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# Realistic Image Synthesis

- Rendering Equation -

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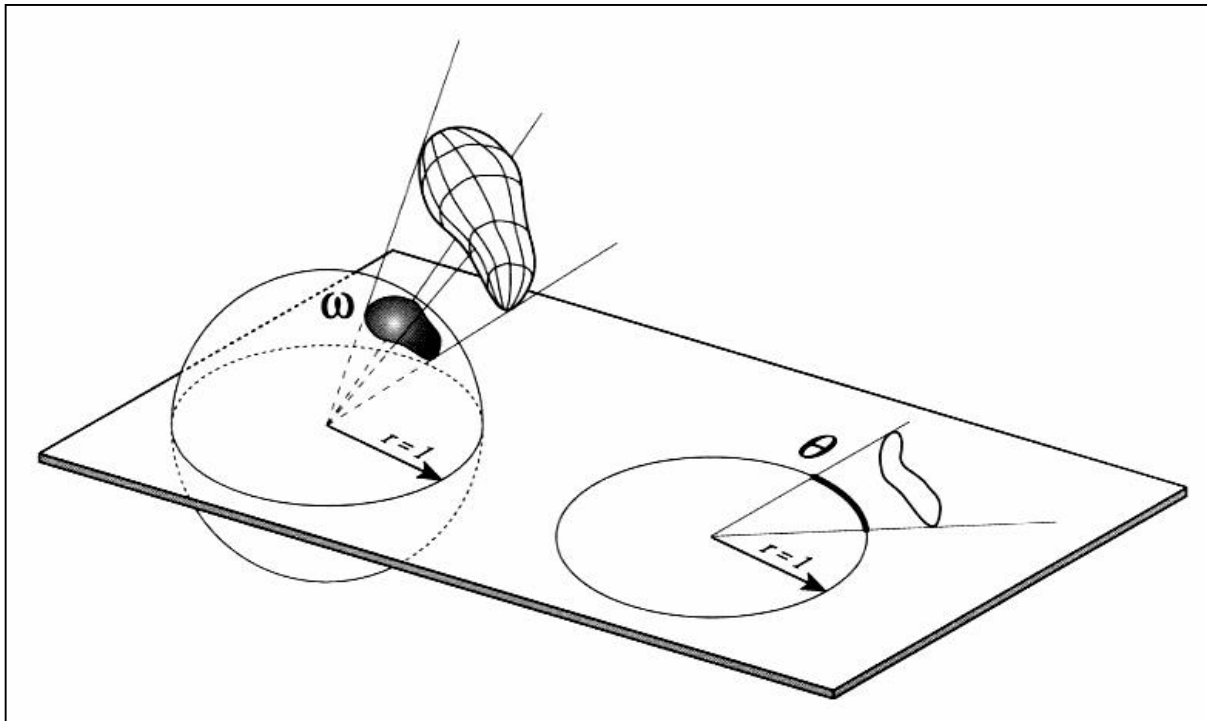
# Angle and Solid Angle

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$\theta$  the angle subtended by a curve in the plane is the length of the projected arc on the unit circle.

$\Omega, \omega$  the solid angle subtended by an object is the surface area of its projection onto the unit sphere

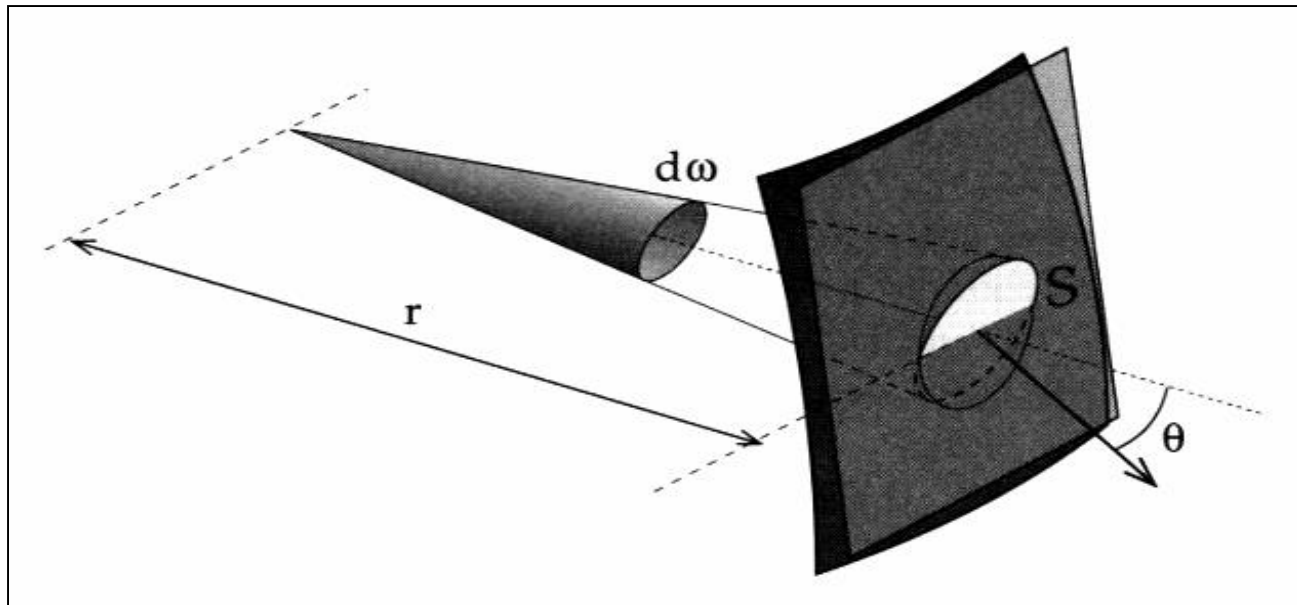
**Solid angle units: steradians [sr]**



# Solid Angle for a Small Area

The solid angle subtended by an (infinitely) small surface patch  $S$  with area  $dA$  is obtained by dividing the projected area  $dA \cos \theta$  by the square of the distance to the origin:

$$d\omega, d\Omega = \frac{dA \cos \theta}{r^2}$$



# Solid Angle in Spherical Coordinates

- **Infinitesimally small solid angle**

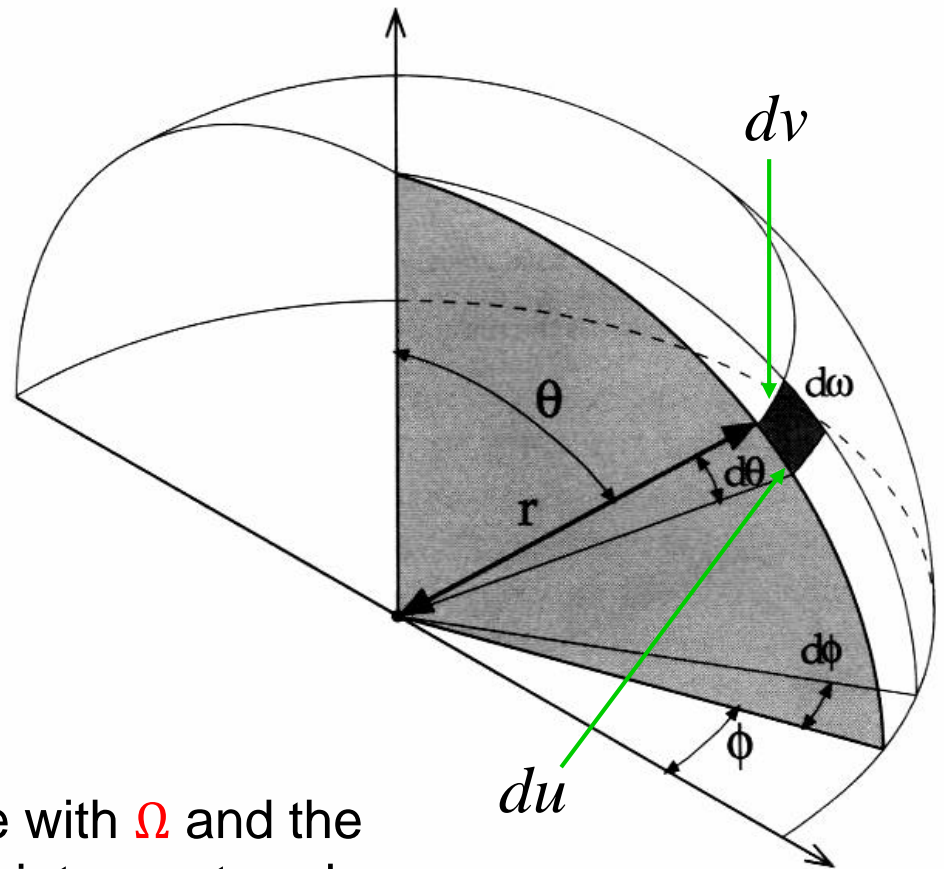
- $du = r d\theta$
- $dv = r \sin \theta d\phi$
- $dA = du dv = r^2 \sin \theta d\theta d\phi$
- $\Rightarrow d\omega = \frac{dA}{r^2} = \sin \theta d\theta d\phi$

- **Finite solid angle of an surface S**

- $\omega = \int_S \sin \theta d\theta d\phi$

- **Definition:**

- We denote the entire Sphere with  $\Omega$  and the (positive) hemisphere at a point x centered around its normal vector with  $\Omega_+$



# Radiometry

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- Radiometry is the science of measuring radiant energy transfers. Radiometric quantities have physical meaning and can be directly measured using proper equipment such as spectral photometers.
- **Radiometric Quantities**
  - Energy [watt second]  $n \cdot h\nu$
  - Radiant power (total flux) [watt]  $\Phi, P$
  - **Radiance** [watt/(m<sup>2</sup> sr)]  $L$
  - Irradiance (flux density) [watt/m<sup>2</sup>]  $E$
  - Radiosity (flux density) [watt/m<sup>2</sup>]  $B$

