$$L_o = L_e + \int_\Omega L_i \, f \cos\theta_i \, d\omega_i$$

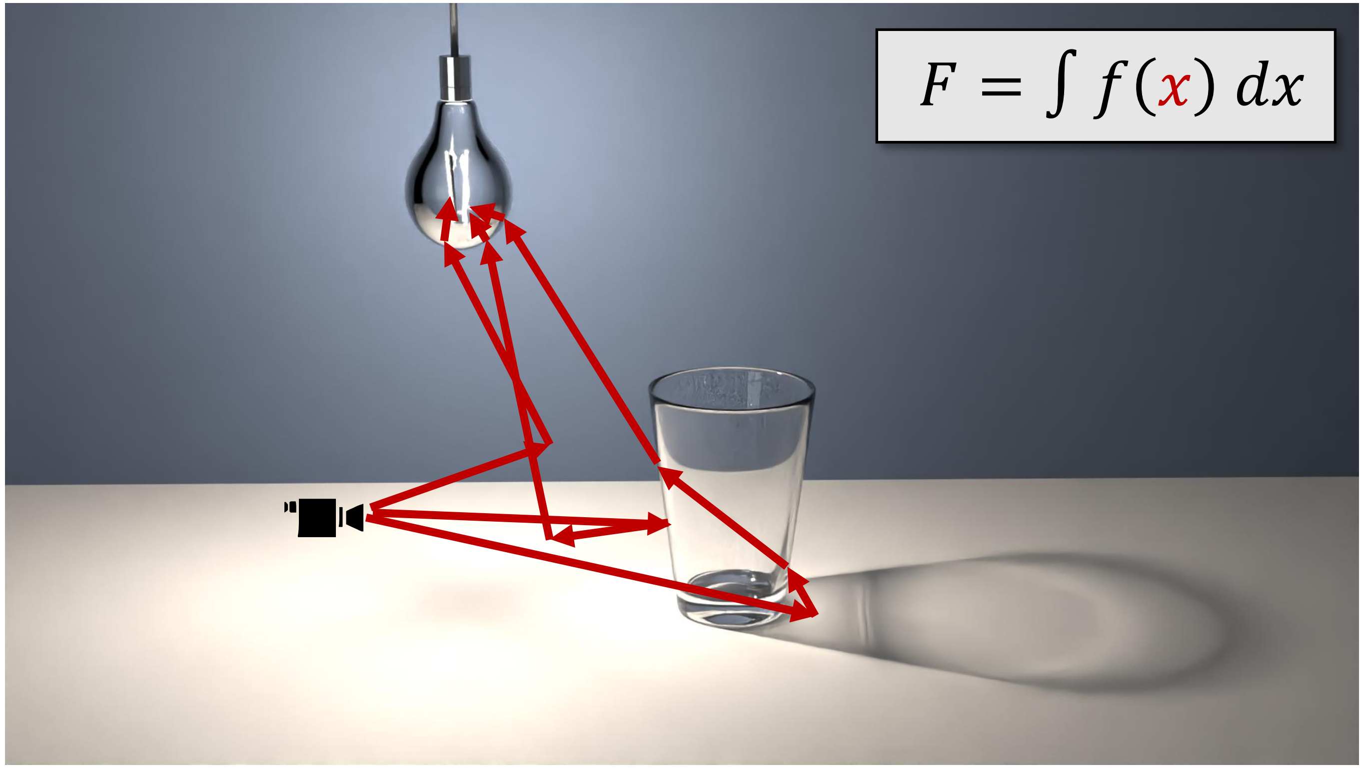
Monte Carlo Integration

The mathematical foundation of light transport simulation

Pascal Grittmann

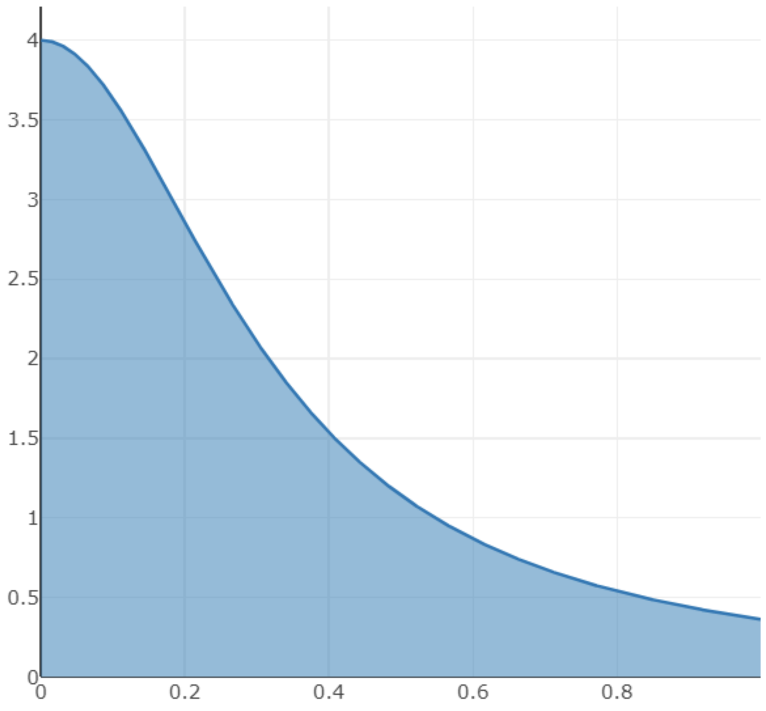
Recap

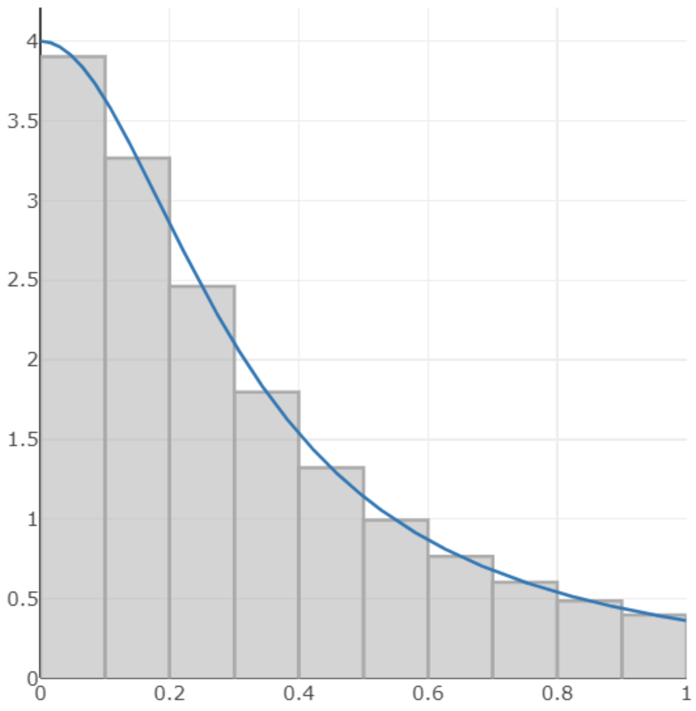
- Goal: compute pixel value
- Integral over all possible paths connecting the pixel to a light source



How to compute an integral?

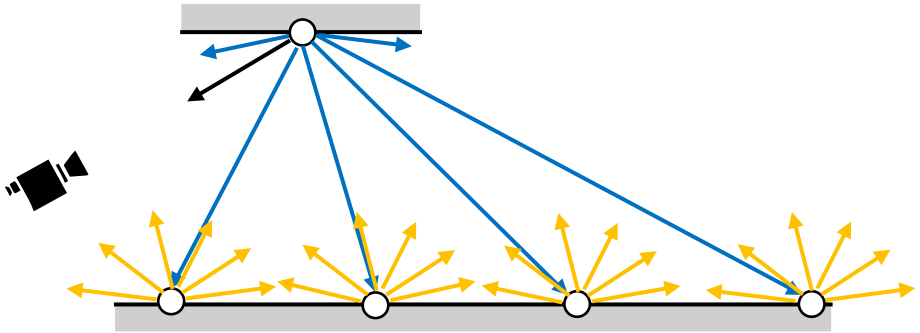
1. Analytically (usually not possible in rendering)
2. Numerically, e.g., quadrature with midpoint rule

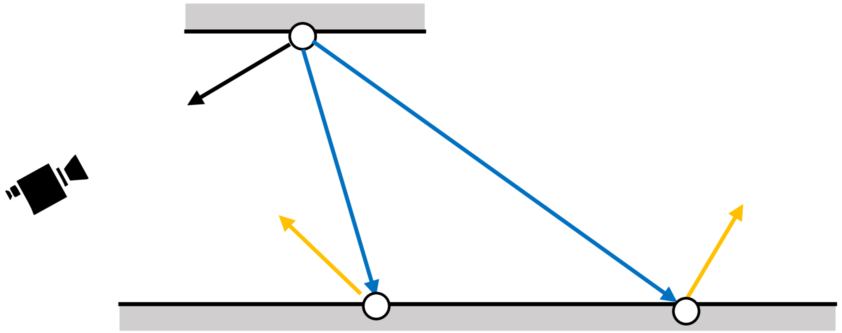




Problem: High dimensionality

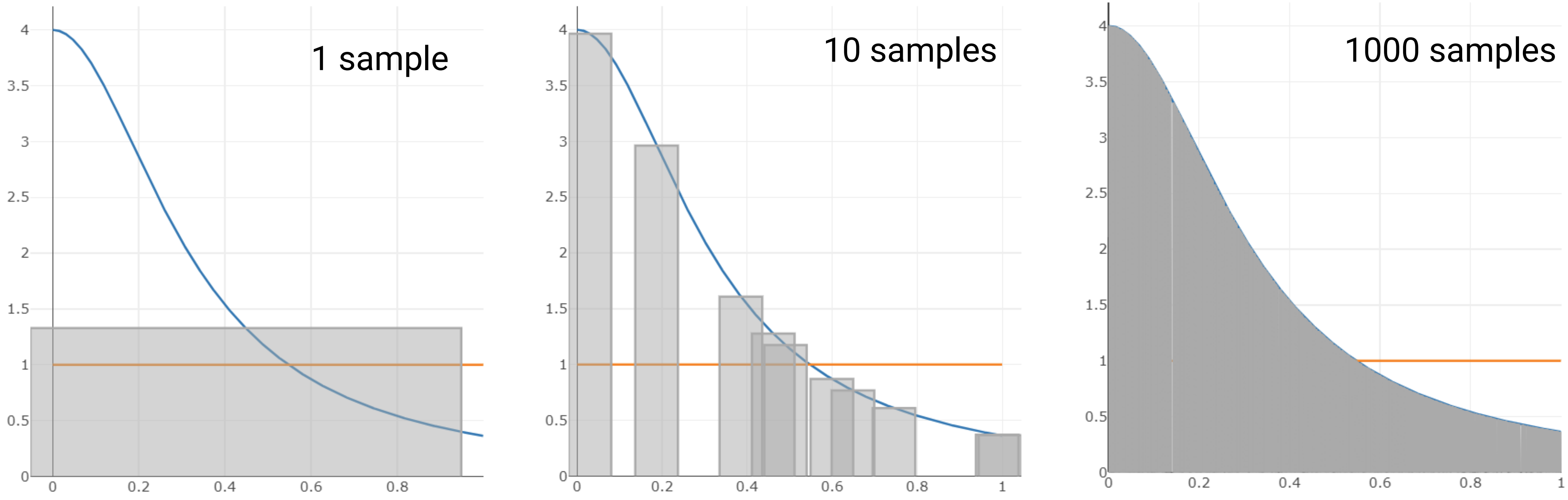
- Recursive! → High (infinite) dimensionality
- Simple quadrature: exponential cost
- Monte Carlo: one n-dimensional sample at a time





Monte Carlo integration: Estimate via random samples

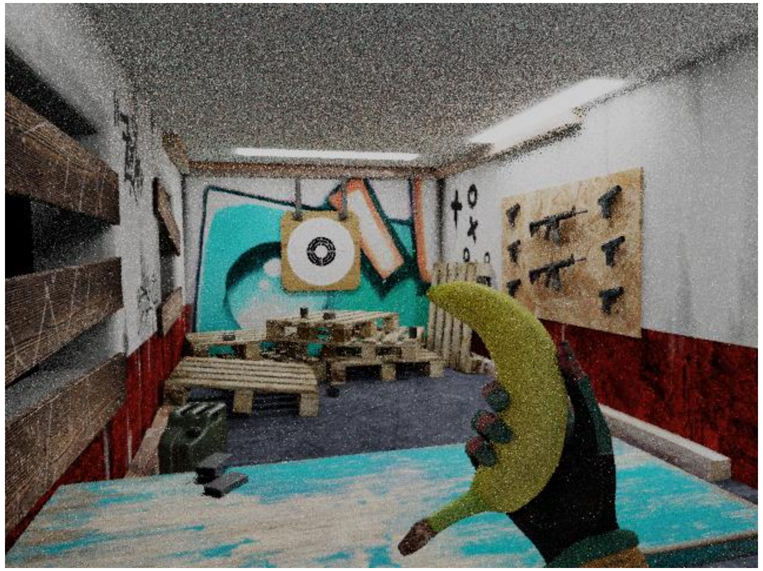
- Choose midpoints at random
- Converges to the integral
- Can be done **one sample at a time** – that's why it scales well!



