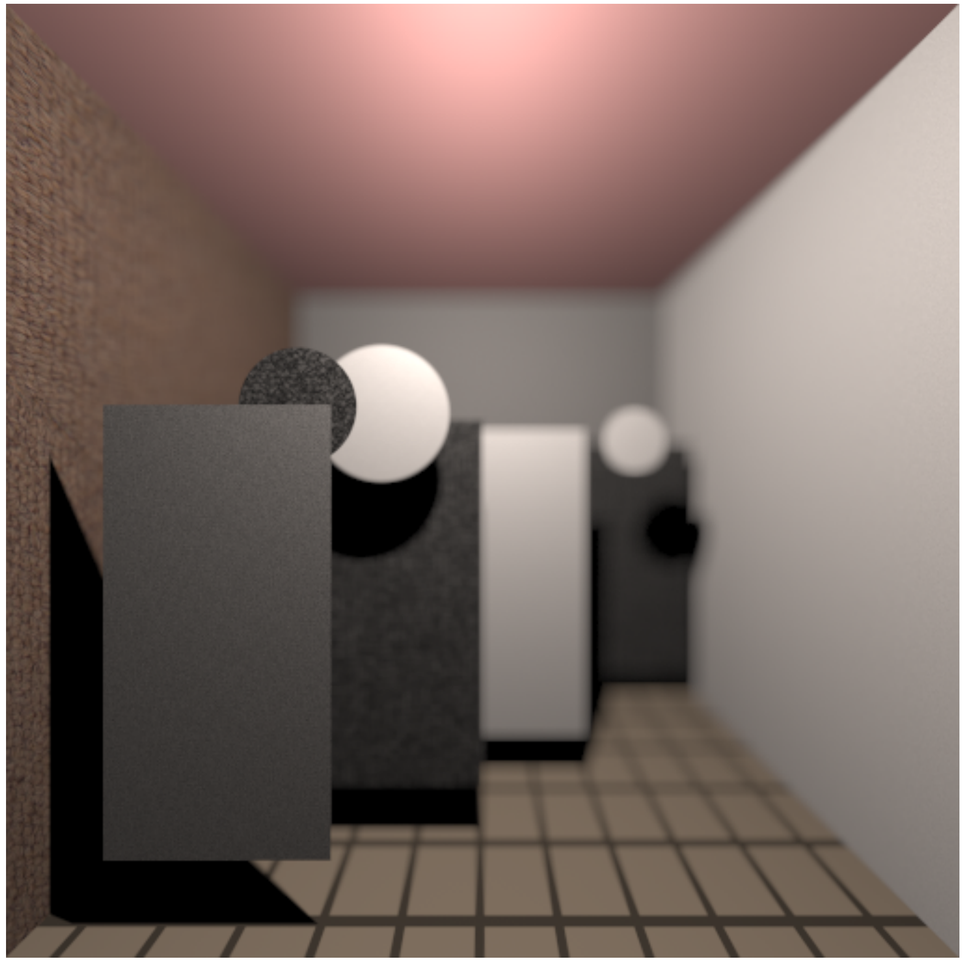
Advance Sampling: Fourier Analysis of Variance in Monte Carlo Integration

Gurprit Singh
Philipp Slusalek Karol Myszkowski

Recap: Advanced Sampling

Monte Carlo Integration



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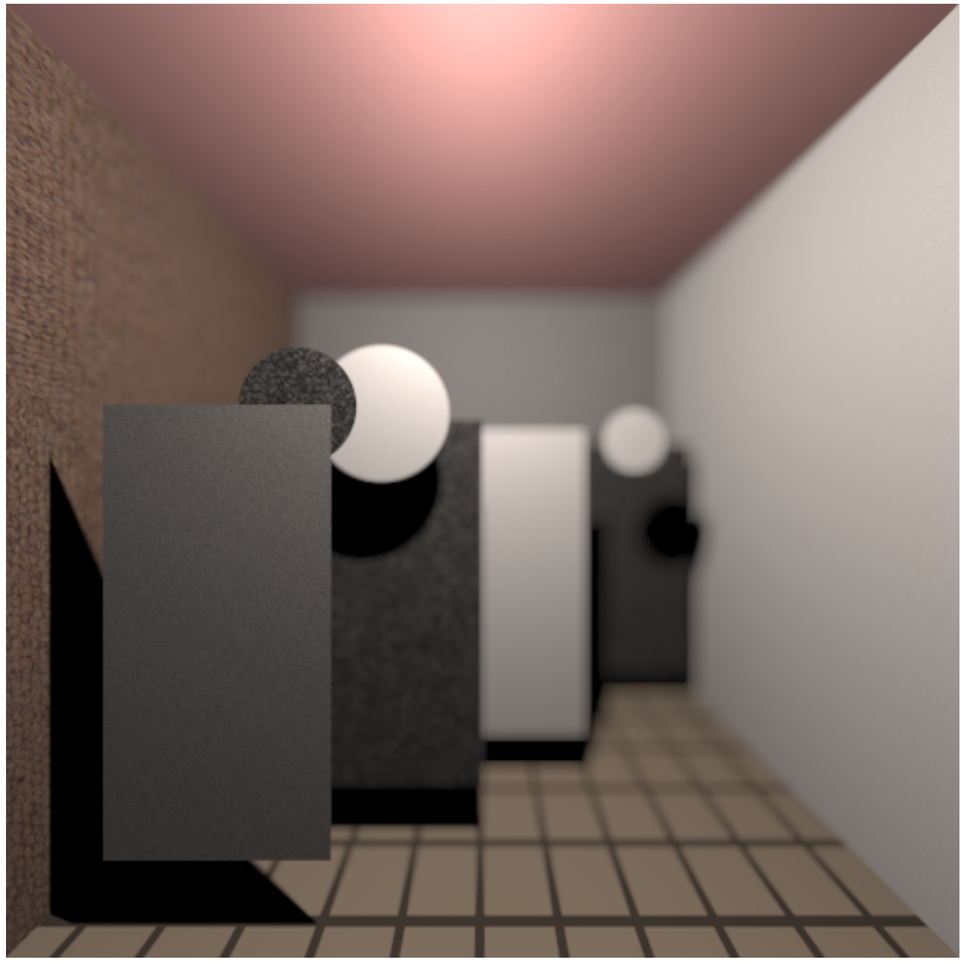

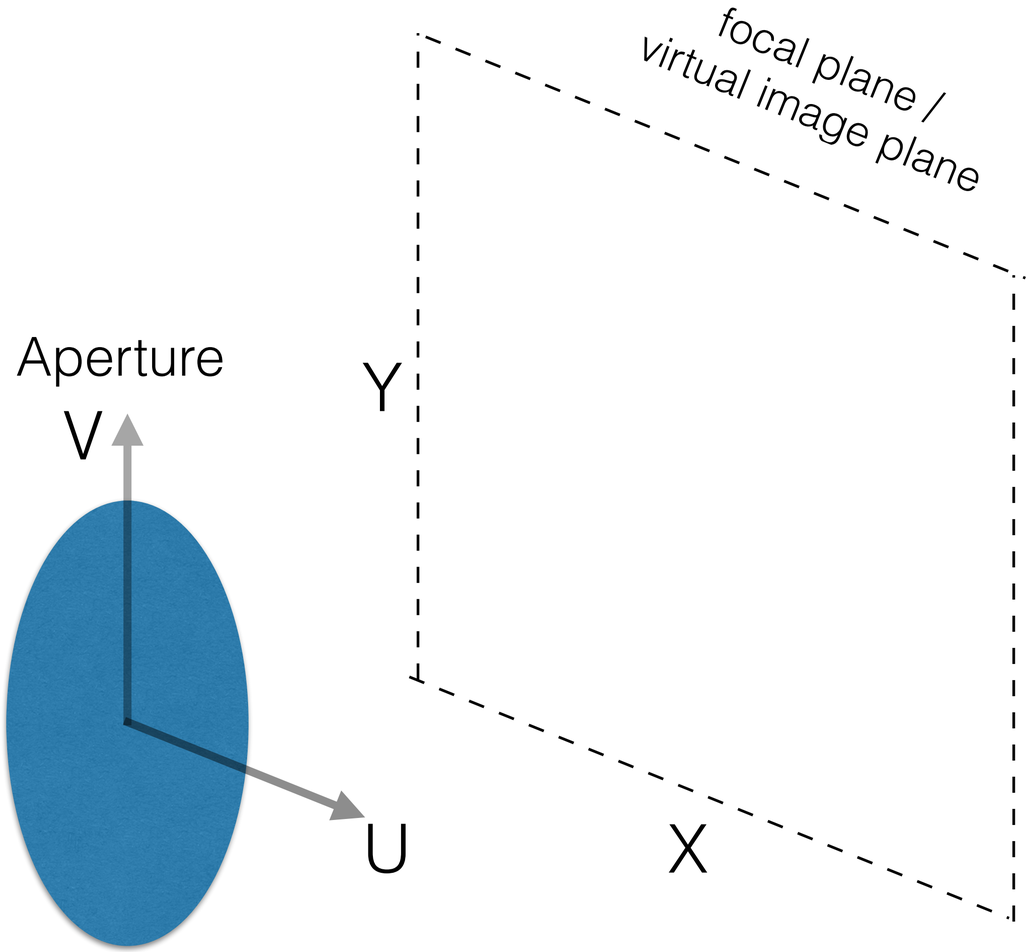

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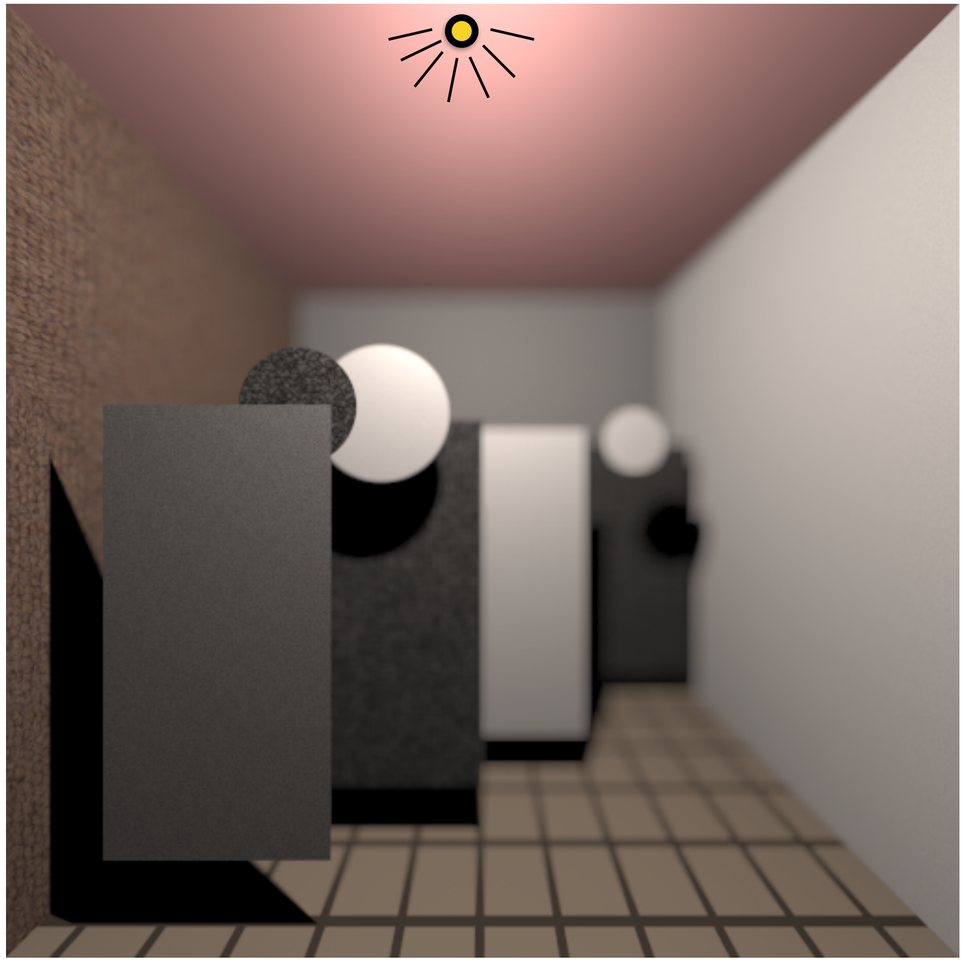

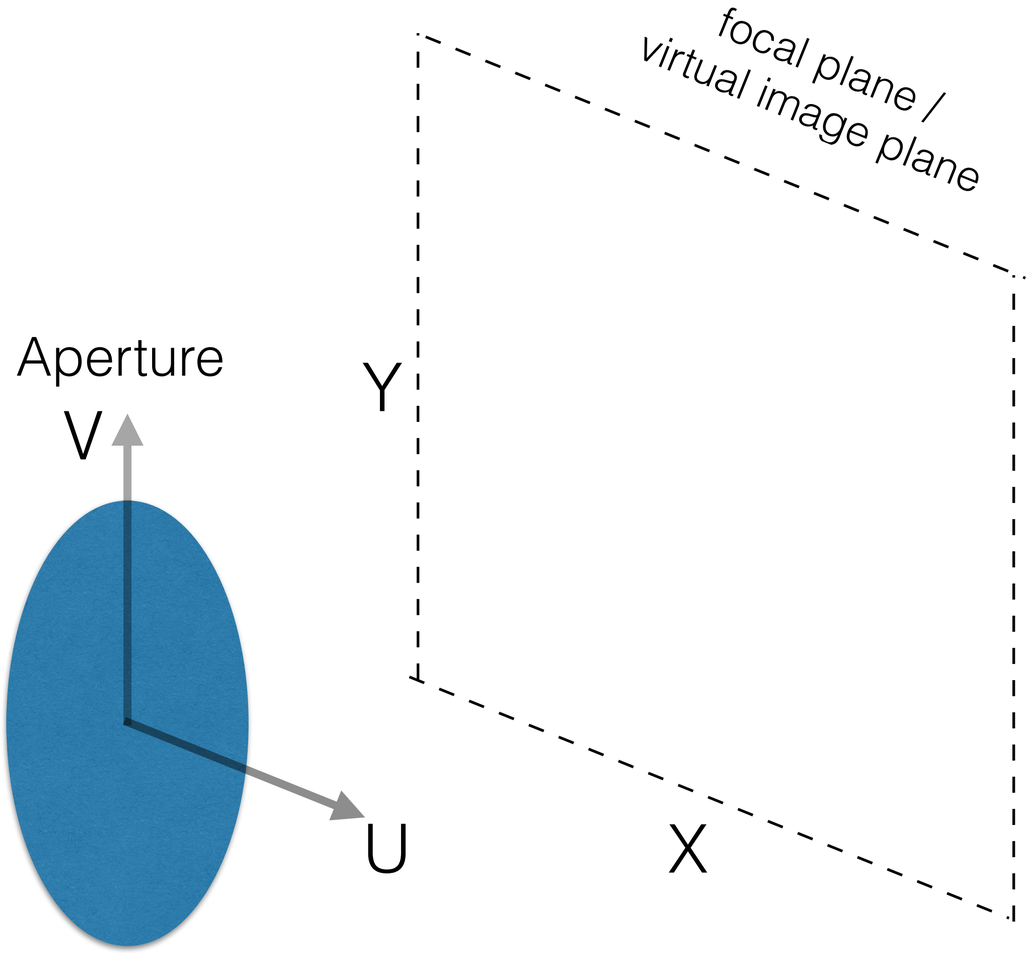

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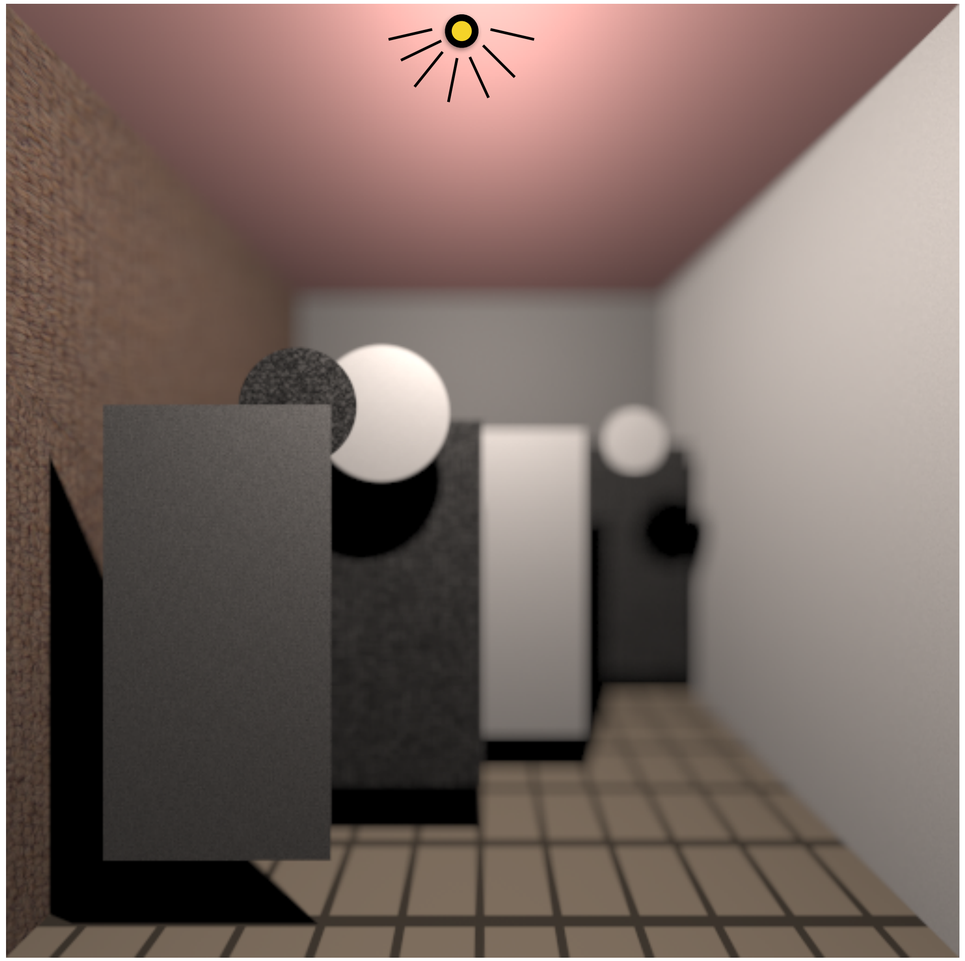
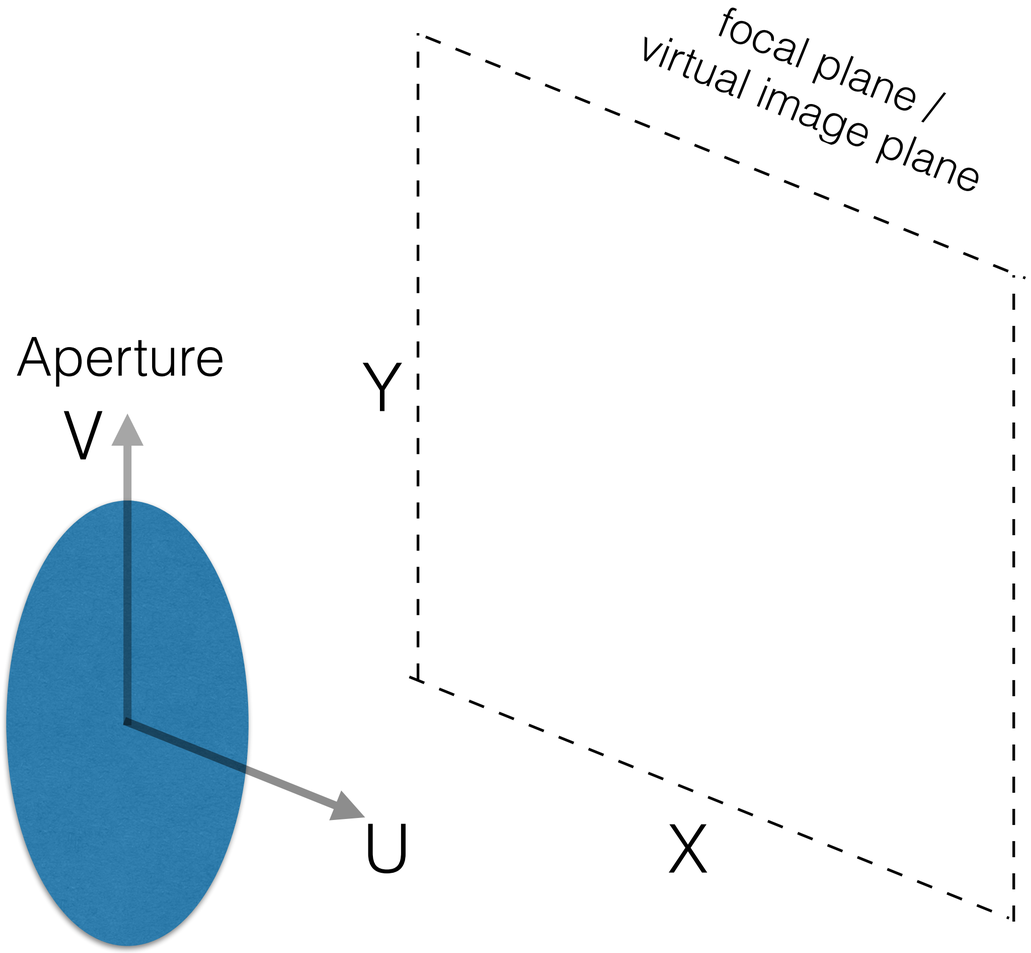
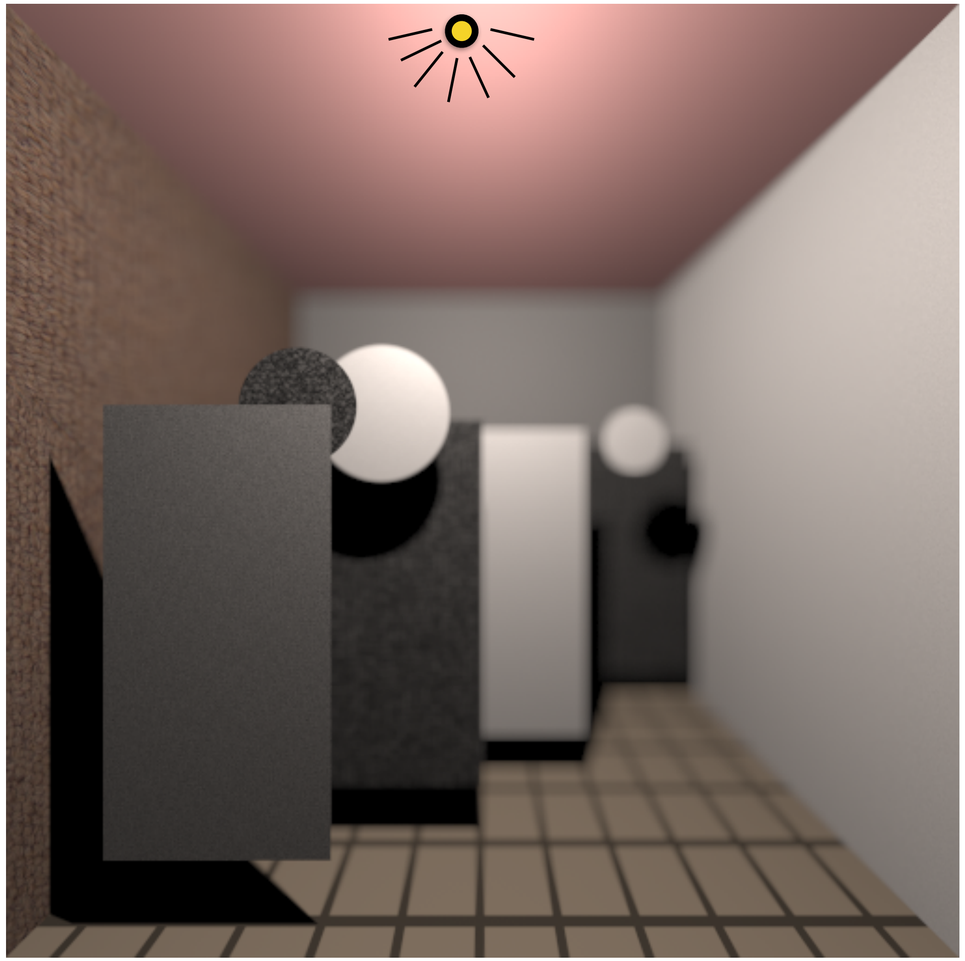
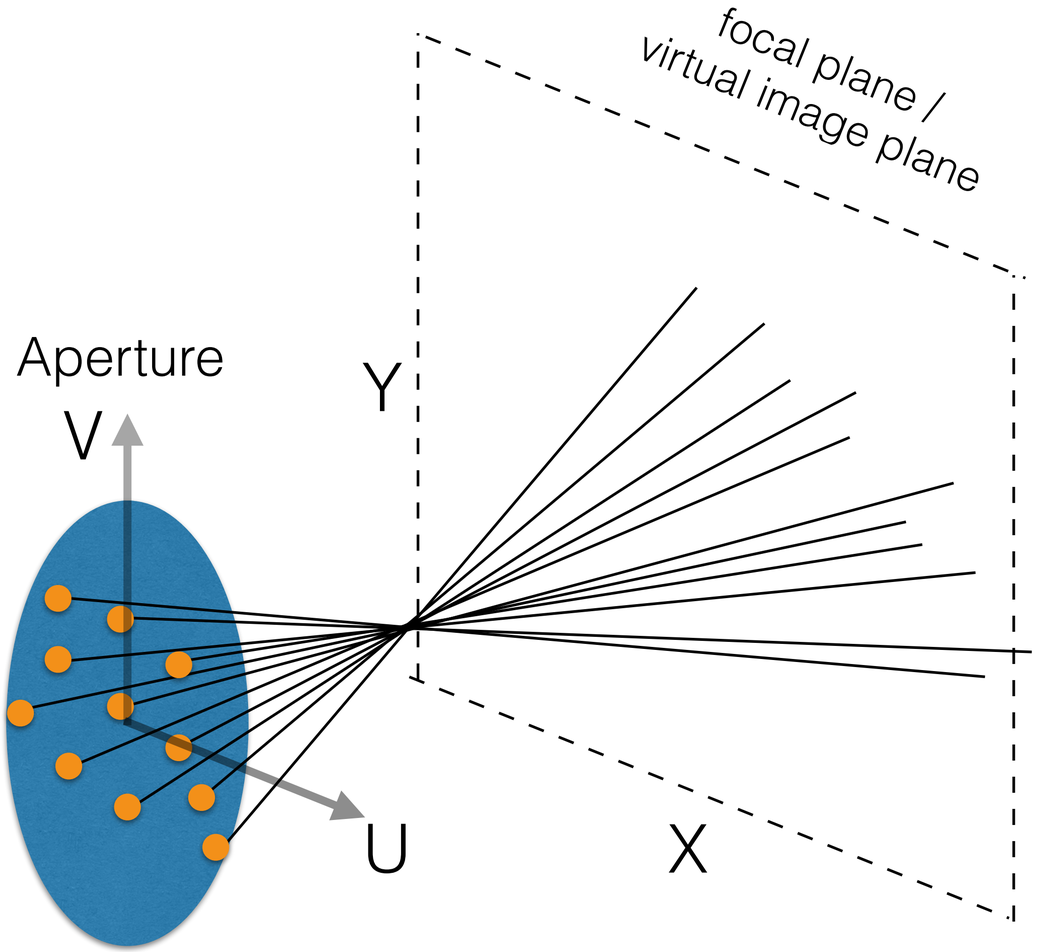




