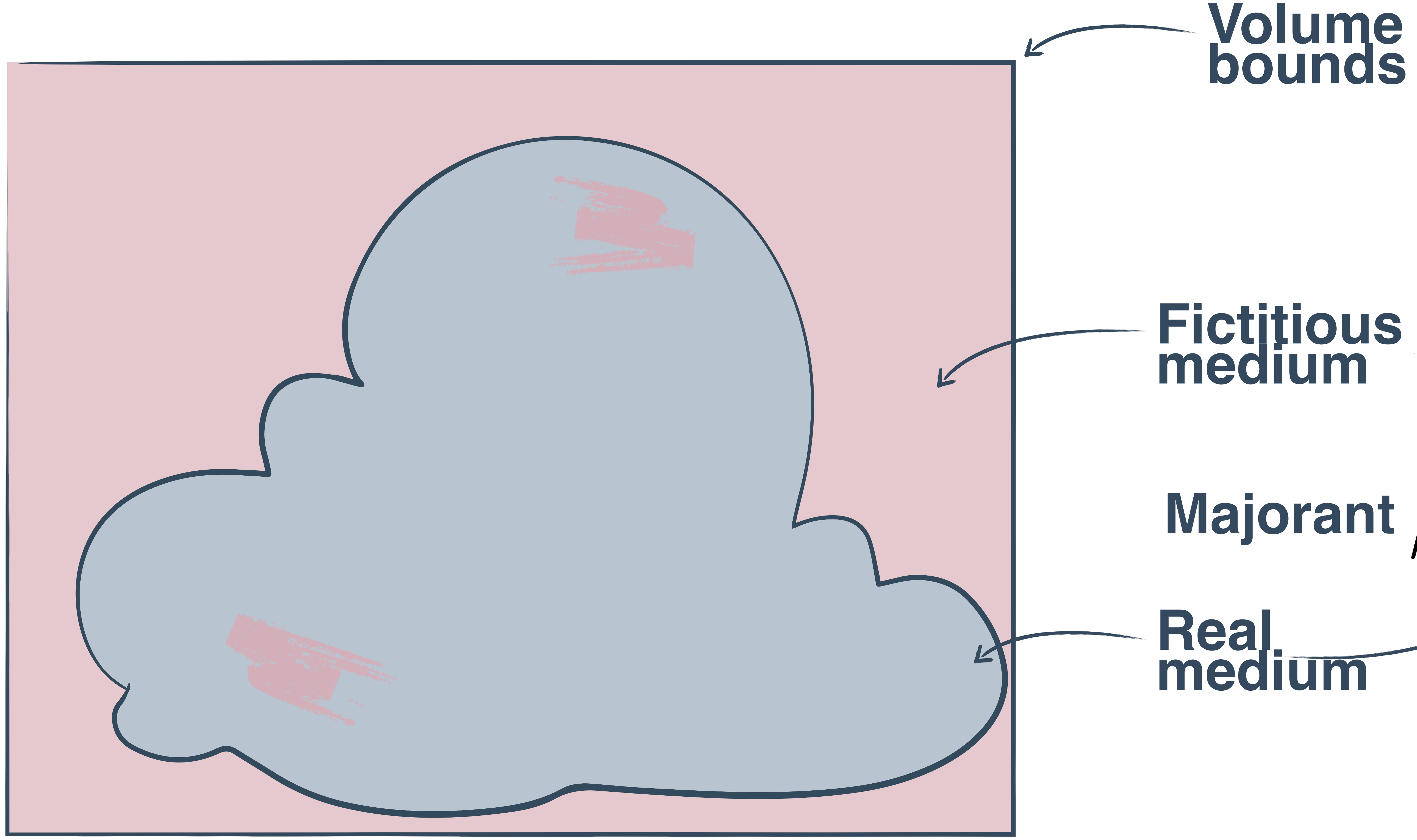

Majorant $\bar{\mu} = \mu_{\mathrm{t}}(\mathbf{x}) + \mu_{\mathrm{n}}(\mathbf{x})$

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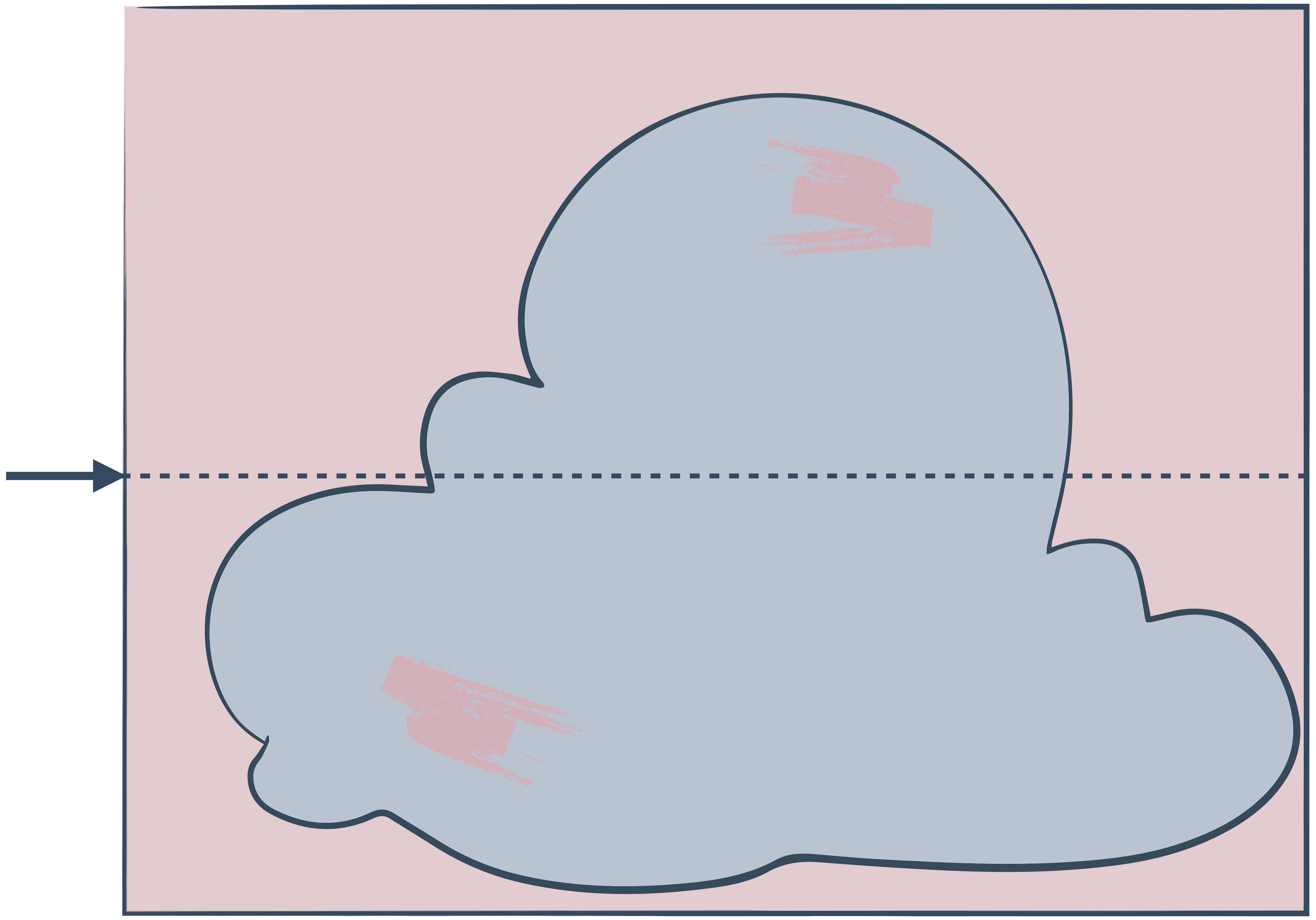