Advantages of Monte Carlo integration

- Scales well: One sample at a time
- Converges to the correct solution
- Early iterations are noisy, but no systematic error (bias)

1 sample



10 samples

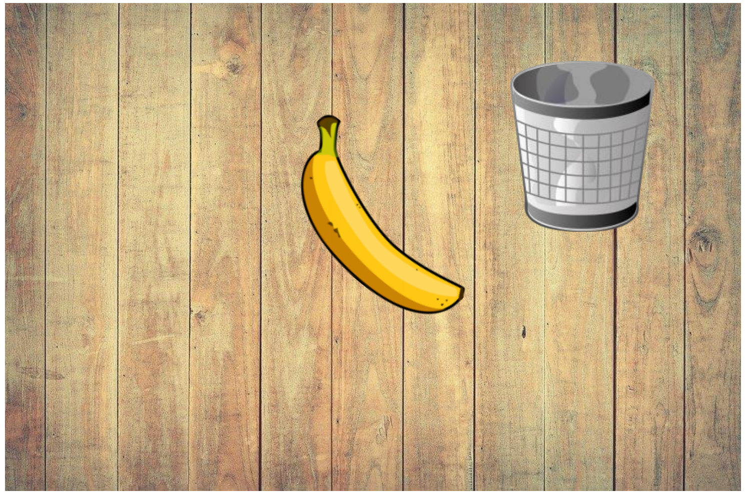
100 samples

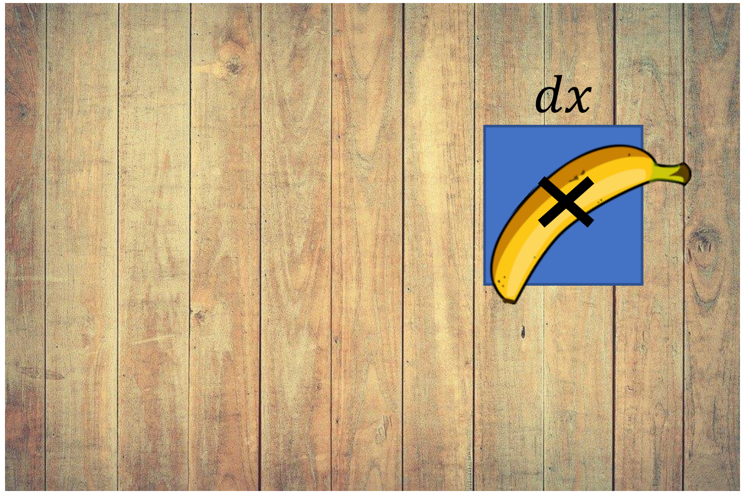
Applied to rendering

- Sample random path between camera and light, e.g., via
  - Recursive path tracing from the camera (“Path Tracer”)
  - Recursive path tracing from the light (“Light Tracer”)
  - Combine both (“BDPT”, “VCM”, …)
- Discussed in more depth over the next lectures!

A bit of math to back it all up

Probability density (PDF)





Discrete probability:

Throw a banana. Either lands in the bin or lands on the floor.

$$P(bin) \leq 1$$
$$P(floor) \leq 1$$
$$P(bin) + p(floor) = 1$$

Continuous probability:

Throw a banana. At what point on the floor will it land?

- Infinitely many possibilities (real numbers)
- Any exact position has zero probability
- $p(x)dx$ is the probability that the banana lands in the differential area dx
- $\int p(x)dx = 1$
- Unit: $[p(x)] = [dx]^{-1} = m^{-2}$

Basic properties much like discrete probabilities

- Joint PDF:
  - $p(x, y)$
- Can be written based on the conditional PDF
  - $p(x, y) = p(x|y)\, p(y)$
- We can obtain the marginal PDF
  - $p(x) = \int_Y p(x, y) dy = \int_Y p(x|y)\, p(y) dy$
- Useful for transforming samples (as we will see later)

Expected value

- For discrete random variables: Sum over all possible values F_i, multiplied by their probability

$$E[F] = \sum F_i \, P(F_i)$$

- Continuous case: Integral over all possible values $f(x)$ times their PDF $p(x)$

$$E[F] = \int f(x) \, p(x) \, dx$$

Monte Carlo Integration

Integral as an expected value

- We want to compute

$$F = \int_X f(x) dx$$

- Idea of MC integration: rewrite as

$$F = \int_X f(x) \frac{p(x)}{p(x)} dx = E\left[\frac{f(x)}{p(x)}\right]$$

- Where $p(x)$ is an arbitrary PDF, with $p(x) \neq 0$ whenever $f(x) \neq 0$

Primary estimator

- Sample a random x distributed according to $p(x)$ and compute

$$\langle F \rangle_1 = \frac{f(x)}{p(x)}$$

- The expected value is the integrand we are looking for

$$E[\langle F \rangle_1] = E\left[\frac{f(x)}{p(x)}\right] = F$$

The Monte Carlo estimator

- Average many primary estimators

$$\langle F \rangle_n = \frac{1}{n} \sum_{i=1}^{n} \frac{f(x_i)}{p(x_i)}$$

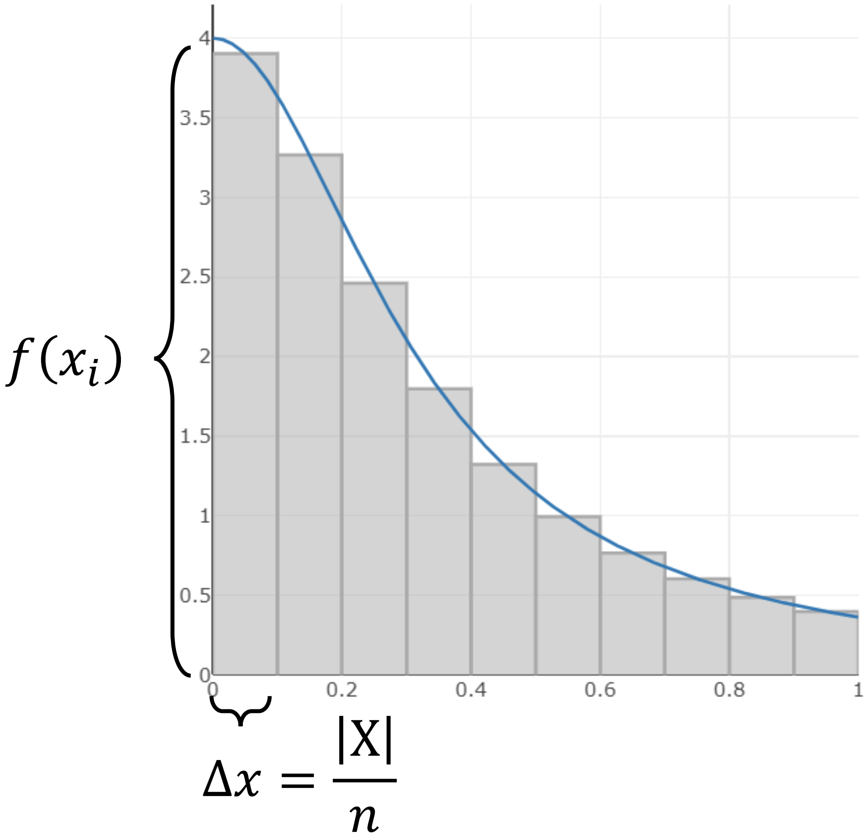
- Due to the law of large numbers

$$\lim_{n \to \infty} \langle F \rangle_n = E[\langle F \rangle_1]$$

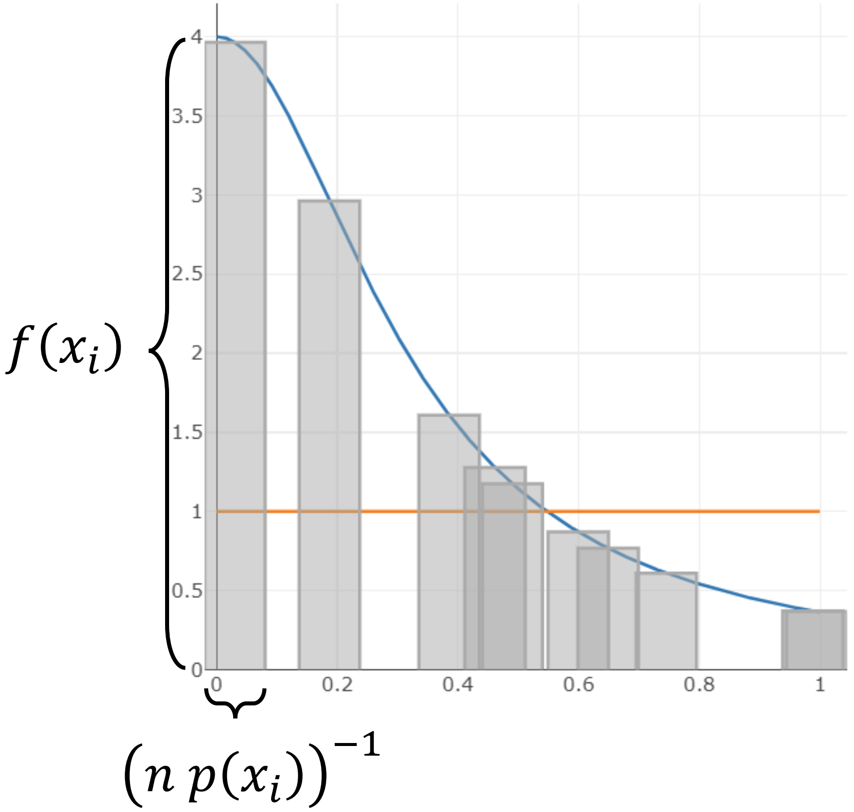
- Converges to the desired integral!

Graphical interpretation

Regular quadrature: $F \approx \sum_{i=1}^{n} f(x_i)\Delta x$



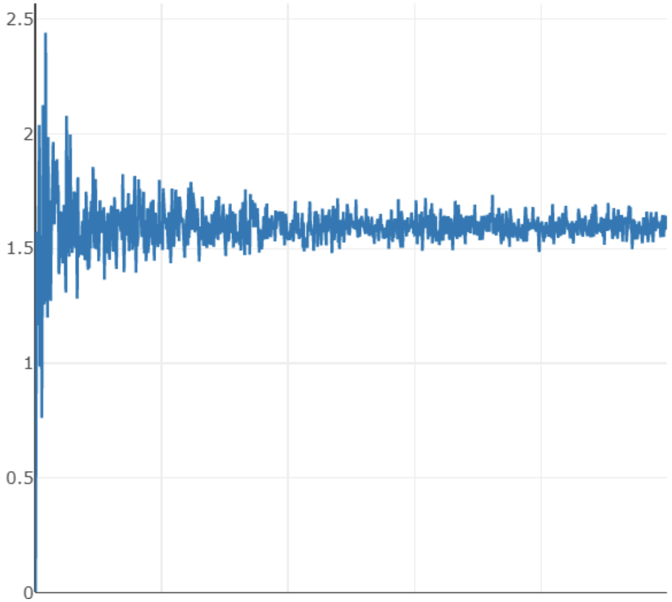
Monte Carlo: $\langle F \rangle_n = \sum_{i=1}^{n} \frac{f(x_i)}{n\, p(x_i)}$



A simple example

- Here, we use a uniform PDF $p(x) = 1$

```
int numSamples = 1_000_000;
double estimate = 0;
for (int i = 0; i < numSamples; ++i) {
    x = rng.NextDouble();
    estimate += Integrand(x) / numSamples;
}
```