$$\int_x \int_y \int_u \int_v f(x, y, u, v)\, dv\, du\, dy\, dx$$

Monte Carlo Integration





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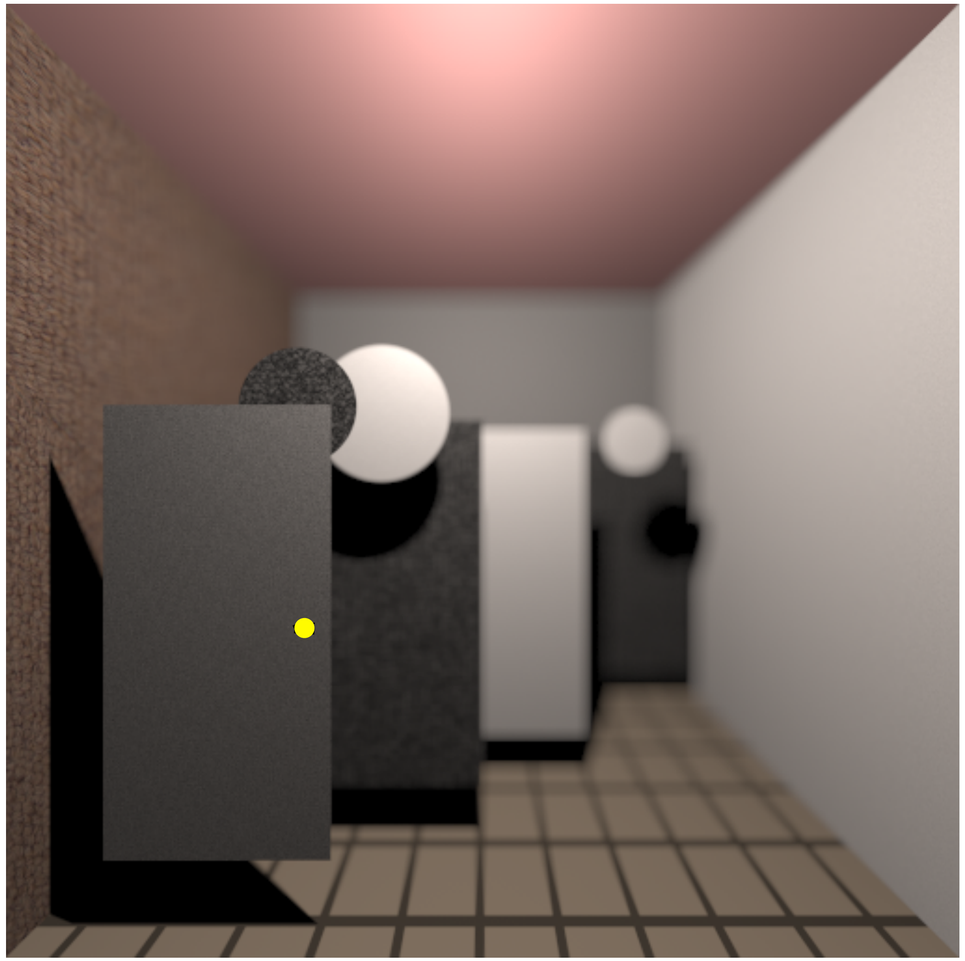

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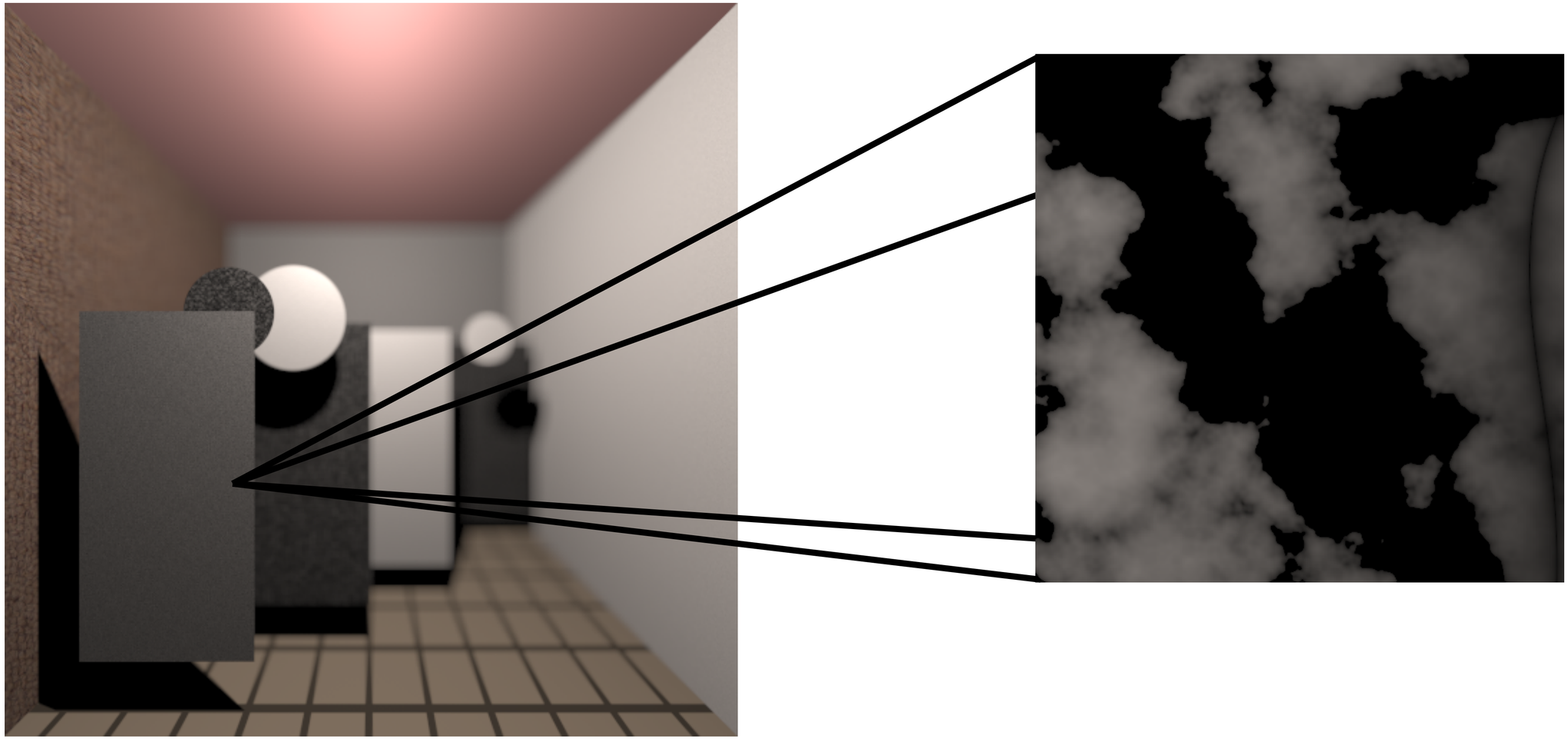

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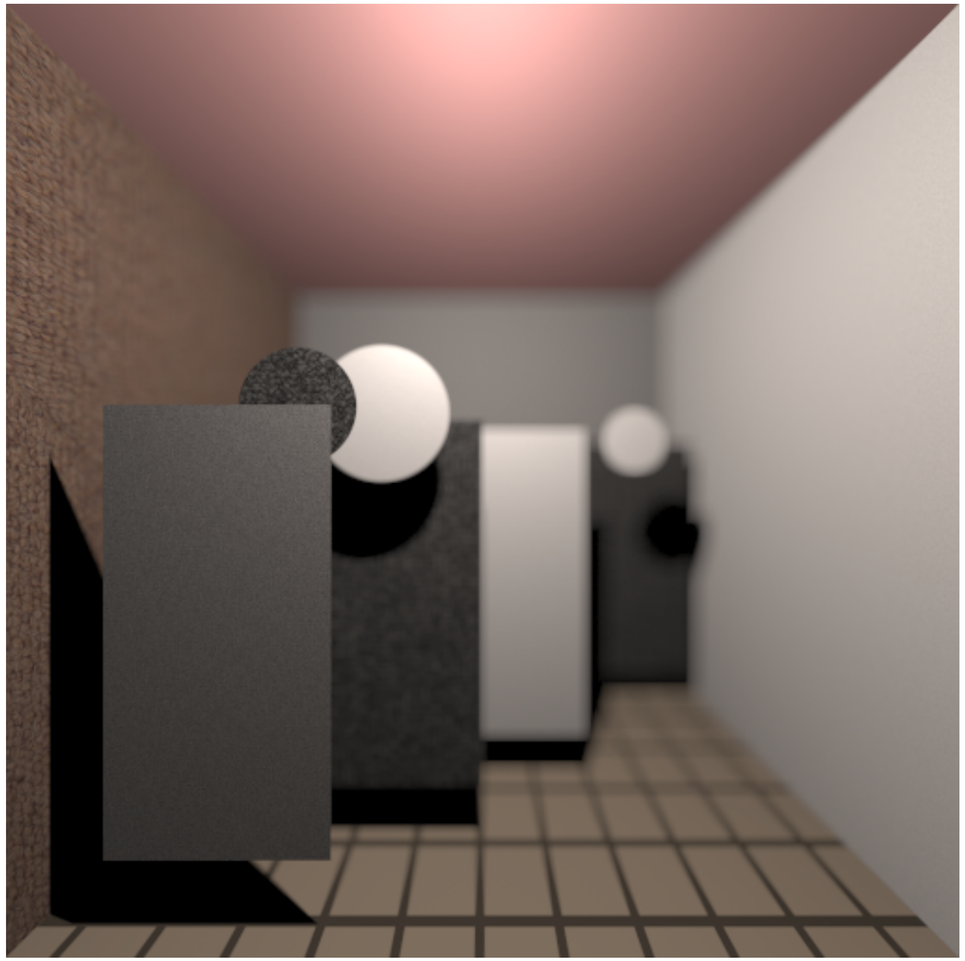
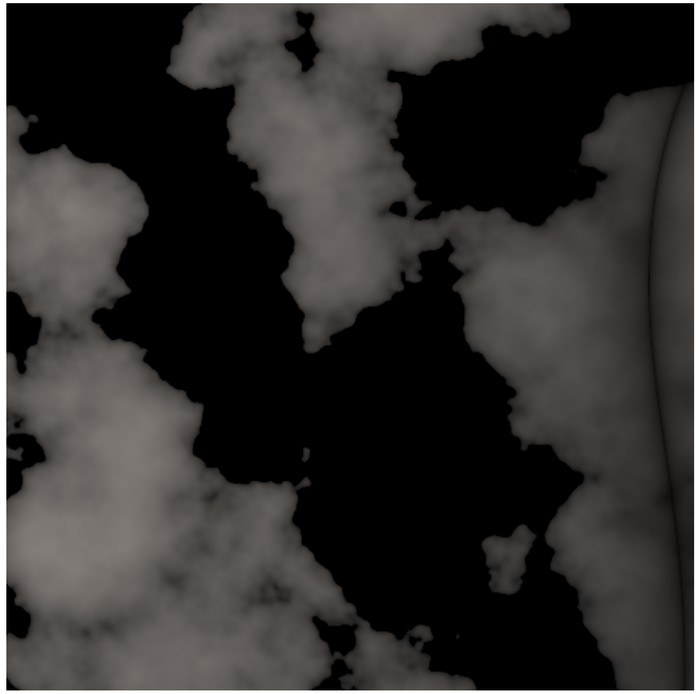
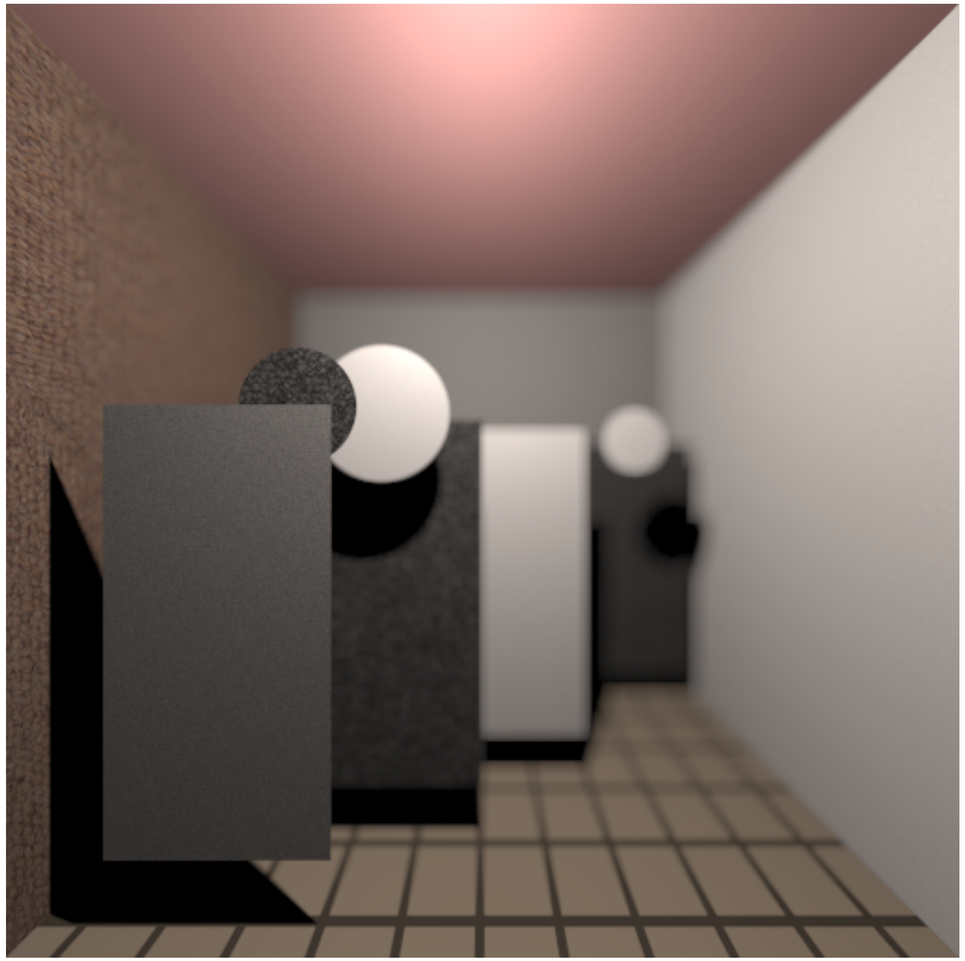


$f(\vec{x})$



Monte Carlo Integration



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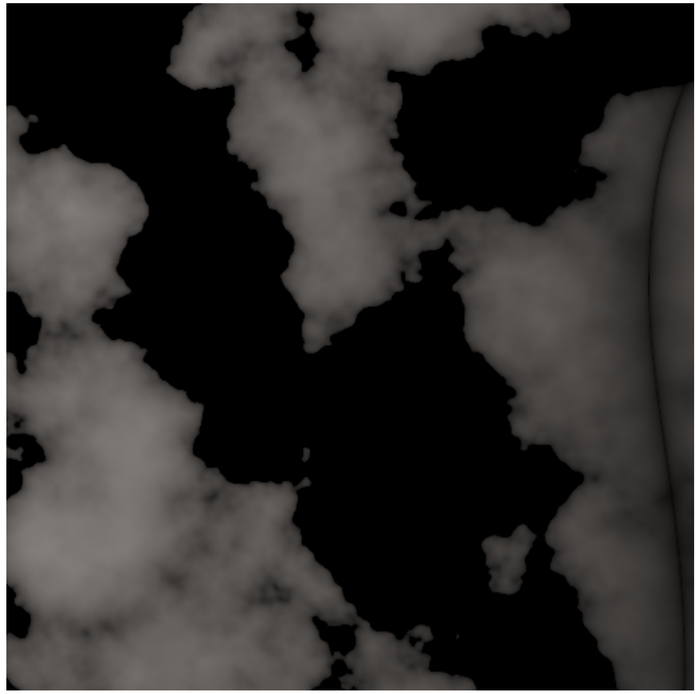
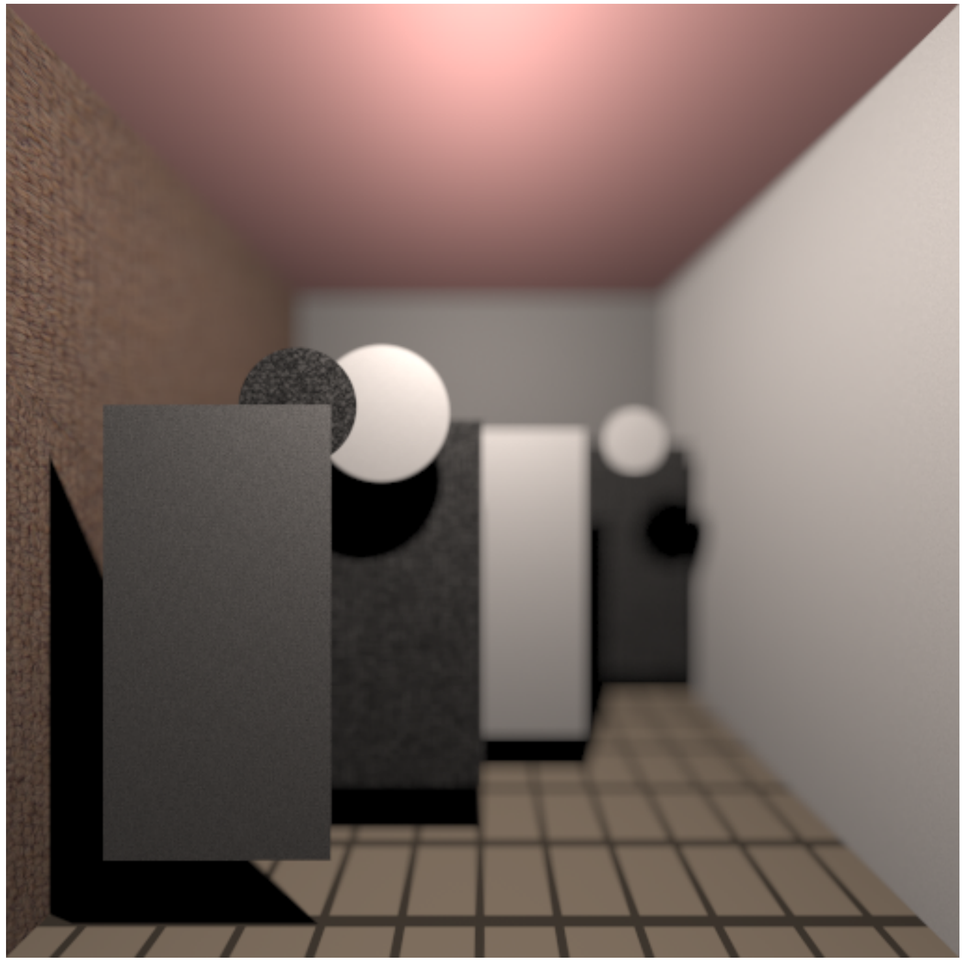


$$I = \int_0^1 f(\vec{x}) d\vec{x}$$

Monte Carlo Integration



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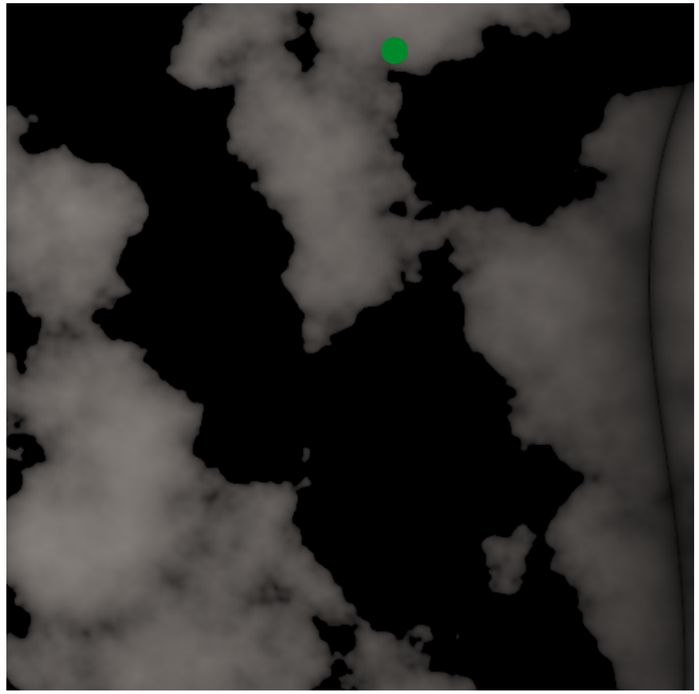

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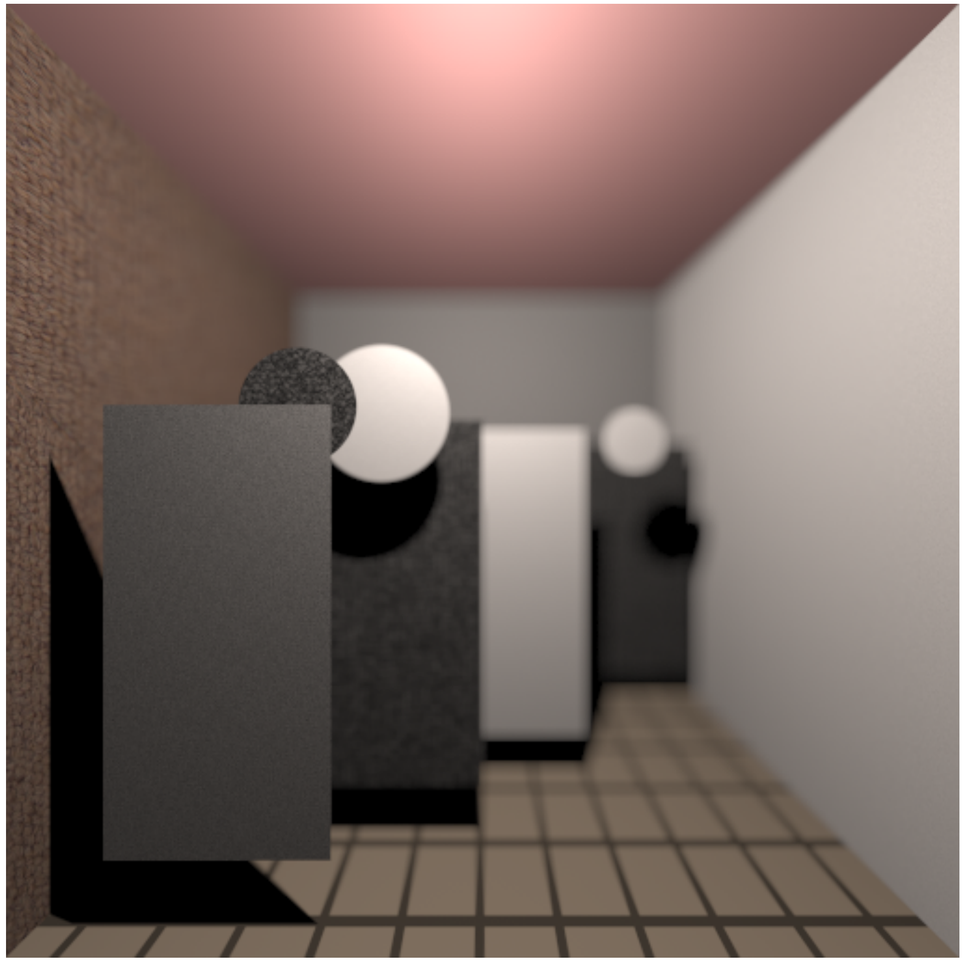
Monte Carlo Integration



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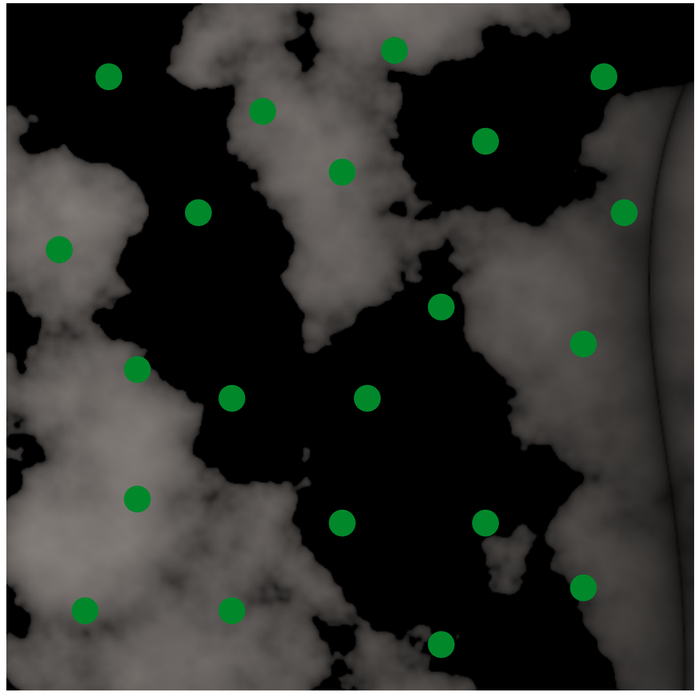