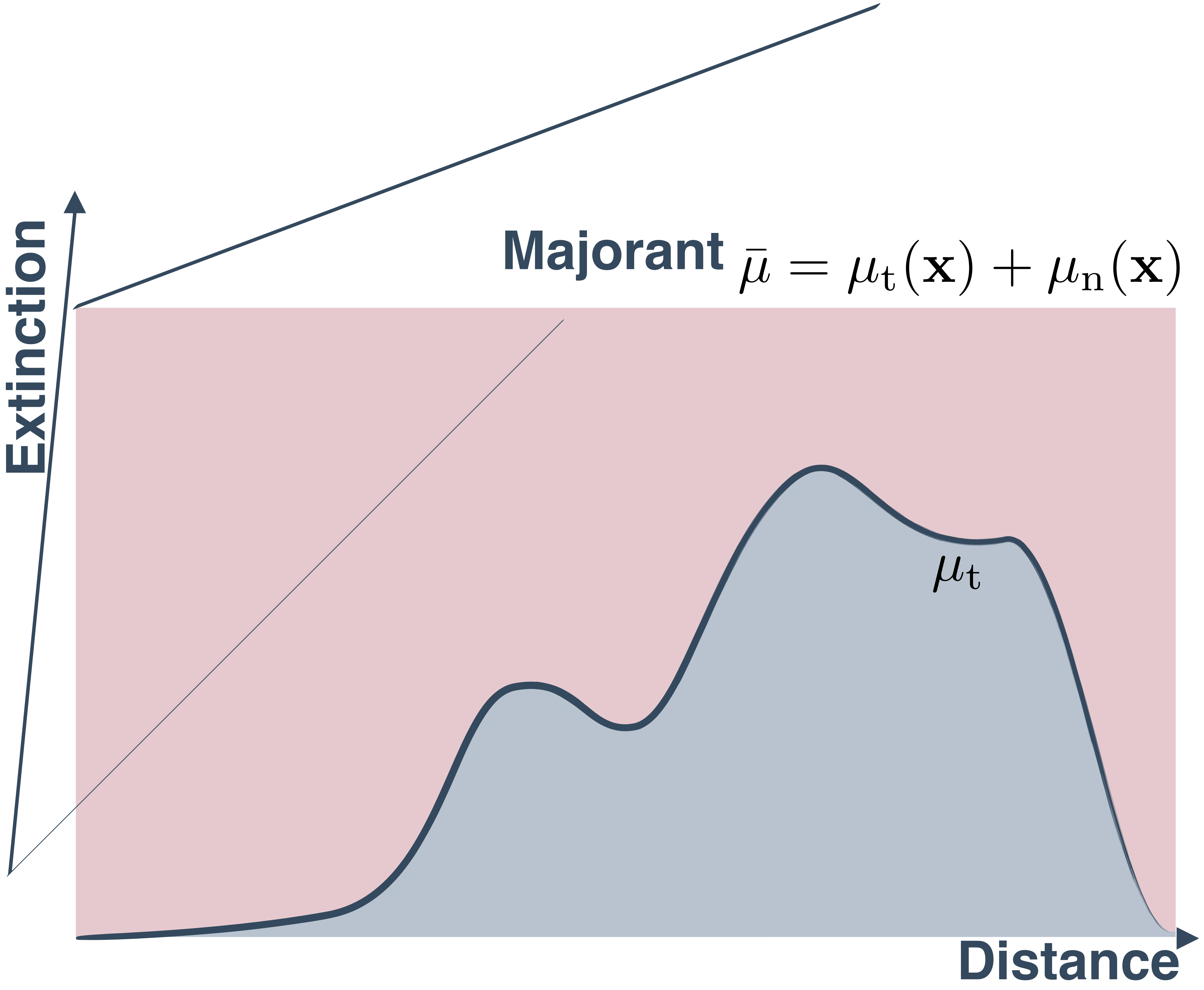

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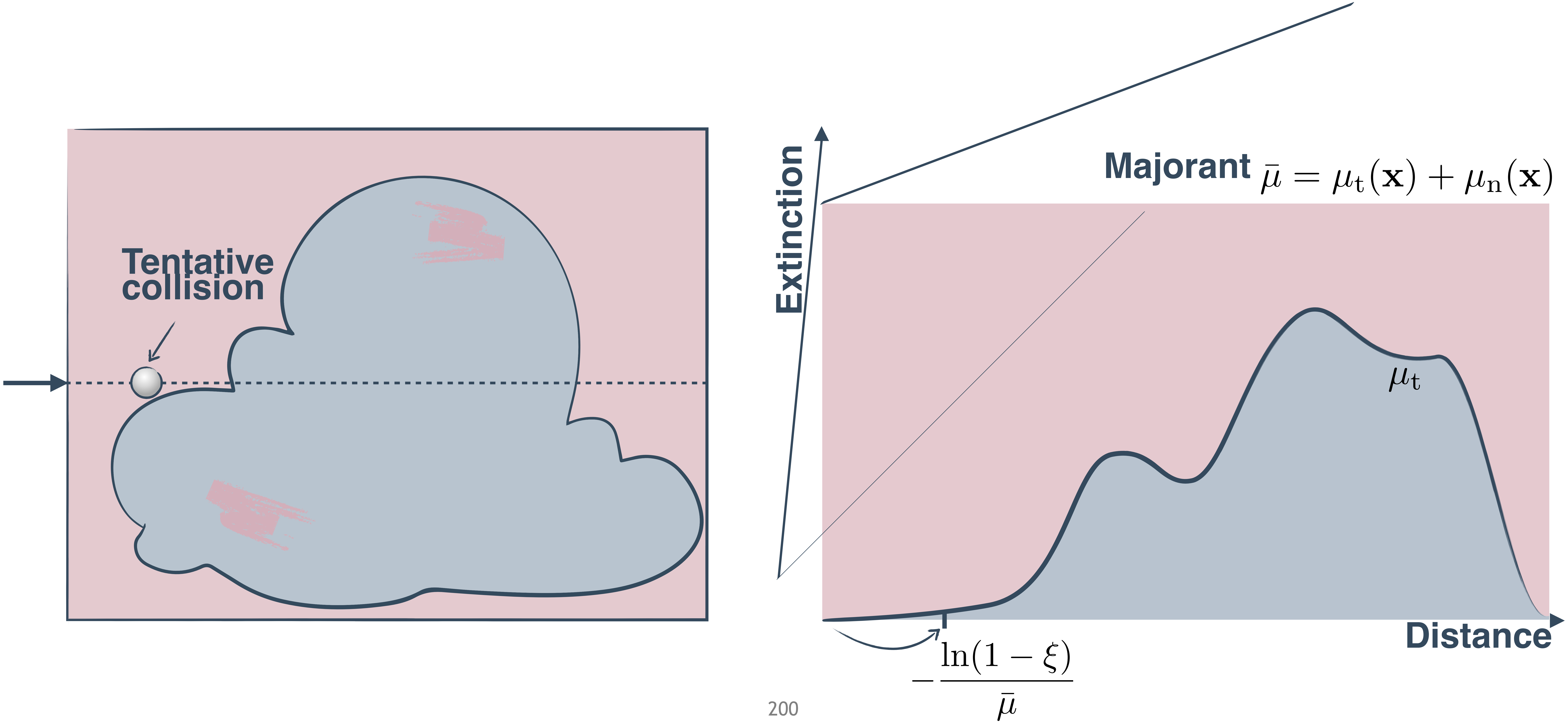

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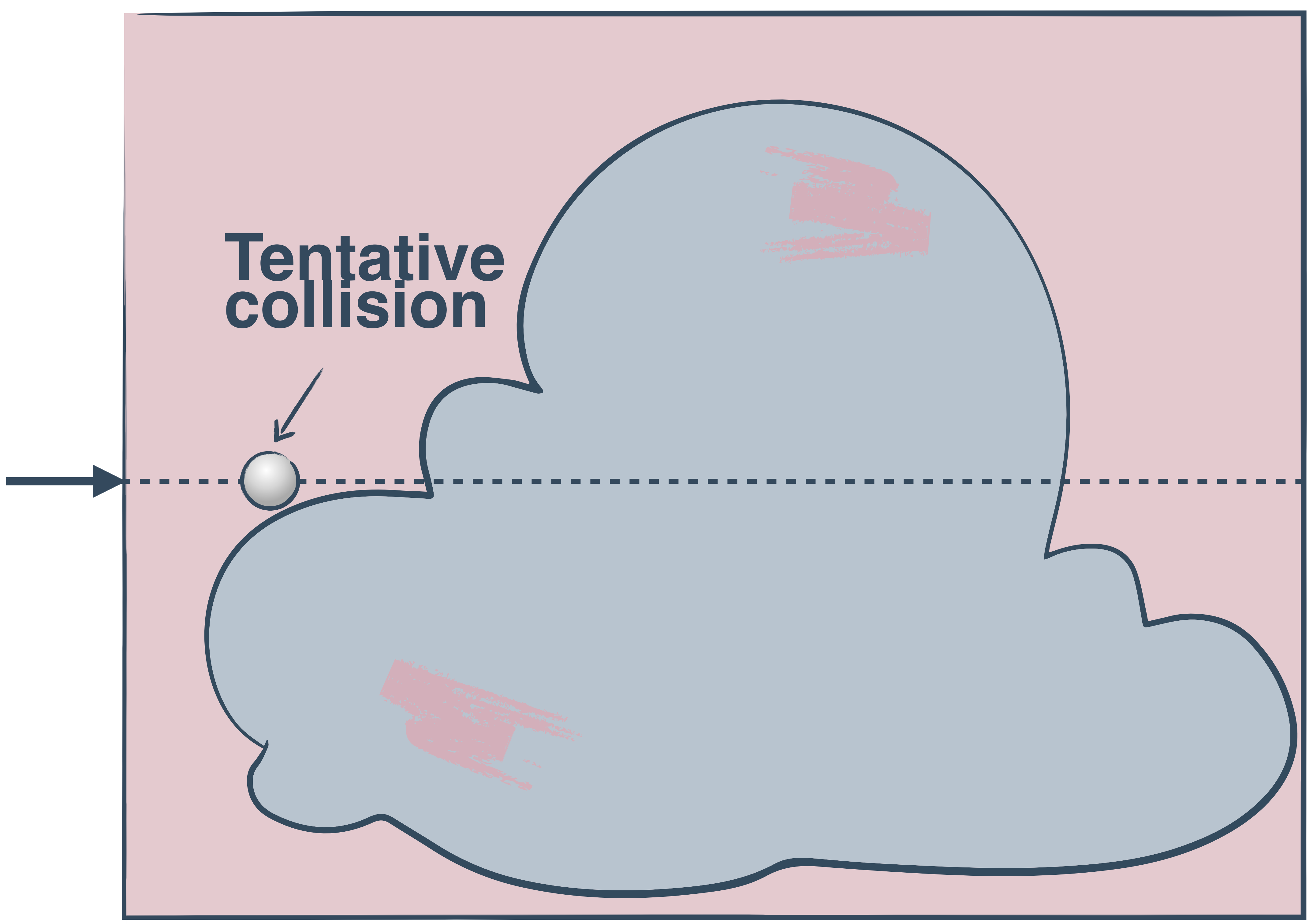

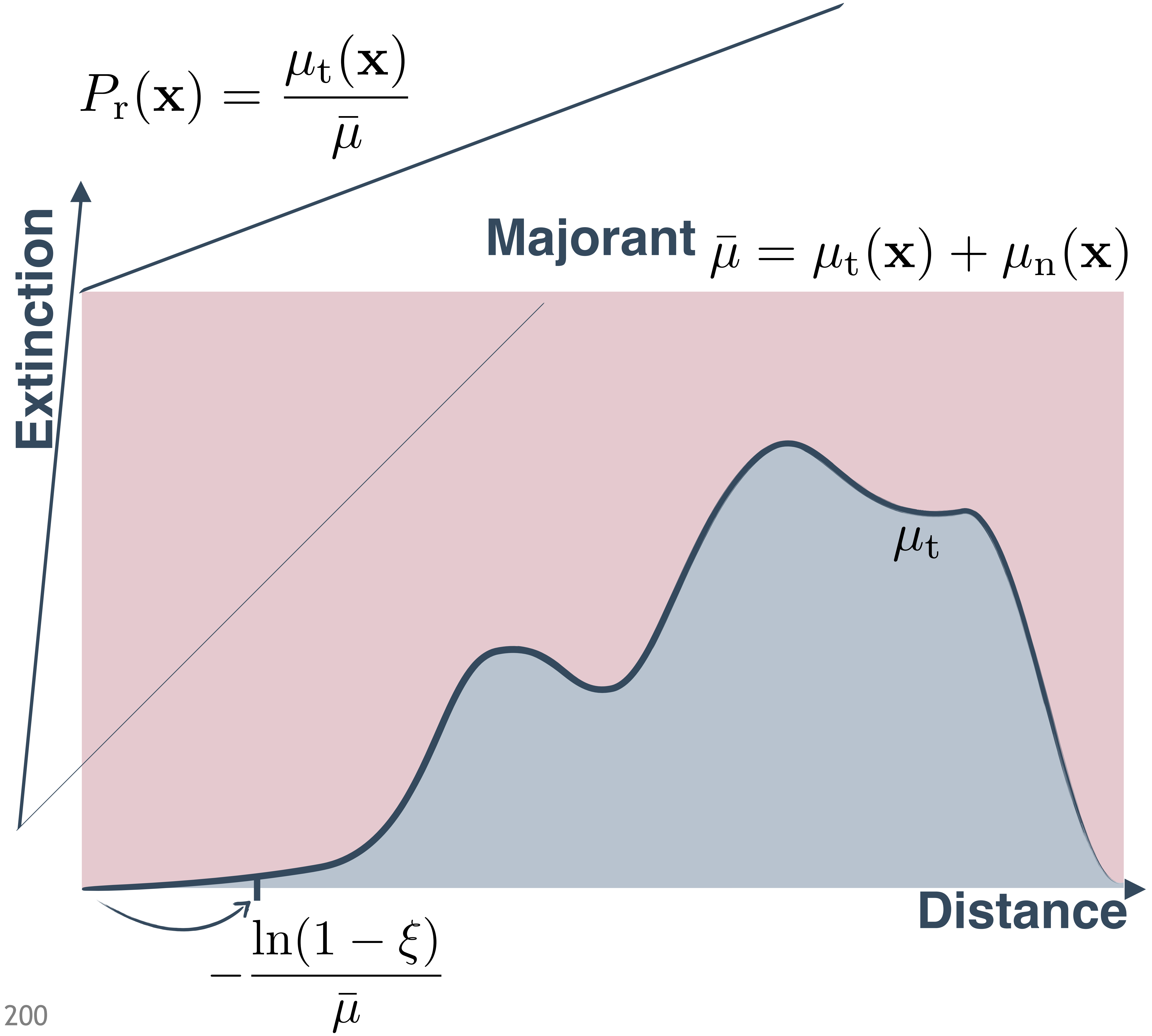

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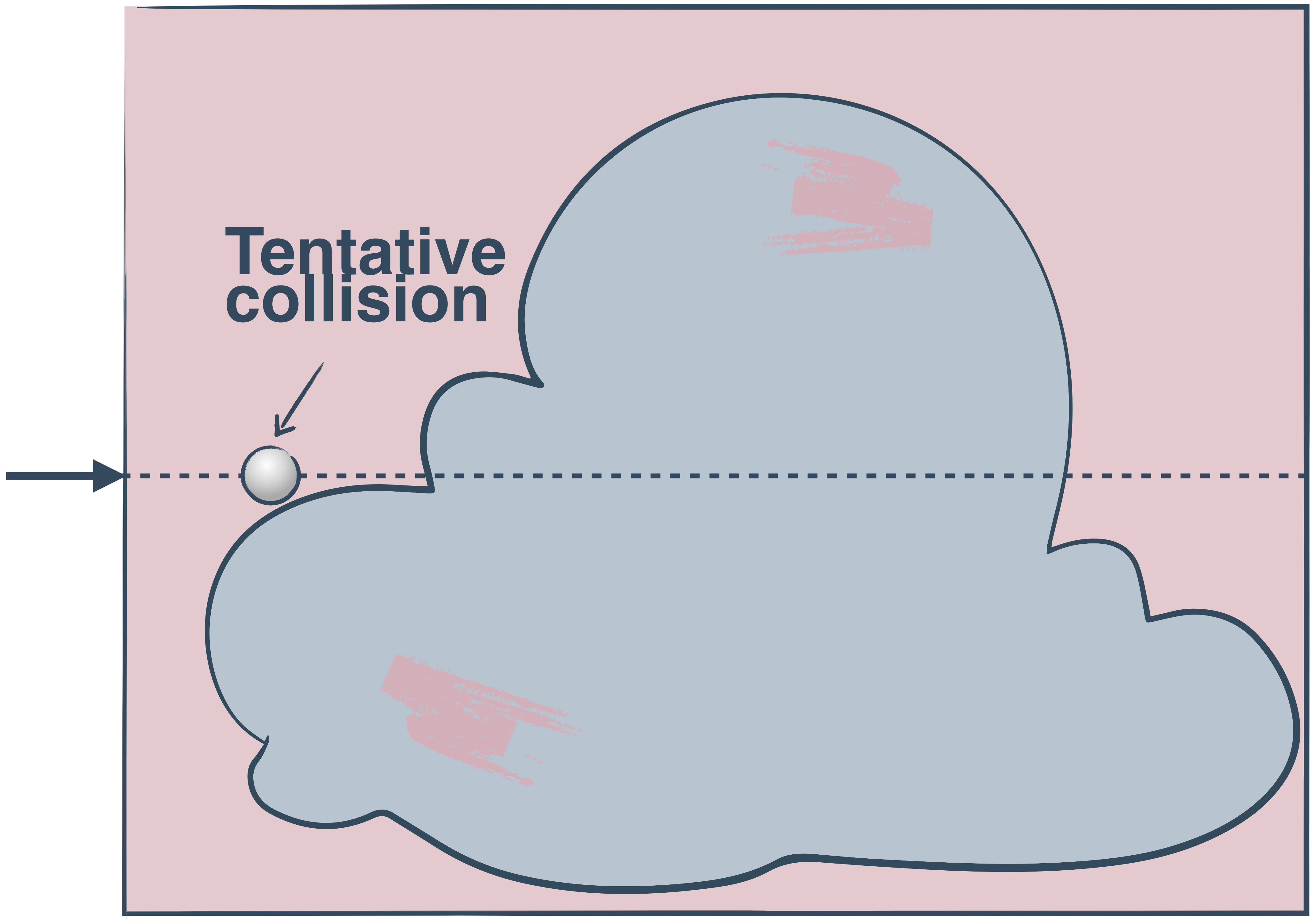