# Radiometric Quantities: Radiance

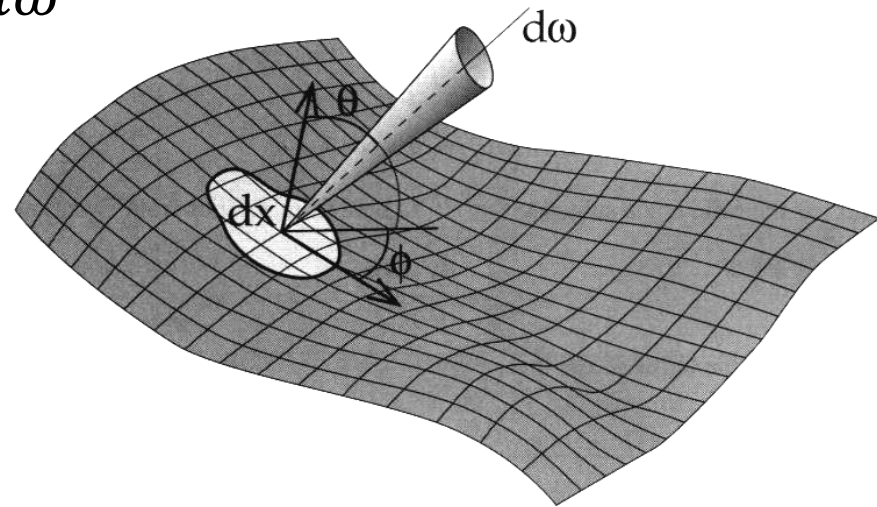
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- Radiance is used to describe radiant energy transfer.
- Radiance  $L$  is defined as the total flux (radiant power) traveling at some point  $x$  in a specified direction  $\omega$ , per unit area perpendicular to the direction of travel, per unit solid angle.
- Thus, the differential flux  $d^2\Phi$  radiated through the differential solid angle  $d\omega$ , from the projected differential area  $dA \cos \theta$  is:

$$d^2\Phi = L(x, \omega) dA \cos \theta d\omega$$

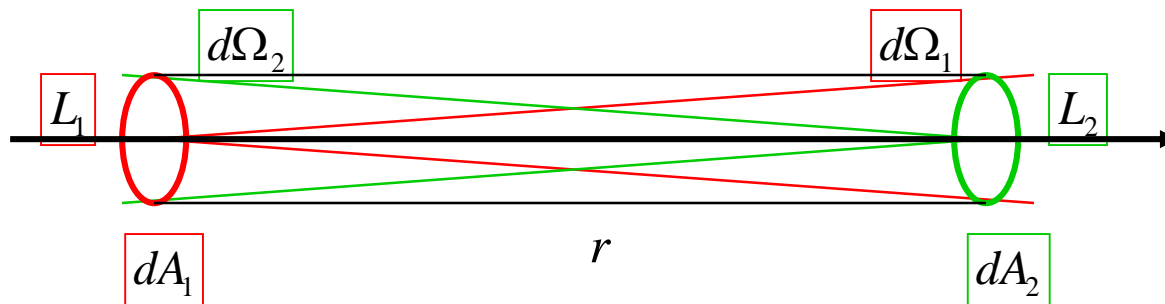
or

$$L(x, \omega) = \frac{d^2\Phi}{dA \cos \theta d\omega}$$



- **From here on we distinguish between the direction  $\omega$  and the (differential) solid angle  $d\omega$  !!!**

# Radiance in Space



Flux leaving surface 1 must be equal to flux arriving on surface 2

$$L_1 \cdot d\Omega_1 \cdot dA_1 = L_2 \cdot d\Omega_2 \cdot dA_2$$

From geometry follows

$$d\Omega_1 = \frac{dA_2}{r^2} \quad d\Omega_2 = \frac{dA_1}{r^2}$$

Def: Ray **Throughput**

$$T = d\Omega_1 \cdot dA_1 = d\Omega_2 \cdot dA_2 = \frac{dA_1 \cdot dA_2}{r^2} \quad \Longrightarrow \quad L_1 = L_2$$

The radiance in the direction of a light ray remains constant as it propagates along the ray.

Sensors response is proportional to radiance (human eye, camera)

# Radiometric Quantities: Irradiance

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- Irradiance  $E$  is the total radiant power per unit area (flux density) *incident* onto a surface with a fixed orientation. To obtain the total flux incident to  $dA$ , the incoming radiance  $L_i$  is integrated over the upper hemisphere  $\Omega_+$  above the surface:

$$E \equiv \frac{d\Phi}{dA}$$

$$d\Phi = \left[ \int_{\Omega_+} L_i(x, \theta, \phi) \cos \theta d\omega \right] dA$$

$$E = \int_{\Omega_+} L_i(x, \theta, \phi) \cos \theta d\omega = \int_0^{2\pi} \int_0^{\pi/2} L_i(x, \theta, \phi) \cos \theta \sin \theta d\theta d\phi$$



# Radiometric Quantities: Radiosity

- Radiosity  $B$  is defined as the total radiant power per unit area (flux density) *leaving* a surface. To obtain the total flux radiated from  $dA$ , the outgoing radiance  $L_o$  is integrated over the upper hemisphere  $\Omega_+$  above the surface.

$$B \equiv \frac{d\Phi}{dA}$$

$$d\Phi = \left[ \int_{\Omega_+} L_o(x, \theta, \phi) \cos \theta d\omega \right] dA$$

$$B = \int_{\Omega_+} L_o(x, \theta, \phi) \cos \theta d\omega = \int_0^{2\pi} \int_0^{\pi/2} L_o(x, \theta, \phi) \cos \theta \sin \theta d\theta d\phi$$