Estimate with increasing n

Error and convergence

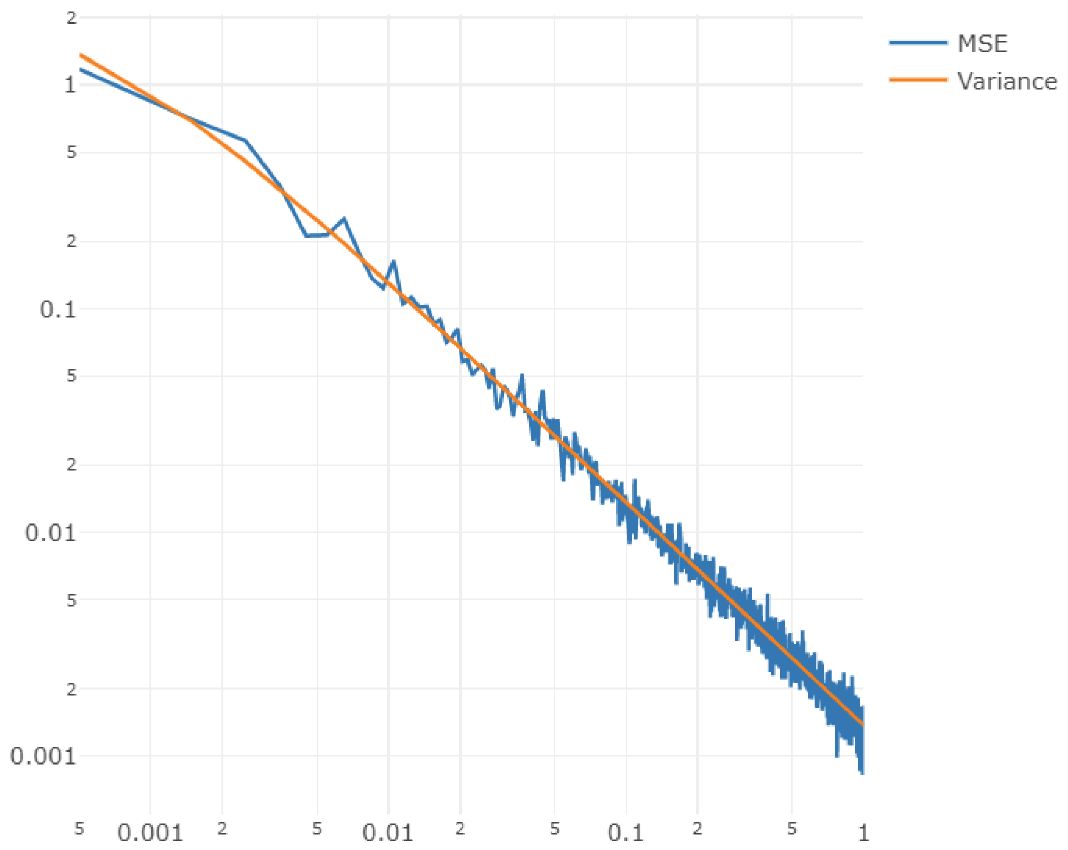
- Measured by variance: Expected squared error

$$\mathrm{V}[\langle F \rangle] = \mathrm{E}[(\langle F \rangle - F)^2] = \mathrm{E}[\langle F \rangle^2] - \mathrm{E}[\langle F \rangle]^2$$

- Reduces with increasing n

$$\mathrm{V}\left[\sum_{i=1}^{n} \frac{f(x_i)}{n\, p(x_i)}\right] = \frac{1}{n} \mathrm{V}\left[\frac{f(x)}{p(x)}\right]$$

- Can be reduced by choosing $p(x)$ intelligently



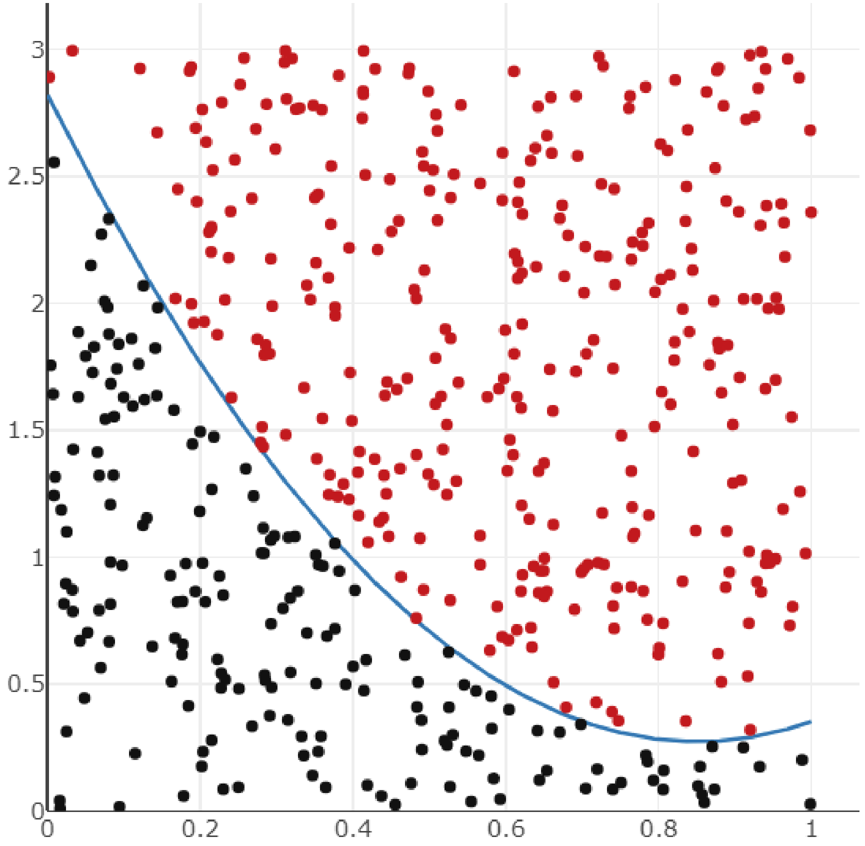
Sample transformation

How to sample according to a non-uniform PDF $p(x)$?

Rejection sampling

- Uniformly sample pair (x_i, y_i) from the envelope of $p(x)$
- Only keep points (x_i, y_i) below the PDF
- Repeat until desired number of accepted points found

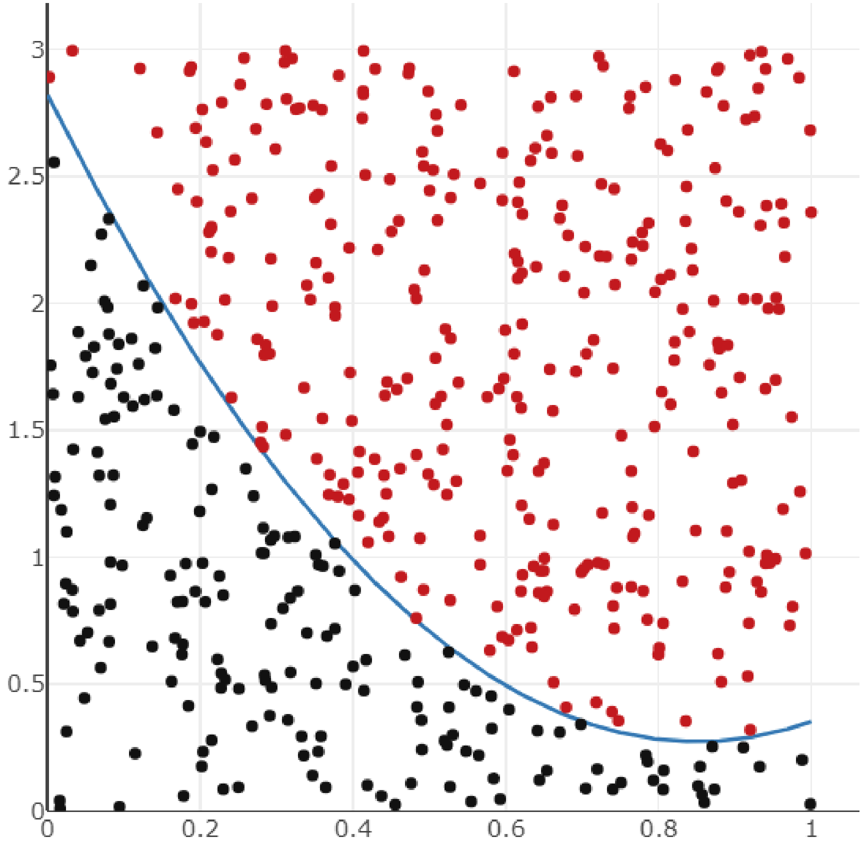
```
double RejectionSample() {
    while (true) {
        double x = rng.NextDouble();
        double y = rng.NextDouble() * yrange;
        if (y < Pdf(x)) return x;
    }
}
```



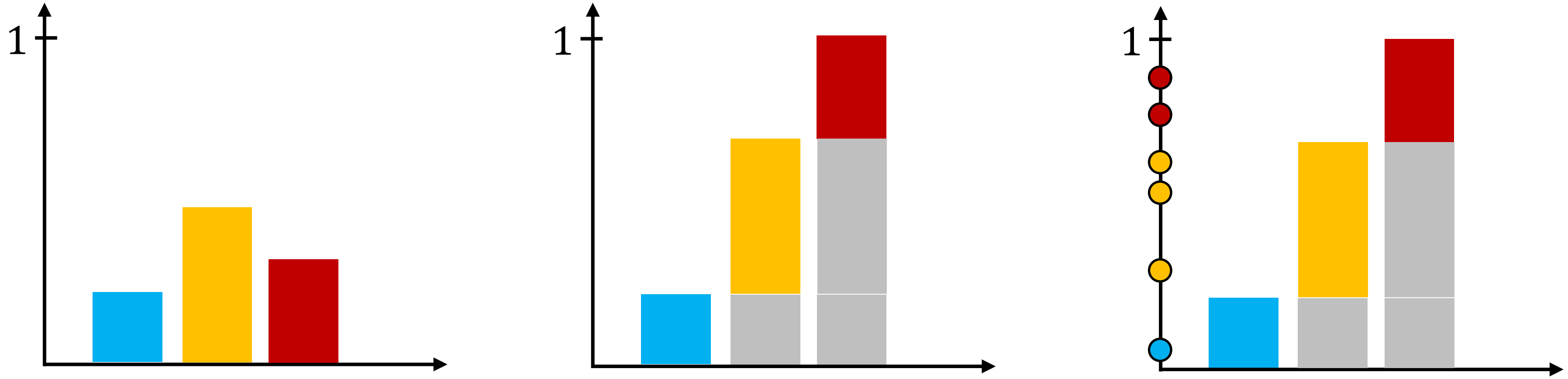
Rejection sampling

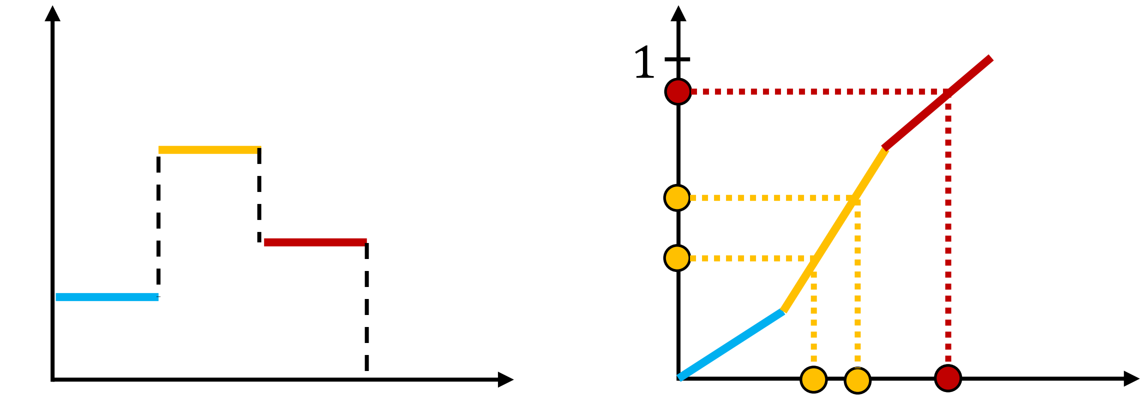
- Pros:
  - Easy to implement

- Cons:
  - Can be inefficient (if many samples are rejected)
  - Prevents sample stratification / jittered sampling