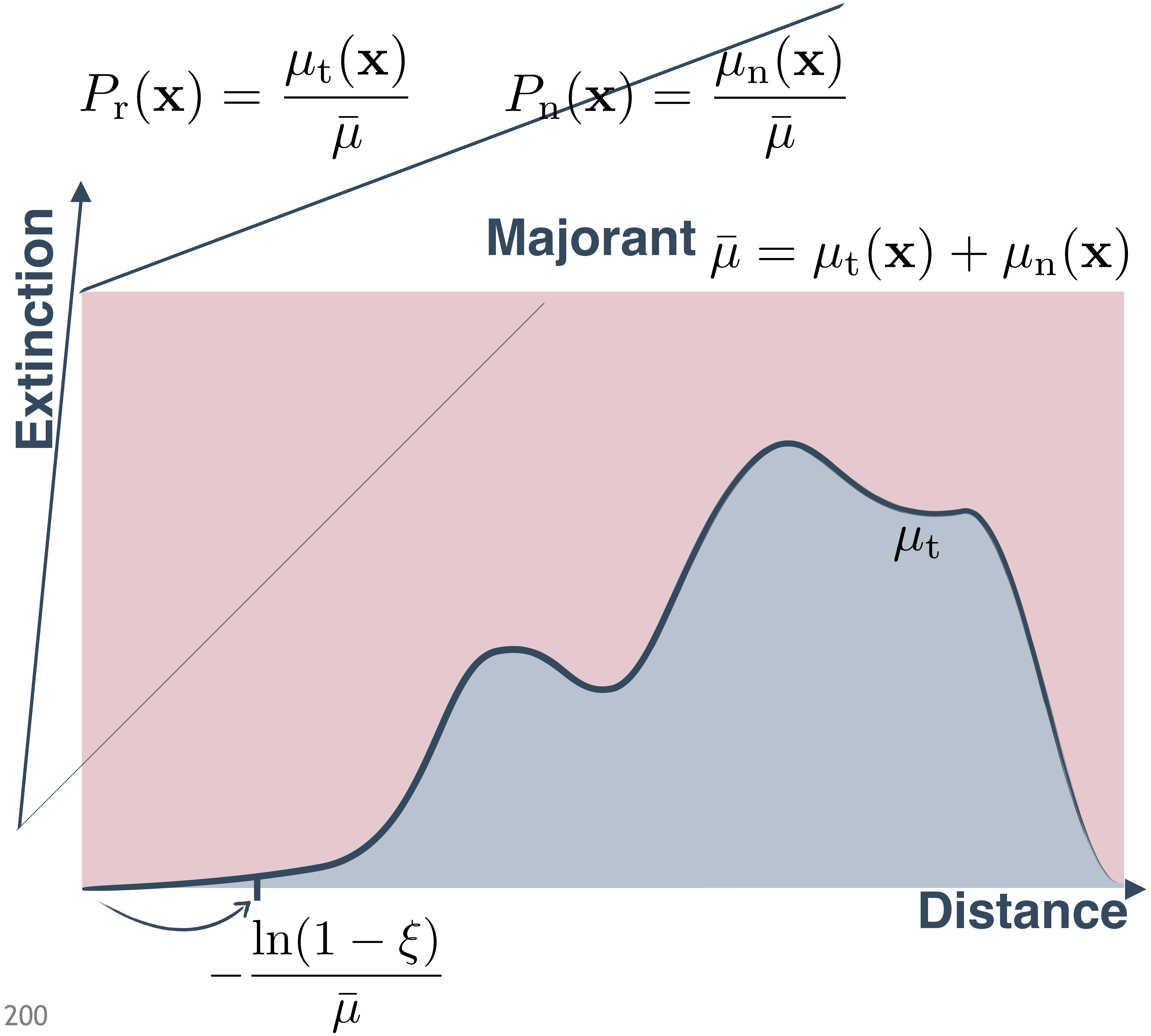

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$$-\frac{\ln(1-\xi)}{\bar{\mu}}$$

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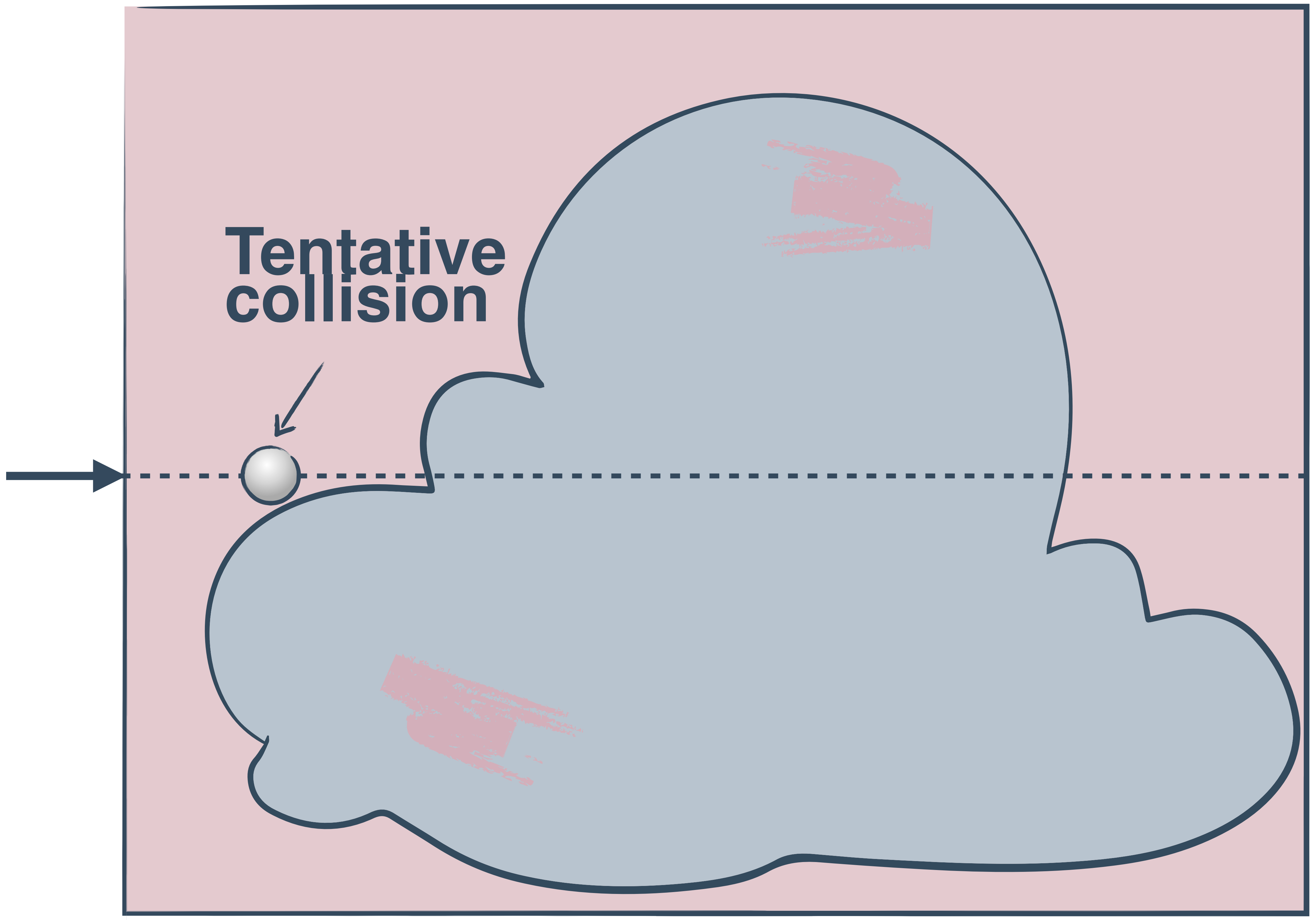