# Bidirectional Reflectance Distribution Function

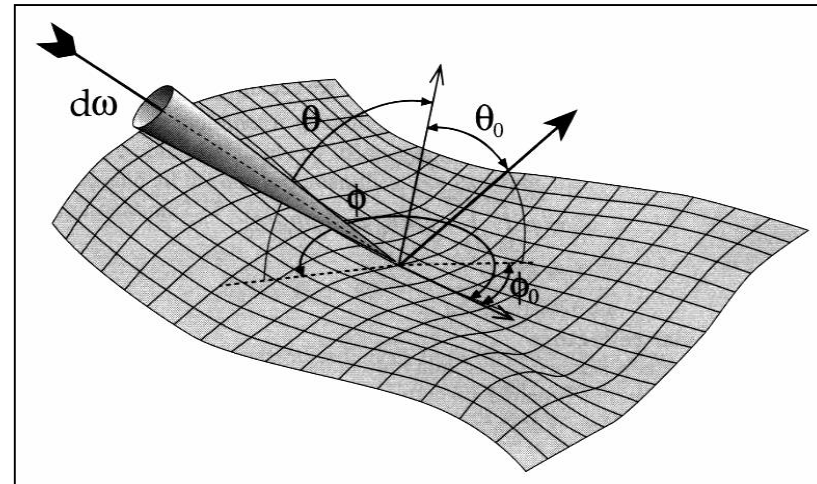
- **BRDF  $f_r$  describes surface reflection at a point  $x$  for light incident from direction  $\omega_i = (\theta_i, \varphi_i)$  reflected into direction  $\omega_o = (\theta_o, \varphi_o)$**
- **Bidirectional (six dimensional function)**
  - Depends on two directions  $\omega_i$  and  $\omega_o$  (2D plus 2D = 4D)
  - Also depends on location  $x$  (2D)
- **Distribution function**
  - Can be infinite but integrates to finite value
  - Strictly positive (physics!)

## • **Definition of BRDF:**

- Outgoing radiance per incident irradiance

$$- f_r(\omega_i, x, \omega_o) = \frac{L_o(x, \omega_o)}{dE_i}$$

$$- f_r(\omega_i, x, \omega_o) = \frac{L_o(x, \omega_o)}{L_i(x, \omega_i) \cos \theta_i d\omega_i}$$

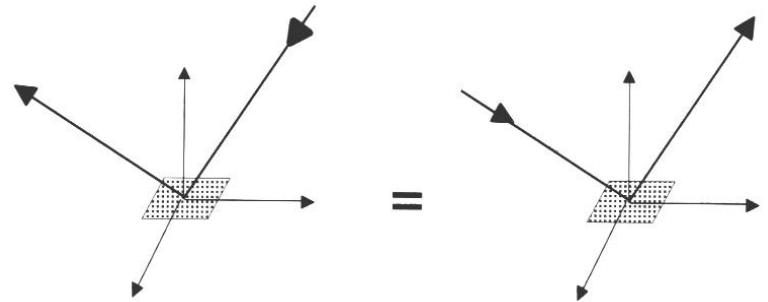


# BRDF Properties

- **Helmholtz reciprocity principle**

- BRDF remains unchanged if incident and reflected directions are interchanged
- Due to physics (linearity)

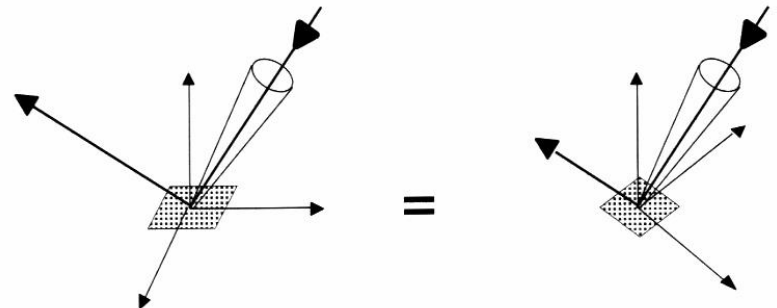
$$f_r(\omega_i, \mathbf{x}, \omega_o) = f_r(\omega_o, \mathbf{x}, \omega_i)$$



- **Smooth surface: Isotropic BRDF**

- Reflectivity is independent of rotation around surface normal
- BRDF directional dependence has only 3 instead of 4 degrees of freedom

$$f_r(\omega_i, \mathbf{x}, \omega_o) = f_r(\mathbf{x}, \theta_i, \theta_o, \varphi_i - \varphi_o)$$



# BRDF Properties

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- **Characteristics**

- BRDF units [ $\text{sr}^{-1}$ ]
  - Not very intuitive
- Range of values:
  - From 0 (complete absorption) to
  - $\infty$  (perfect mirror reflection,  $\delta$ -function)
    - Because it relates the density  $L$  to an absolute value
- Energy conservation law
  - Integrating over all outgoing light:
    - No more energy can be reflected than was incoming
  - In other words the directional-hemispherical reflectance must be smaller than 1
    - $\rho_{dh} = \int_{\Omega_+} f_r(\omega_i, x, \omega_o) \cos\theta \, d\omega_o \leq 1, \quad \forall \omega_i$
- Reflection only at the point of entry ( $x_i = x_o$ )
  - Subsurface scattering (e.g. in skin) is not included in this formulation