CDF inversion





Commulative distribution function (CDF) $P(x) = \int_0^x p(y)dy$

Setting

$$x = P^{-1}(y)$$

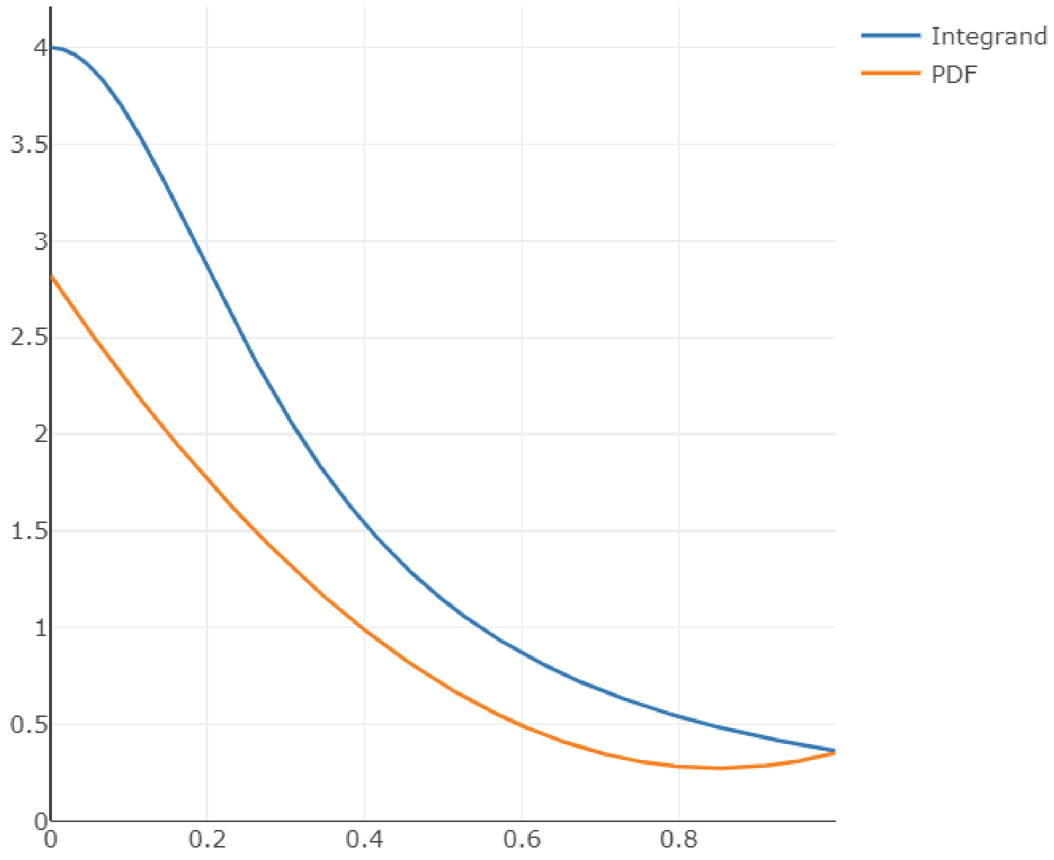
With uniform y results in

$$x \sim p(x)$$

CDF inversion

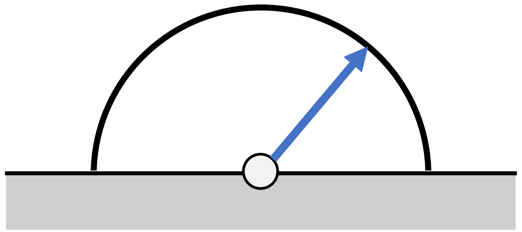
- Pros:
  - Exact sampling possible
  - (Sometimes) efficient to compute

- Cons:
  - Difficult to find invertible CDF (even in our simple 1D case!)
  - (Sometimes) expensive to compute



Example: Sampling the uniform hemisphere

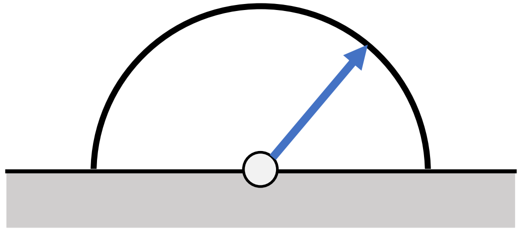
- Target pdf: $p(\omega) \propto 1$
- Normalize so that $\int p(\omega) d\omega = 1$
- We know $\int 1 d\omega = |\Omega| = 2\pi \text{ sr}$
- So $p(\omega) = \frac{1}{2\pi \text{ sr}}$



$$\int_\Omega L_i \, f \cos\theta \, d\omega_i$$

CDF inversion for the uniform hemisphere

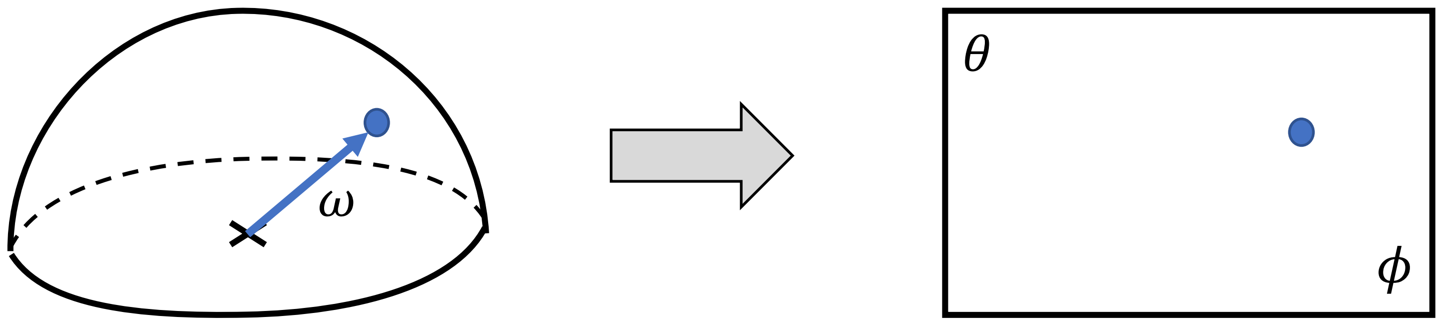
- Goal: Sample according to $p(\omega) = \frac{1}{|\Omega|} = \frac{1}{2\pi \text{ sr}}$
- $p(\omega)$ is a 3D density, we don't want to try to invert its CDF directly
- Instead, express in spherical coordinates: $p(\theta, \phi)$
- Separate into two 1D PDFs: $p(\theta, \phi) = p(\phi)p(\theta|\phi)$
- First sample ϕ then, conditionally, θ



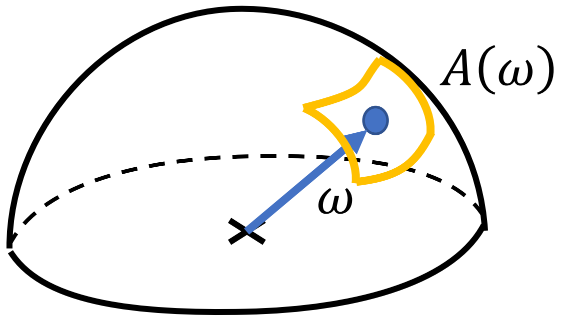
$$\int_{\Omega} L_i \, f \cos\theta \, d\omega_i$$

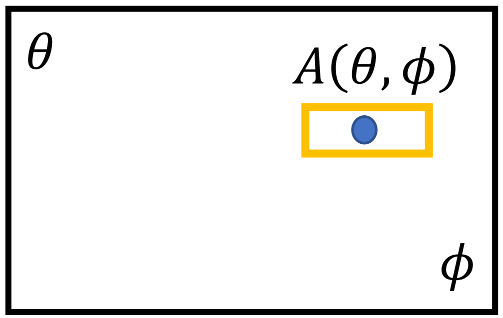
Mapping a direction to spherical coordinates

$$x = \cos\phi \sin\theta$$
$$y = \sin\phi \sin\theta$$
$$z = \cos\theta$$



How does the density change? (Intuition)





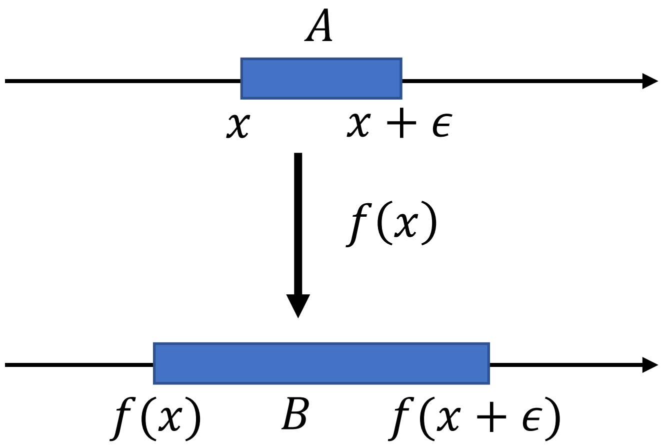
Larger area -> lower density

Ratio of densities = Inverse ratio of areas

$$\frac{p(\theta, \phi)}{p(\omega)} = \frac{A(\omega)}{A(\theta, \phi)}$$

(intuitively; equality only holds for differential areas $d\omega$ and $d\theta d\phi$)

Even simpler: change of length in 1D



$$\frac{B}{A} = \frac{f(x+\epsilon) - f(x)}{x+\epsilon-x}$$

$$\lim_{\epsilon \to 0} \frac{B}{A} = \frac{df(x)}{dx}$$

Derivative!

In multiple dimensions: Jacobian determinant

- Mapping from spherical coordinates to cartesian coordinates:

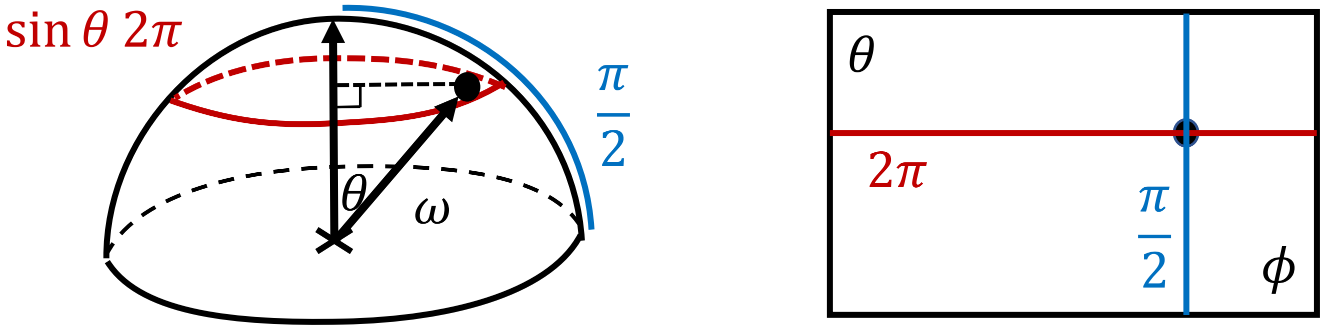
$$f(\theta, \phi, r) = \begin{pmatrix} r\cos\phi\sin\theta \\ r\sin\phi\sin\theta \\ r\cos\theta \end{pmatrix}$$

- Jacobian determinant:

$$\begin{vmatrix} \frac{d}{d\theta} r\cos\phi\sin\theta & \frac{d}{d\phi} r\cos\phi\sin\theta & \frac{d}{dr} r\cos\phi\sin\theta \\ \frac{d}{d\theta} r\sin\phi\sin\theta & \frac{d}{d\phi} r\sin\phi\sin\theta & \frac{d}{dr} r\sin\phi\sin\theta \\ \frac{d}{d\theta} r\cos\theta & \frac{d}{d\phi} r\cos\theta & \frac{d}{dr} r\cos\theta \end{vmatrix} = \begin{vmatrix} r\cos\phi\cos\theta & -r\sin\phi\sin\theta & \cos\phi\sin\theta \\ r\sin\phi\cos\theta & r\cos\phi\sin\theta & \sin\phi\sin\theta \\ -r\sin\theta & 0 & \cos\theta \end{vmatrix} = r^2\sin\theta$$

- For directions: $r = 1$
- PDF conversion: $p(\theta, \phi) = \sin\theta\, p(\omega)$

Verified geometrically



Area on hemisphere is locally $\sin\theta$ times as large as the corresponding area on the 2D plane of spherical coordinates

$$\Rightarrow d\omega = \sin\theta\, d\theta d\phi$$

$$\Rightarrow p(\theta, \phi) = \sin\theta\, p(\omega)$$

CDF inversion for the uniform hemisphere (continued)

- $p(\omega) = \frac{1}{2\pi} \Rightarrow p(\theta, \phi) = \frac{\sin\theta}{2\pi}$
- Marginal PDF: $p(\phi) = \int_0^{\frac{\pi}{2}} \frac{\sin\theta}{2\pi} d\theta = \frac{1}{2\pi}$
- No big surprise: it's uniform
- Conditional PDF: $p(\theta|\phi) = \frac{p(\theta,\phi)}{p(\phi)} = \sin\theta$
- The CDF is $P(\theta|\phi) = \int_0^{\theta} \sin x \, dx = 1 - \cos\theta$
- And its inverse $P^{-1}(y) = \cos^{-1}(1-y)$

Sampling the uniform hemisphere

Input: 2 uniform random numbers in [0,1]