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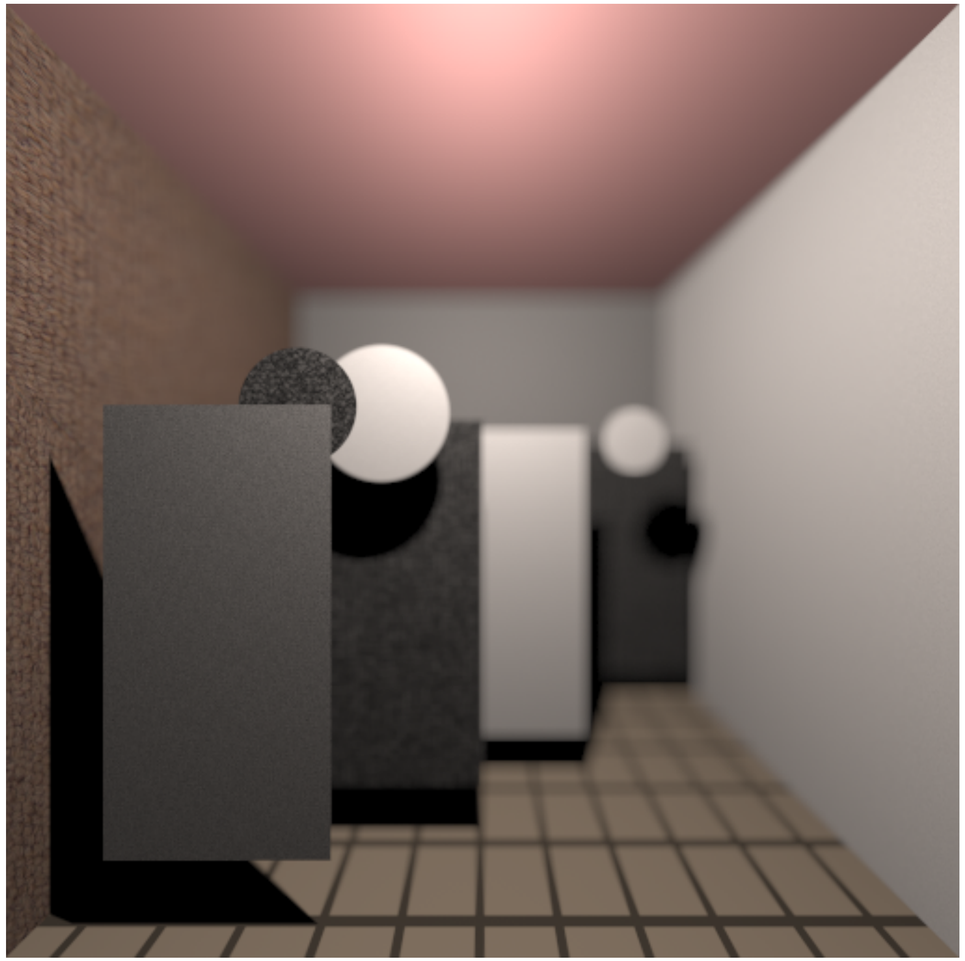
Monte Carlo Integration



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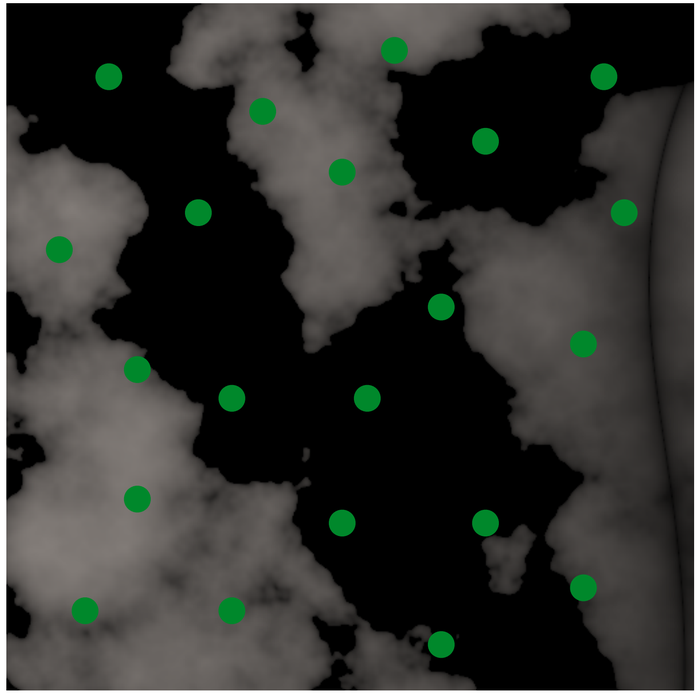
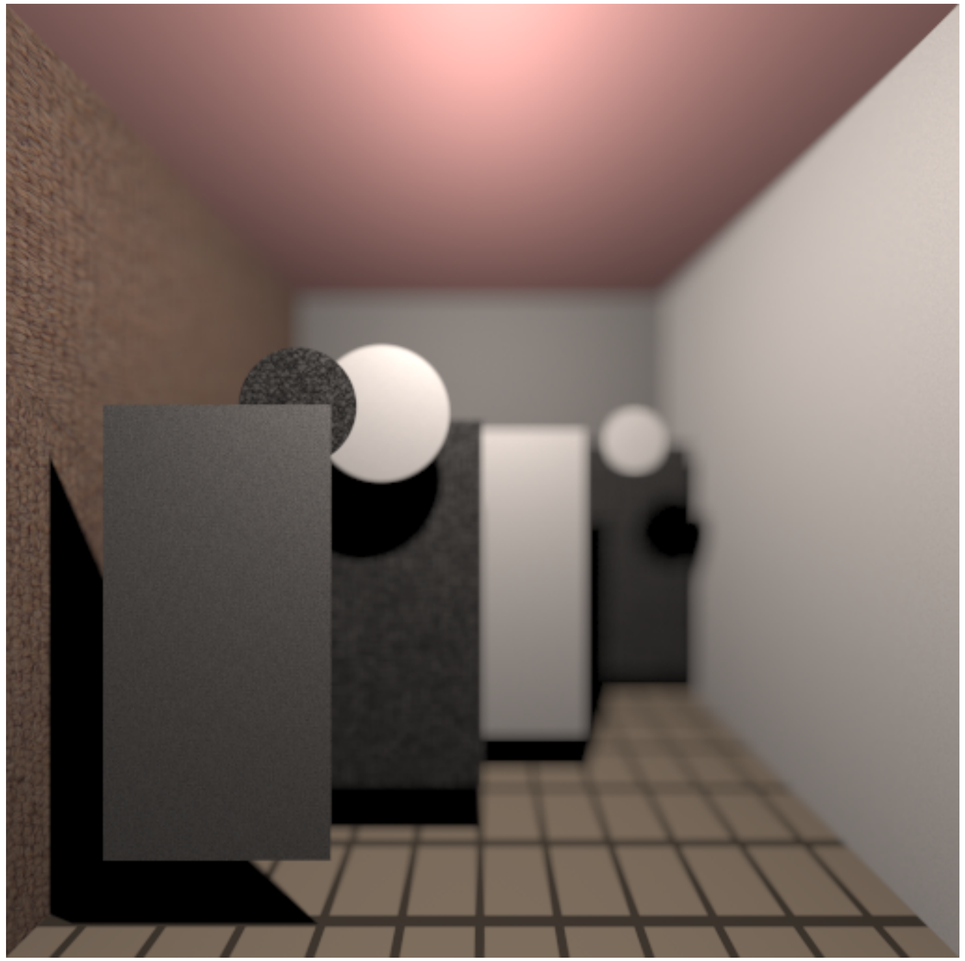


$$I_N = \frac{1}{N} \sum_{k=1}^{N} \frac{f(\vec{x}_k)}{p(\vec{x}_k)}$$

Monte Carlo Integration



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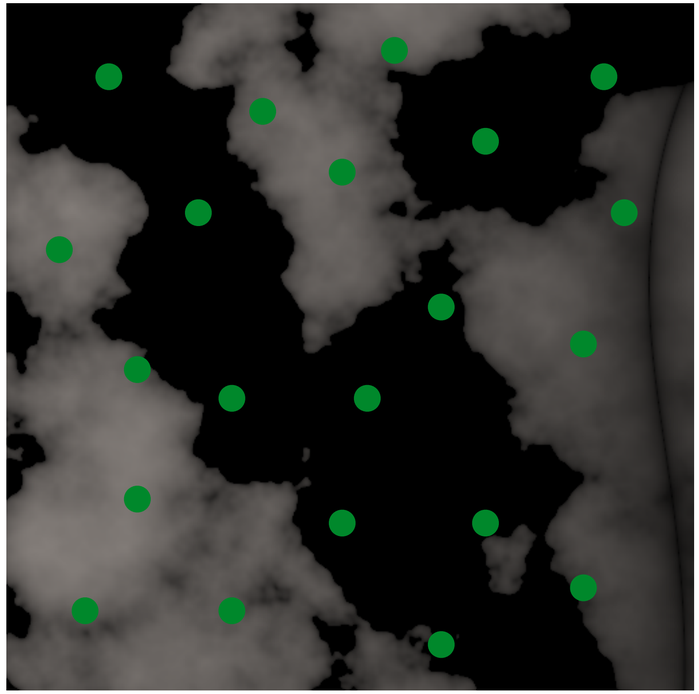
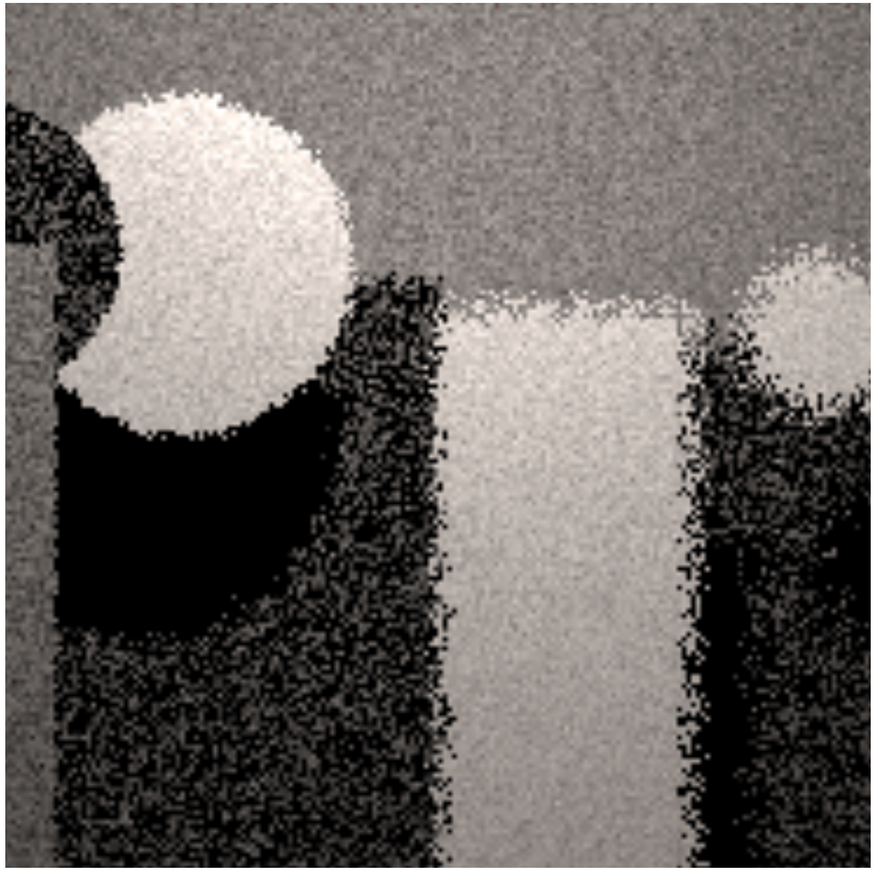


$$I_N = \frac{1}{N} \sum_{k=1}^{N} \frac{f(\vec{x}_k)}{p(\vec{x}_k)}$$

Variance



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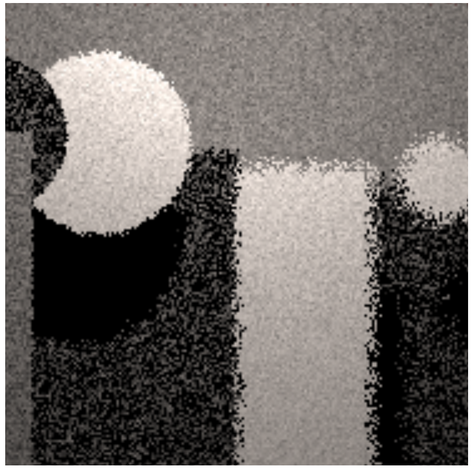

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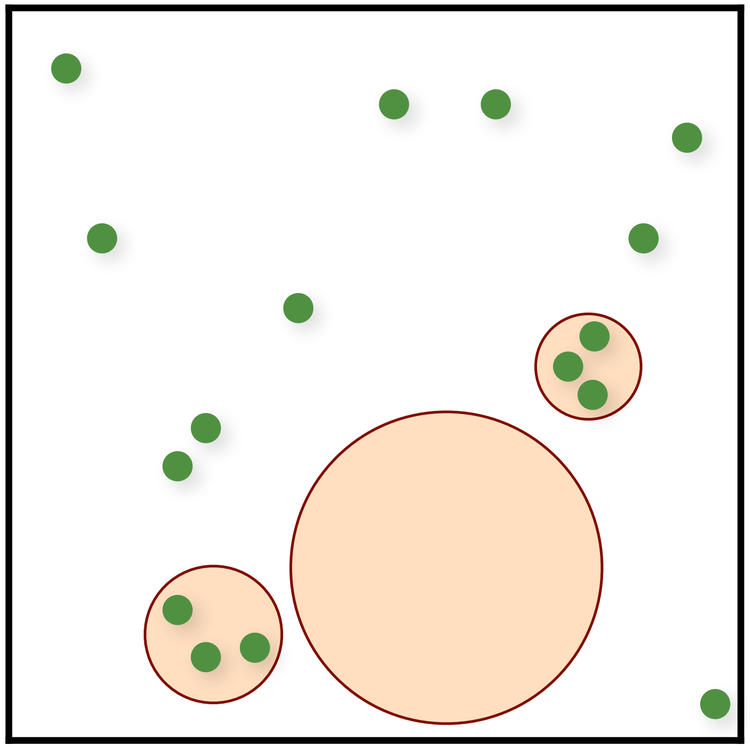

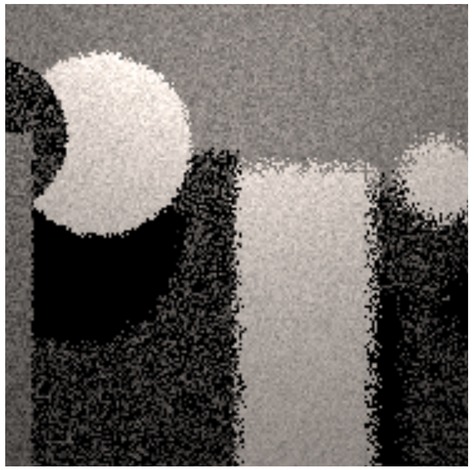

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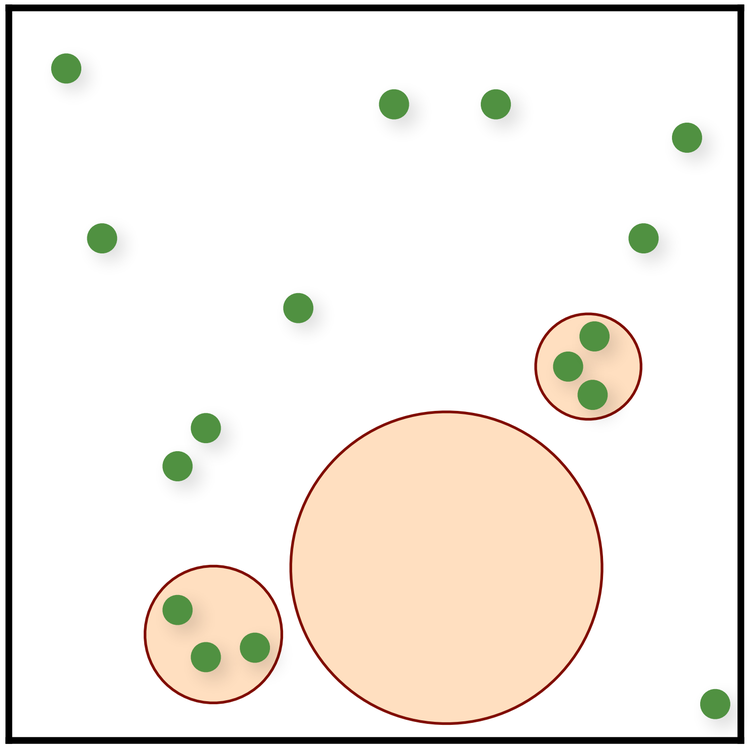

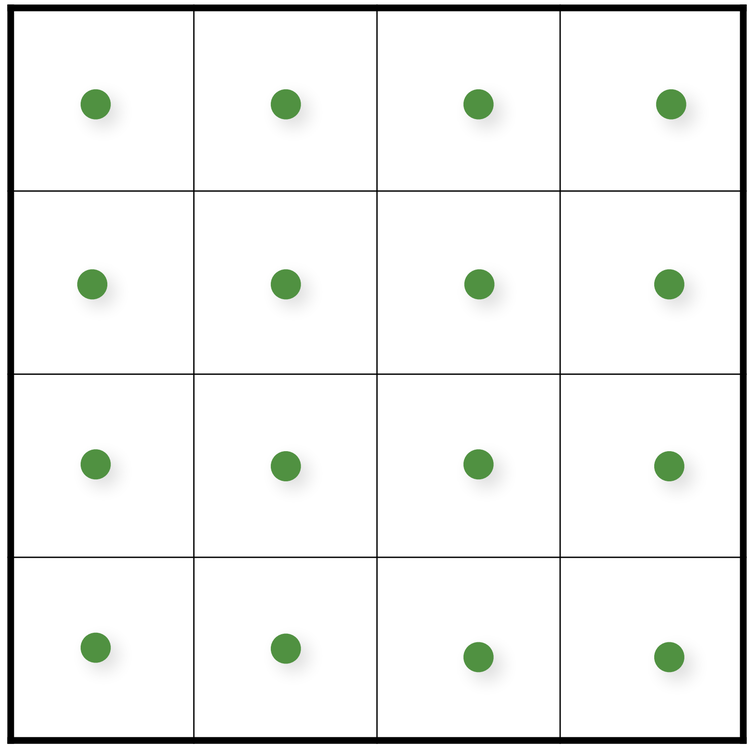

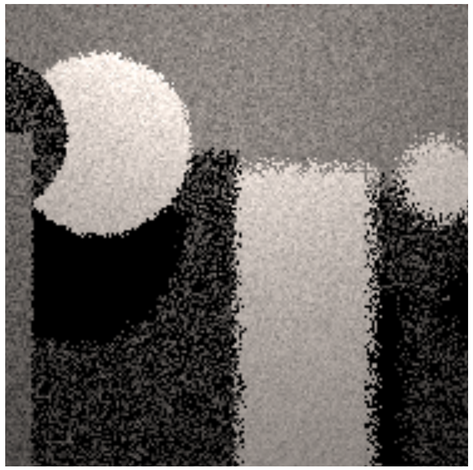

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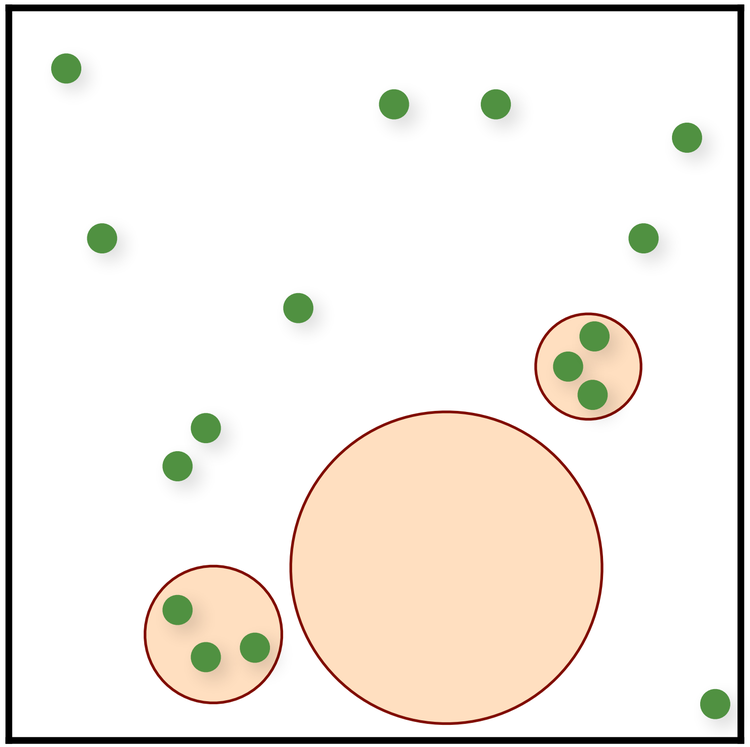
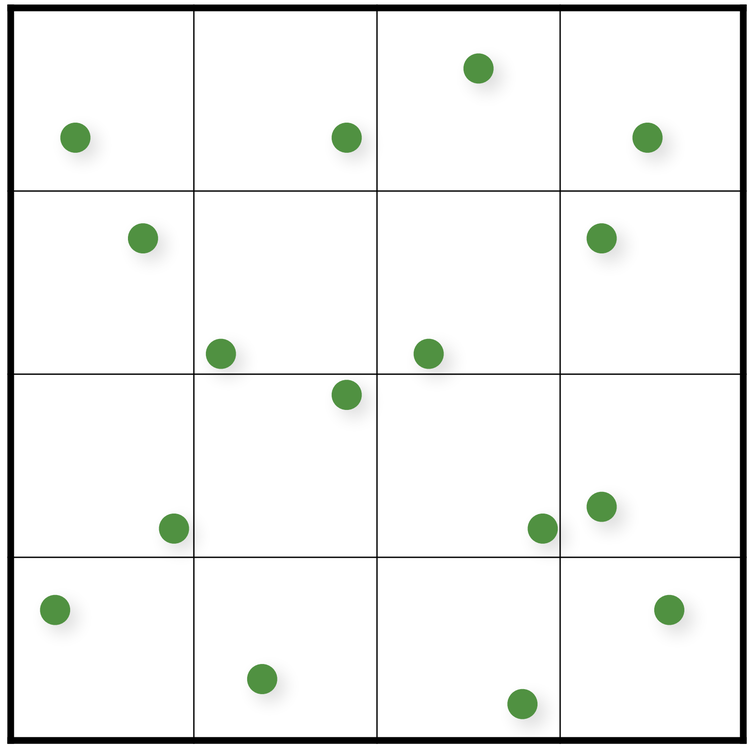
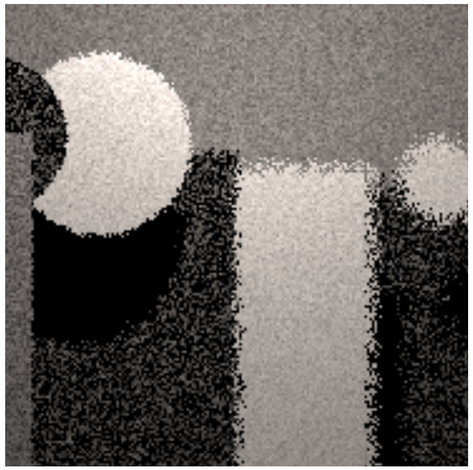
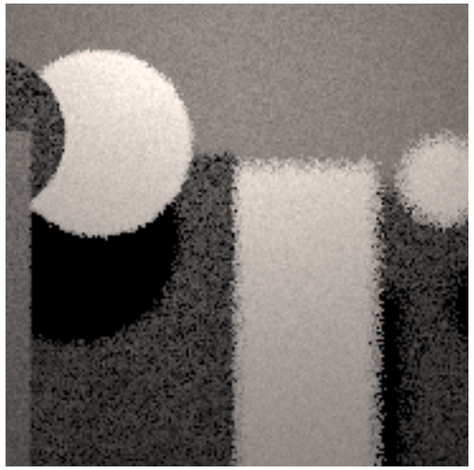


Jitter







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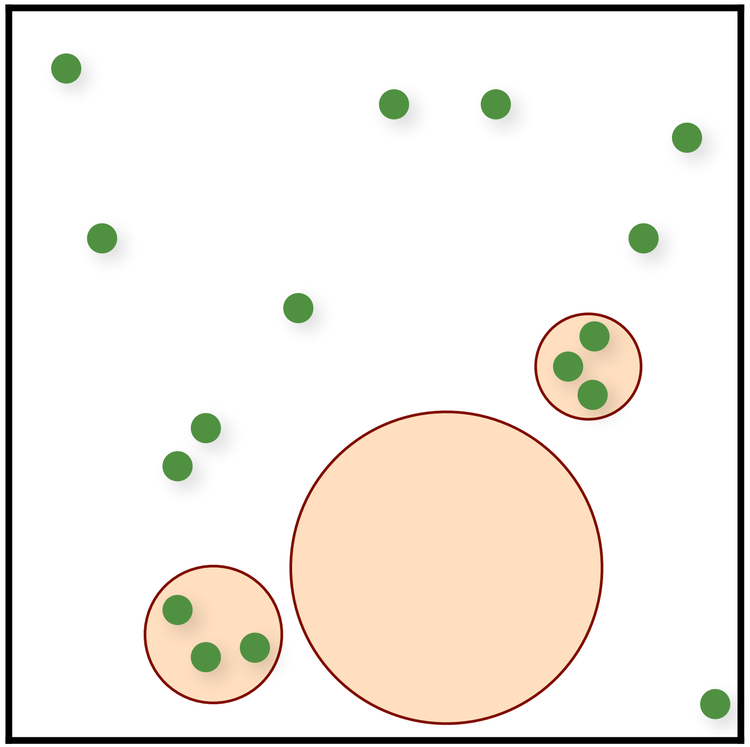