# Directional Hemispherical Reflectance

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- More intuitive measure of reflectance is the **directional-hemispherical reflectance**:
  - The fraction of the incident radiant flux density incoming from a given direction that is reflected by the surface in all possible directions.
  - Dimensionless number in  $[0,1]$
  - Can change with the angle of incidence

$$\rho_{dh}(\omega_i) = \frac{dB}{dE(\omega_i)} = \frac{\int_{\Omega_+} L_o(x, \omega_o) \cos \theta_o d\omega_o}{dE(\omega_i)} = \int_{\Omega_+} f_r(\omega_i, x, \omega_o) \cos \theta_o d\omega_o$$



$$\frac{L_o(x, \omega_o)}{dE(\omega_i)} = f_r(\omega_i, x, \omega_o)$$

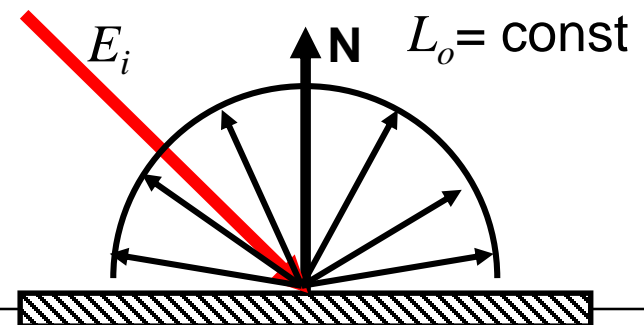
# Lambertian Diffuse Reflection

- Diffuse reflectance is modeled by assuming that light is equally likely to be scattered in any direction, regardless of the incident direction
- Therefore the BRDF and reflected radiance are constant:
  - $f_r(\omega_i, x, \omega_o) = \rho$  and  $L_o = \text{const}$
- Also, directional-hemispherical reflectance  $\rho_d$  becomes independent of direction. This dimensionless constant, which corresponds to the intuitive meaning of reflectance, is then called the diffuse reflectance  $\rho_d$ :

$$- \rho_d = \int_{\Omega_+} \rho \cos \theta_o d\omega_o = \rho \int_0^{2\pi} \int_0^{\pi/2} \cos \theta_o \sin \theta_o d\theta_o d\varphi_o = \pi\rho$$

- Irradiance  $E$  and radiosity  $B$  for the Lambertian surface are related as:

$$- \rho_d = \frac{B}{E} \quad \leftarrow B = \int_{\Omega} L_o(x, \theta, \phi) \cos \theta d\omega = L_o \cdot \pi$$



# Reflection Equation

- **Putting at all together:**

- The light reflected at a point  $x$  in direction  $\omega$  is given as

$$L_r(x, \omega_o) = \int_{\omega_i \in \Omega_+} f_r(\omega_i, x, \omega_o) L_i(x, \omega_i) \cos \theta_i d\omega_i$$

- **Visible surface radiance**

- Surface position
- Outgoing direction
- Incoming illumination direction

$$L_r(x, \omega_o)$$

$$x$$

$$\omega_o$$

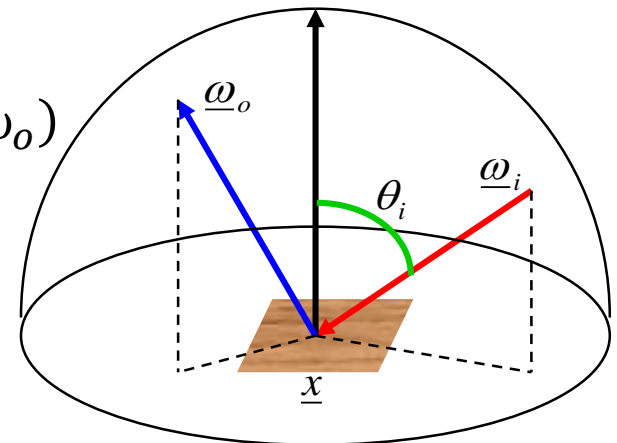
$$\omega_i$$

- **Reflected light**

- Incoming radiance
- Direction-dependent reflectance

$$L_i(x, \omega_i)$$

$$f_r(\omega_i, x, \omega_o)$$



# Reflection Equation: Properties

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- **Reflection operator is linear**
  - Superposition holds
  - Solution could be computed separately for each light source
    - And be accumulated
- **BRDF is a six-dimensional function**
  - Difficult to represent and compute accurately
  - Measurements are expensive and need much storage
    - But often compresses well



# Light Transport in a Scene

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- **Scene**
  - Lights (emitters)
  - Object surfaces (partially absorbing)
- **Illuminated object surfaces become emitters, too !**
  - Radiosity = Irradiance minus absorbed photon flux
    - Radiosity: photons per second per  $m^2$  leaving surface
    - Irradiance: photons per second per  $m^2$  incident on surface
- **Light bounces between all mutually visible surfaces**
- **Invariance of radiance in free space (vacuum)**
  - No absorption in-between objects
  - Hold also in clean air (approximately!)
- **Dynamic Energy Equilibrium**
  - Emitted photons = absorbed photons (+ photons escaping scene)

**Global Illumination Problem**

# Definition: Rendering Equation

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- **Light exiting at some point**