Output: cartesian coordinates (z axis up) of the direction in the hemisphere

$$p(\phi) = \frac{1}{2\pi} \Rightarrow \phi = 2\pi x$$

$$\theta = \cos^{-1}(1 - y)$$

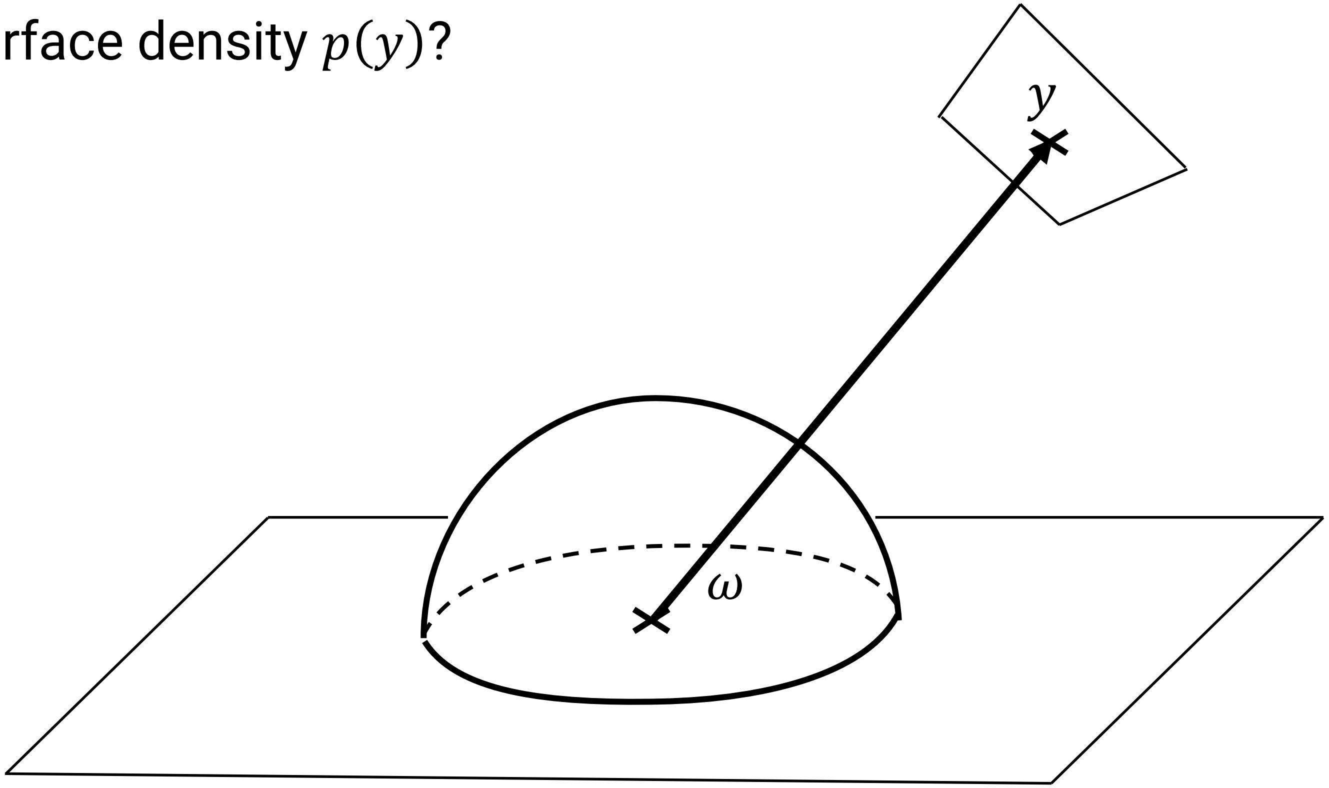
```
Vector3 ToUniformHemisphere(float x, float y) {
    float phi = 2 * MathF.PI * x;

    float cosTheta = 1 - y;
    float sinTheta = MathF.Sqrt(1 - cosTheta * cosTheta);

    return new Vector3(
        sinTheta * MathF.Cos(phi),
        sinTheta * MathF.Sin(phi),
        cosTheta
    );
}
```

Sampling directions via points on a surface

- Sometimes, we rather sample points on surfaces than directions
- Example: connecting directly to a point on a light
- How can we compute $p(\omega)$ from a surface density $p(y)$?

$$p(\omega)d\omega = p\big(y(\omega)\big)dy(\omega)$$

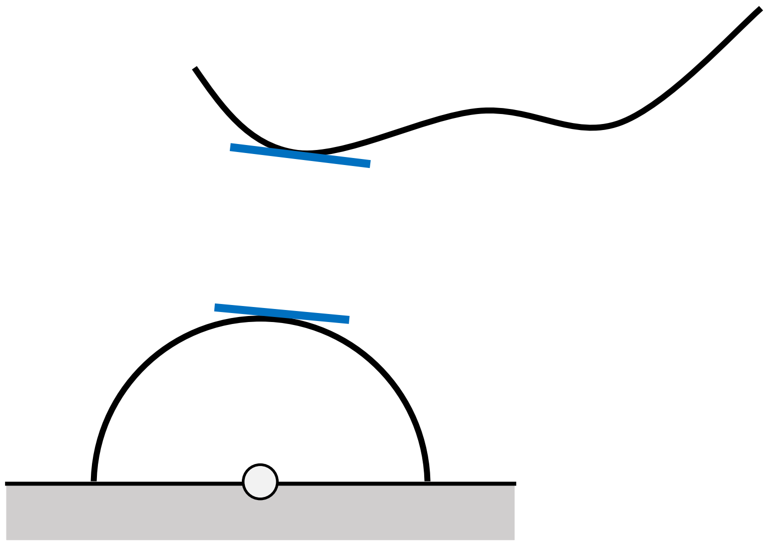
$$\Leftrightarrow p(\omega) = p\big(y(\omega)\big)\boxed{\frac{dy(\omega)}{d\omega}}$$

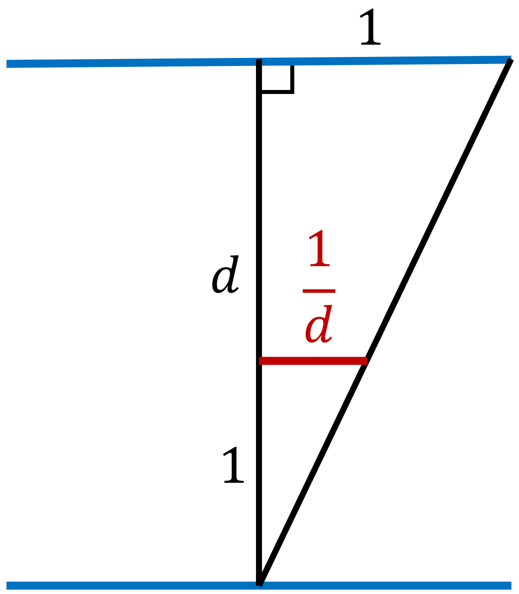


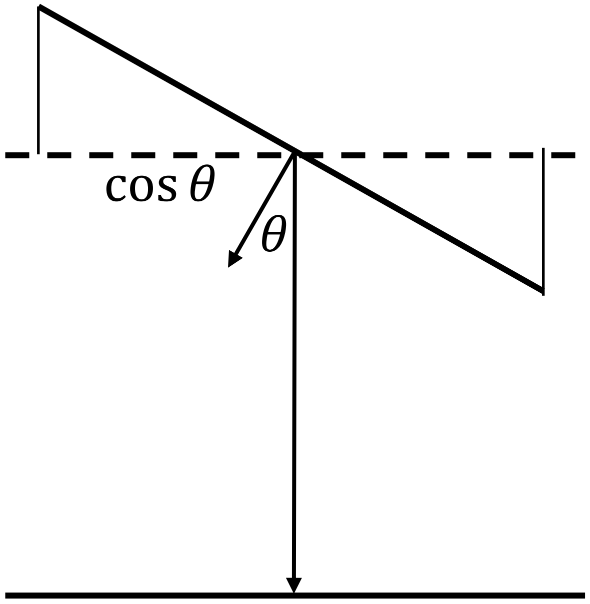
Geometry term

- Surfaces are 2D manifolds
- They locally resemble a plane

$$\frac{d\omega}{dy} = \frac{\cos\theta}{d^2}$$

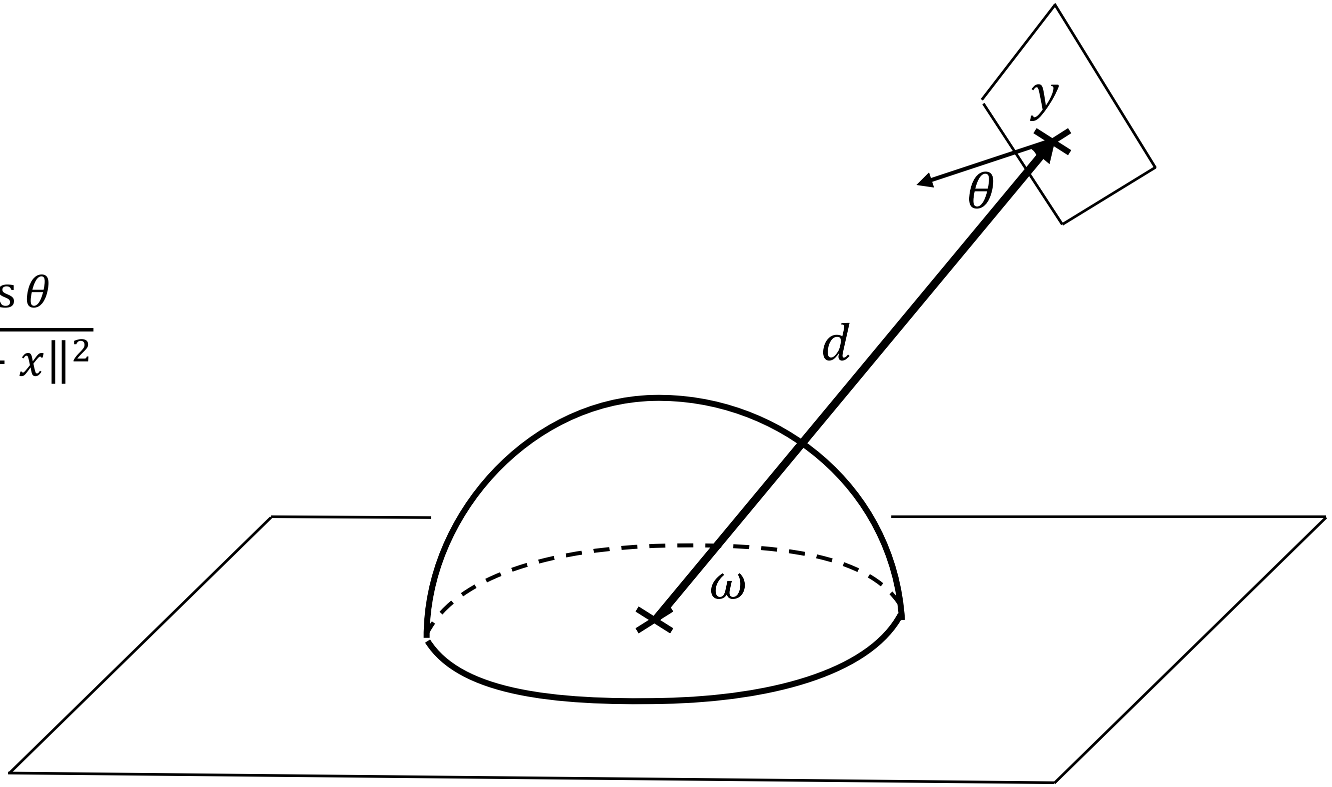






Sampling directions via points on a surface

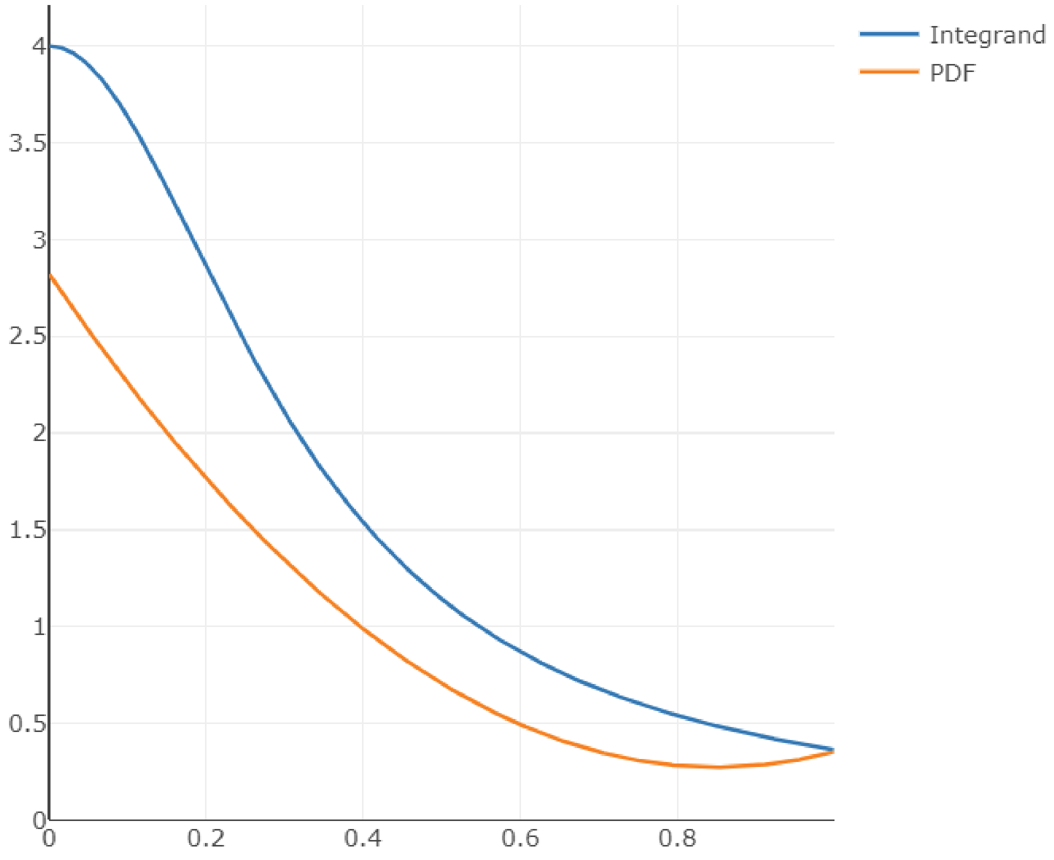
1. Sample a point y on the surface (e.g., light source)
2. Compute
$$p(\omega) = p(y)\frac{\|y-x\|^2}{\cos\theta}$$
3. Monte Carlo estimate:
$$\frac{f\big(x,\omega(x,y)\big)}{p(\omega)} = \frac{f\big(x,\omega(x,y)\big)}{p(y)}\frac{\cos\theta}{\|y-x\|^2}$$