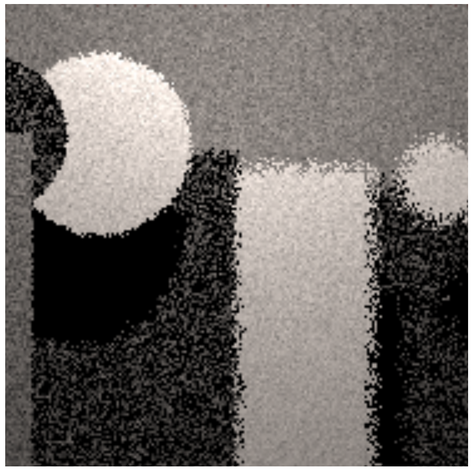

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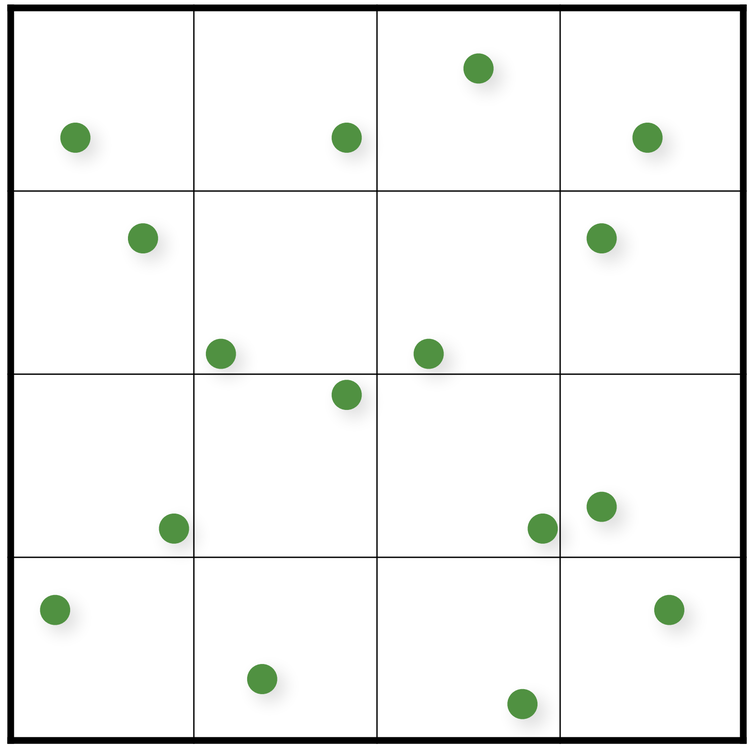

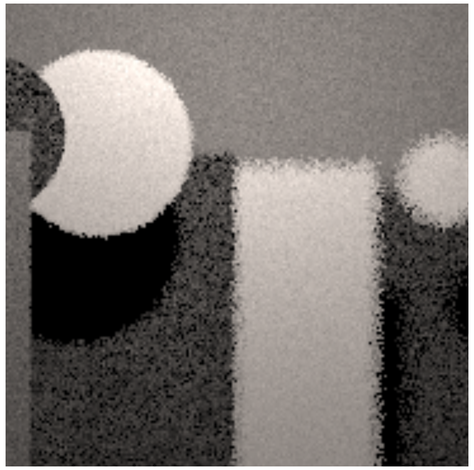

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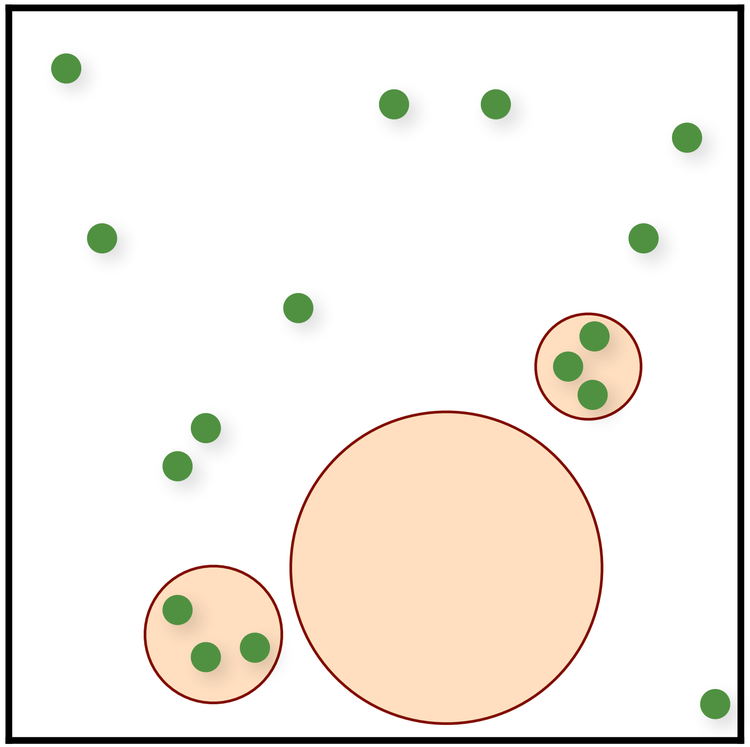

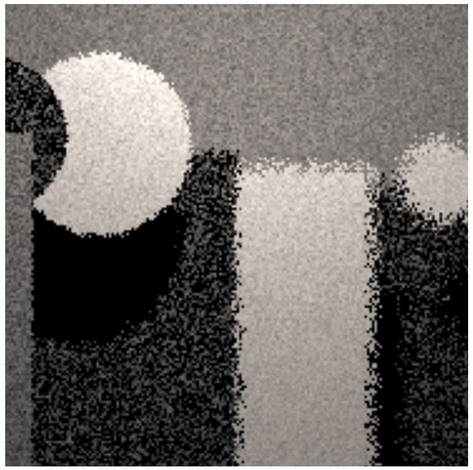

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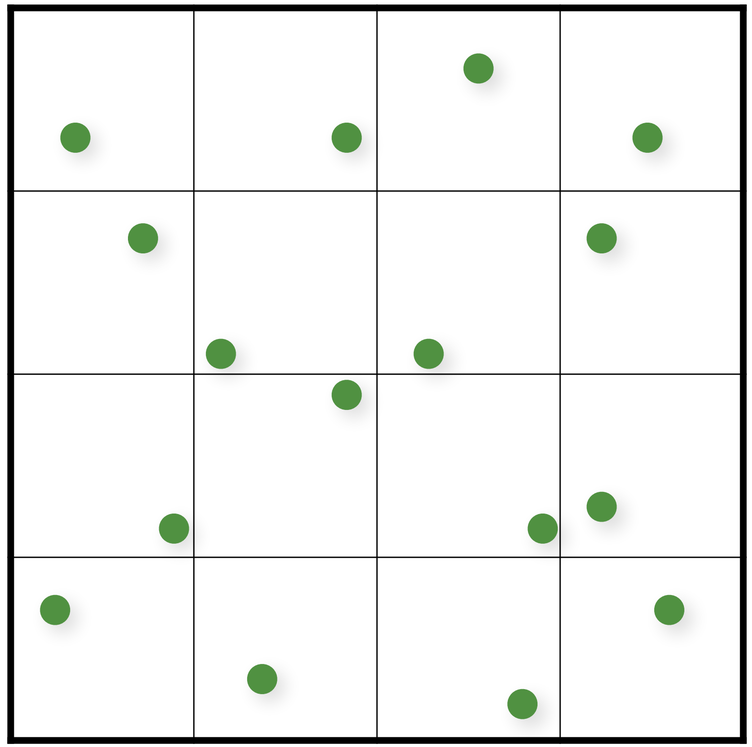

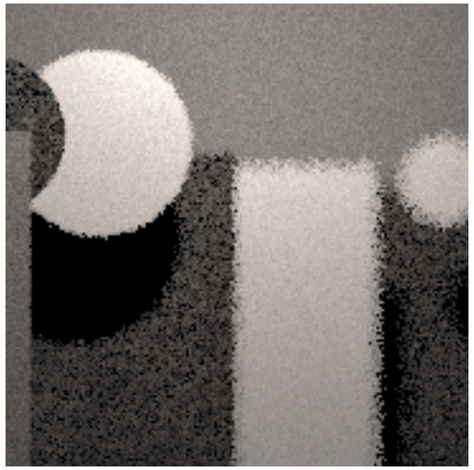

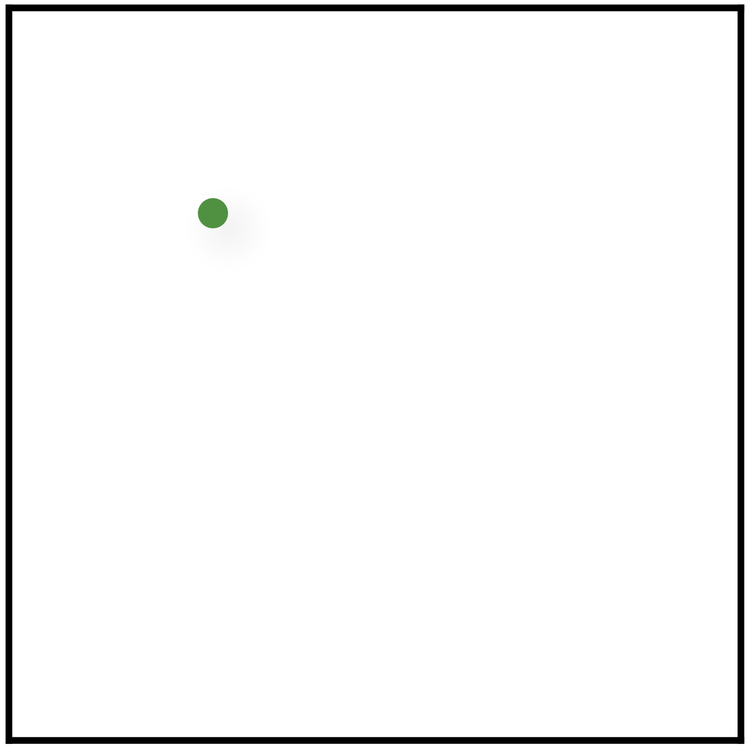

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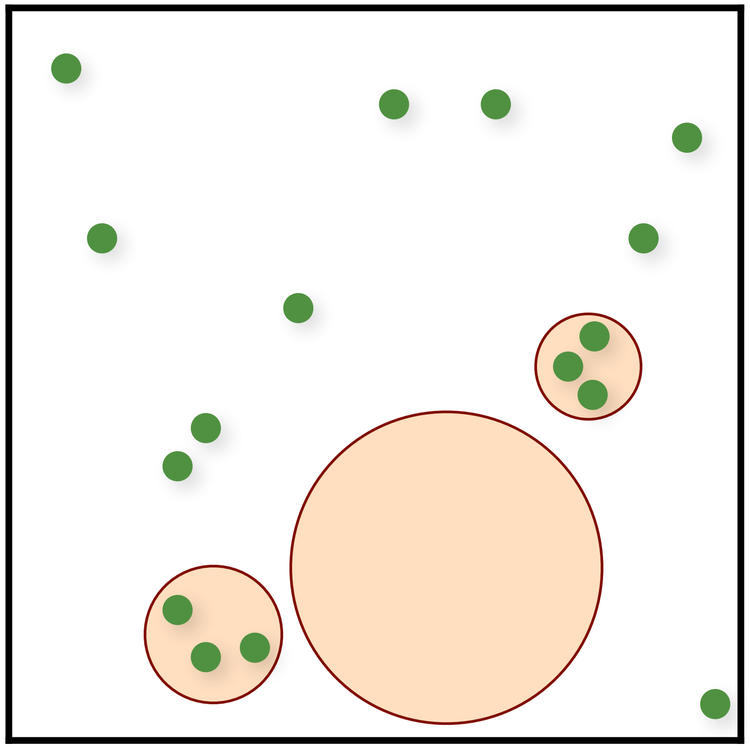

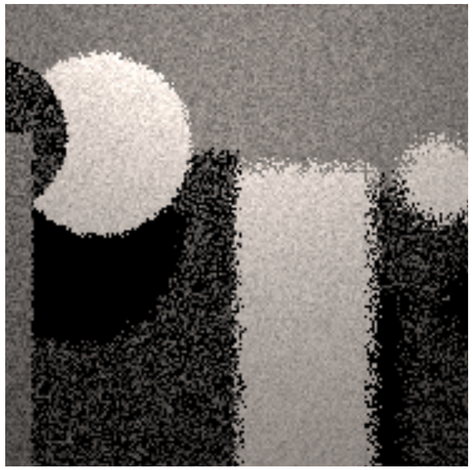

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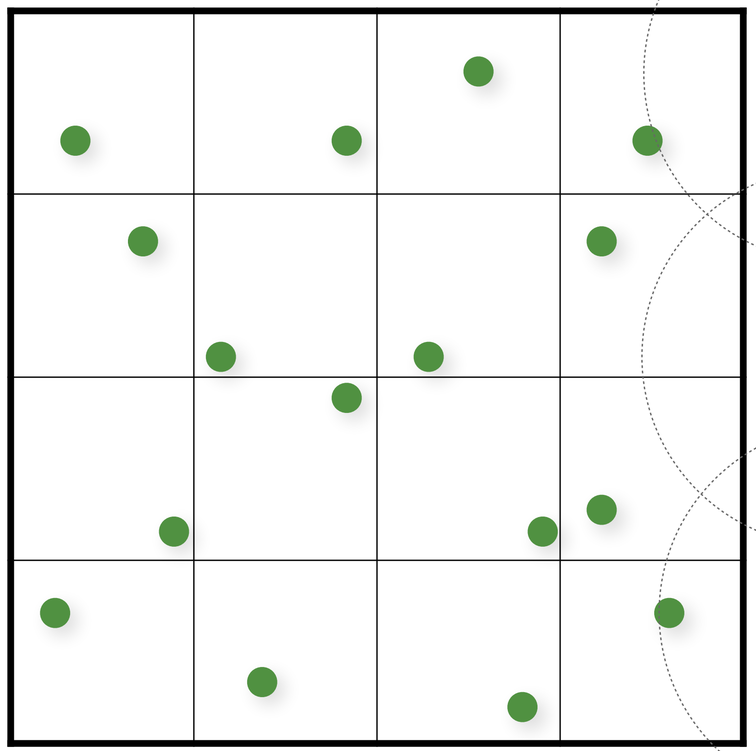

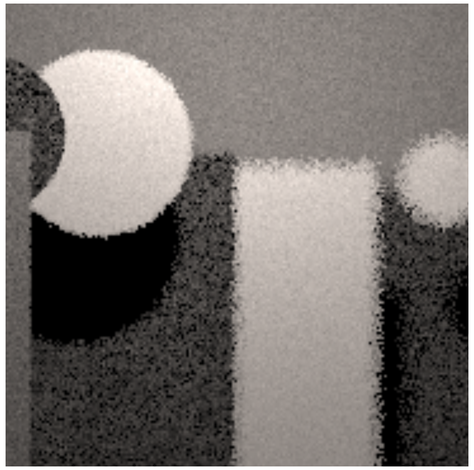

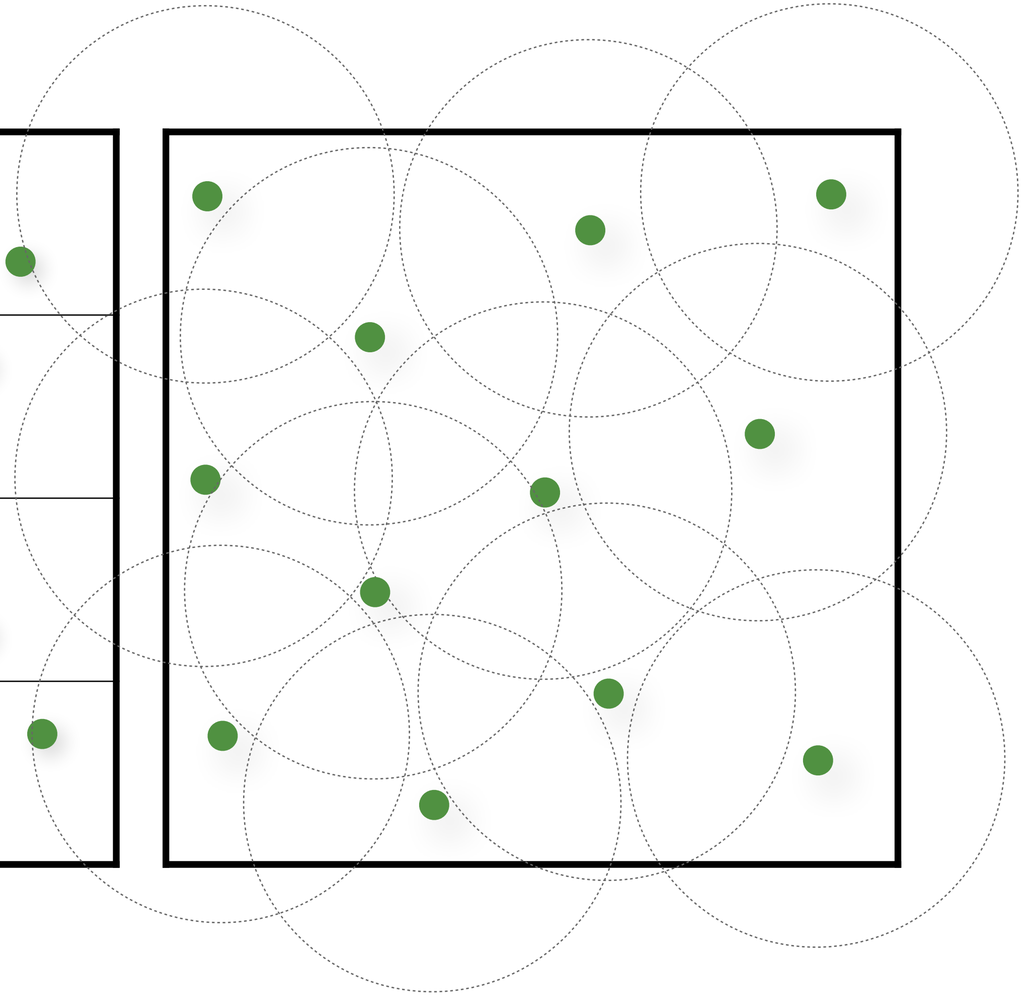

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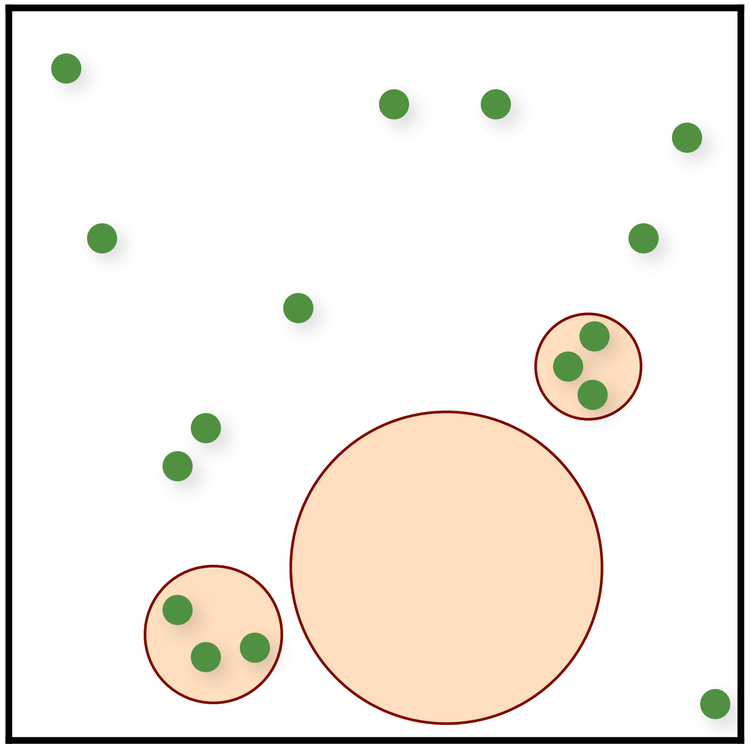

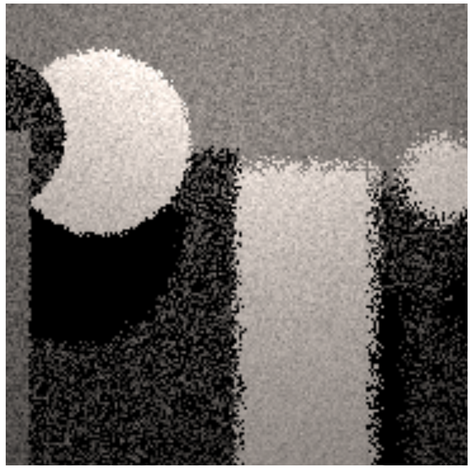

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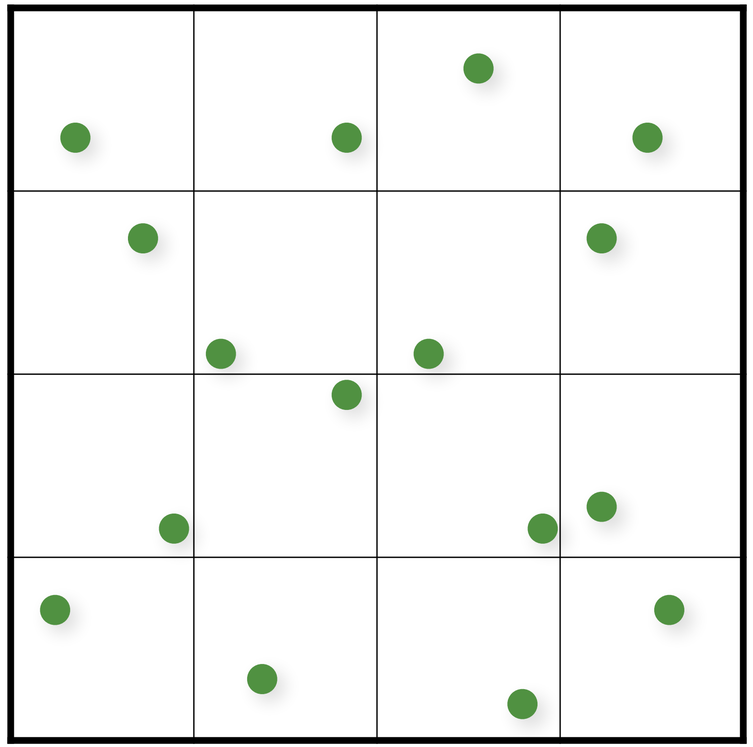
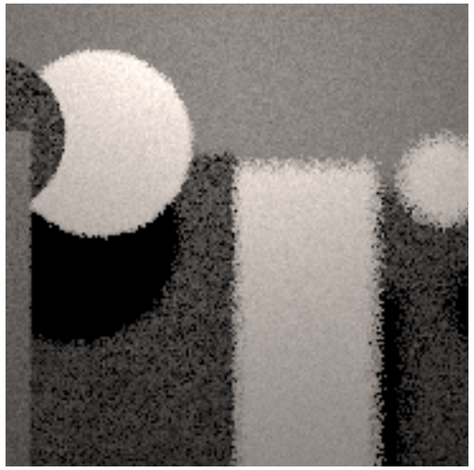
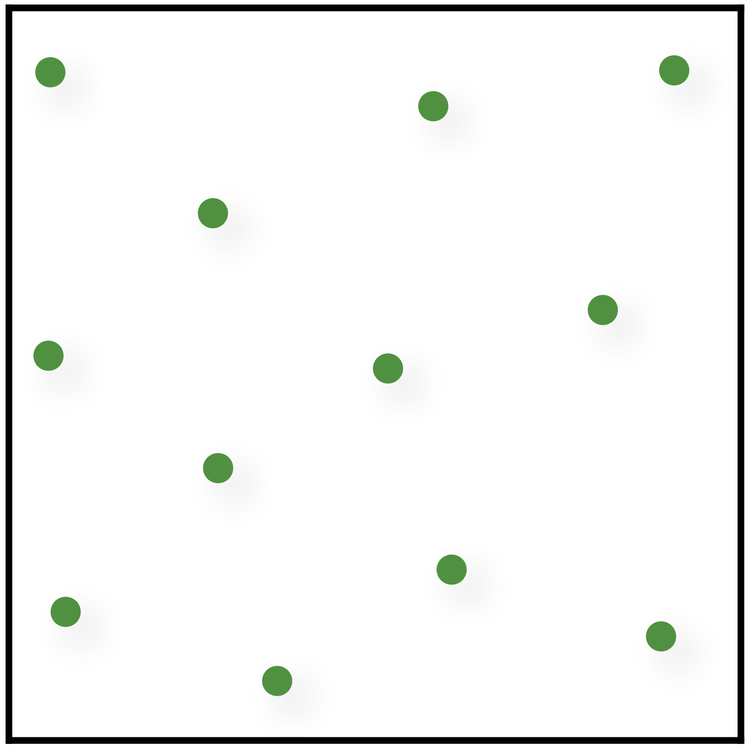
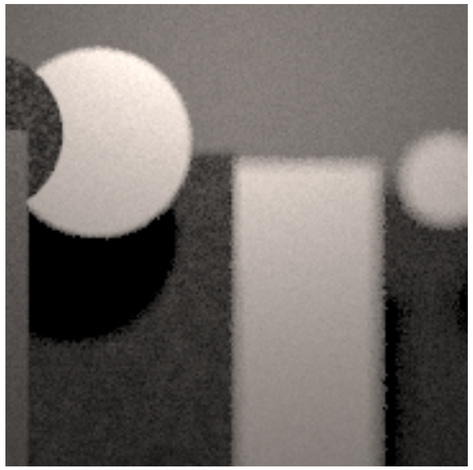