- Given by emitted light plus reflected incoming light at  $x$

- $L_o(x, \omega_o) = L_e(x, \omega_o) + L_r(x, \omega_o)$

$$= L_e(x, \omega_o) + \int_{\omega_i \in \Omega_+} f_r(\omega_i, x, \omega_o) L_i(x, \omega_i) \cos \theta_i d\omega_i$$

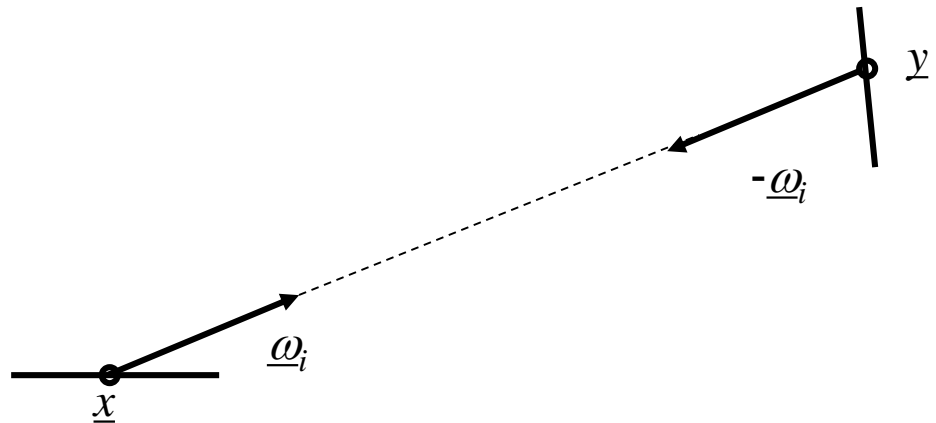
- **Coupling output back to input**

- Light incident at  $x$  is the light exiting at some other point  $y$

- $L_i(x, \omega_i) = L_o(y, -\omega_i) = L_o(RT(x, \omega_i), -\omega_i)$

- With the visibility or ray-tracing operator  $RT$

- $y = RT(x, \omega_i)$



# Definition: Rendering Equation

- **Rendering Equation**

- Parameterized by direction

$$L_o(x, \omega_o) = L_e(x, \omega_o) + \int_{\omega_i \in \Omega_+} f_r(\omega_i, x, \omega_o) L_i(x, \omega_i) \cos \theta_i d\omega_i$$

- Parameterized by position over all surfaces  $S$

$$L_o(x, \omega_o)$$

$$= L_e + \int_{y \in S} f_r(\omega_i, x, \omega_o) L_o\left(y, \frac{x - y}{\|x - y\|}\right) V(x, y) G(x, y) dA_y$$

- with  $V(x, y)$  giving visibility between  $x$  and  $y$ ,
- and the Geometric Term  $G$  given by

- $d\omega_i = dA_y \frac{\cos \theta_y}{\|x - y\|^2}$

- $G(x, y) = \frac{\cos \theta_x \cos \theta_y}{\|x - y\|^2}$

# Rendering Equation

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$$L_o(x, \omega_o) = L_e(x, \omega_o) + \int_{y \in S} f_r(\omega_i, x, \omega_o) L_o(y(x, \omega_i), -\omega_i) V(x, y) G(x, y) dA_y$$

- **Properties**

- Mathematical: Fredholm equation of the 2-nd kind
- Global coupling of illumination
  - Each point potentially influences each other point
  - Often still a sparse operator due to occlusion
- Linear transport operator **T**
  - Solution can be computed separately for each light source
    - And accumulated
    - Dimmed lights result in dimmed solutions
- Volume effects are not considered !!

► **Lighting Simulation == Solving the Rendering Equation**

# RE: In Operator Form

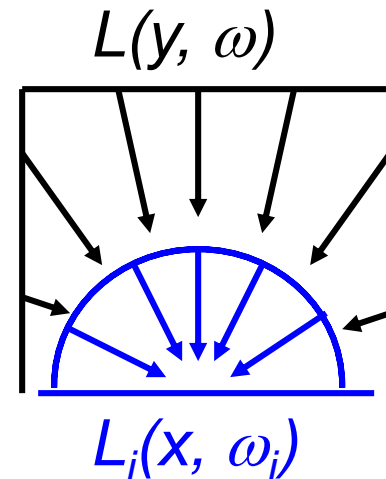
- **Transport operator  $T$**

- Built from **reflection operator  $S$**  and **propagation operator  $H$**

- $L = L_o = L_e + TL = L_e + (S \circ H)L$

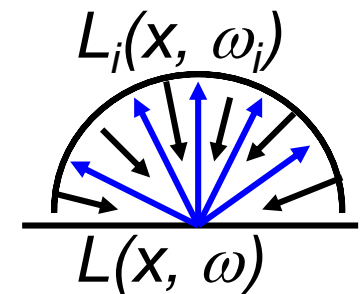
- **Propagation operator  $H$**

- Computes light incident at points  $L_i(x_i, \omega_i)$  from excitant light at other locations  $L(y_i, \omega_i)$
- Evaluation of **ray tracing operator**
- **Global operator: needs essentially entire scene**