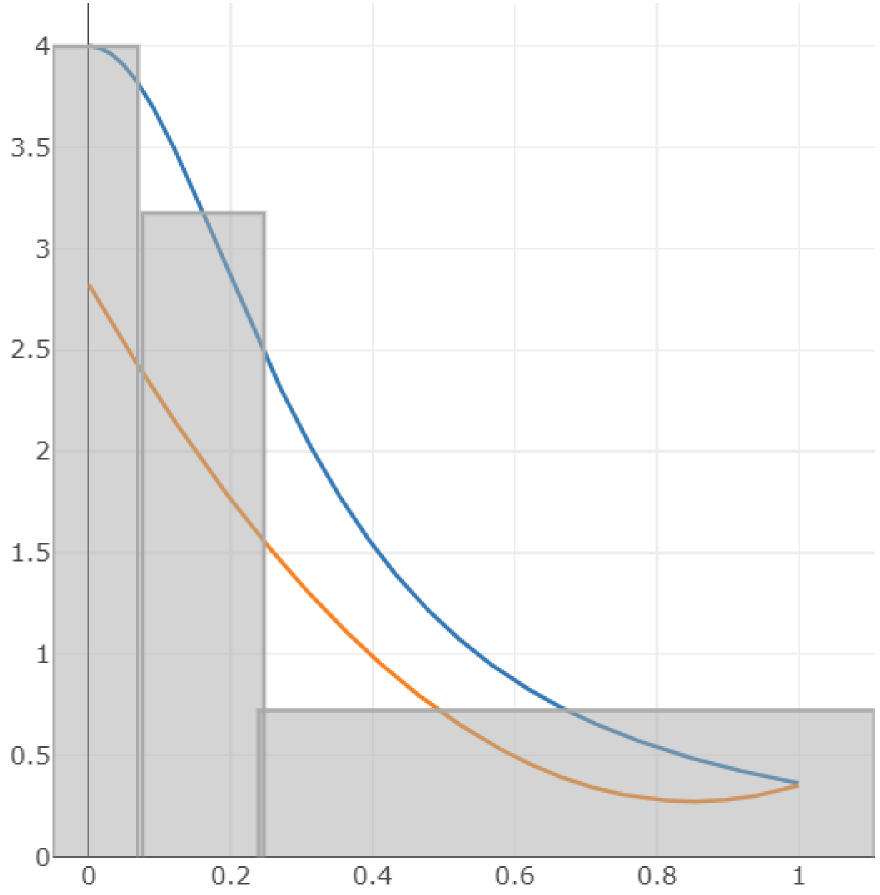


Variance Reduction

Importance sampling

- Choosing $p(x)$ to focus on “important” regions





Zero variance (intuition)

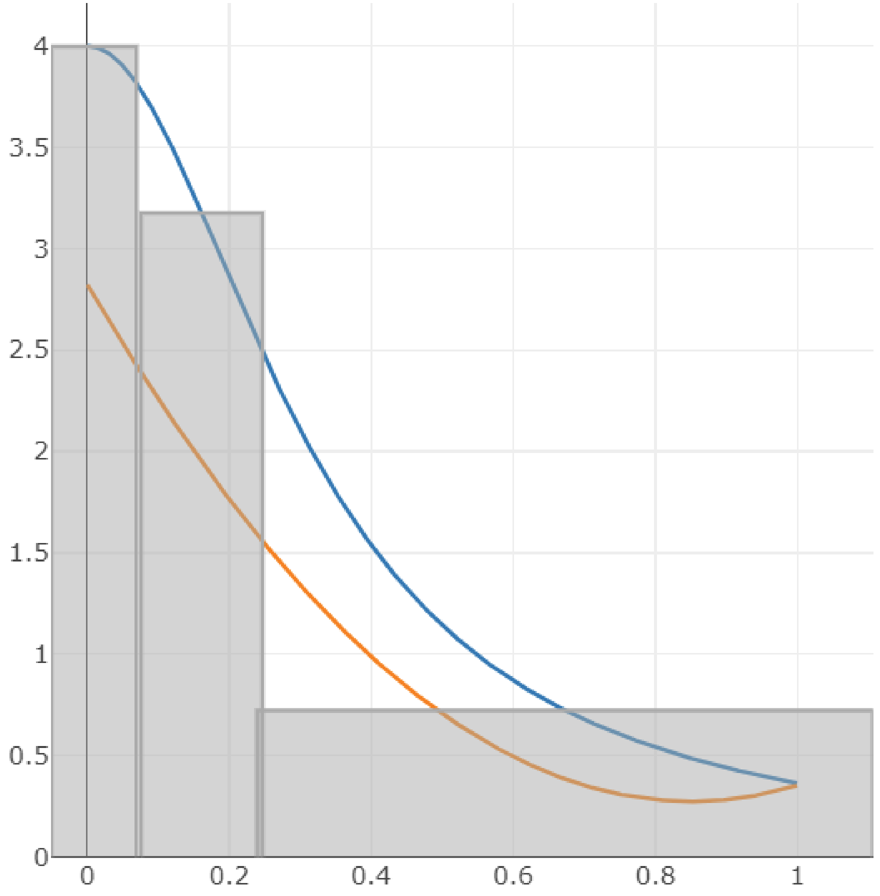
- Ideally: area of every “box” should be: $A = \int f(x)dx$
- Then, a single sample gives the correct result

$$A = \frac{f(x)}{p(x)}$$

$$\Rightarrow \frac{f(x)}{p(x)} = \int f(x)dx$$

$$\Leftrightarrow p(x) = \frac{f(x)}{\int f(x)dx} \propto f(x)$$

- Of course, that requires we know $\int f(x)dx$ already…



Zero variance (formal)

- The variance is $\sigma^2 = E[\langle F \rangle^2] - F^2 = \int \frac{f^2(x)}{p(x)} dx - F^2$
- We want the minimizing $p(x)$, constrained by $\int p(x)dx = 1$, via a Lagrangian multiplier

$$p_{opt}(x) = \arg\min_{p(x)} \left(\int \frac{f^2(x)}{p(x)} dx - \lambda \left(\int p(x)dx - 1 \right) \right)$$

$$\frac{\delta}{\delta p} \left(\int \frac{f^2(x)}{p(x)} dx - \lambda \left(\int p(x)dx - 1 \right) \right) = 0$$

$$\Leftrightarrow -\frac{f^2(x)}{p^2(x)} - \lambda = 0$$

$$\Leftrightarrow p(x) = \frac{f(x)}{\sqrt{-\lambda}} \longrightarrow$$

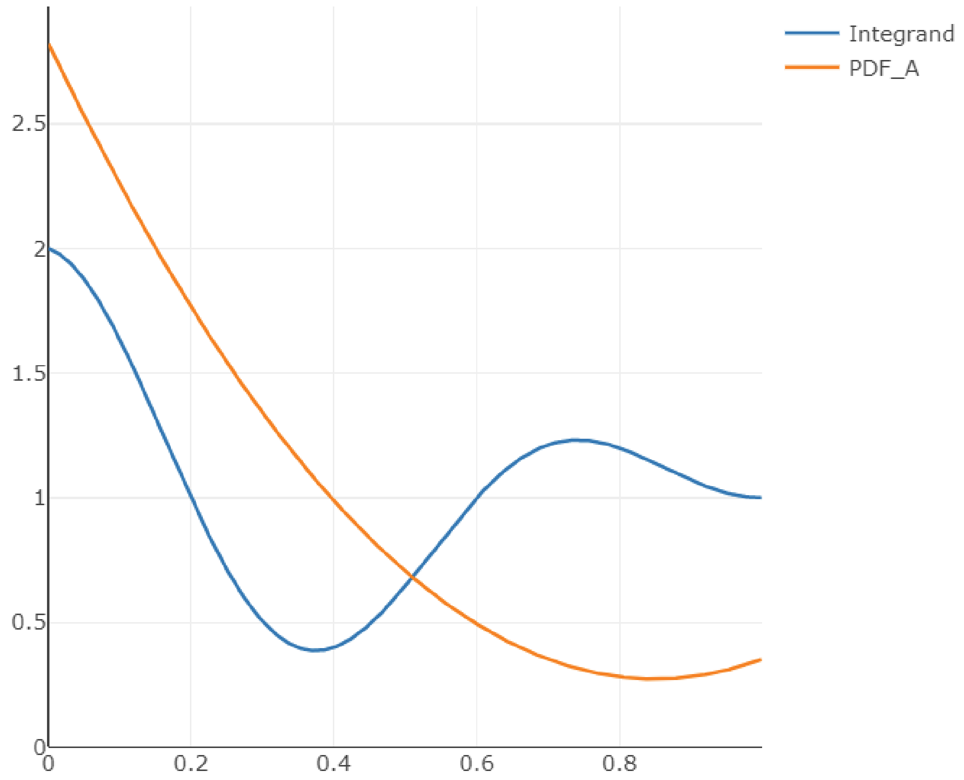
$$\Rightarrow \int \frac{f(x)}{\sqrt{-\lambda}} dx = 1 \Leftrightarrow \sqrt{-\lambda} = \int f(x) dx$$

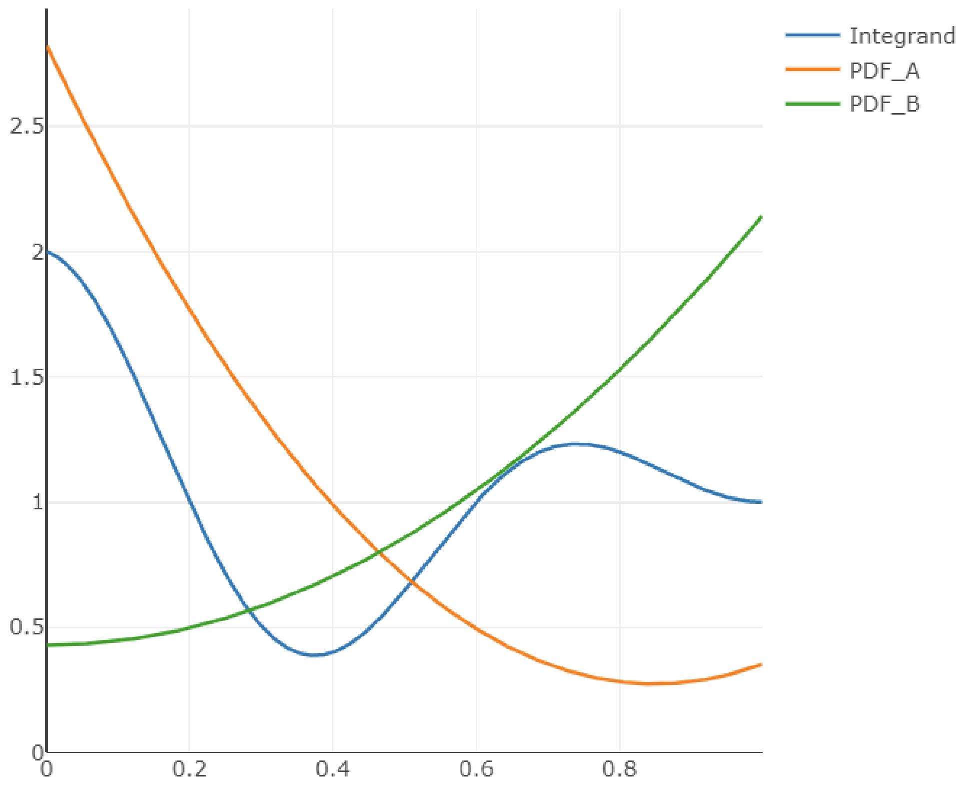
$$\frac{\delta}{\delta \lambda}(\dots) = 0 \Leftrightarrow \int p(x)dx = 1 \nearrow$$

$$\Rightarrow p(x) = \frac{f(x)}{\int f(x)dx}$$

Multiple importance sampling (MIS)