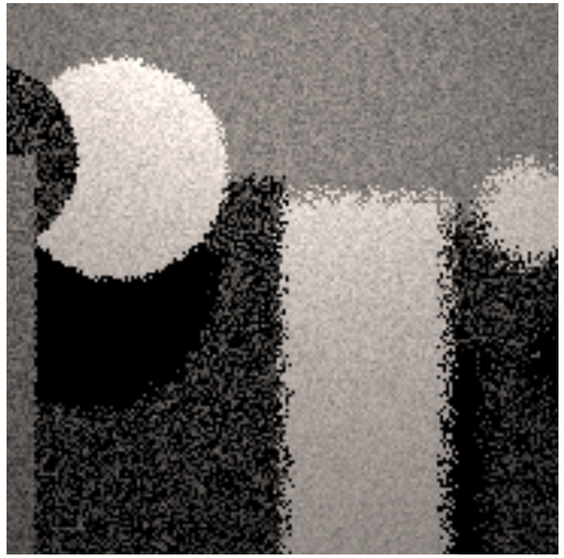


Poisson Disk





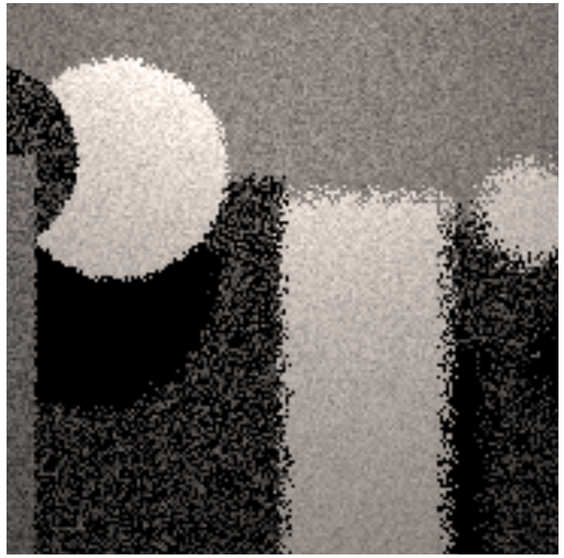
Variance Convergence Rate of Samplers

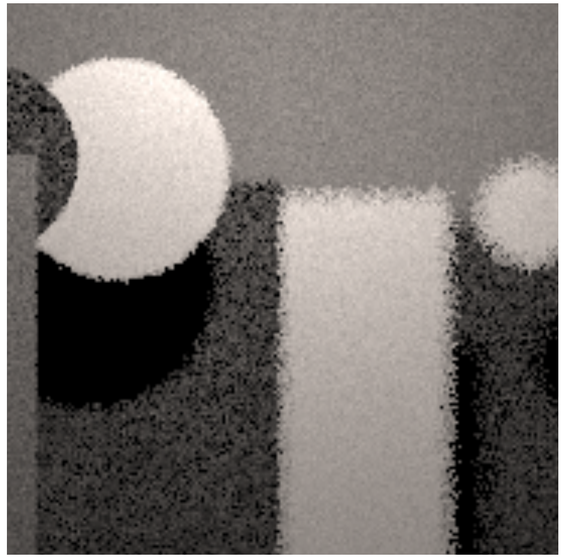


Variance

Number of Samples

Variance Convergence Rate of Samplers





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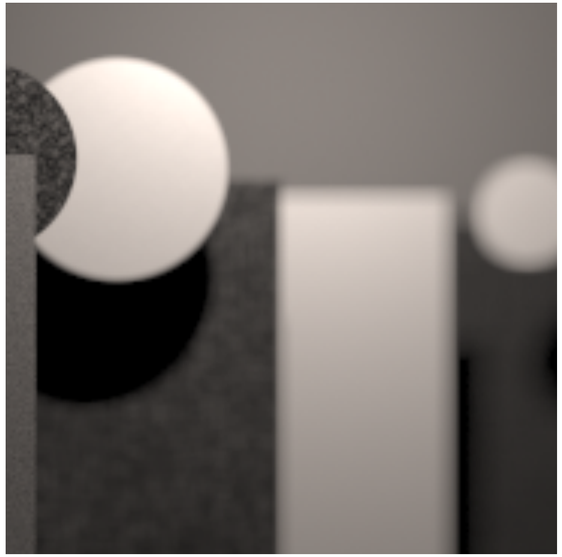

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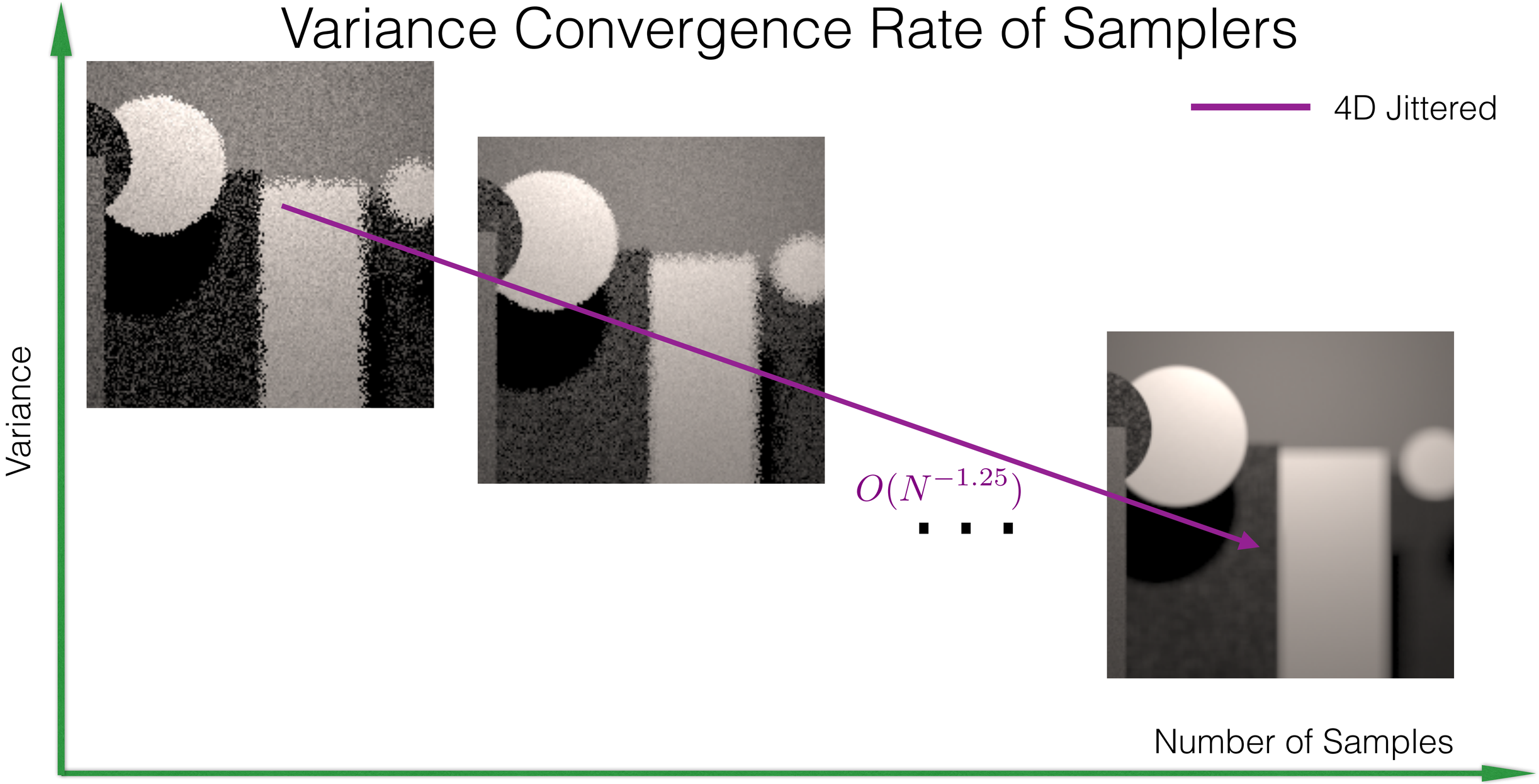
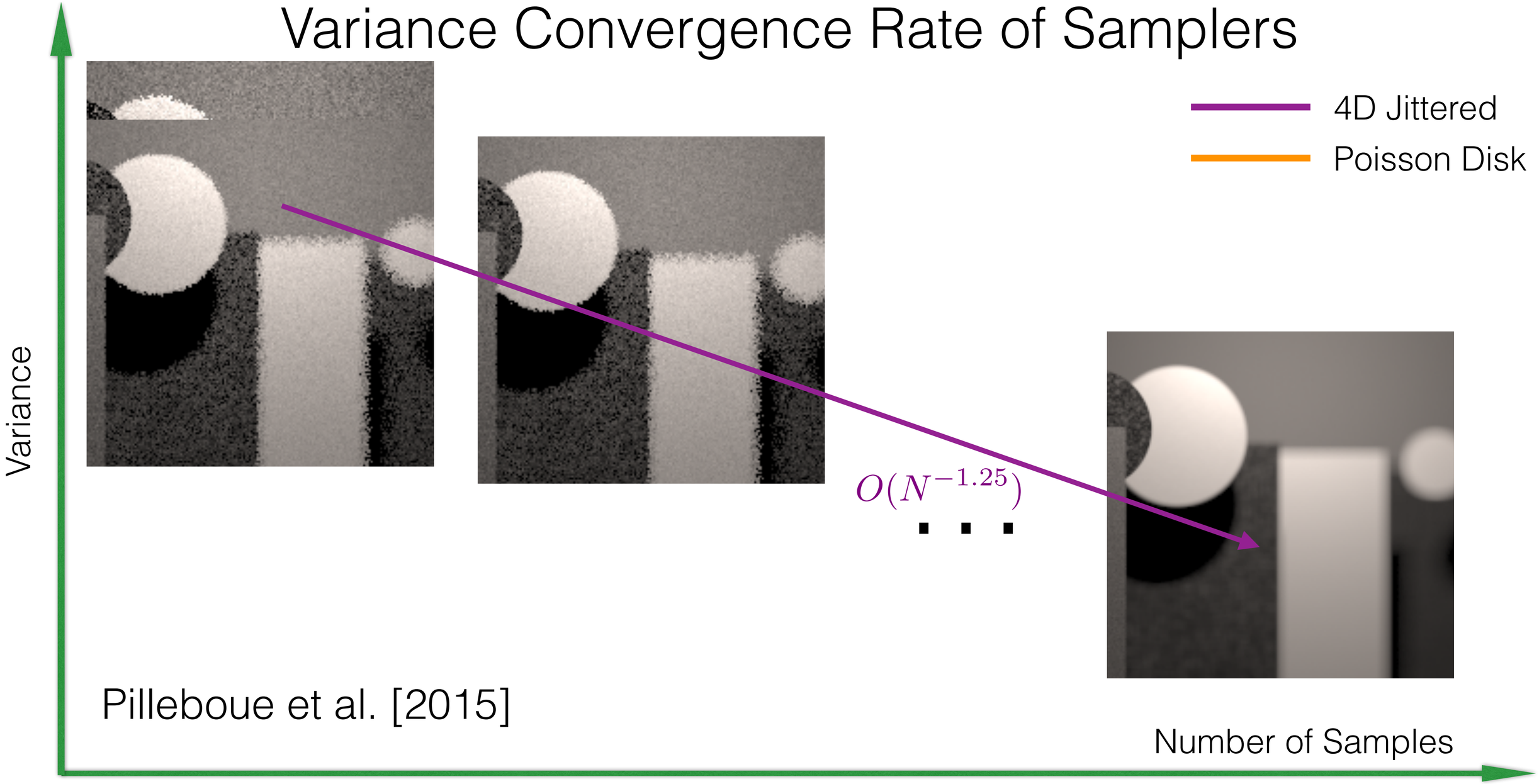
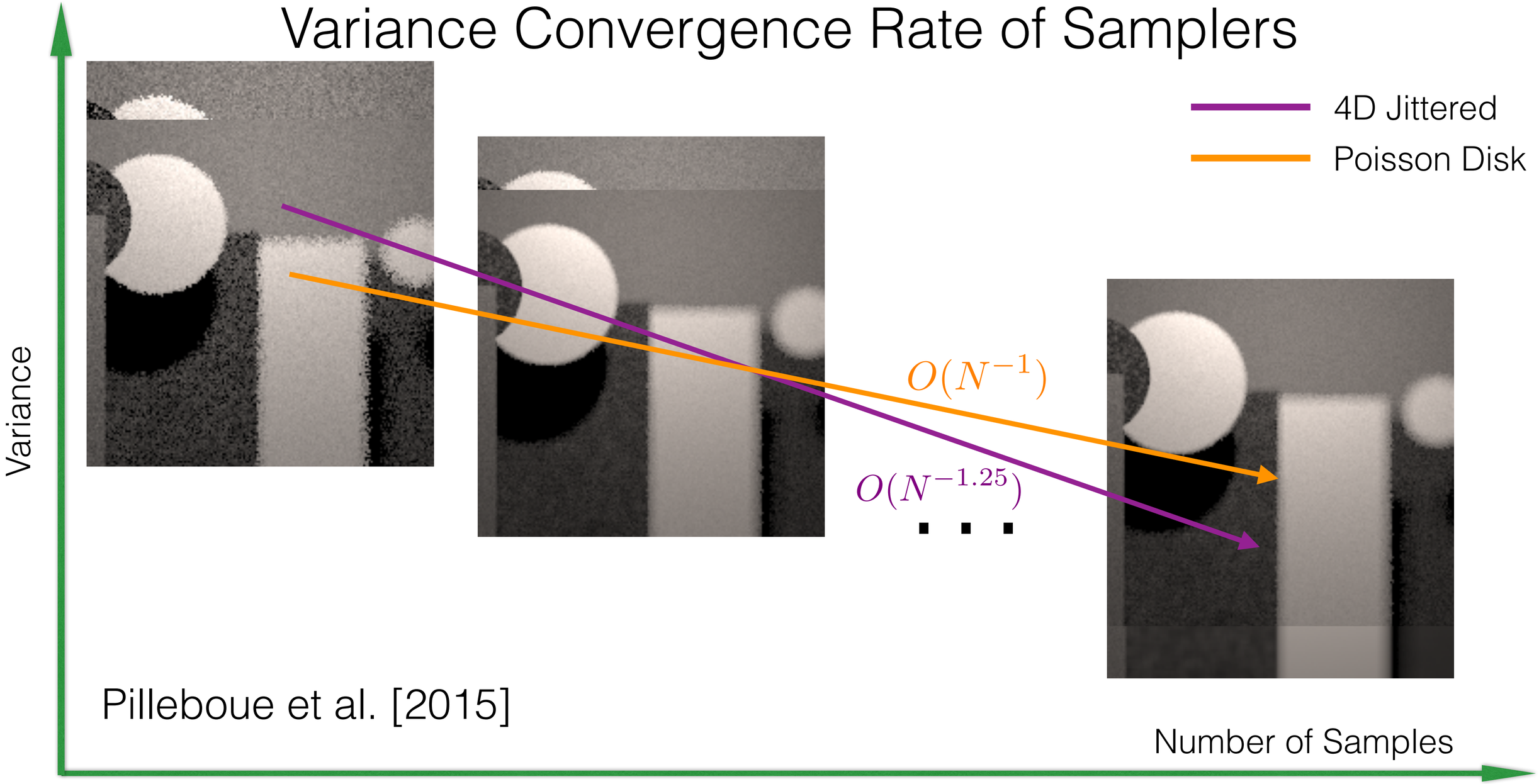


Variance Convergence Rate of Samplers



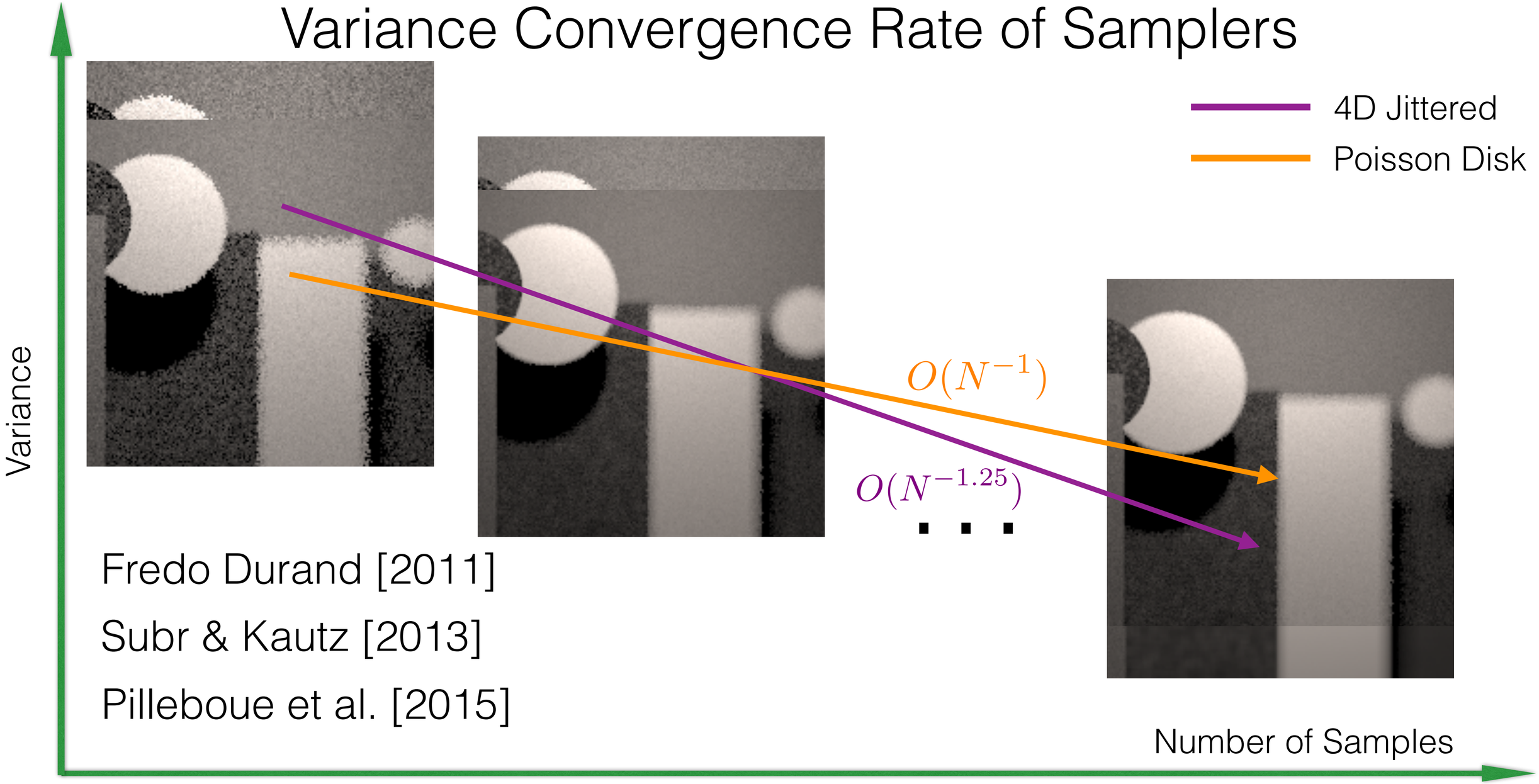
Pilleboue et al. [2015]

Variance Convergence Rate of Samplers



Pilleboue et al. [2015]

Variance Convergence Rate of Samplers



Fredo Durand [2011]

Subr & Kautz [2013]

Pilleboue et al. [2015]

Monte Carlo Estimator

$$I_N = \frac{1}{N} \sum_{k=1}^{N} f(\vec{x}_k)$$

Fredo Durand [2011]

Monte Carlo Estimator

$$I_N = \frac{1}{N}\sum_{k=1}^{N} f(\vec{x}_k) = \int_0^1 \frac{1}{N}\sum_{k=1}^{N} \delta(\vec{x} - \vec{x}_k)\, f(\vec{x})\, d\vec{x}$$

Fredo Durand [2011]

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Fredo Durand [2011]

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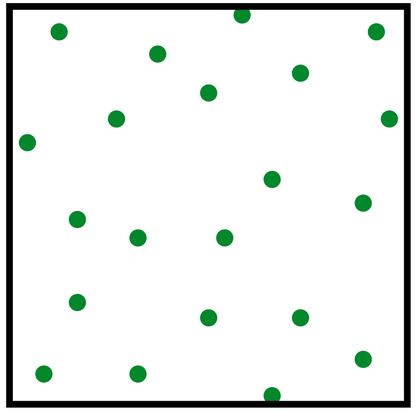

Fredo Durand [2011]

Monte Carlo Estimator

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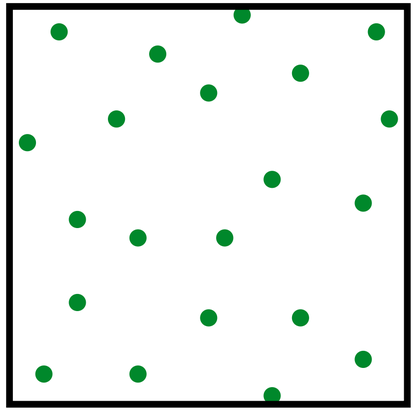


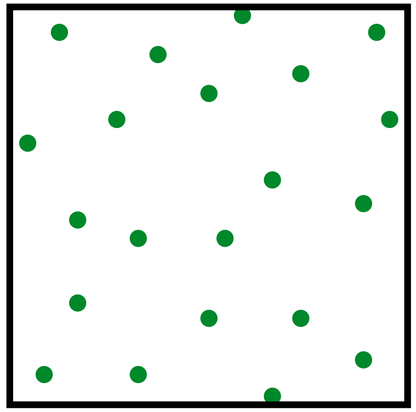
Fredo Durand [2011]

Samples Power Spectrum

$$I_N = \int_0^1 S_N(\vec{x}) f(\vec{x})\, d\vec{x}$$

Samples

Poisson Disk



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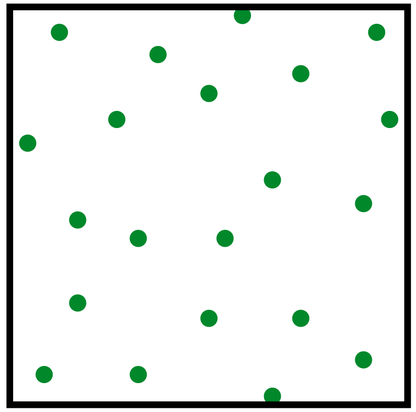
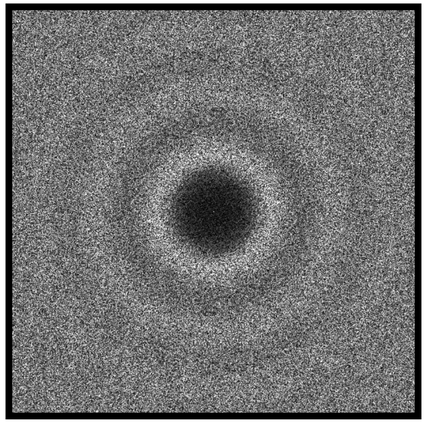


Spectrum



$$S_N(\vec{x}) = \frac{1}{N} \sum_{k=1}^{N} \delta(\vec{x} - \vec{x}_k)$$

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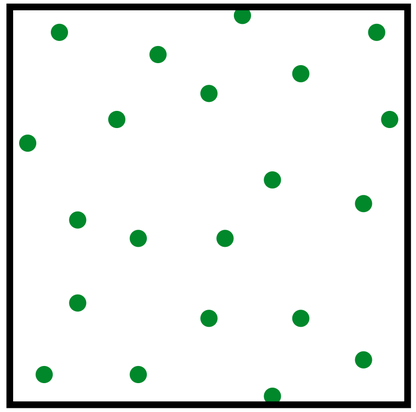
Poisson Disk



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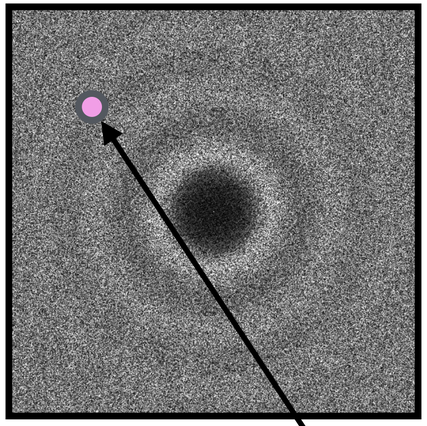

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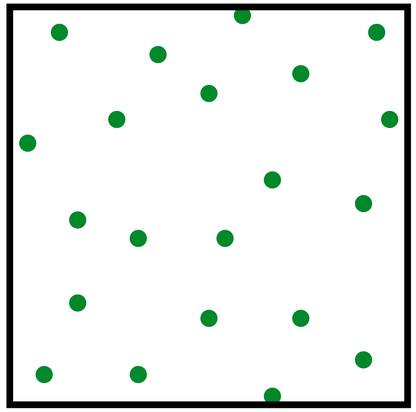
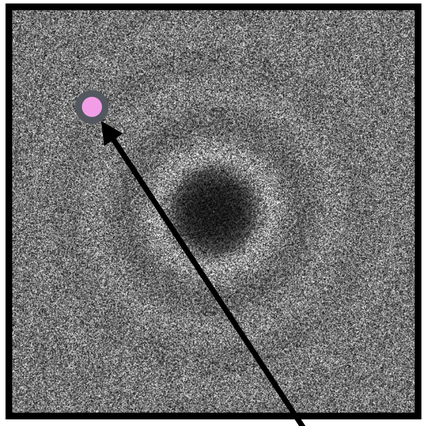
Samples

Poisson Disk



Spectrum



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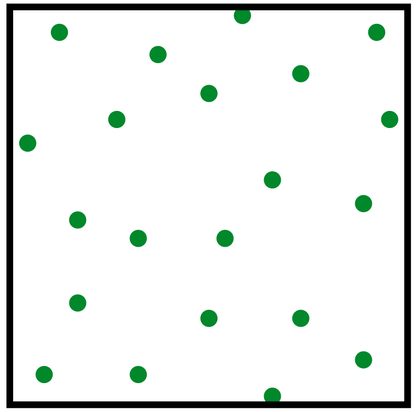
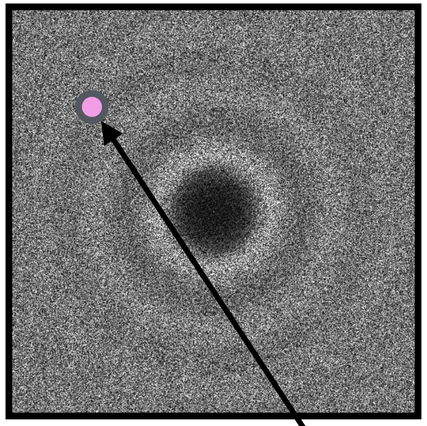
Samples

Poisson Disk



Spectrum



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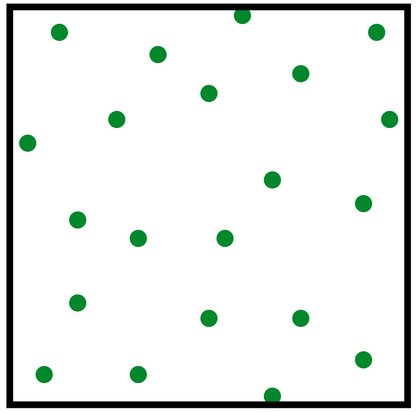
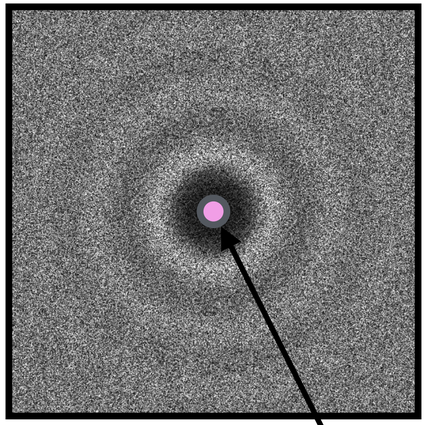
Samples