- **Reflection (scattering) operator  $S$**

- Computes reflected light field  $L(y_i, \omega_i)$  from incident light  $L_i(x_i, \omega_i)$  evaluating **reflection equation**
- Evaluates BRDF for entire incident light field
- **Local operator: operates at one point only**



# Rendering Equation

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- **Solution Approaches**

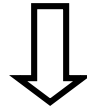
- Monte Carlo technique (and extensions)
  - Point-wise evaluation of multi-dimensional integral equation
  - Efficient solution for the general case
  - Can cause noise through variance of random evaluation
  - No bias and correlation (in approach)
- Finite Element technique
  - Projection of infinite dimensional equation into function space with finite dimensions
    - Solution is represented as combination of basis functions
    - Constant basis functions in the simplest case
  - Leads to solution of a linear system of equations
  - Efficient for smooth, slowly varying illumination and reflection
  - Causes bias through correlation between solution of neighboring points

# Discretization of Rendering Equation

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- **Simplification of the rendering equation**

$$L_o(x, \theta_o, \varphi_o) = L_e(x, \theta_o, \varphi_o) + \int_{\Omega} \rho_{bd}(x, \theta_o, \varphi_o, \theta, \phi) L_i(x, \theta, \varphi) \cos \theta d\omega$$



$$B_i = E_i + \rho_i \sum_{j=1}^N B_j F_{ij}$$

- All surfaces in the scene are Lambertian
  - Equation expressed in terms of the radiosity quantities
  - Integration domain split into  $N$  pieces corresponding to discrete patches in the scene
  - Constant radiosity and reflectance assumptions for each patch
- **We are going to discuss all these steps in detail**

# Lambertian Diffuse Reflection

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- **Diffuse reflectance is modeled by assuming that light is equally likely to be scattered in any direction, regardless of the incident direction:**

$$\rho_{bd}(x, \theta_o, \varphi_o, \theta, \varphi) = \rho(x)$$

- **Directional-hemispherical reflectance  $\rho_d$  becomes independent of direction:**

$$\rho_d(x) = \int_{\Omega} \rho(x) \cos \theta_o d\omega_o = \rho(x) \int_0^{2\pi} \int_0^{\pi/2} \cos \theta_o \sin \theta_o d\theta_o d\varphi_o = \pi \rho(x)$$

$$\rho(x) = \frac{\rho_d(x)}{\pi}$$

- **Then the rendering equation simplifies to:**

$$L_o(x, \theta_o, \varphi_o) = L_e(x, \theta_o, \varphi_o) + \frac{\rho_d(x)}{\pi} \int_{\Omega} L_i(x, \theta, \varphi) \cos \theta d\omega$$



# Further Simplifications

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- **For diffuse surfaces**

- the radiance  $L_o(x, \theta_o, \varphi_o) \equiv L_o(x)$  does not depend on the outgoing direction,
- the incoming radiance  $L_i$  still depends on the incoming direction

$$L_o(x) = L_e(x) + \frac{\rho_d(x)}{\pi} \int_{\Omega} L_i(x, \theta, \varphi) \cos \theta d\omega$$

- **Now let us replace radiances by radiosities:**

$$B(x) = \int_{\Omega} L(x) \cos \theta d\omega = \int_0^{2\pi} \int_0^{\pi/2} L(x) \cos \theta \sin \theta d\theta d\phi = \pi L(x)$$

$$\pi L_o(x) = \pi L_e(x) + \pi \frac{\rho_d(x)}{\pi} \int_{\Omega} L_i(x, \theta, \varphi) \cos \theta d\omega$$

$$B(x) = E(x) + \rho_d(x) \int_{\Omega} L_i(x, \theta, \varphi) \cos \theta d\omega$$

# Transforming the Hemispherical Integral into a Surface Integral

- The invariance of radiance along a line of sight states that:

$$L_i(x, \theta, \varphi) = L(y, \theta', \varphi')$$

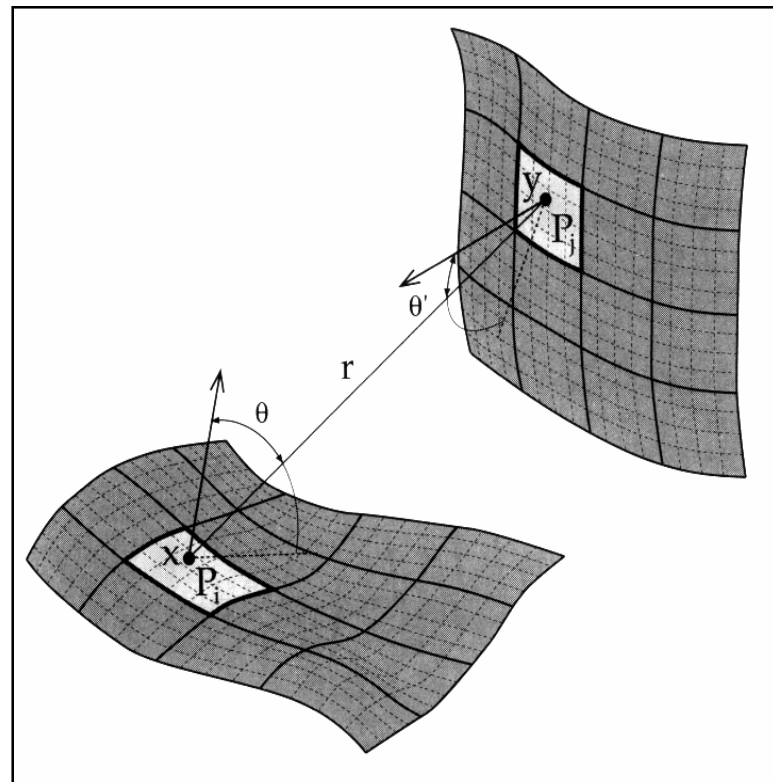
$$L(y, \theta', \varphi') = \frac{B(y)}{\pi}$$