- Idea: use multiple densities that match different regions well





Simple average

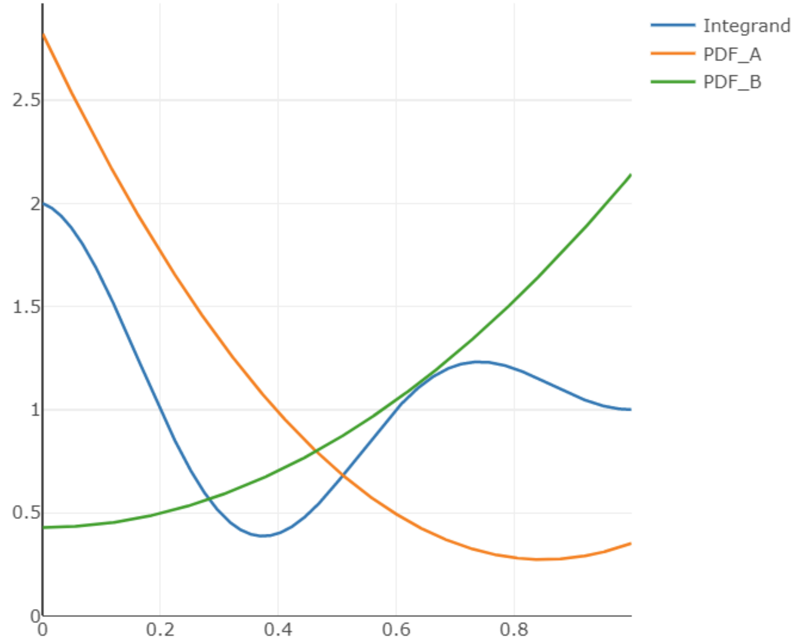
- The simplest way: average both estimators

$$\langle F \rangle_{\text{avg}} = \frac{1}{2} \sum_{i=1}^{n_A} \frac{f(x_i)}{n_A\, p_A(x_i)} + \frac{1}{2} \sum_{i=1}^{n_B} \frac{f(x_i)}{n_B\, p_B(x_i)}$$

- The resulting variance is also the weighted sum:

$$V\big[\langle F \rangle_{\text{avg}}\big] = \frac{1}{4} V[\langle F \rangle_A] + \frac{1}{4} V[\langle F \rangle_B]$$

- If either (or both) individual variances are high, this is still bad!



MIS

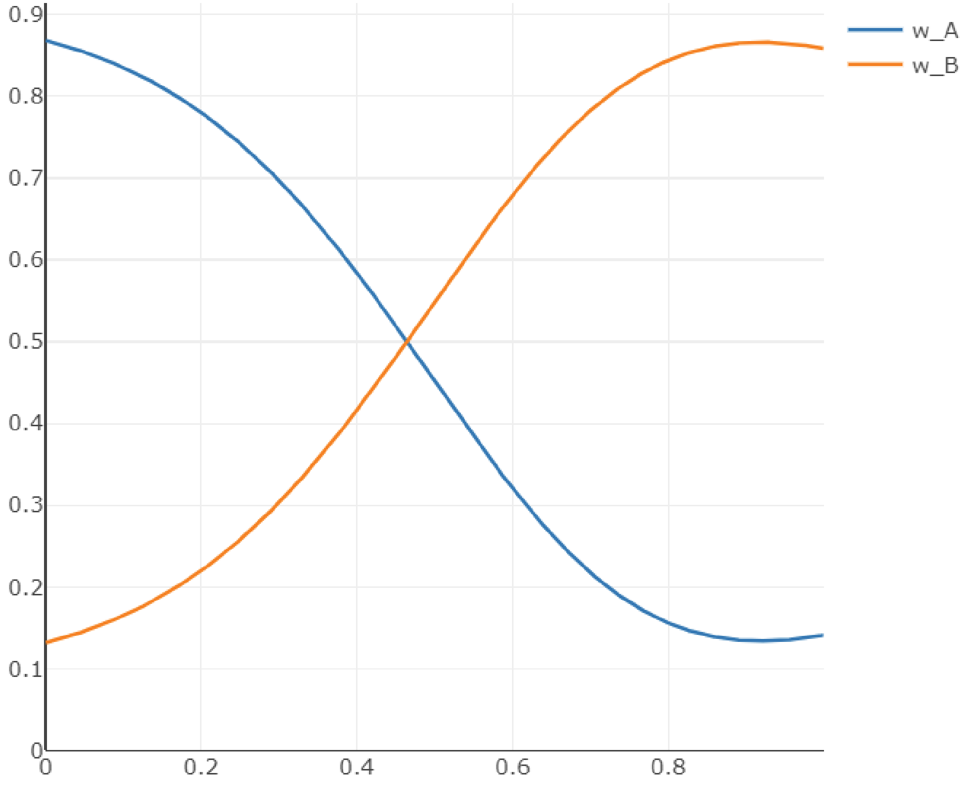
- Form a linear combination
- Weighting function $w_t(x)$ for each sampling technique

$$\langle F \rangle_{\text{MIS}} = \sum_t \sum_{i=1}^{n_t} w_t(x_{t,i}) \frac{f(x_{t,i})}{n_t p_t(x_{t,i})}$$

- Unbiased estimator is achieved if for all x where $f(x) \neq 0$:

$$\sum_t w_t(x) = 1$$

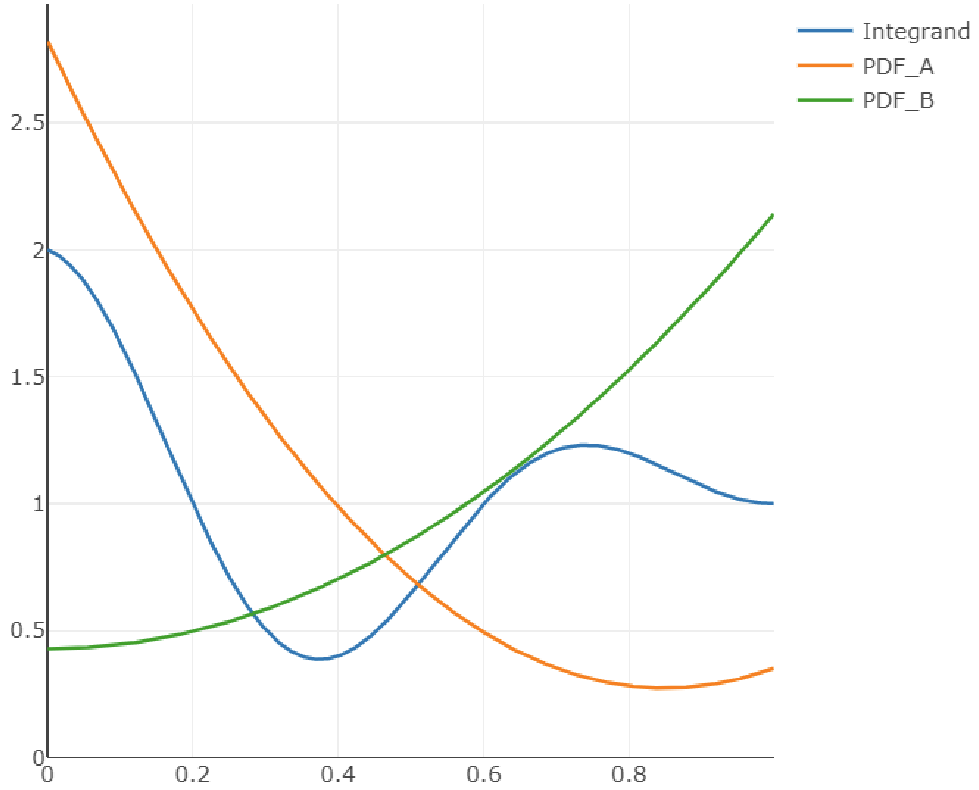
$$p_t(x) = 0 \Rightarrow w_t(x) = 0$$



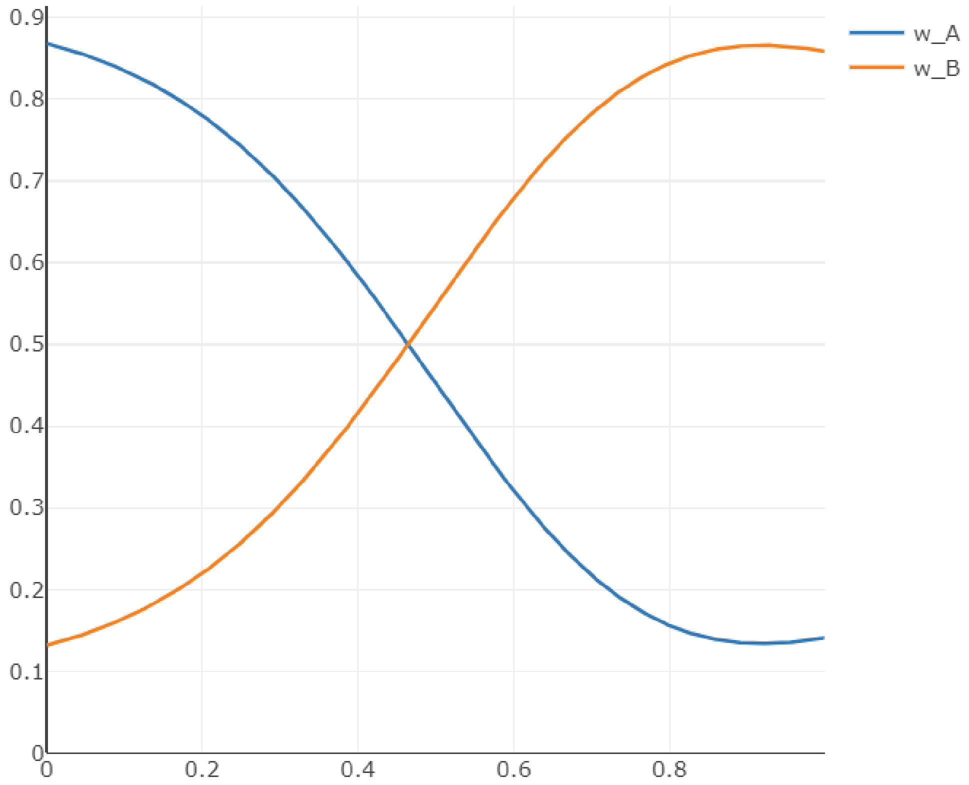
MIS weights

Balance heuristic

$$w_t(x) = \frac{n_t p_t(x)}{\sum_k n_k p_k(x)}$$



Integrand and densities



Balance heuristic weights

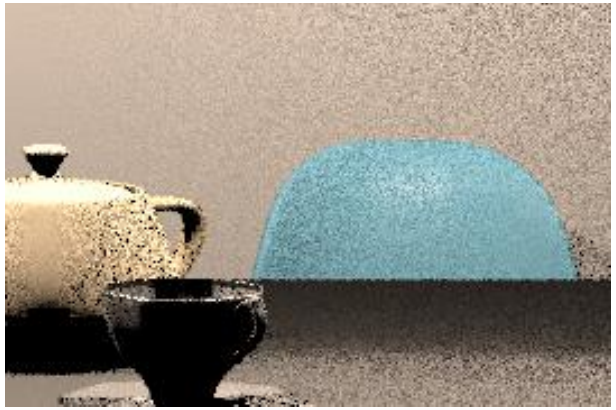
Balance heuristic and optimality

- Provably good but not optimal
- Minimizes an upper bound of the variance
- Ignores sample / technique correlation
- Performs poorly if some techniques have low variance