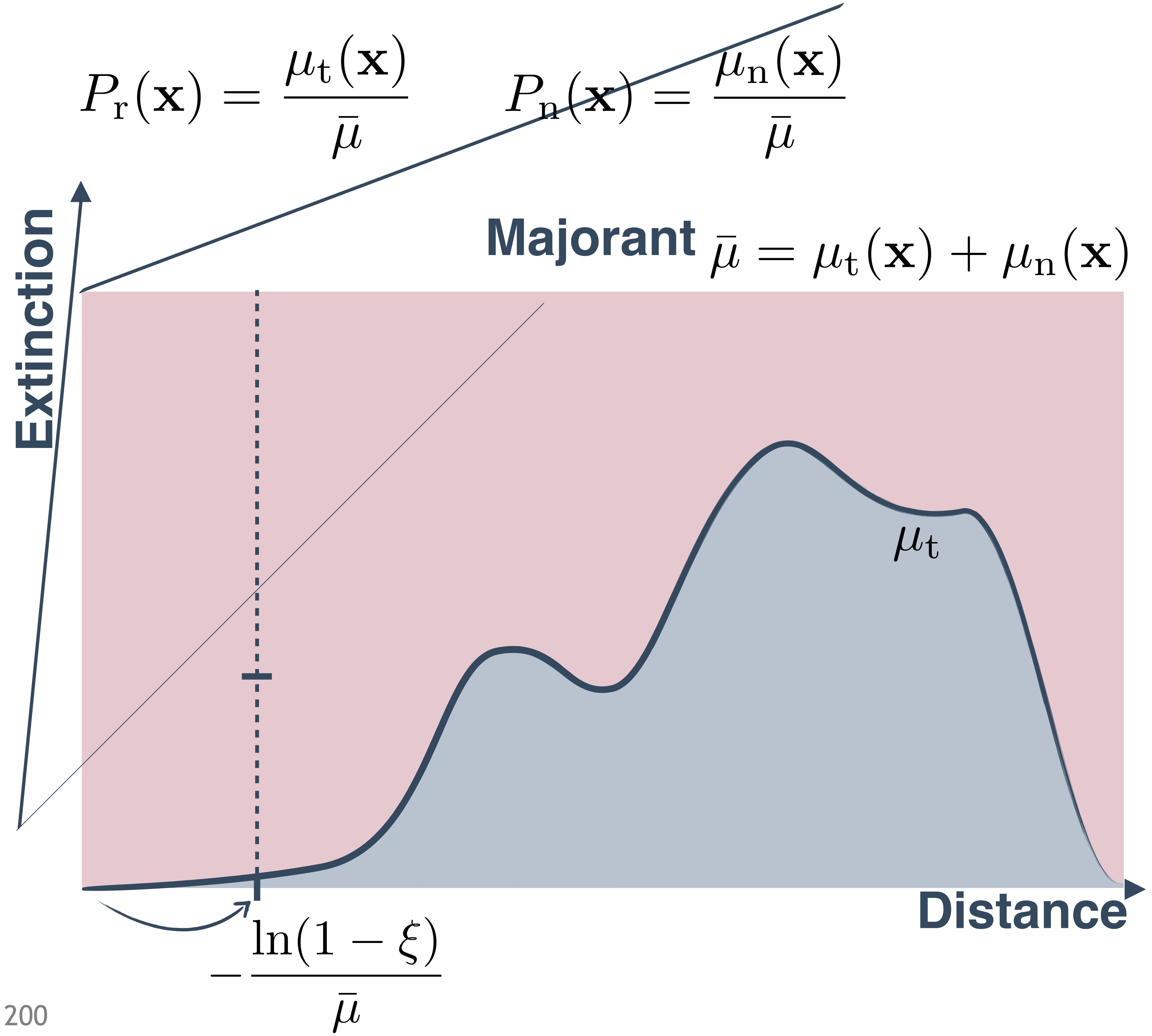

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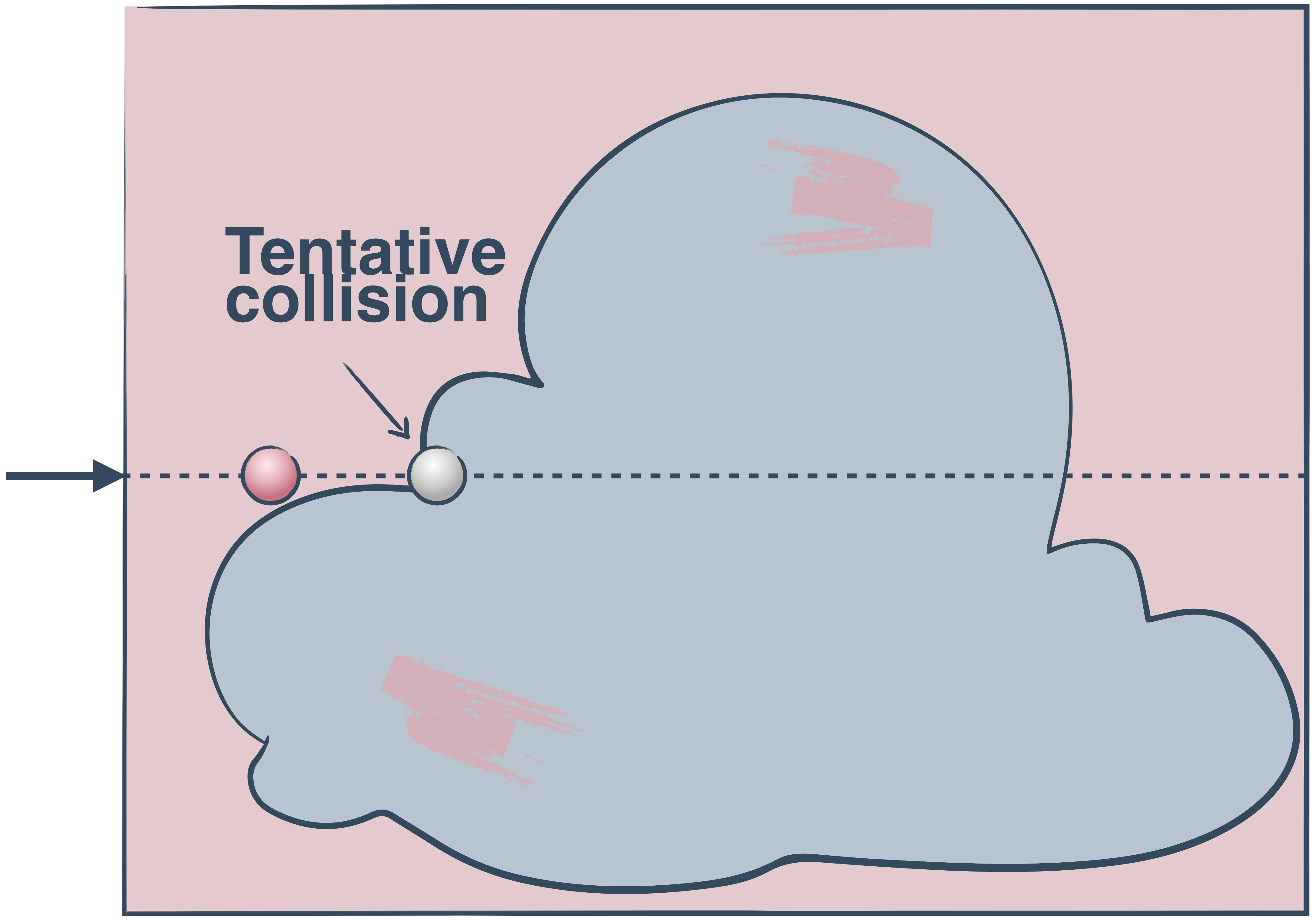
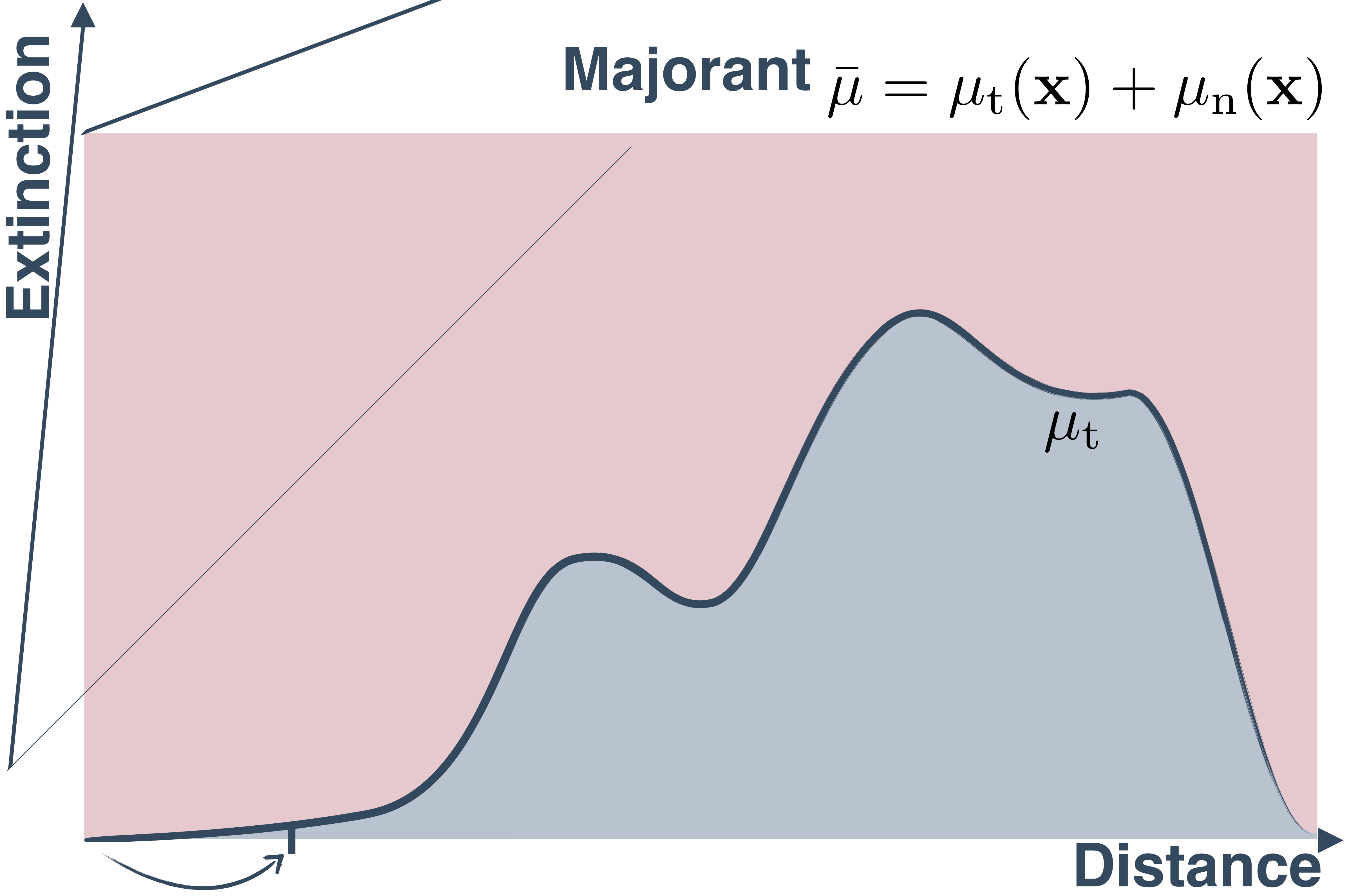
Majorant $\bar{\mu} = \mu_{\mathrm{t}}(\mathbf{x}) + \mu_{\mathrm{n}}(\mathbf{x})$

$$-\frac{\ln(1-\xi)}{\bar{\mu}}$$

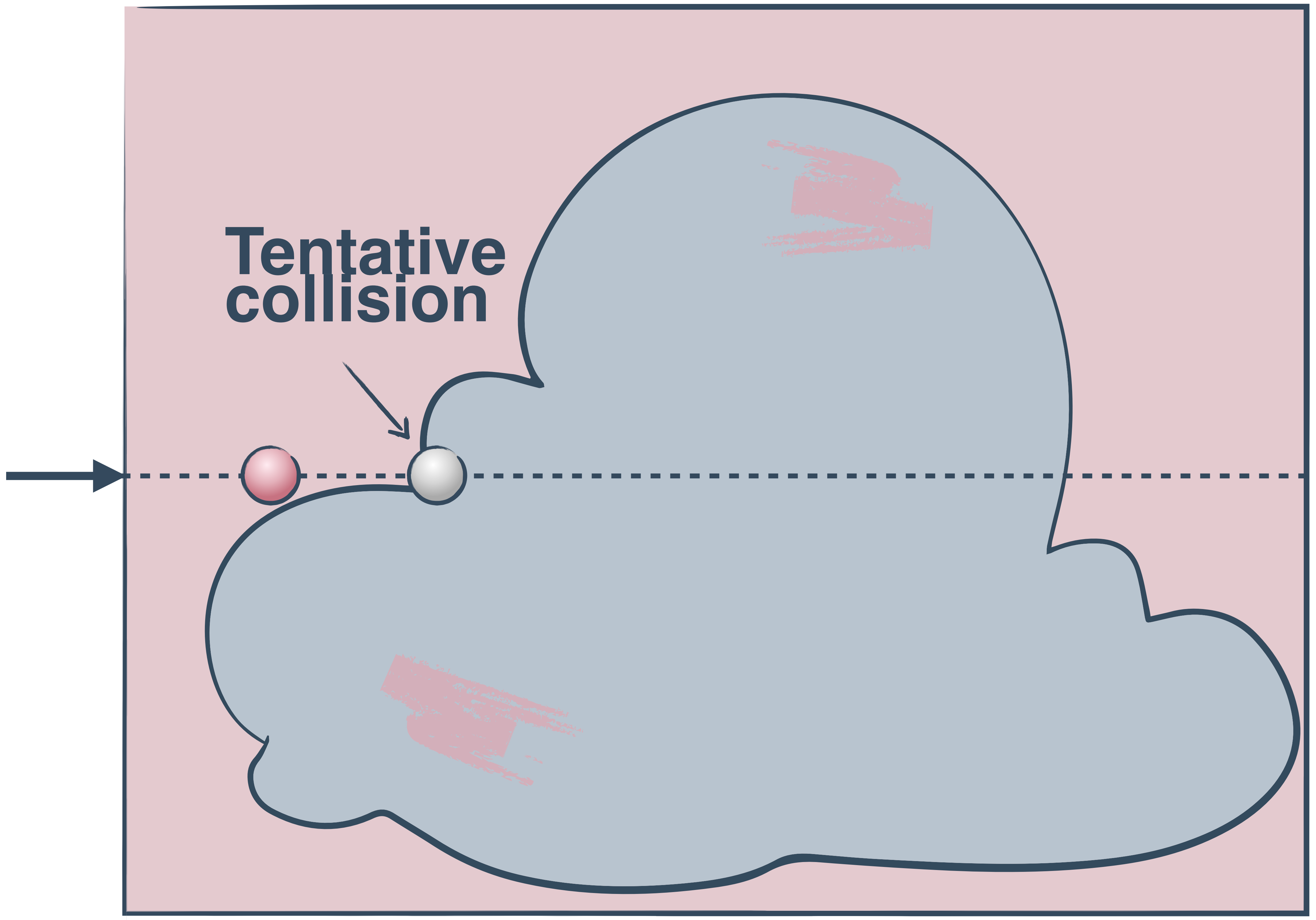
Stochastic Sampling



$$P_{\mathrm{r}}(\mathbf{x}) = \frac{\mu_{\mathrm{t}}(\mathbf{x})}{\bar{\mu}} \qquad P_{\mathrm{n}}(\mathbf{x}) = \frac{\mu_{\mathrm{n}}(\mathbf{x})}{\bar{\mu}}$$