Poisson Disk



Spectrum



$\nu = 0$ DC frequency

$$S_N(\vec{x}) = \frac{1}{N} \sum_{k=1}^{N} \delta(\vec{x} - \vec{x}_k)$$

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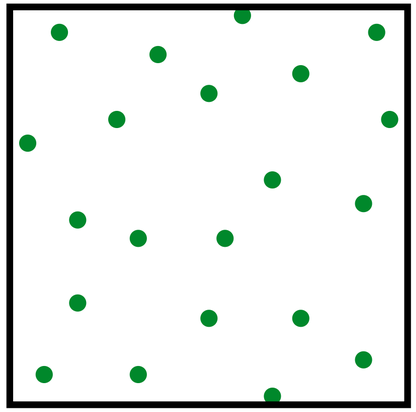
Expected Sampling Power Spectra

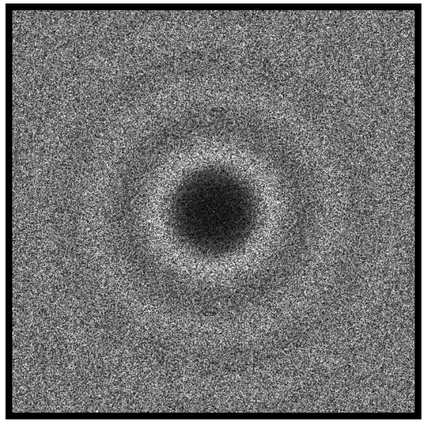
$$I_N = \int_0^1 S_N(\vec{x}) f(\vec{x})\, d\vec{x}$$

Samples

Poisson Disk



Spectrum



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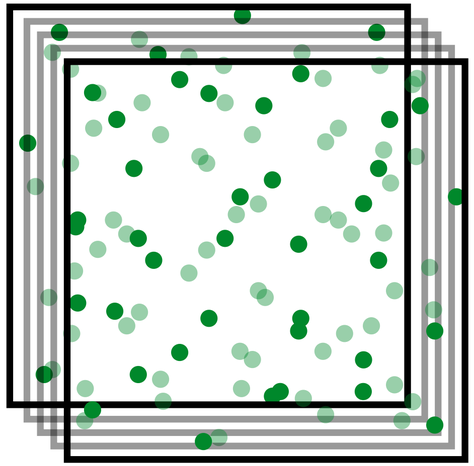
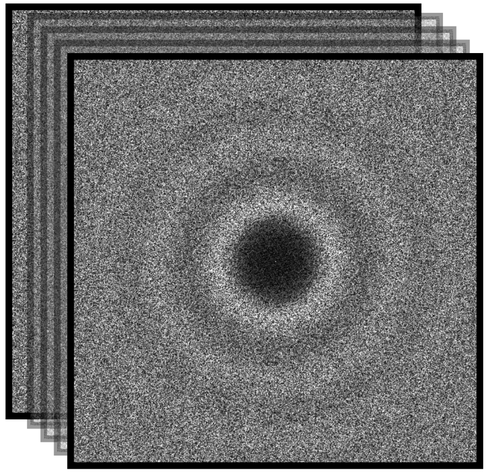


Spectrum



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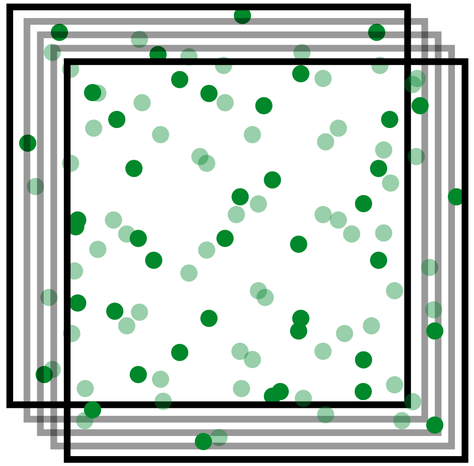
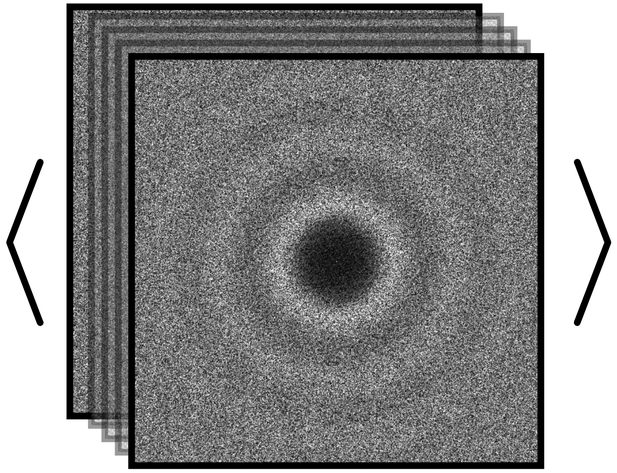
Poisson Disk



Spectrum



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Expected Sampling Power Spectra

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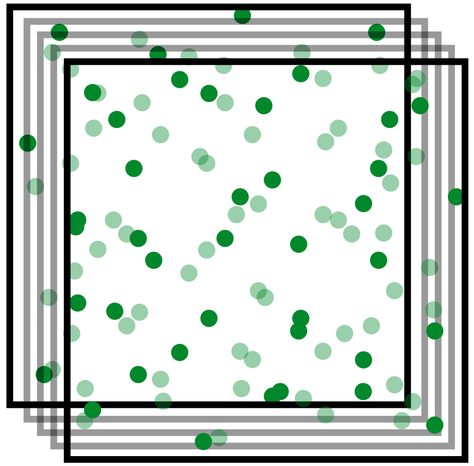
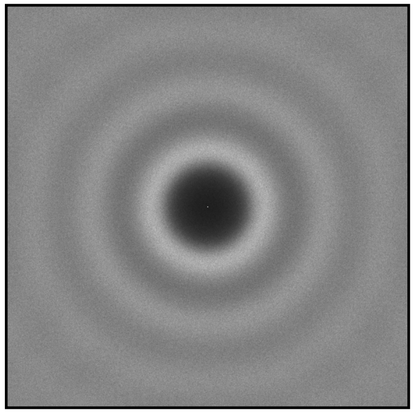
Samples

Poisson Disk



Expected Spectrum



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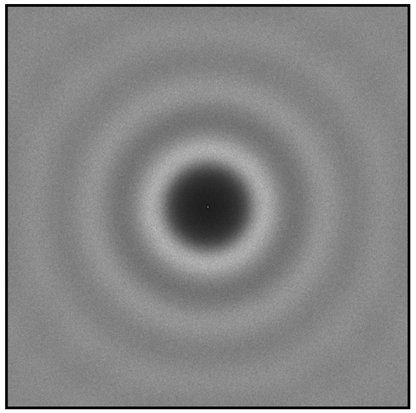
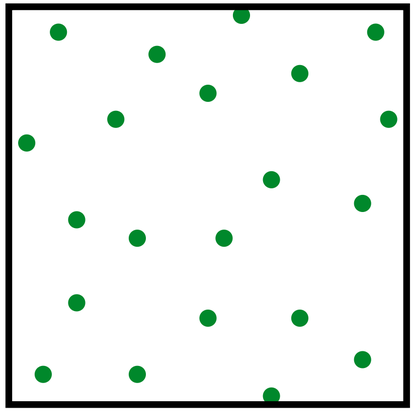
Variance of Monte Carlo Estimator

$\langle \mathcal{P}_{S_N}(\nu) \rangle$

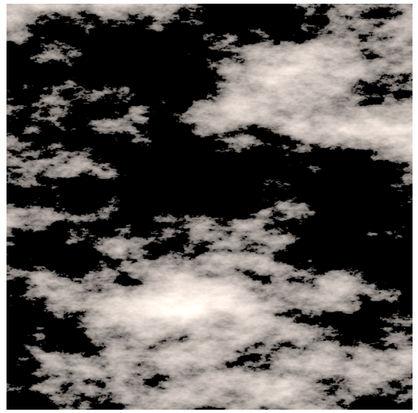


$S_N(\vec{x})$

Poisson Disk



$f(\vec{x})$

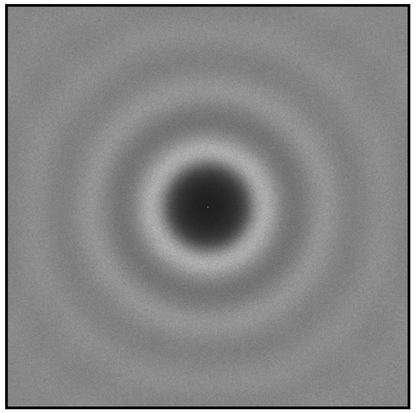


Fredo Durand [2011]

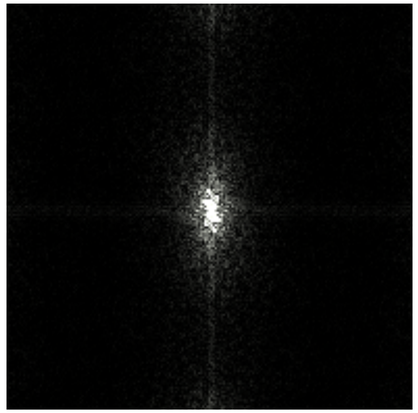
Subr & Kautz [2013]

Variance of Monte Carlo Estimator

$\langle \mathcal{P}_{S_N}(\nu) \rangle$

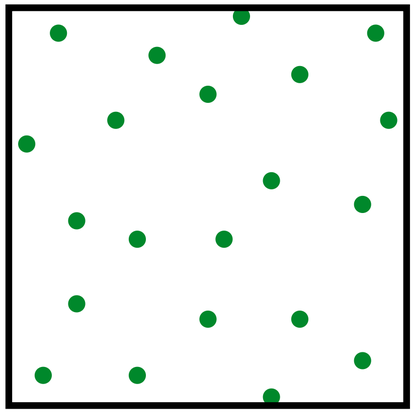


$\mathcal{P}_f(\nu)$

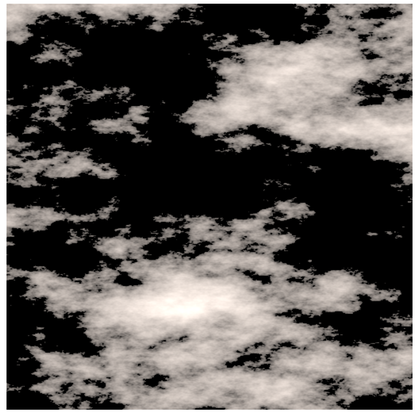


$S_N(\vec{x})$

Poisson Disk



$f(\vec{x})$

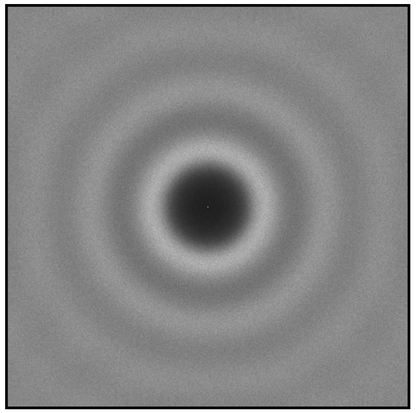


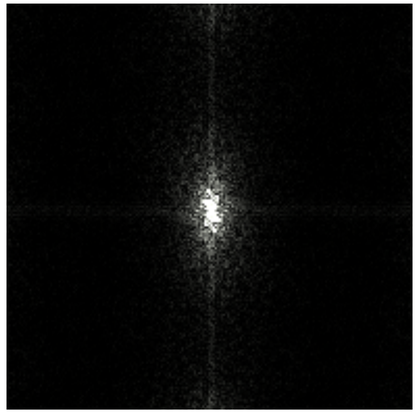
Fredo Durand [2011]

Subr & Kautz [2013]

Variance of Monte Carlo Estimator

$$\mathrm{Var}(I_N) = \int_\Omega \langle \mathcal{P}_{S_N}(\nu) \rangle \times \mathcal{P}_f(\nu)\, d\nu$$



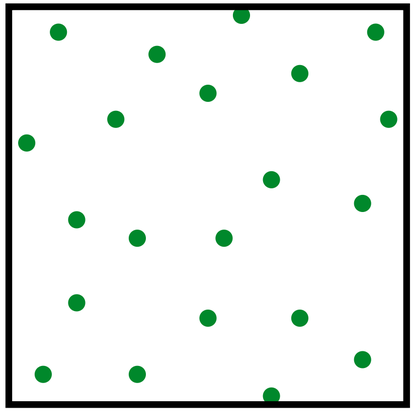


Fredo Durand [2011]

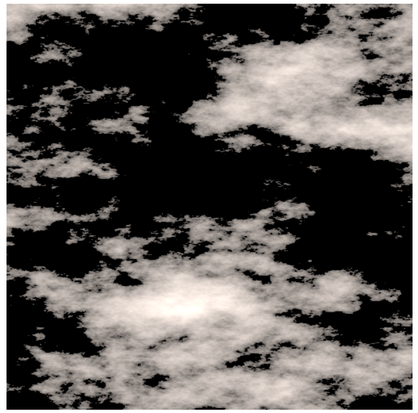
Subr & Kautz [2013]

$S_N(\vec{x})$

Poisson Disk

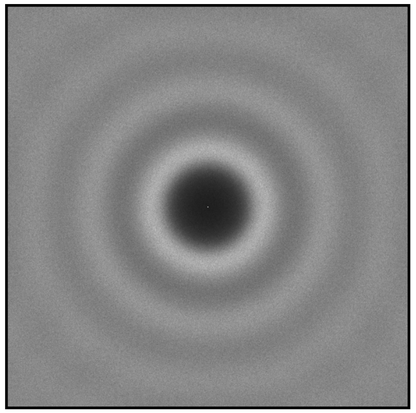


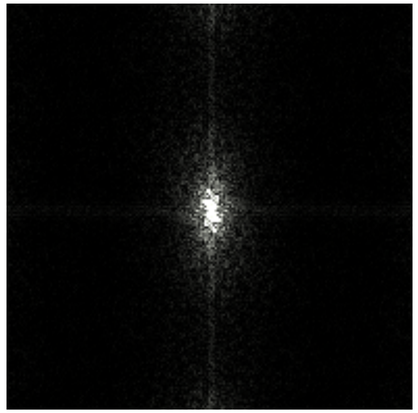
$f(\vec{x})$



Variance of Monte Carlo Estimator

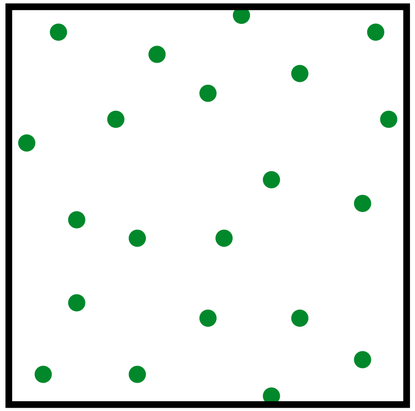
$$\mathrm{Var}(I_N) = \int_\Omega \langle \mathcal{P}_{S_N}(\nu) \rangle \times \mathcal{P}_f(\nu)\, d\nu$$



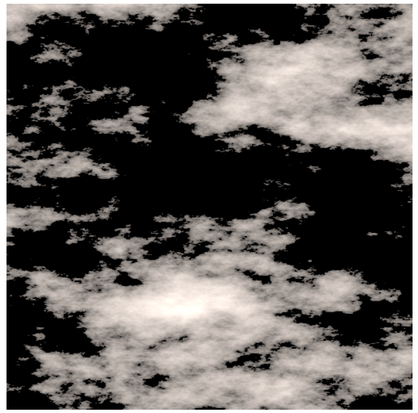


$S_N(\vec{x})$

Poisson Disk



$f(\vec{x})$



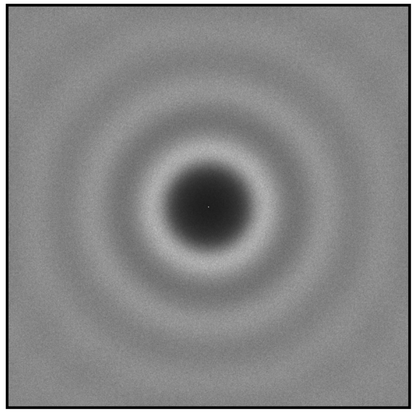
Fredo Durand [2011]

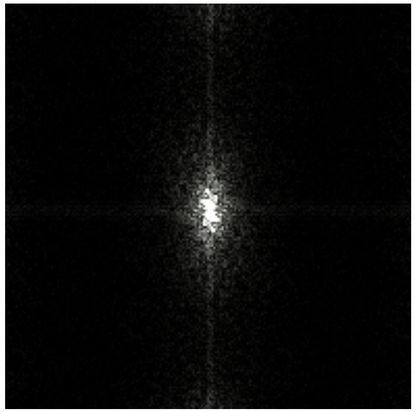
Subr & Kautz [2013]

Pilleboue et al. [2015]

Variance of Monte Carlo Estimator in Polar Coordinates

$$\mathrm{Var}(I_N) = \int_0^\infty \rho^{d-1} \int_{\mathcal{S}^{d-1}} \left\langle \tilde{\mathcal{P}}_{S_N}(\rho\mathbf{n}) \right\rangle \times \mathcal{P}_f(\rho\mathbf{n})\, d\mathbf{n}\, d\rho$$

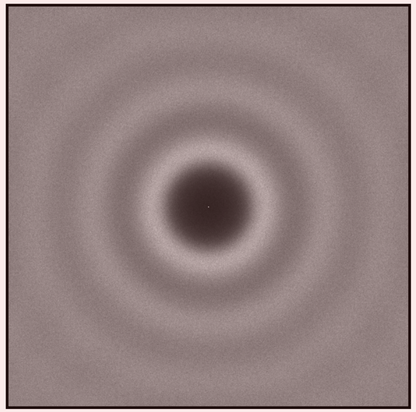


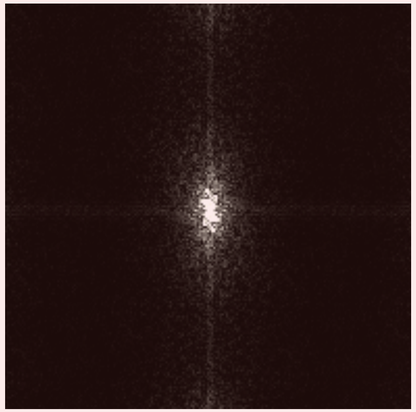


Pilleboue et al. [2015]

Variance of Monte Carlo Estimator in Polar Coordinates

$$\mathrm{Var}(I_N) = \int_0^\infty \rho^{d-1} \int_{\mathcal{S}^{d-1}} \overset{\left\langle \tilde{\mathcal{P}}_{S_N}(\rho\mathbf{n}) \right\rangle}{\boxed{}} \times \overset{\mathcal{P}_f(\rho\mathbf{n})}{\boxed{}} \, d\mathbf{n}\, d\rho$$

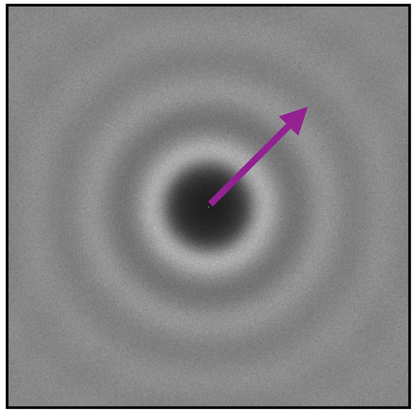


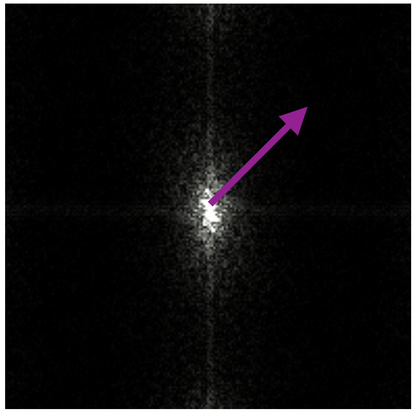


Pilleboue et al. [2015]

Variance of Monte Carlo Estimator in Polar Coordinates

$$\mathrm{Var}(I_N) = \int_0^\infty \rho^{d-1} \int_{\mathcal{S}^{d-1}} \left\langle \tilde{\mathcal{P}}_{S_N}(\rho\mathbf{n}) \right\rangle \times \mathcal{P}_f(\rho\mathbf{n}) \, d\mathbf{n} \, d\rho$$

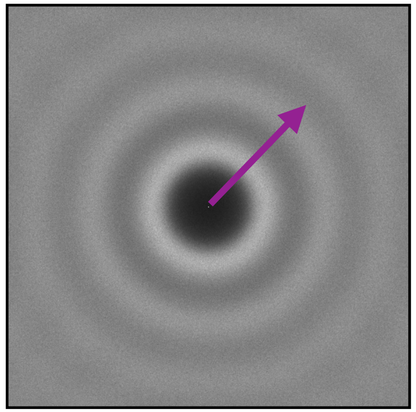


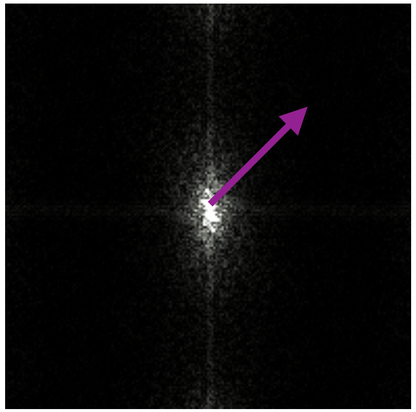


Pilleboue et al. [2015]

Variance of Monte Carlo Estimator in Polar Coordinates

$$\mathrm{Var}(I_N) = \int_0^\infty \rho^{d-1} \int_{\mathcal{S}^{d-1}} \left\langle \tilde{\mathcal{P}}_{S_N}(\rho\mathbf{n}) \right\rangle \times \mathcal{P}_f(\rho\mathbf{n}) \, d\mathbf{n} \, d\rho$$

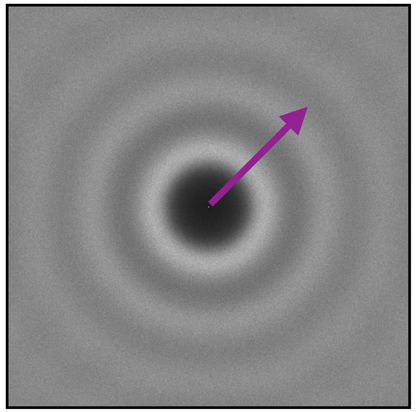


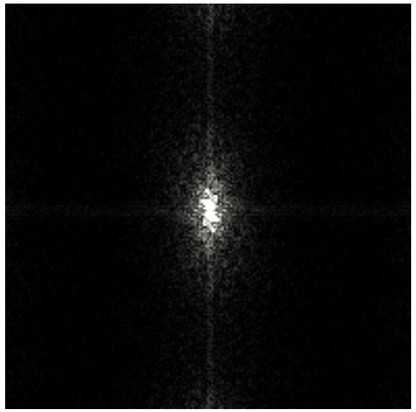


Pilleboue et al. [2015]

Variance of Monte Carlo Estimator for Isotropic Sampling Spectra

$$\mathrm{Var}(I_N) = \int_0^\infty \rho^{d-1} \int_{\mathcal{S}^{d-1}} \left\langle \tilde{\mathcal{P}}_{S_N}(\rho\mathbf{n}) \right\rangle \times \mathcal{P}_f(\rho\mathbf{n}) \, d\mathbf{n} \, d\rho$$

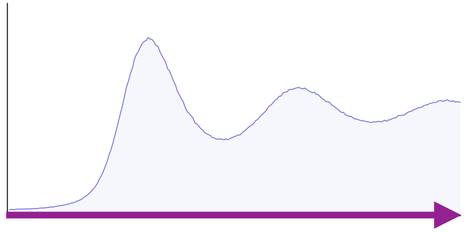


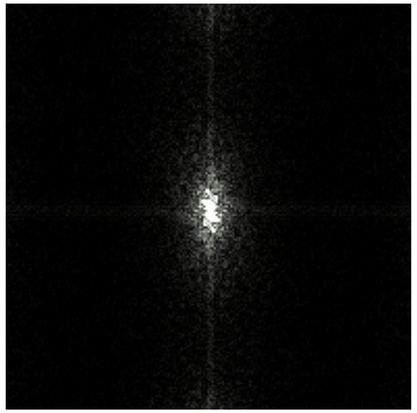


Pilleboue et al. [2015]

Variance of Monte Carlo Estimator for Isotropic Sampling Spectra

$$\mathrm{Var}(I_N) = \int_0^\infty \rho^{d-1} \tilde{\mathcal{P}}_{S_N}(\rho) \int_{\mathcal{S}^{d-1}} \mathcal{P}_f(\rho \mathbf{n})\, d\mathbf{n}\, d\rho$$



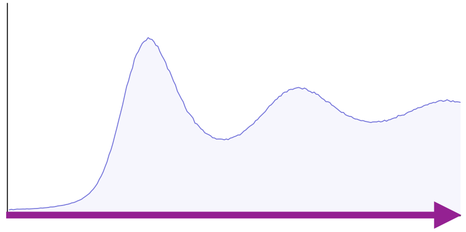


Pilleboue et al. [2015]

Variance of Monte Carlo Estimator for Isotropic Sampling Spectra

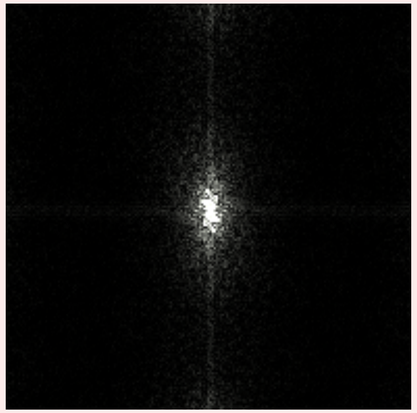
$$\mathrm{Var}(I_N) = \int_0^\infty \rho^{d-1}$$