- Now let us replace integration over the hemisphere by integration over all surfaces  $y$  taking into account their visibility from  $x$ :

$$d\omega = \frac{\cos \theta' dy}{r^2}$$

$$V(x, y) = \begin{cases} 1 & \text{if } x \text{ and } y \text{ are mutually visible} \\ 0 & \text{otherwise} \end{cases}$$

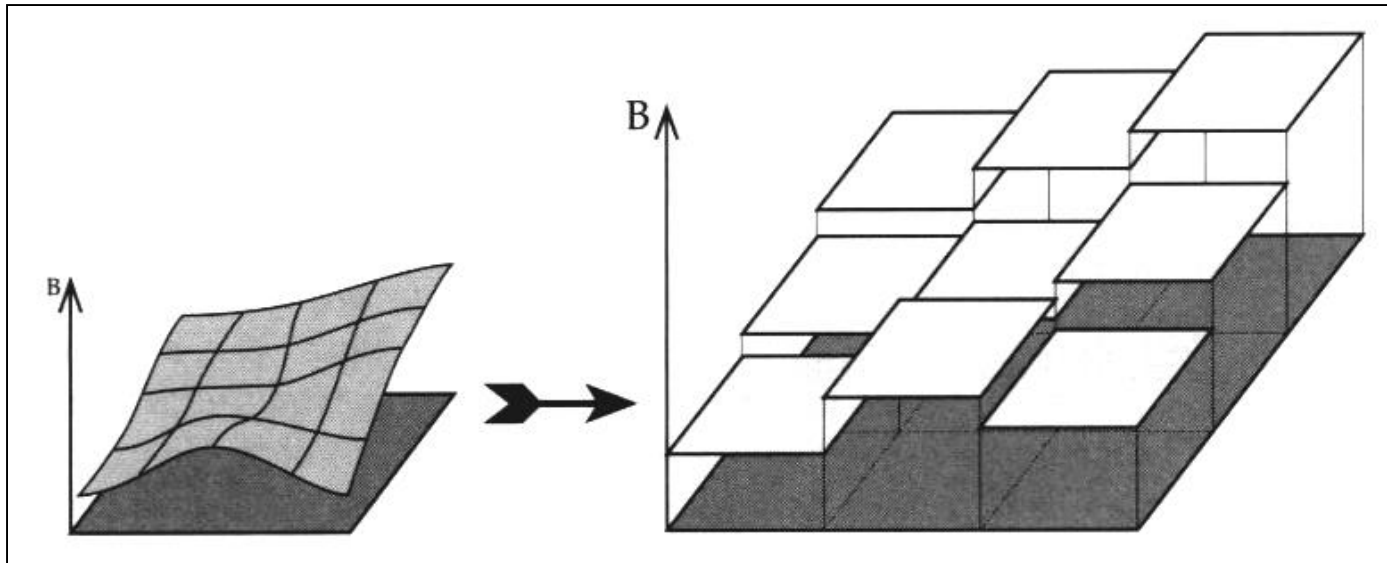
$$B(x) = E(x) + \rho_d(x) \int_{y \in S} B(y) \frac{\cos \theta \cos \theta'}{\pi r^2} V(x, y) dy$$



# Discrete Formulation

- The integral over all surfaces in the scene in the previous slide is broken into  $N$  pieces, each corresponding to a discrete patch.
- It is assumed that each patch has a uniform radiosity at each point  $y$  in patch  $P_j$ .

$$B(x) = E(x) + \rho_d(x) \sum_{j=1}^N B_j \int_{y \in P_j} \frac{\cos \theta \cos \theta'}{\pi r^2} V(x, y) dy$$



# Radiosity Equation and Form Factors

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- The constant radiosity value for each patch is computed as an area-weighted average of radiosity:

$$B_i = \frac{1}{A_i} \int_{x \in P_i} B(x) dx \quad E_i = \frac{1}{A_i} \int_{x \in P_i} E(x) dx$$

- Then assuming also that reflectance is constant across each patch  $\rho_d(x) = \rho_i$ , the radiosity equation can be formulated as:

$$B_i = E_i + \rho_i \sum_{j=1}^N B_j \frac{1}{A_i} \int_{x \in P_i} \int_{y \in P_j} \frac{\cos \theta \cos \theta'}{\pi r^2} V(x, y) dx dy$$

$$B_i = E_i + \rho_i \sum_{j=1}^N B_j F_{ij}$$

- where  $F_{ij}$  is the form factor:

$$F_{ij} = \frac{1}{A_i} \int_{x \in P_i} \int_{y \in P_j} \frac{\cos \theta \cos \theta'}{\pi r^2} V(x, y) dx dy$$