Stochastic Sampling



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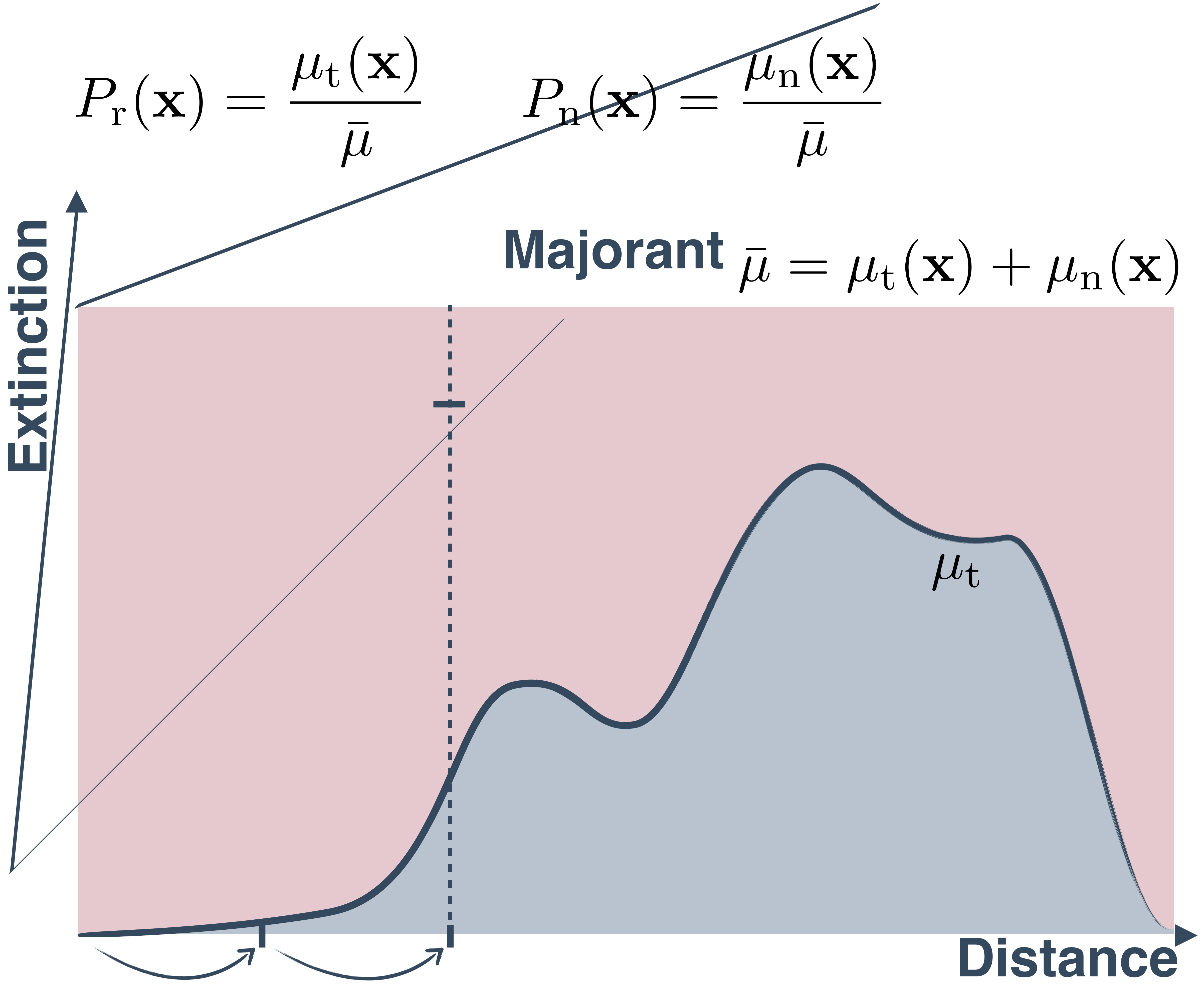
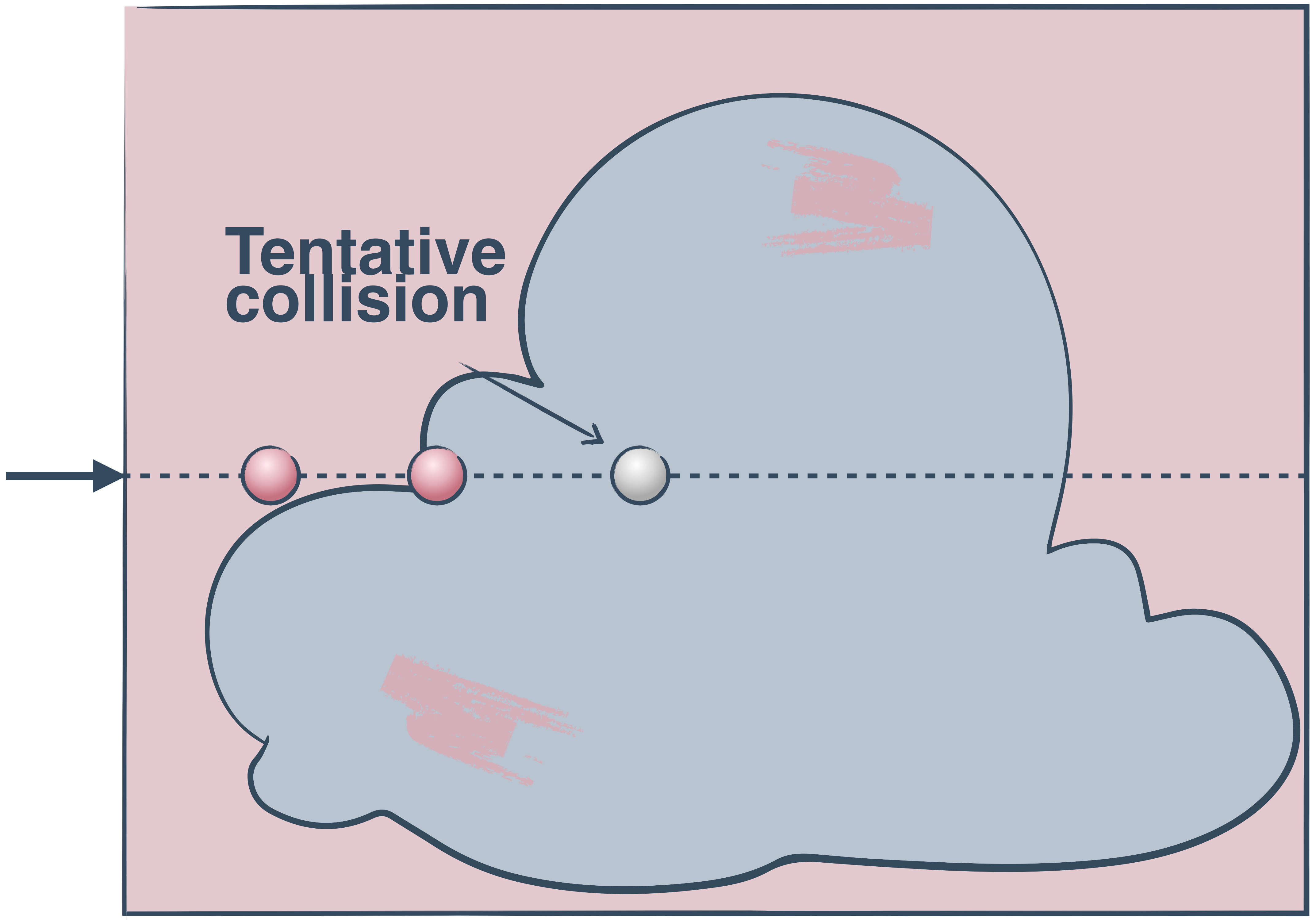


Stochastic Sampling



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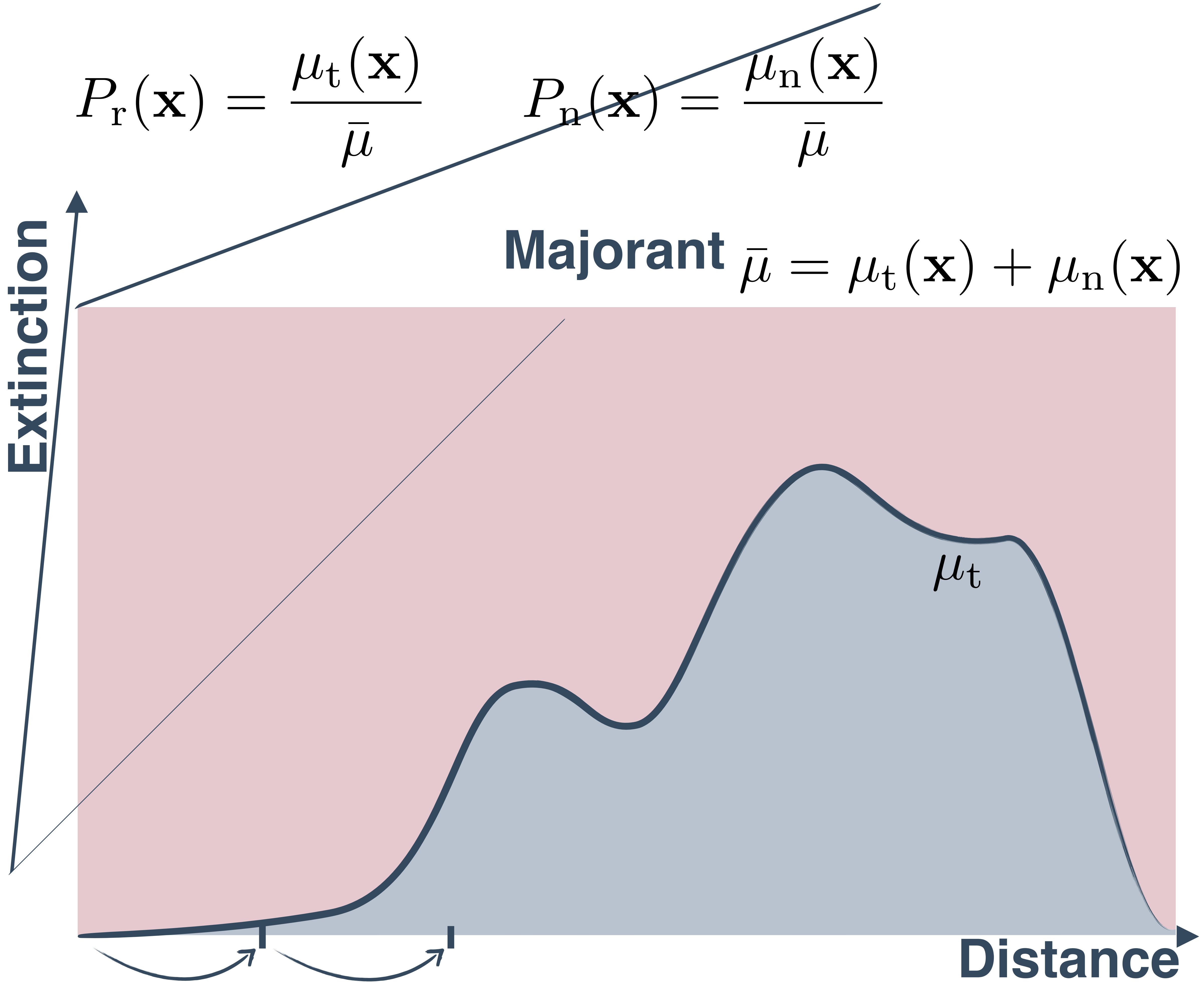
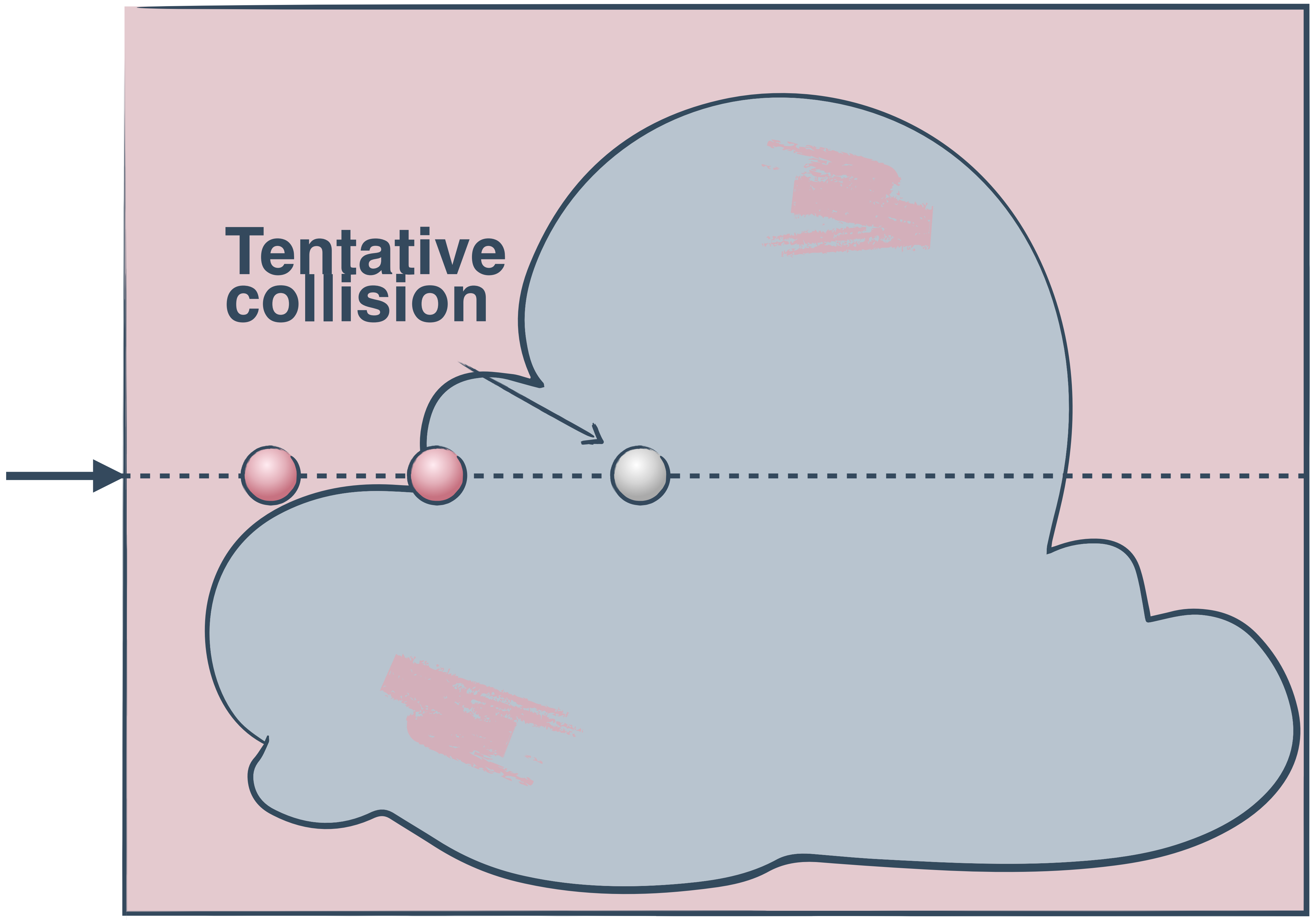