$\tilde{\mathcal{P}}_{S_N}(\rho)$



$\mathcal{P}_f(\rho\mathbf{n})$

$$\int_{\mathcal{S}^{d-1}}$$

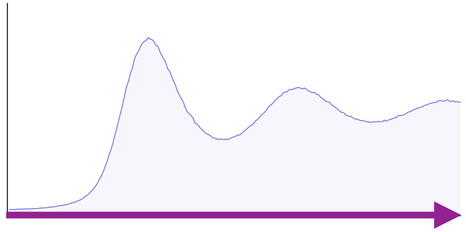


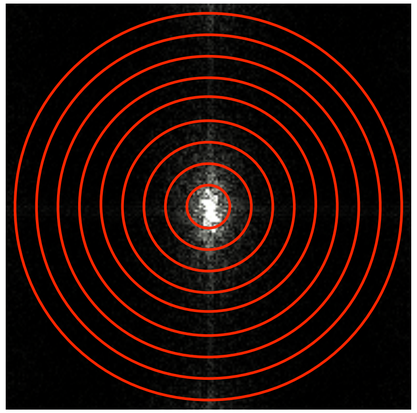
$$d\mathbf{n}\, d\rho$$

Pilleboue et al. [2015]

Variance of Monte Carlo Estimator for Isotropic Sampling Spectra

$$\mathrm{Var}(I_N) = \int_0^\infty \rho^{d-1} \tilde{\mathcal{P}}_{S_N}(\rho) \times \mathcal{P}_f(\rho \mathbf{n})\, d\rho$$

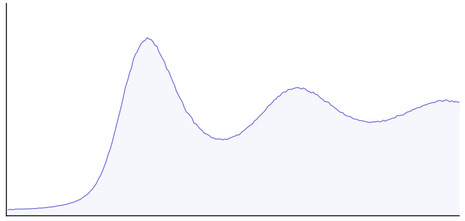


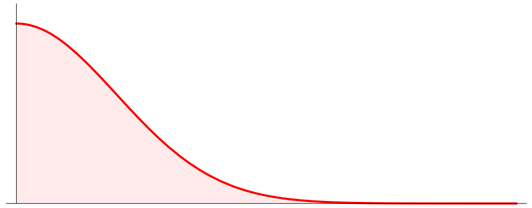


Pilleboue et al. [2015]

Variance of Monte Carlo Estimator for Isotropic Sampling Spectra

$$\mathrm{Var}(I_N) = \int_0^\infty \rho^{d-1} \; \tilde{\mathcal{P}}_{S_N}(\rho) \times \mathcal{P}_f(\rho) \; d\rho$$

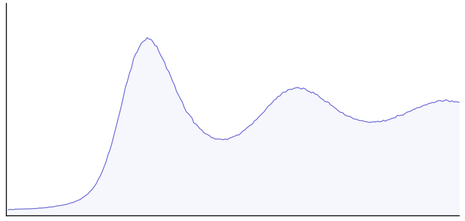


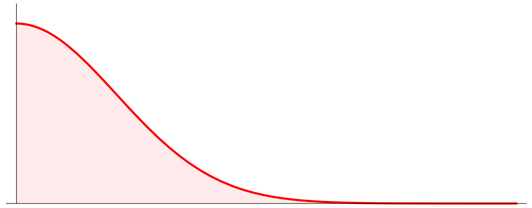


Pilleboue et al. [2015]

Variance of Monte Carlo Estimator for Isotropic Sampling Spectra

$$\mathrm{Var}(I_N) = \int_0^\infty \rho^{d-1} \, \tilde{\mathcal{P}}_{S_N}(\rho) \times \mathcal{P}_f(\rho) \, d\rho$$



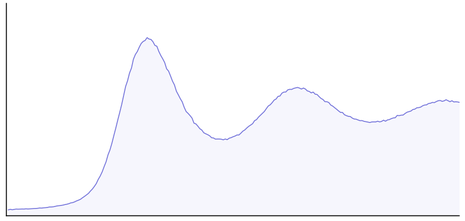


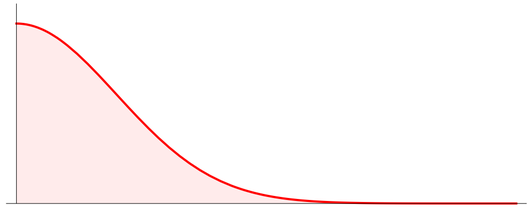
Samplers	Worst Case	Best Case
Random		
Poisson Disk		
CCVT		

Pilleboue et al. [2015]

Variance of Monte Carlo Estimator for Isotropic Sampling Spectra

$$\mathrm{Var}(I_N) = \int_0^\infty \rho^{d-1} \underbrace{\tilde{\mathcal{P}}_{S_N}(\rho)}_{} \times \underbrace{\mathcal{P}_f(\rho)}_{} \, d\rho$$





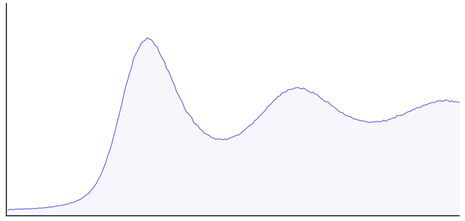
Samplers	Worst Case	Best Case
Random	$\mathcal{O}(N^{-1})$	$\mathcal{O}(N^{-1})$
Poisson Disk	$\mathcal{O}(N^{-1})$	$\mathcal{O}(N^{-1})$
CCVT	$\mathcal{O}(N^{-1.5})$	$\mathcal{O}(N^{-3})$

Pilleboue et al. [2015]

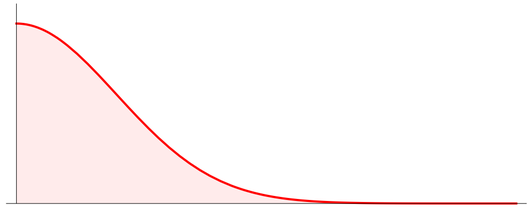
Variance of Monte Carlo Estimator for Isotropic Sampling Spectra

$\tilde{\mathcal{P}}_{S_N}(\rho)$ $\mathcal{P}_f(\rho)$

$$\mathrm{Var}(I_N) = \int_0^\infty \rho^{d-1}$$



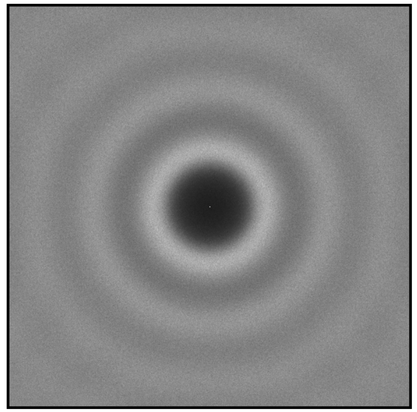
$\times$



$d\rho$

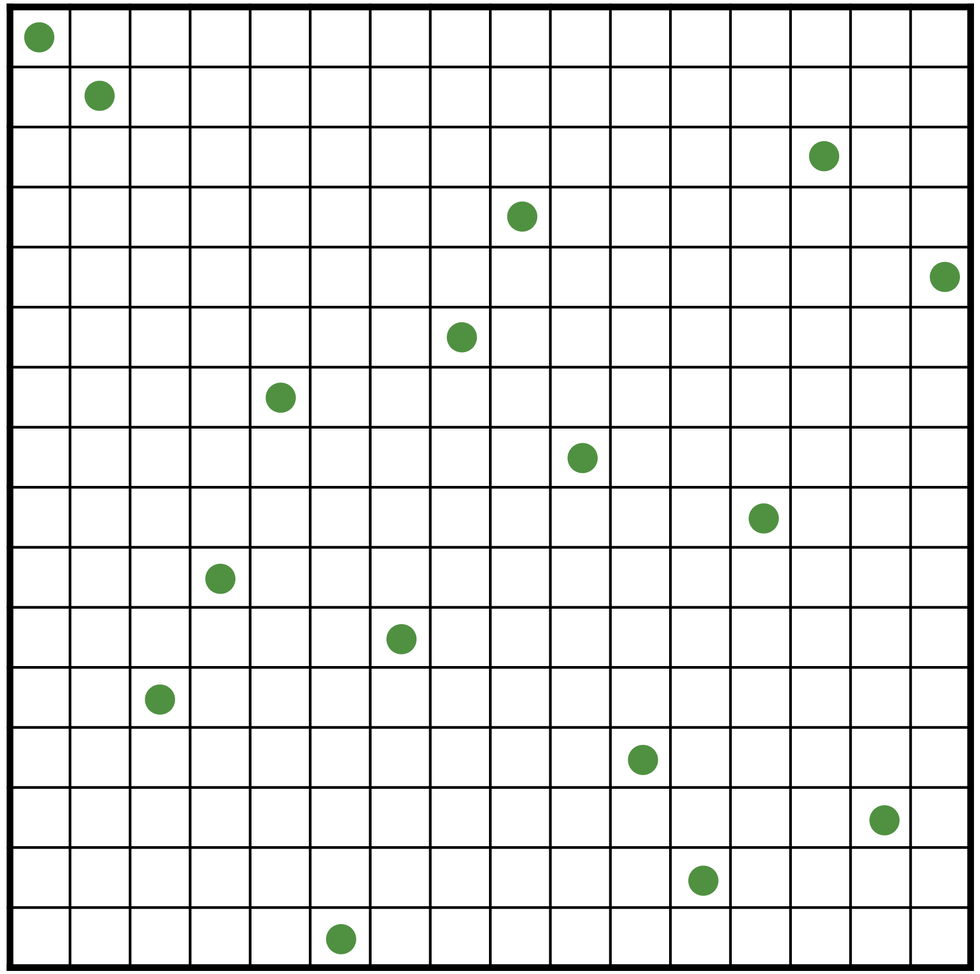
Samplers	Worst Case	Best Case
Random	$\mathcal{O}(N^{-1})$	$\mathcal{O}(N^{-1})$
Poisson Disk	$\mathcal{O}(N^{-1})$	$\mathcal{O}(N^{-1})$
CCVT	$\mathcal{O}(N^{-1.5})$	$\mathcal{O}(N^{-3})$

Isotropic Spectrum
Poisson Disk

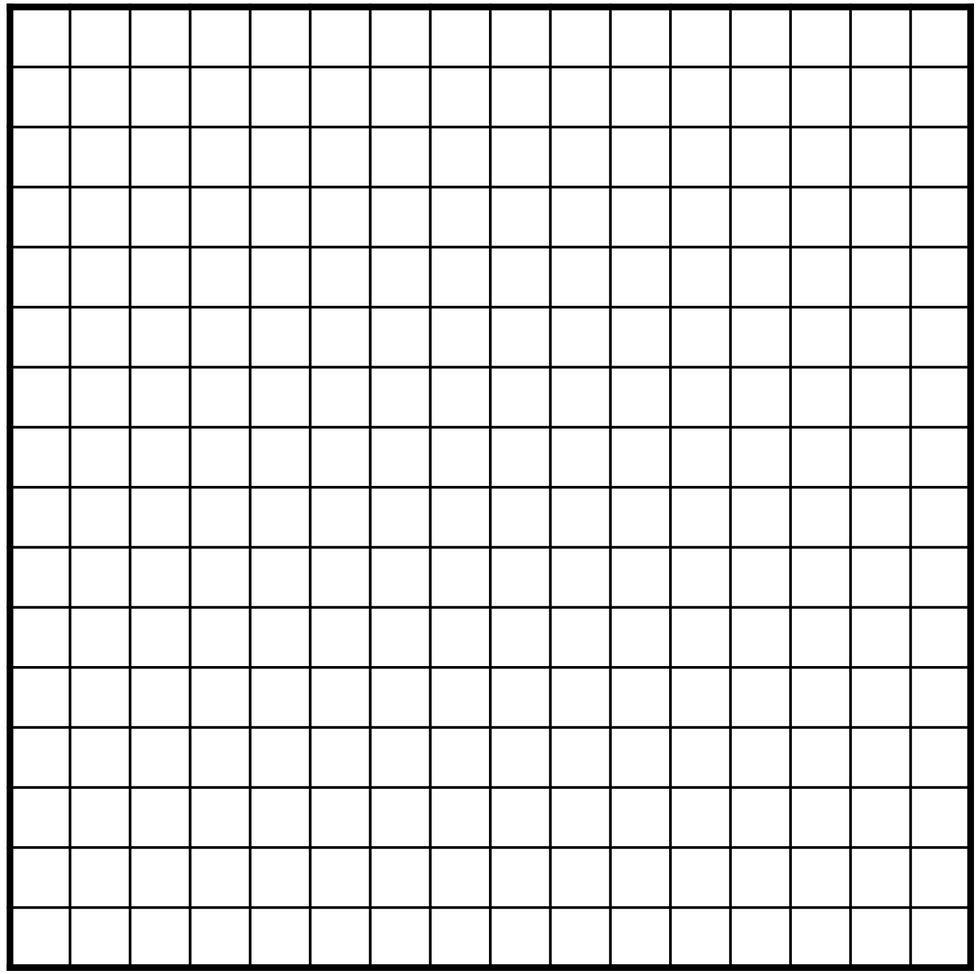


Pilleboue et al. [2015]

Latin Hypercube Sampler (N-rooks)

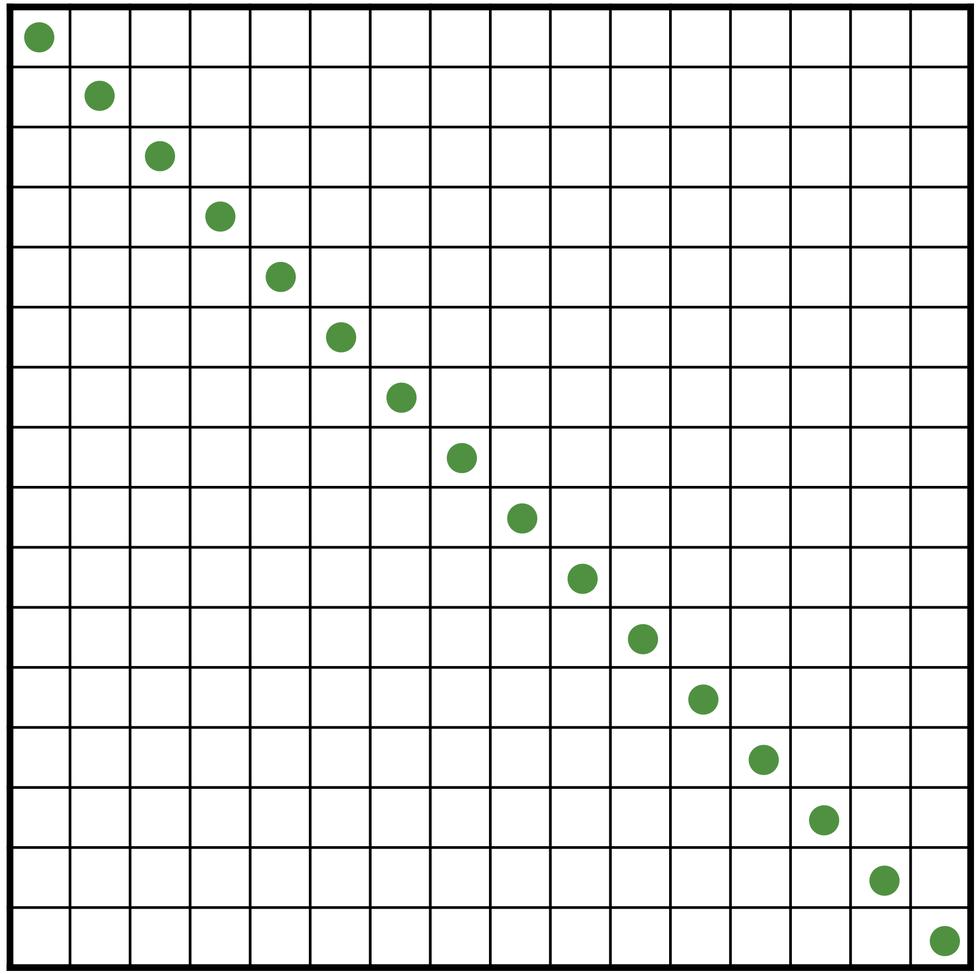


Latin Hypercube Sampler (N-rooks)



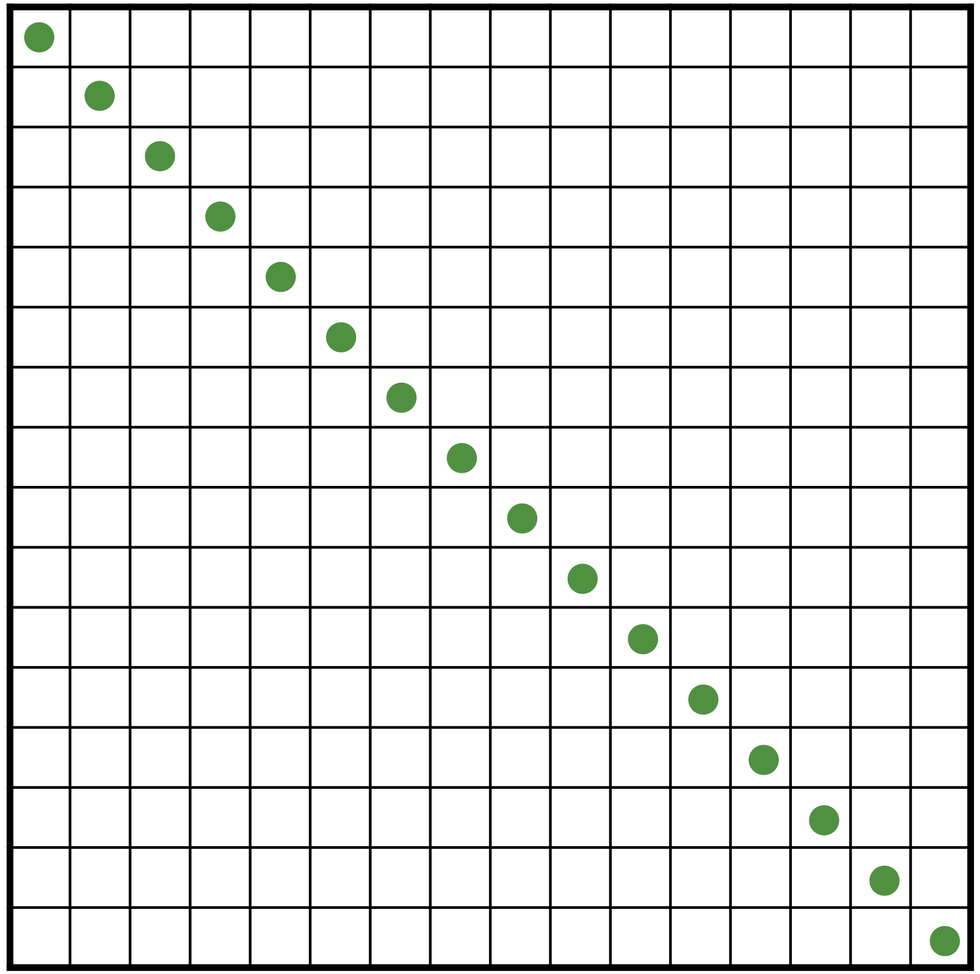
Latin Hypercube Sampler (N-rooks)

Initialize



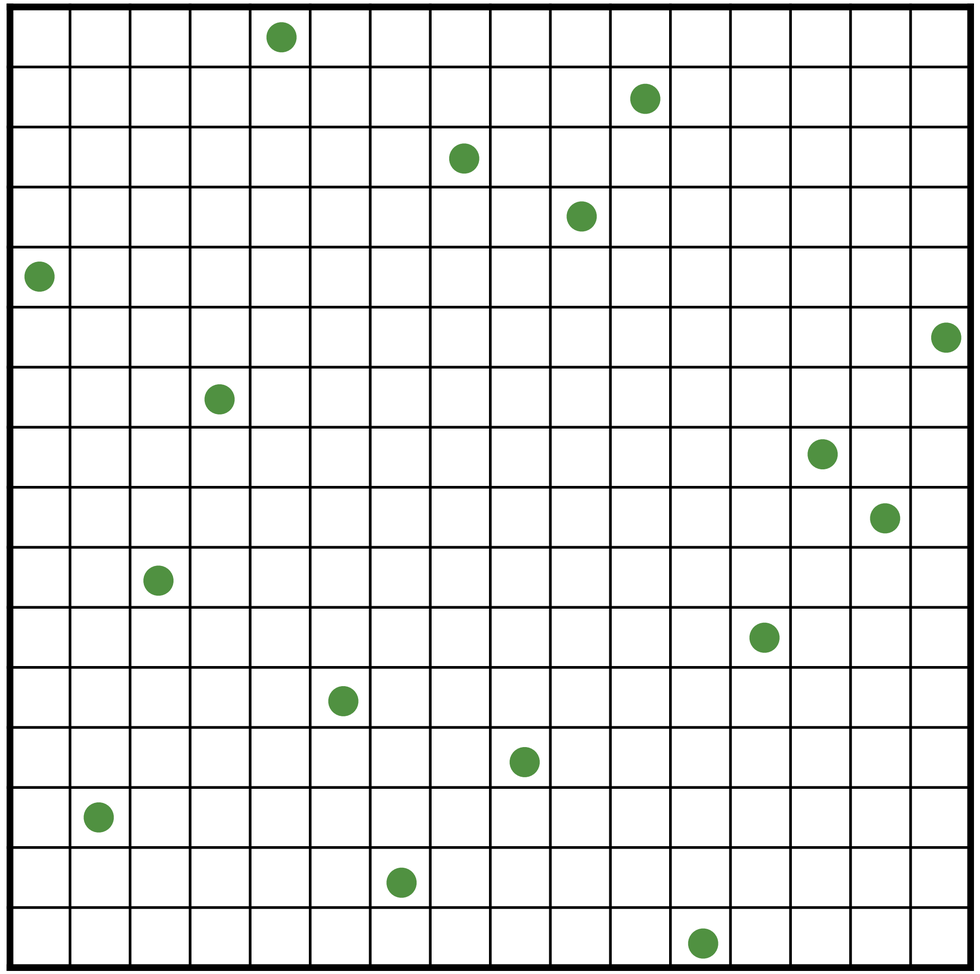
Latin Hypercube Sampler (N-rooks)

Shuffle rows

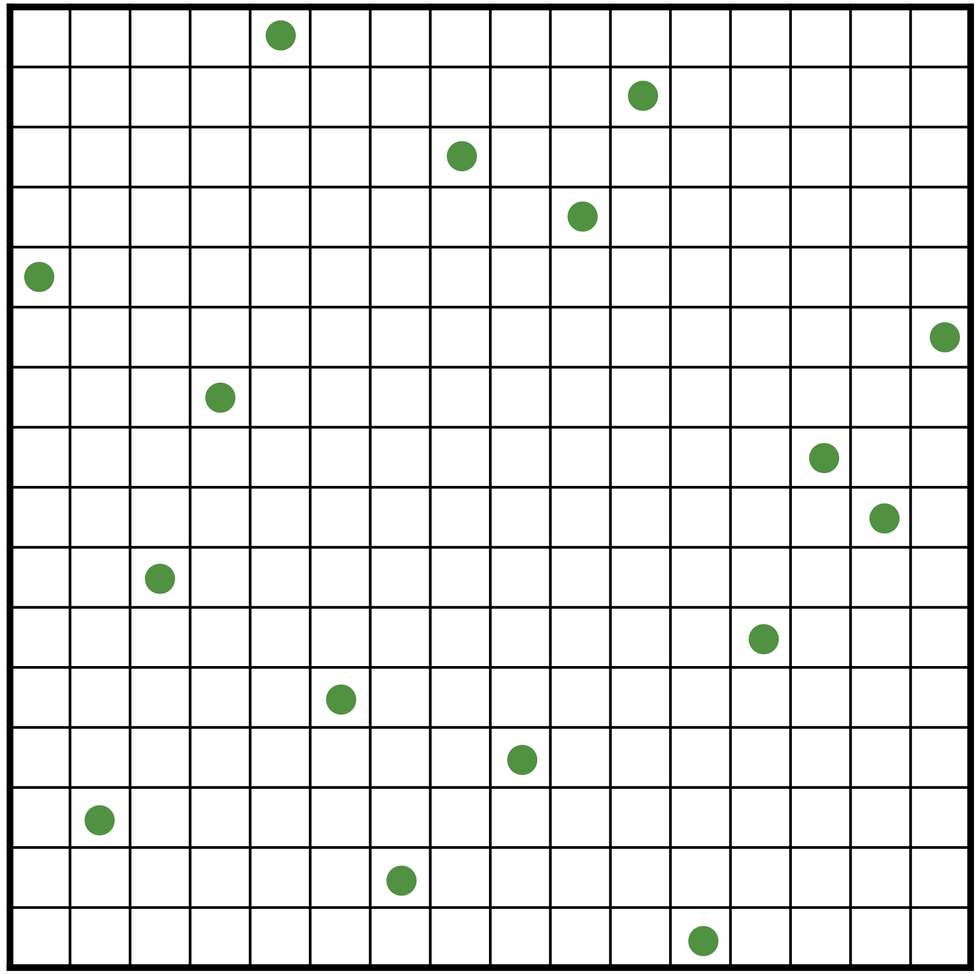


Latin Hypercube Sampler (N-rooks)