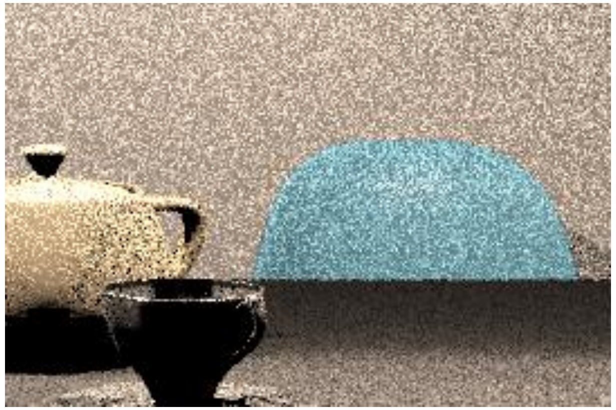
Minimized by optimal weights

$$\sigma_t^2 = \int_\Omega \frac{f^2(x)}{n_t p_t(x)} dx - r_t$$

Minimized by balance



Path tracer



Path tracer **+ Bidir.**
(Balance heuristic)

Power and maximum heuristics

- Amplify weights where one density is higher
- If high density correlates with low variance, that improves the “low variance” issue

Power heuristic

$$w_t(x) = \frac{\left(n_t p_t(x)\right)^2}{\sum_k \left(n_k p_k(x)\right)^2}$$

Maximum heuristic

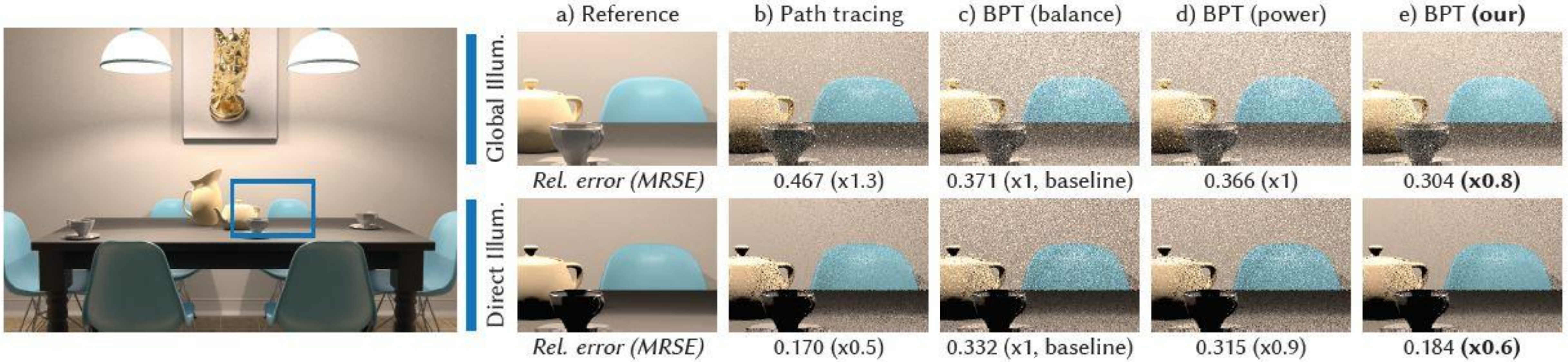
$$w_t(x) = \begin{cases} 1, & n_t p_t(x) > n_k p_k(x)\ \forall k \\ 0, & \text{else} \end{cases}$$

Variance-aware balance heuristic

- Estimate second moment and variance
- Use to offset the weights in the right direction

$$v_t = \frac{\int_\Omega \frac{f^2(x)}{n_t p_t(x)} dx}{\sigma_t^2}$$

$$w_t(x) = \frac{{\color{red}v_t} n_t p_t(x)}{\sum_k {\color{red}v_k} n_k p_k(x)}$$



Optimal MIS weights: complex and expensive, but worth it

Minimized by optimal weights

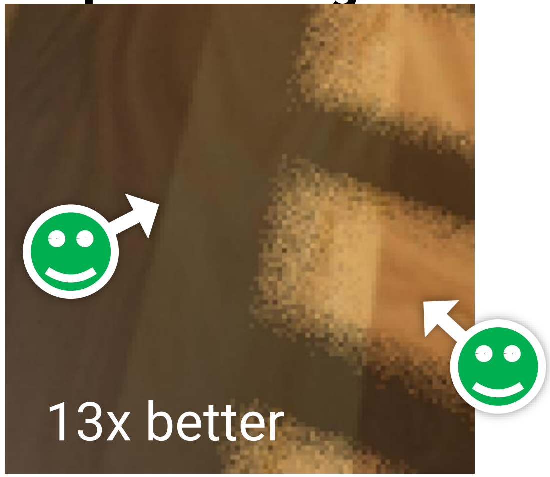
$$\sigma_t^2 = \int_\Omega \frac{f^2(x)}{n_t p_t(x)} dx - r_t$$

Minimized by balance

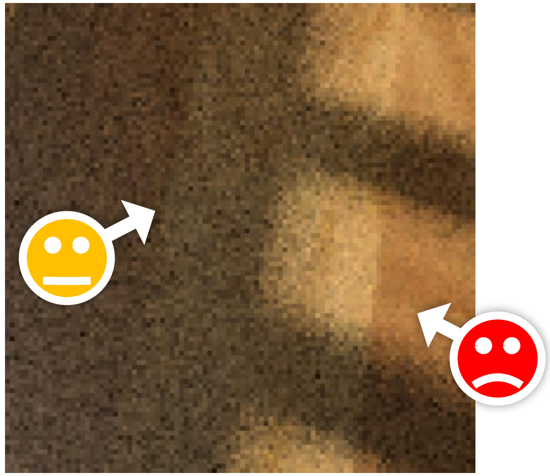
$$a_{ij} = \int \frac{p_i p_j}{\sum n_k p_k} \qquad b_i = \int \frac{p_i f}{\sum n_k p_k}$$

$$\begin{pmatrix} a_{11} & \cdots & a_{1N} \\ \vdots & \ddots & \vdots \\ a_{N1} & \cdots & a_{NN} \end{pmatrix} \begin{pmatrix} \alpha_1 \\ \vdots \\ \alpha_N \end{pmatrix} = \begin{pmatrix} b_1 \\ \vdots \\ b_N \end{pmatrix}$$

Optimal weights

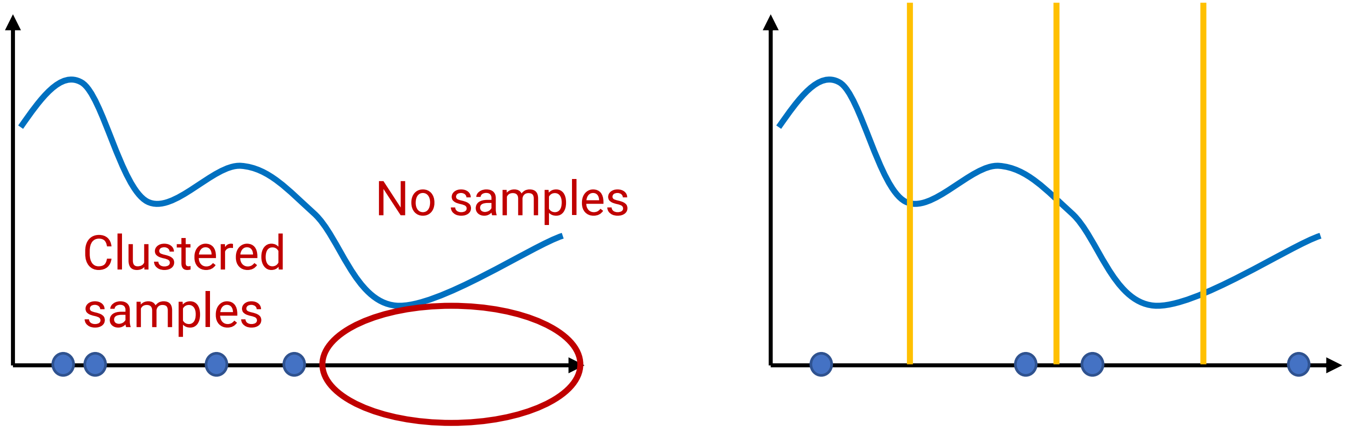


Balance heuristic



Stratified sampling

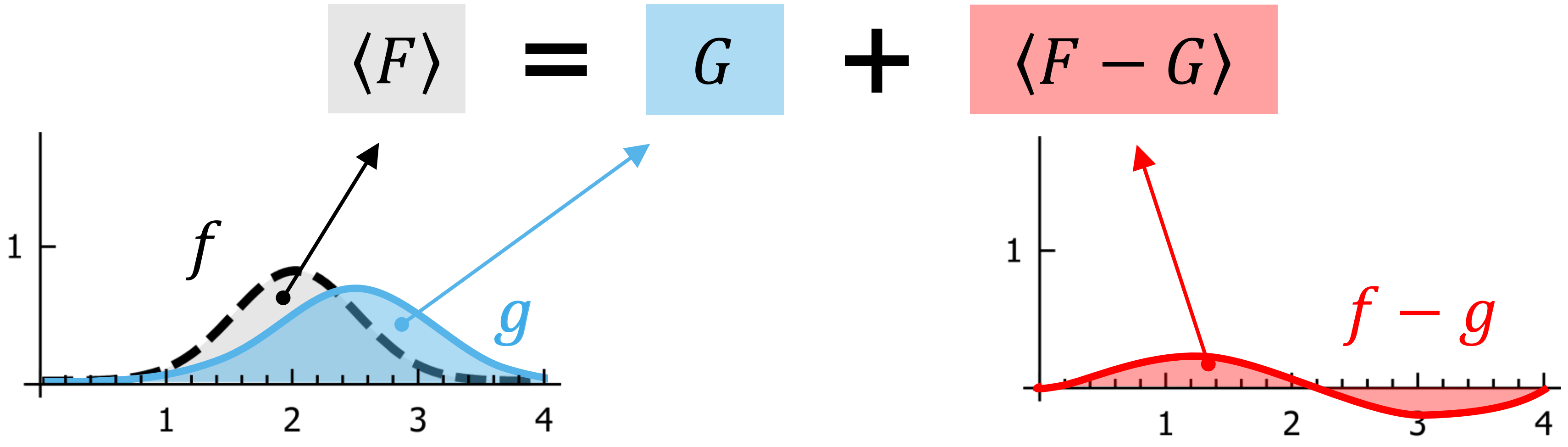
- Subdivide domain into bins
- Sample within each bin
- Less sample clustering → Guarantees that all regions are explored
- Lower variance!



- Discussed in more depth later in the course

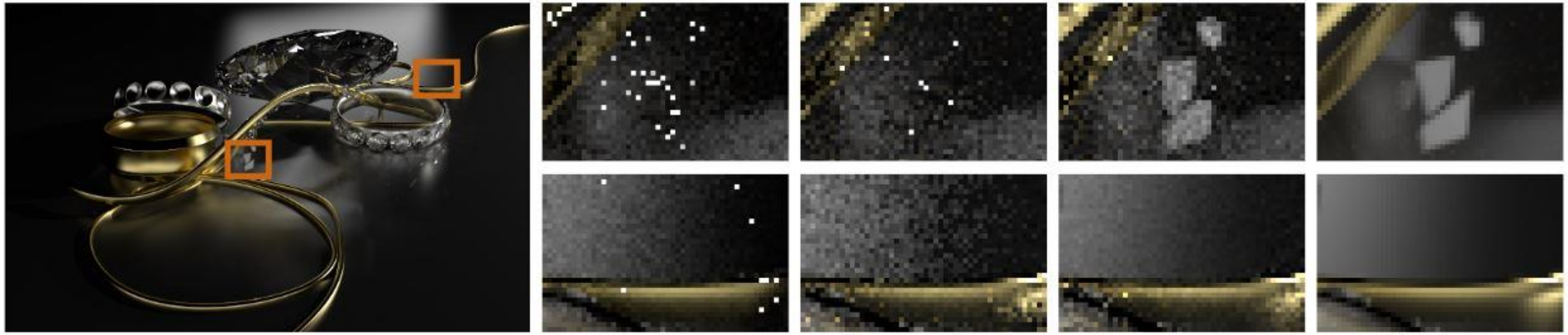
Control Variates

- Start with known integral
- Estimate difference between that and the target
- Can have lower variance if chosen well



Learning to importance sample

- Sample in multiple iterations
- Use data from first iteration(s) to learn a PDF that is closer to the optimal one
- Discussed more in-depth later in the course



NECKLACE | (a) Path tracer 89.9 | (b) Radiance-based 0.4 (baseline) | (c) Our target density **0.16 (2.6x)** | (d) Reference *relMSE*

Summary

What have we learned today?

This lecture

- Rendering equation: recursive integral, infinite dimensionality
- Analytical solution not possible
- Monte Carlo integration: numerical integration method that scales well with dimensionality
- Like “normal” quadrature but with random positions
- Efficient, scales, well, very flexible (many tweak, tricks, improvements possible)

Next up: Apply MC to rendering!

- Path tracing
- Bidirectional path tracing
- Density estimation
- Combinations via MIS
- Learned importance sampling
- Filtering and denoising
- Quasi-Monte Carlo and sampling patterns
- Markov chain Monte Carlo