Majorant $\bar{\mu} = \mu_{\mathrm{t}}(\mathbf{x}) + \mu_{\mathrm{n}}(\mathbf{x})$

Stochastic Sampling



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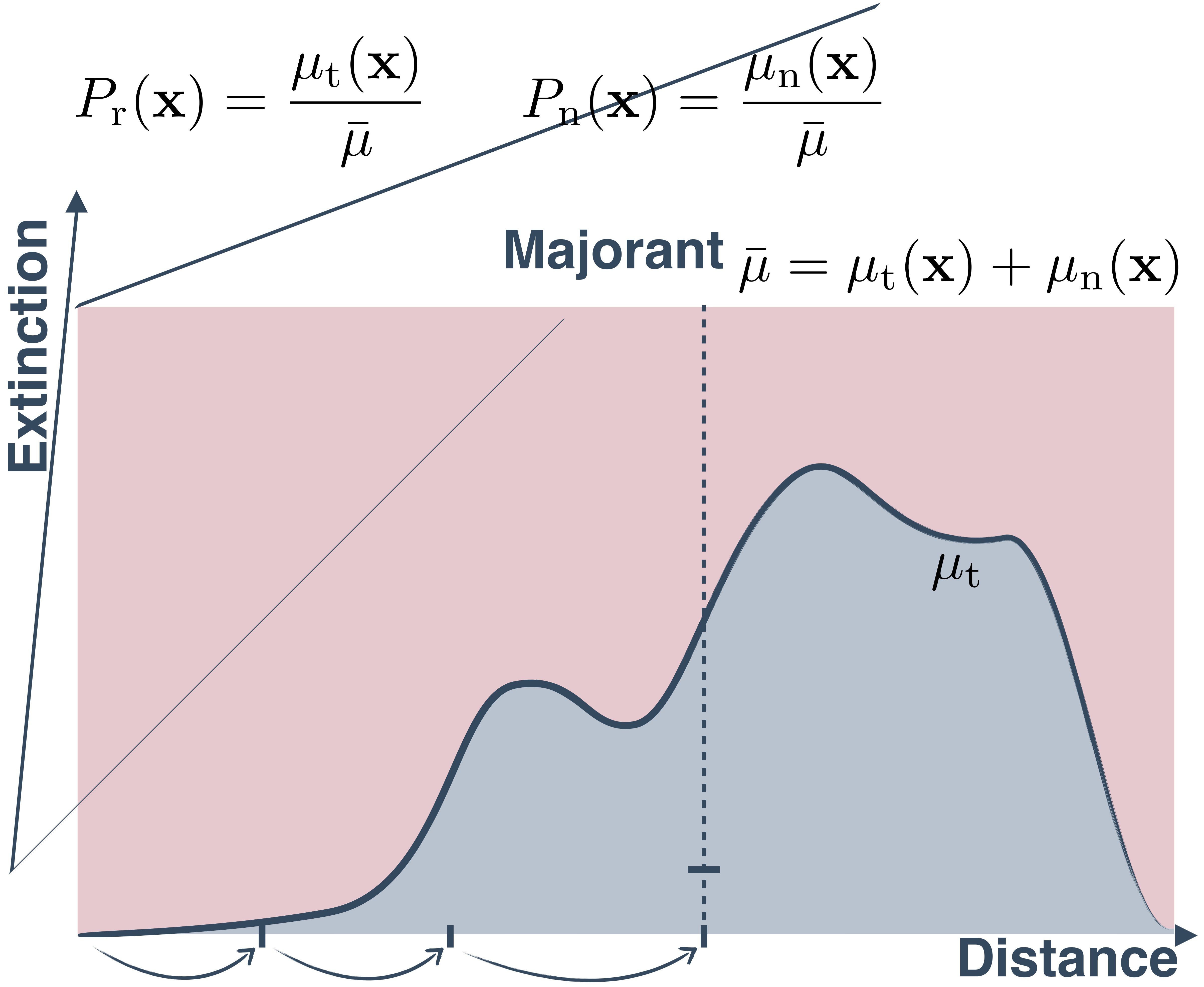
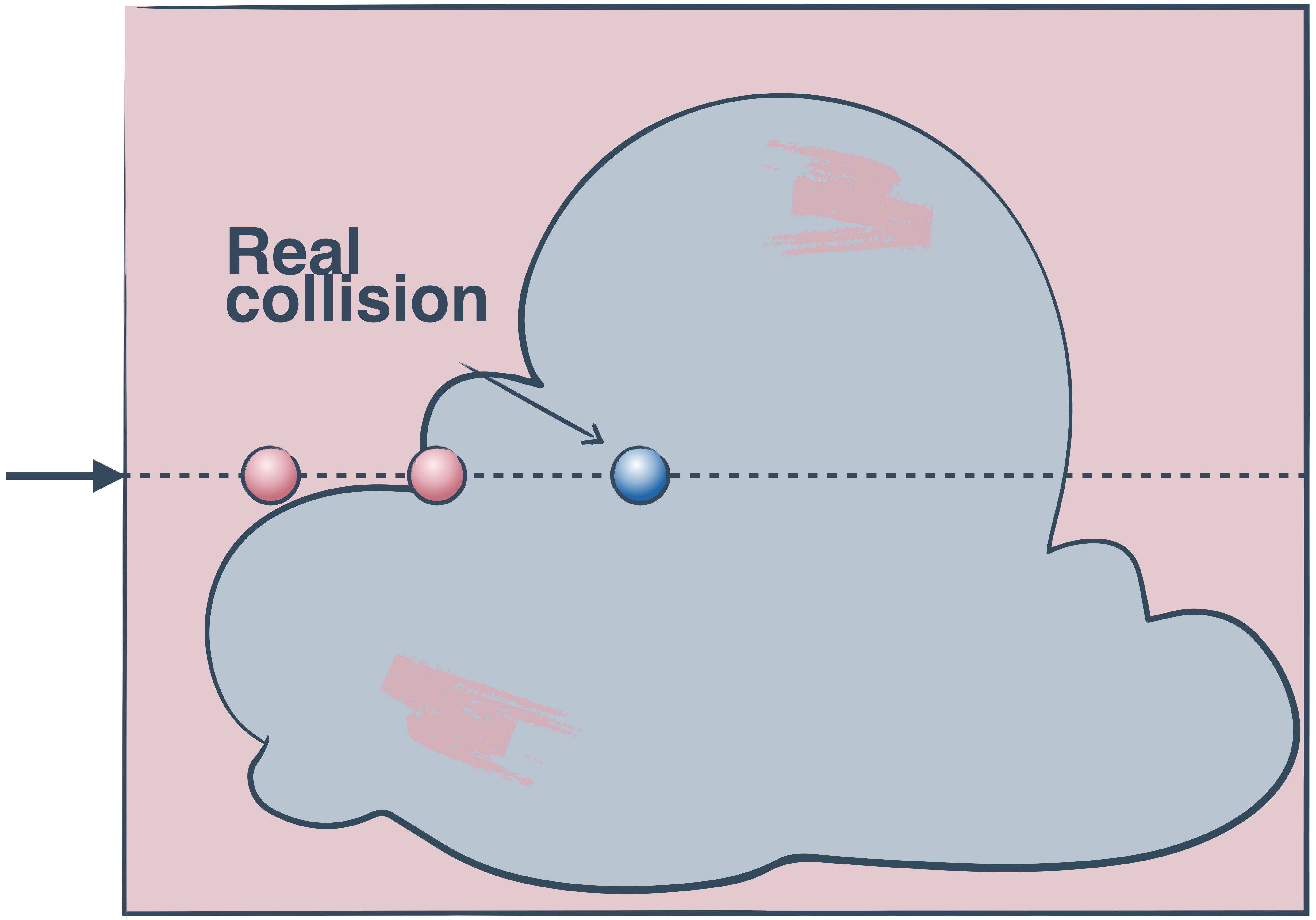
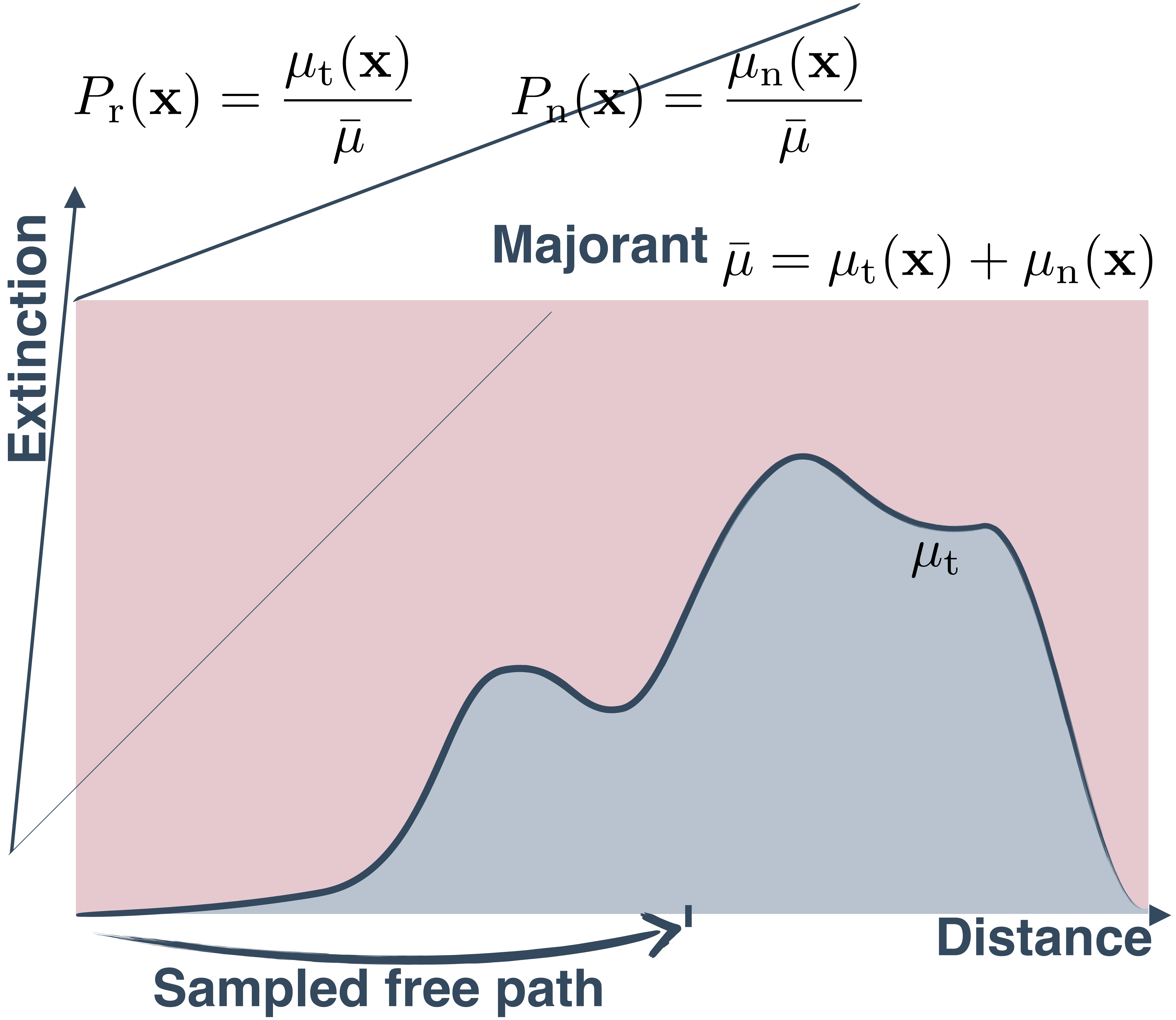


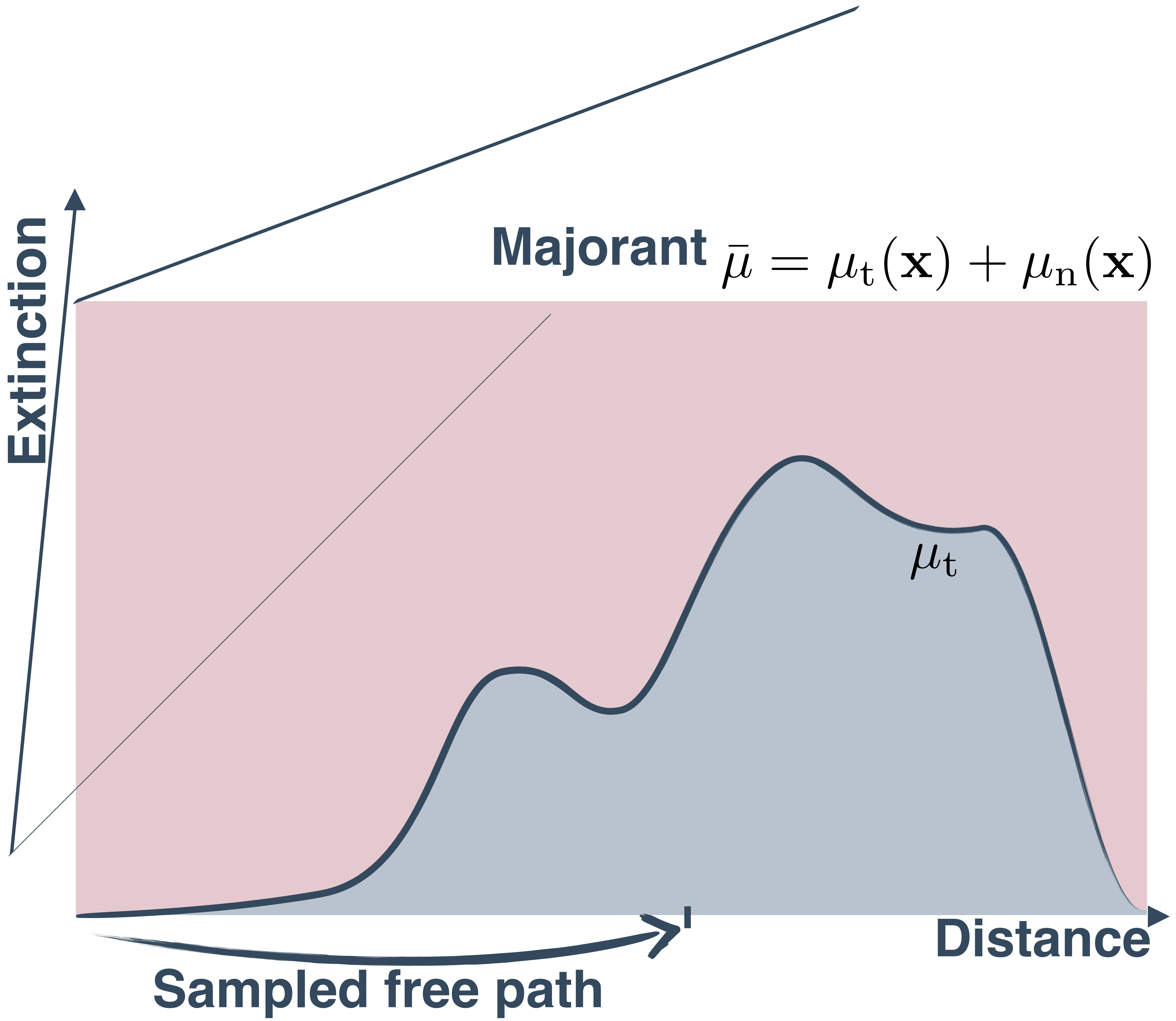
Stochastic Sampling

$$P_{\mathrm{r}}(\mathbf{x}) = \frac{\mu_{\mathrm{t}}(\mathbf{x})}{\bar{\mu}} \qquad P_{\mathrm{n}}(\mathbf{x}) = \frac{\mu_{\mathrm{n}}(\mathbf{x})}{\bar{\mu}}$$

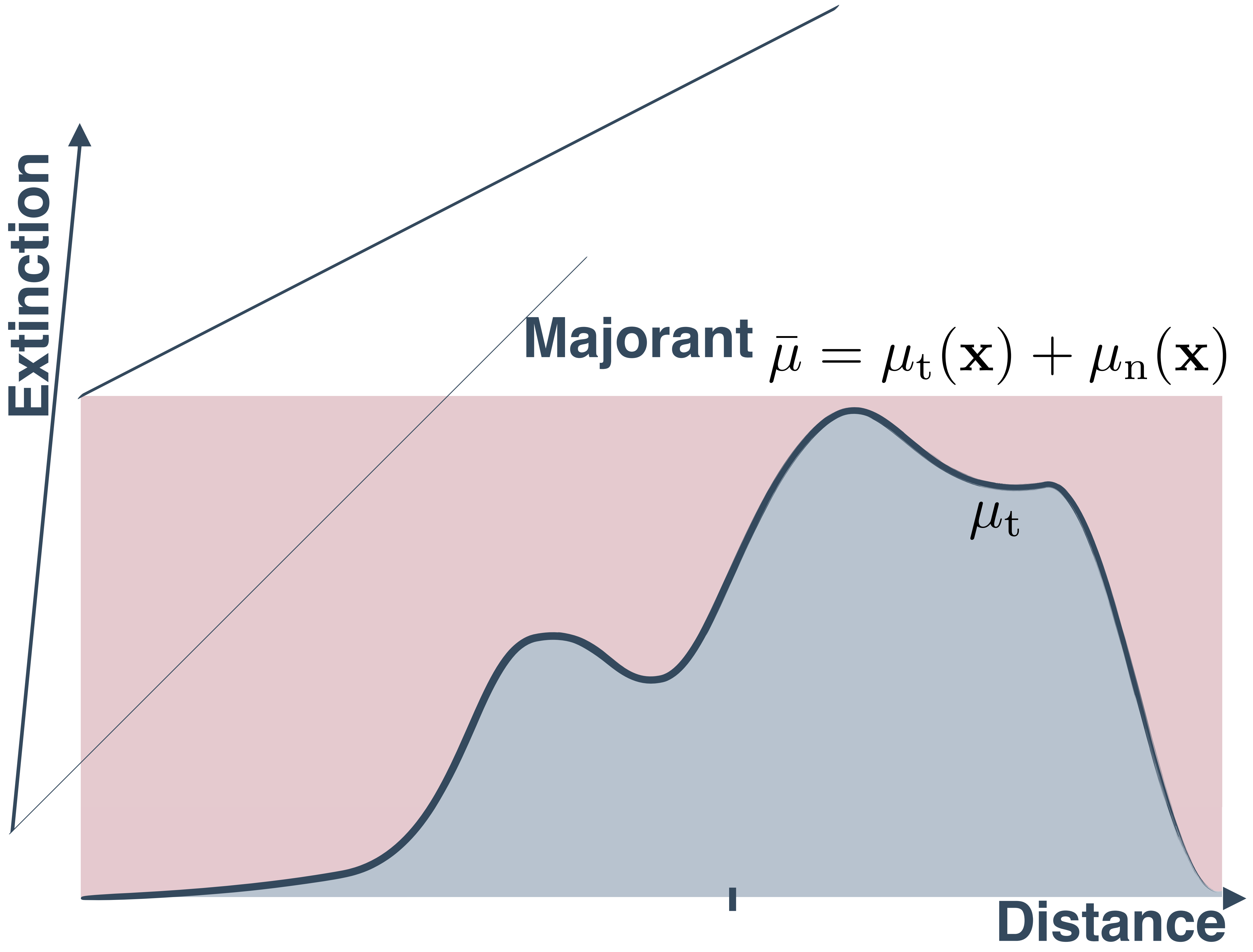




Impact of Majorant

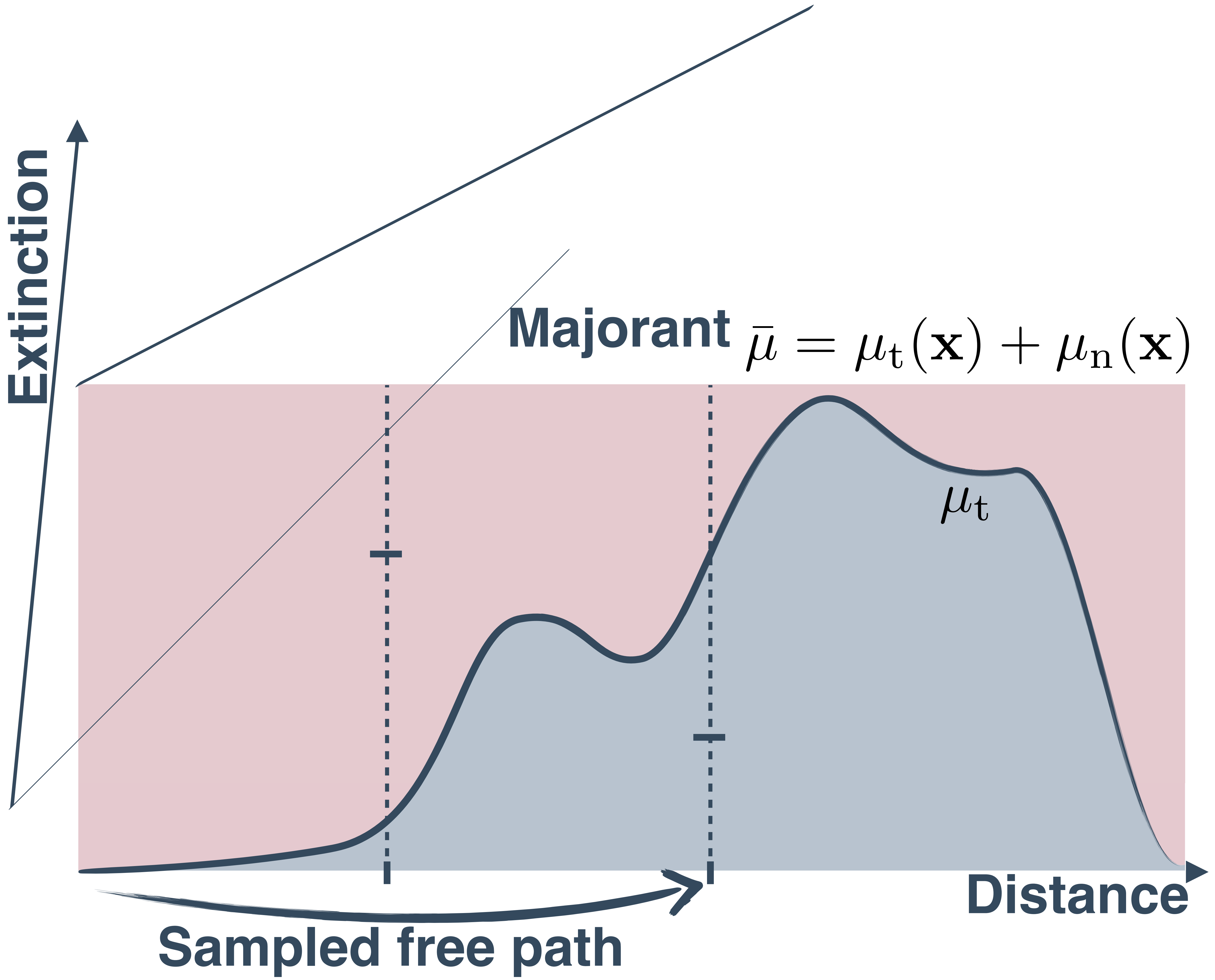


Impact of Majorant

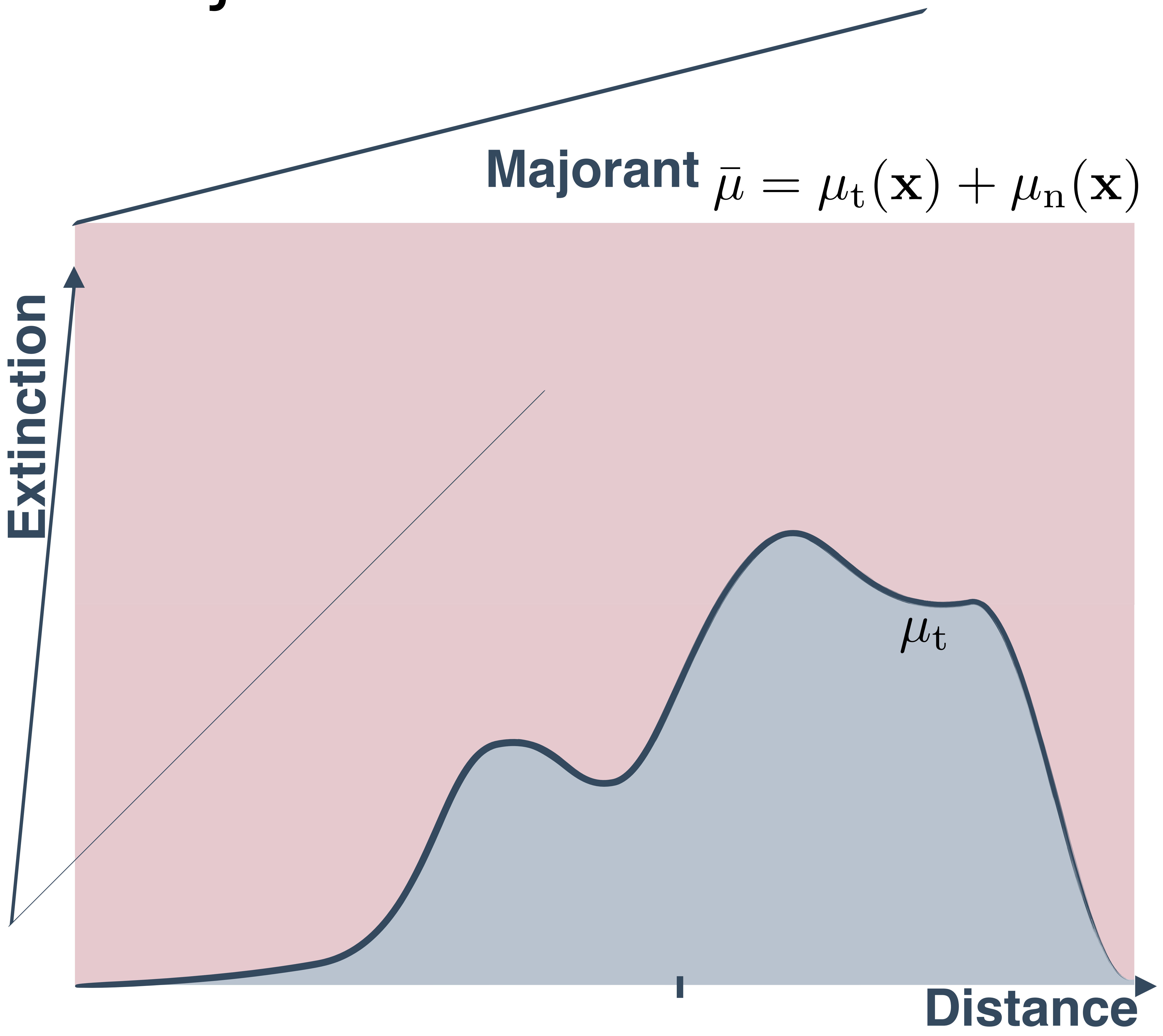


Impact of Majorant

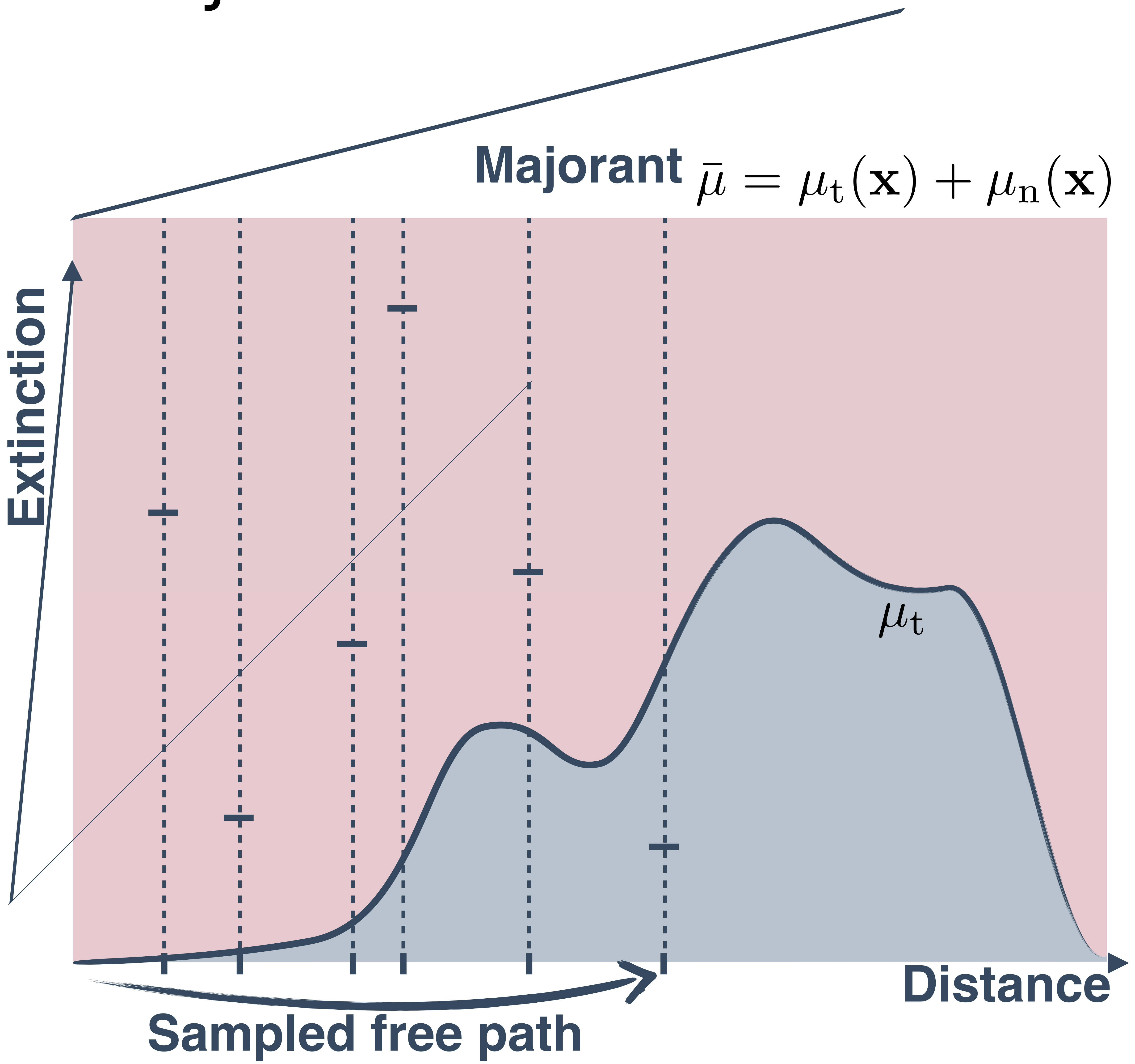
Tight majorant = GOOD
(few rejected collisions)



Impact of Majorant

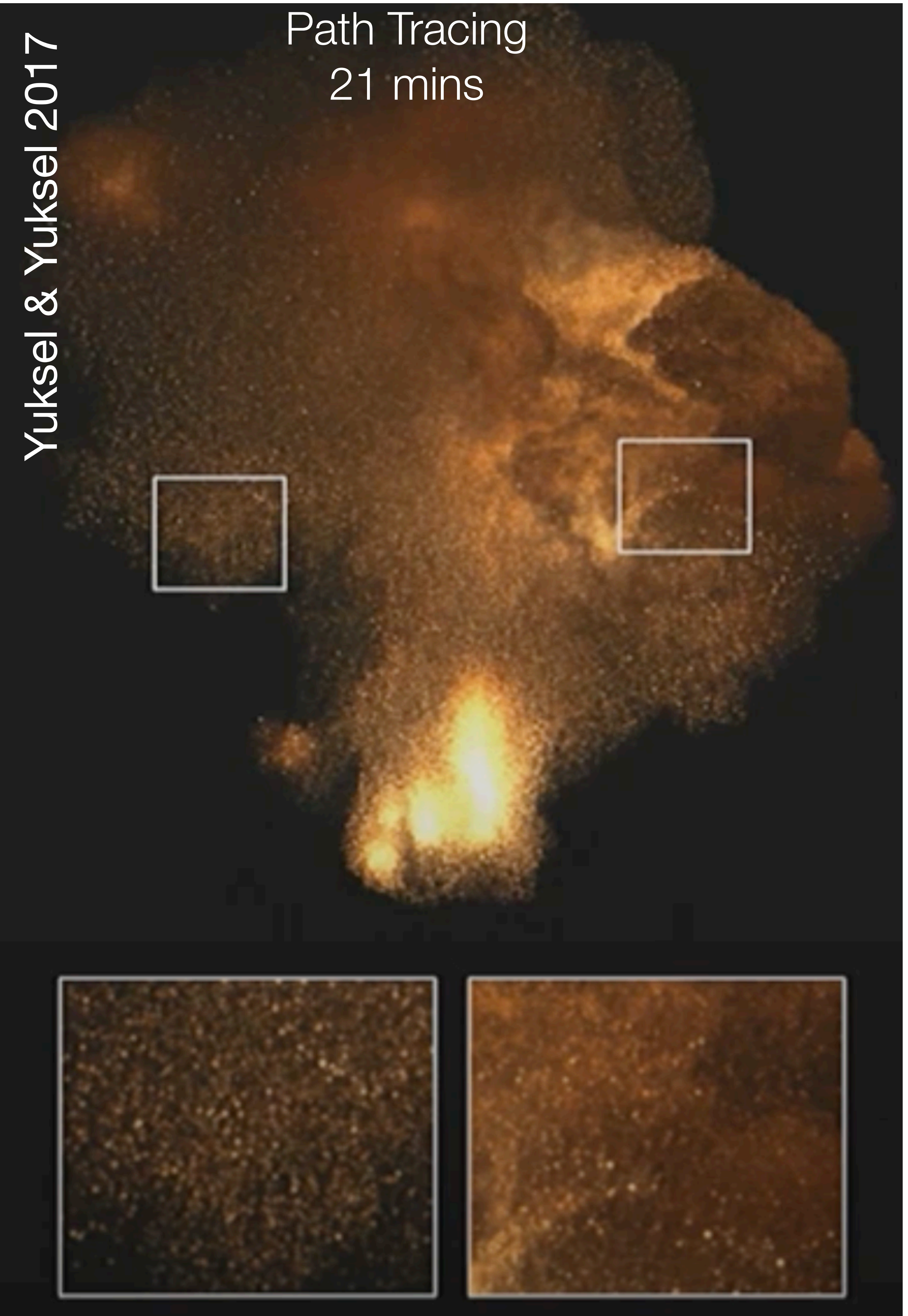


Impact of Majorant

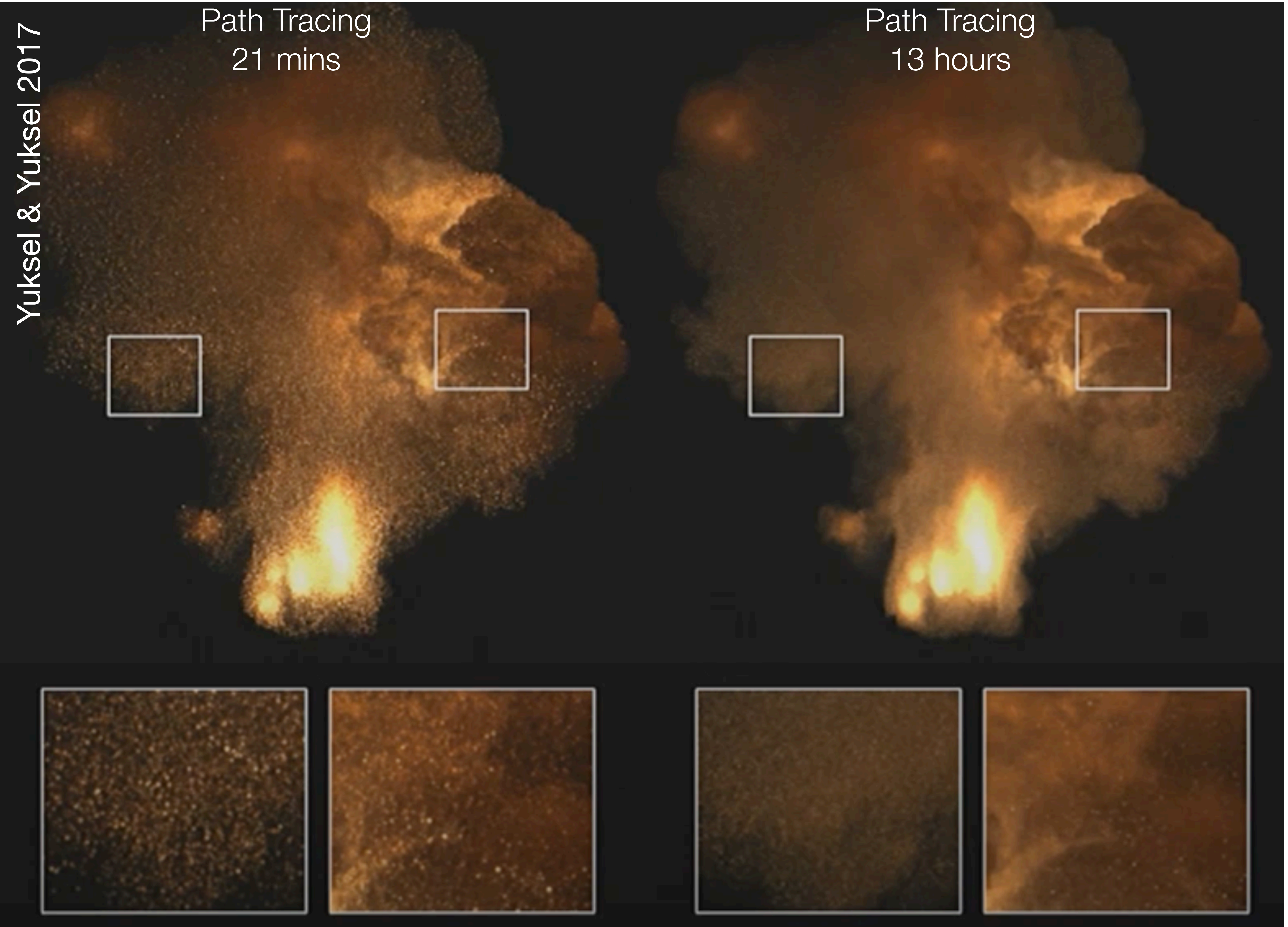


Loose majorant = BAD
(many expensive rejected

Next Lecture: Many-Light Methods



Next Lecture: Many-Light Methods



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Next Lecture: Many-Light Methods

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Acknowledgements

Slides material borrowed from multiple resources.

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