Shuffle rows

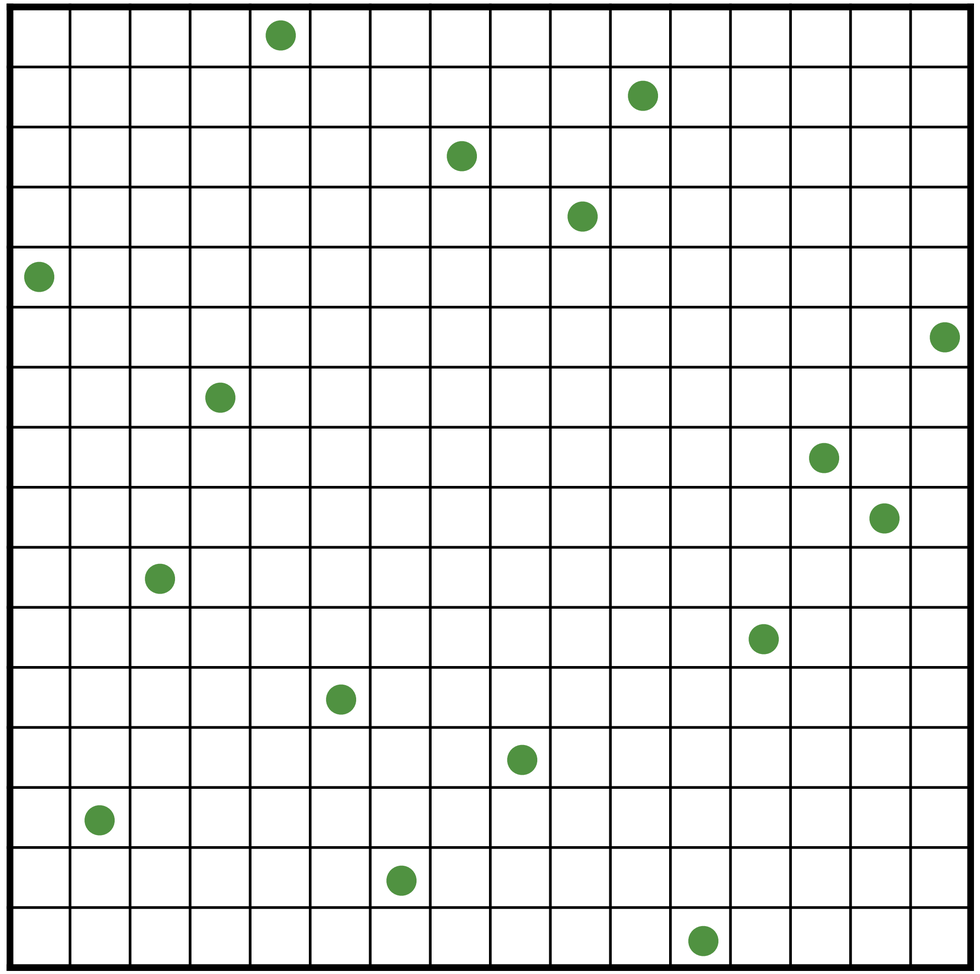


Latin Hypercube Sampler (N-rooks)



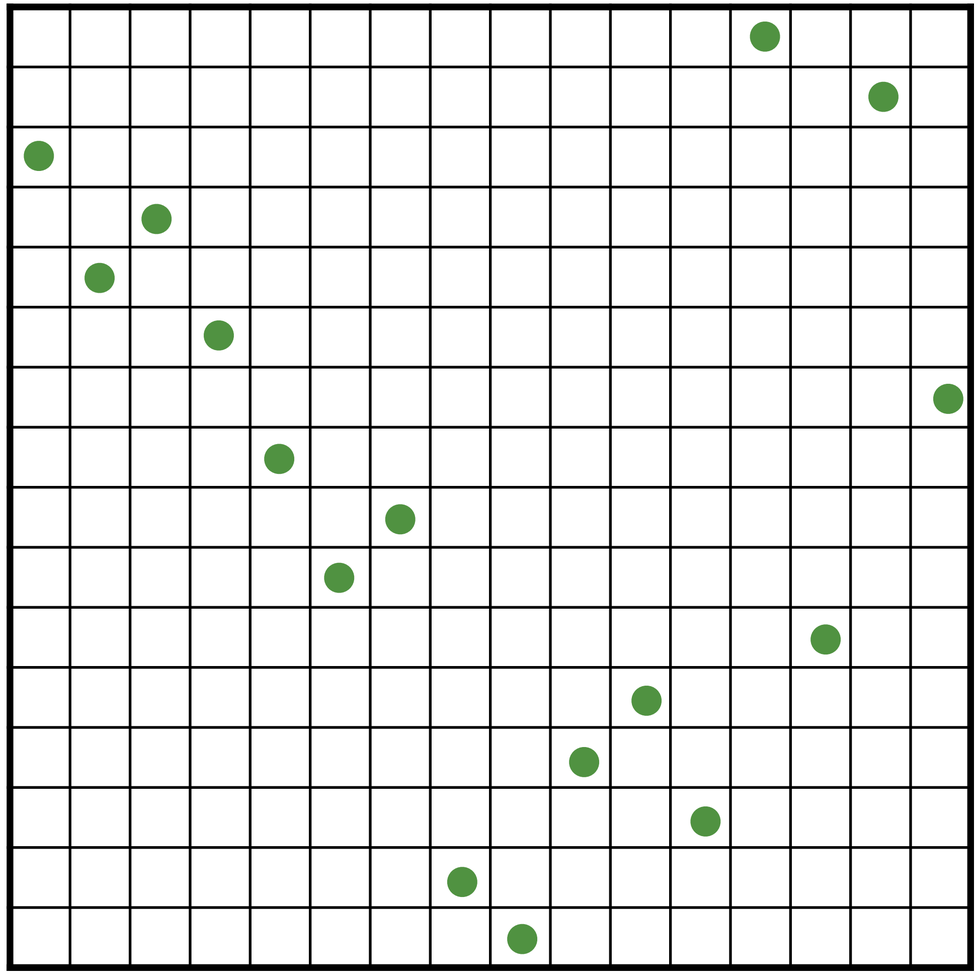
Latin Hypercube Sampler (N-rooks)

Shuffle columns

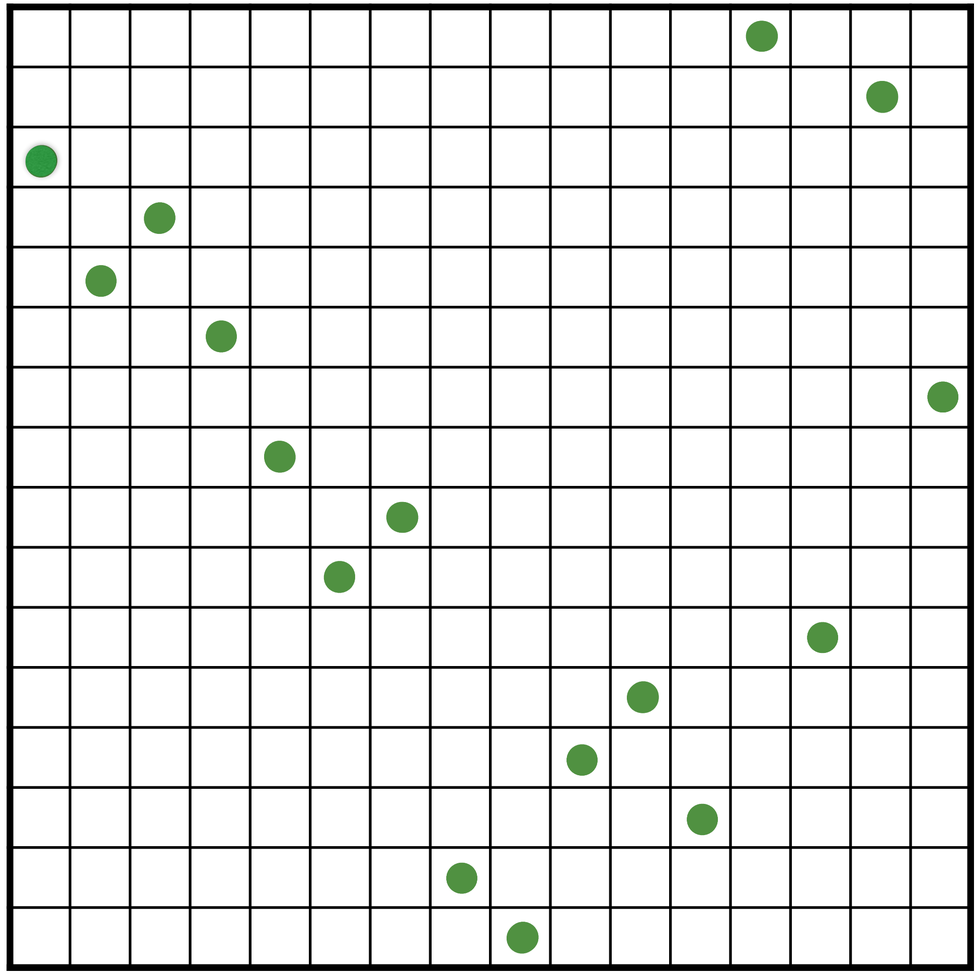


Latin Hypercube Sampler (N-rooks)

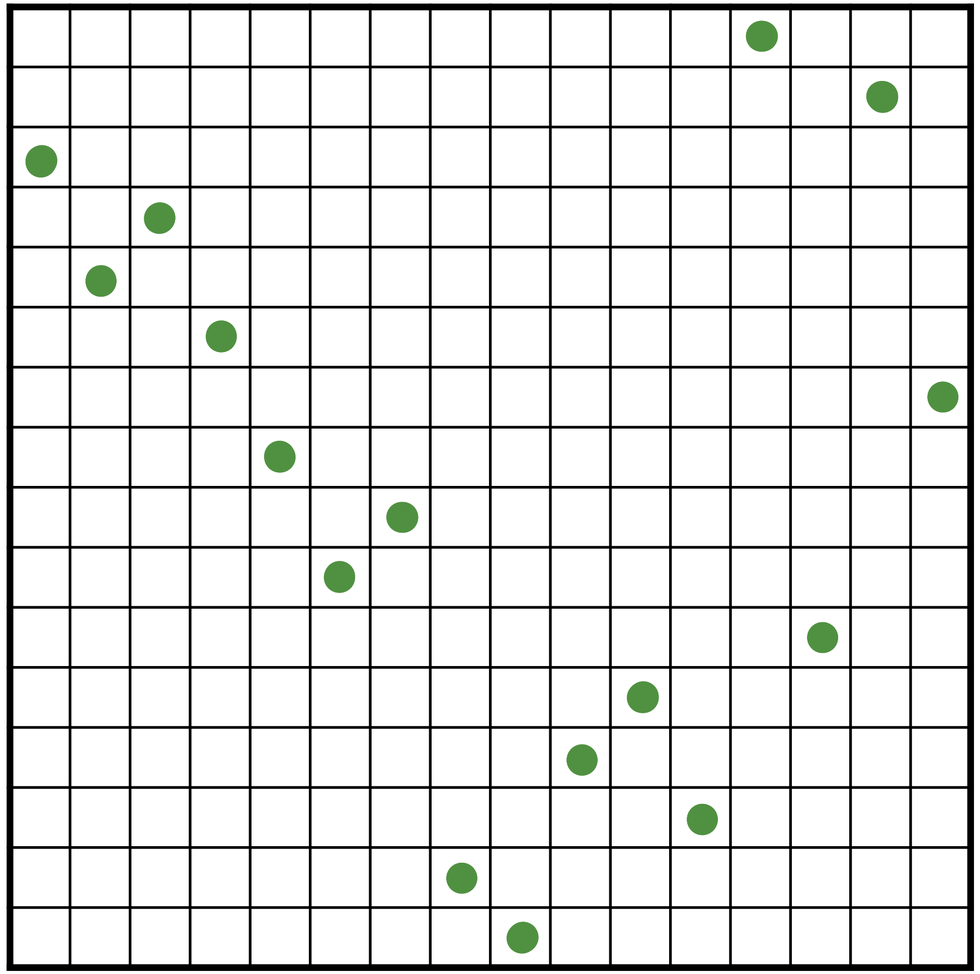
Shuffle columns



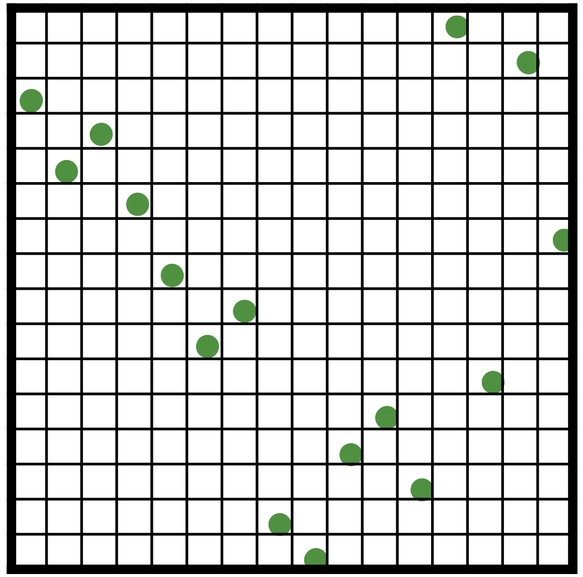
Latin Hypercube Sampler (N-rooks)



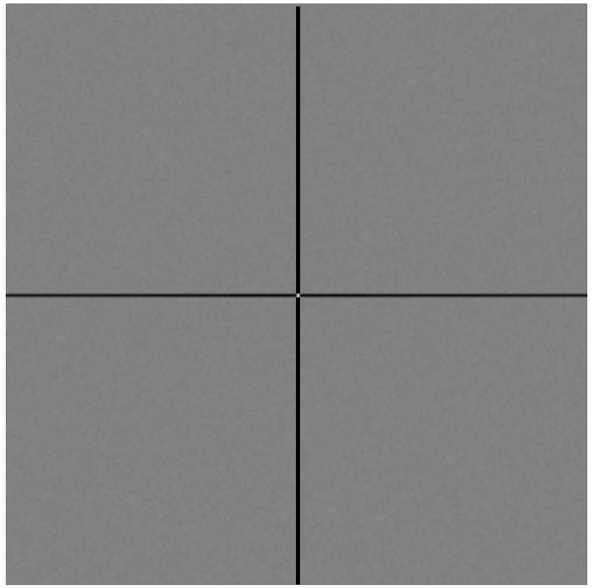
Latin Hypercube Sampler (N-rooks)



Anisotropic Sampling Power Spectra

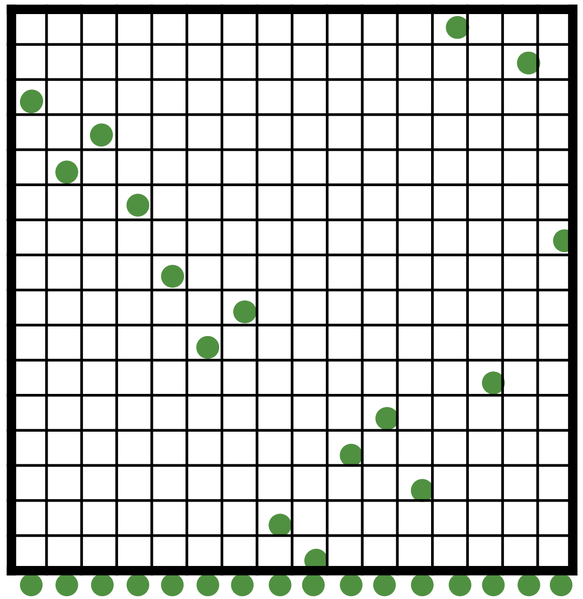


N-rooks /
Latin Hypercube

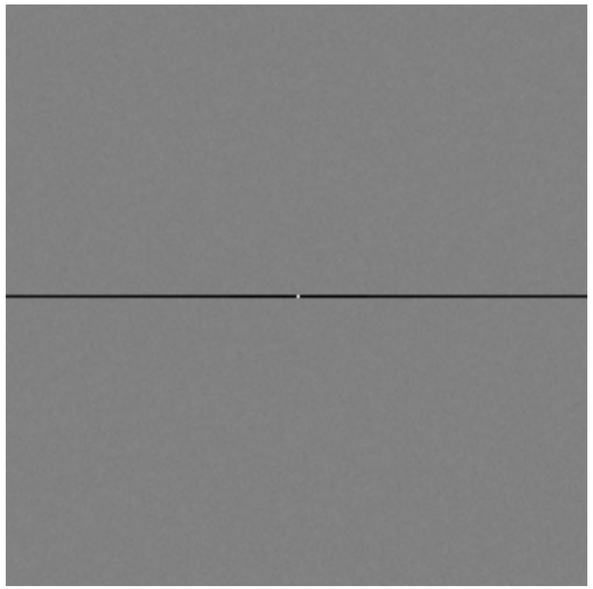


N-rooks
Spectrum

Anisotropic Sampling Power Spectra

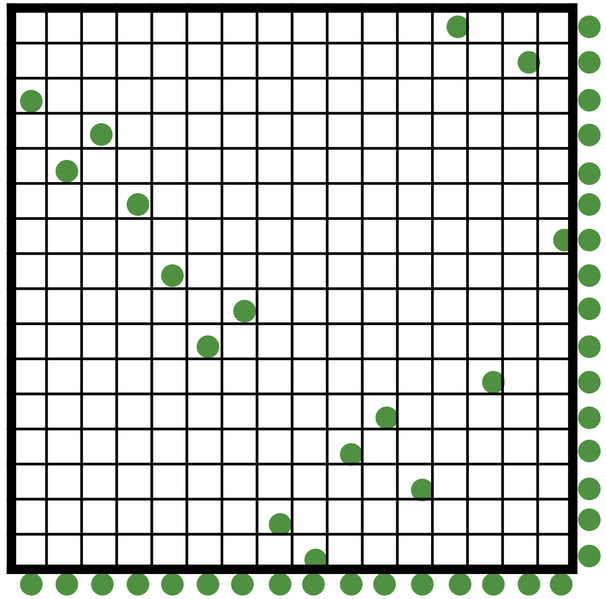


N-rooks /
Latin Hypercube

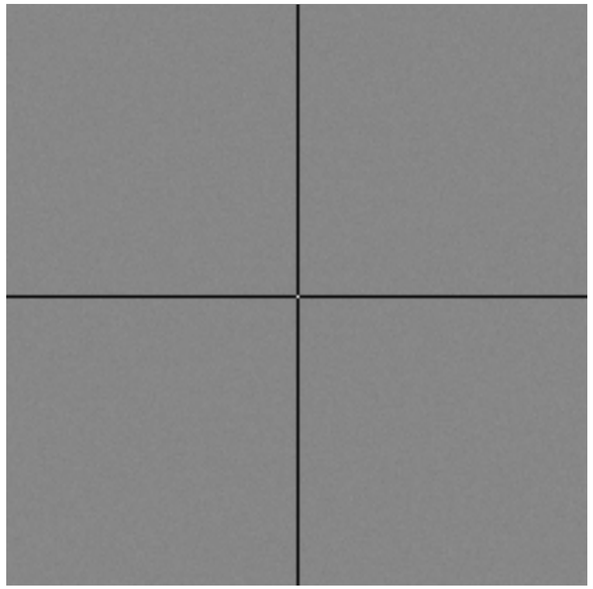


Spectrum

Anisotropic Sampling Power Spectra

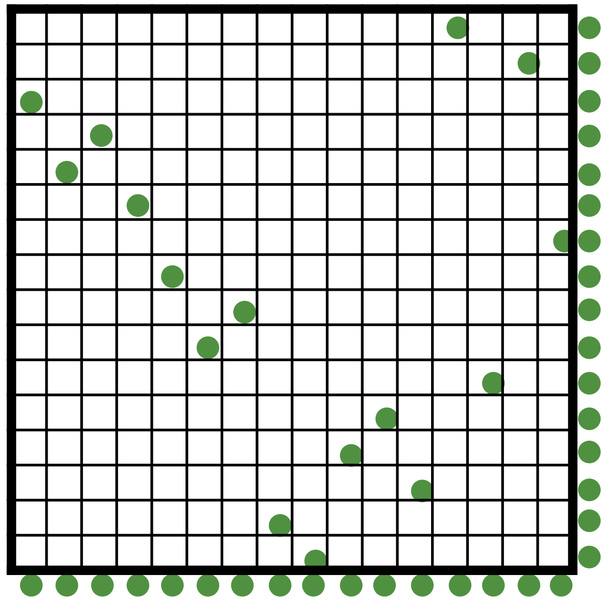


N-rooks /
Latin Hypercube

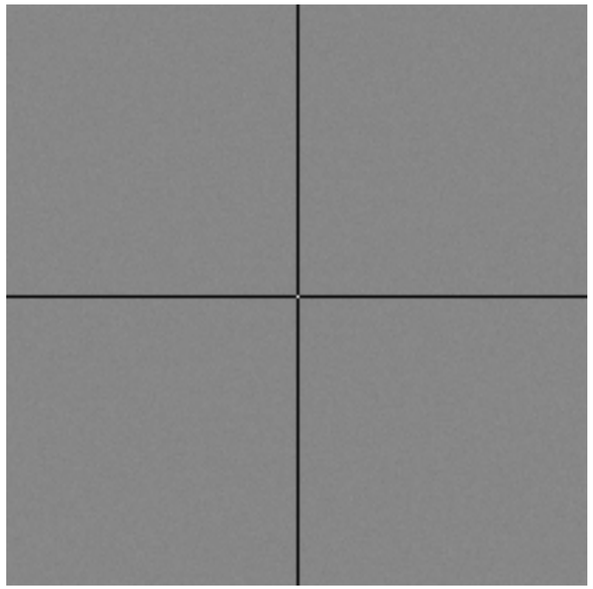


N-rooks
Spectrum

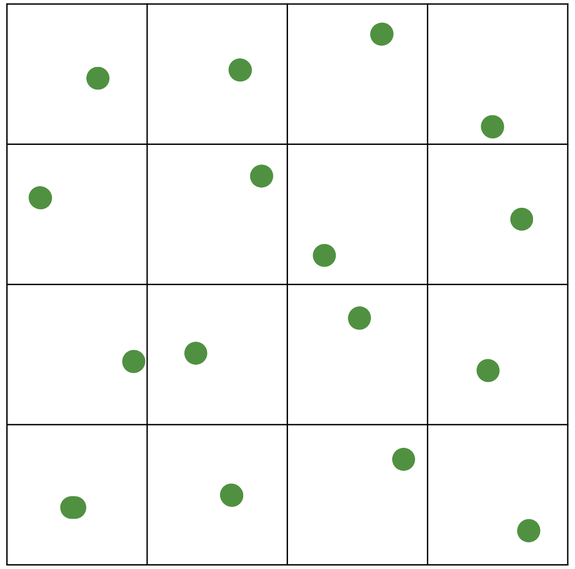
Anisotropic Sampling Power Spectra



N-rooks /
Latin Hypercube

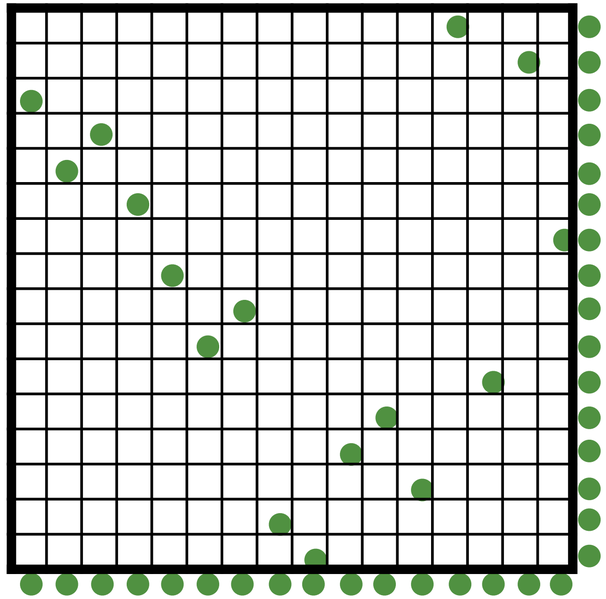


N-rooks
Spectrum

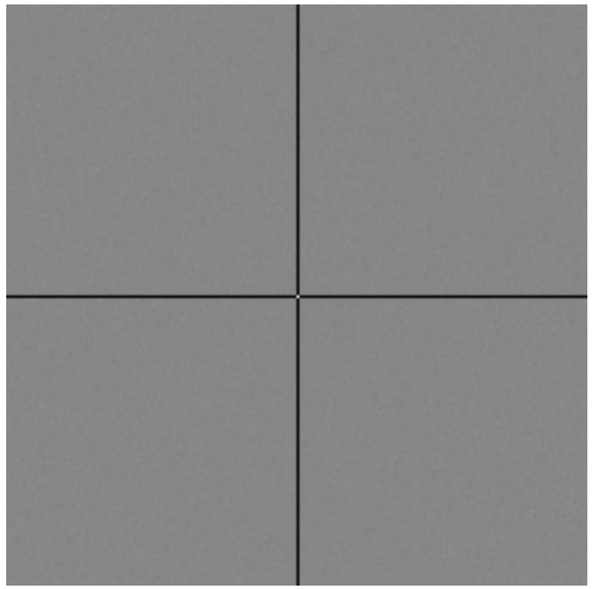


Jitter

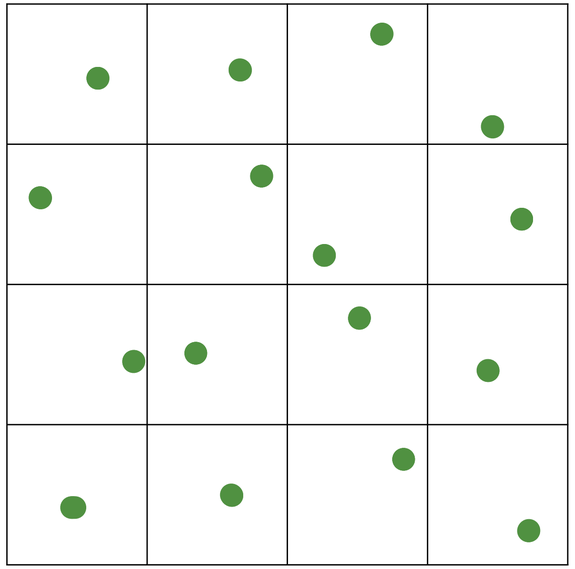
Anisotropic Sampling Power Spectra



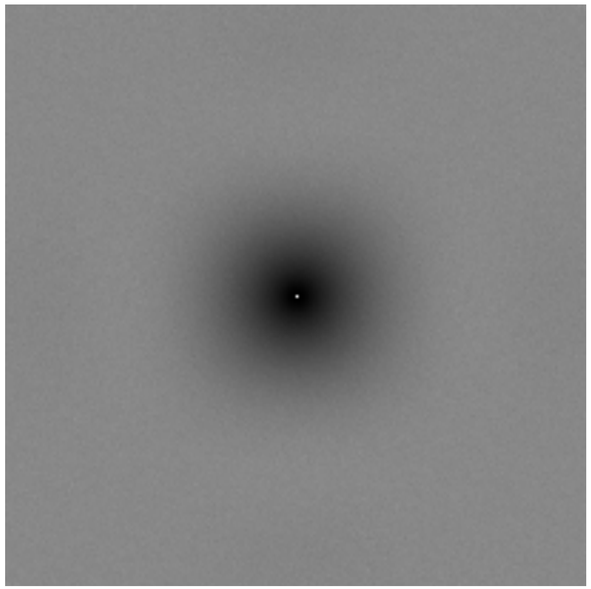
N-rooks /
Latin Hypercube



N-rooks
Spectrum

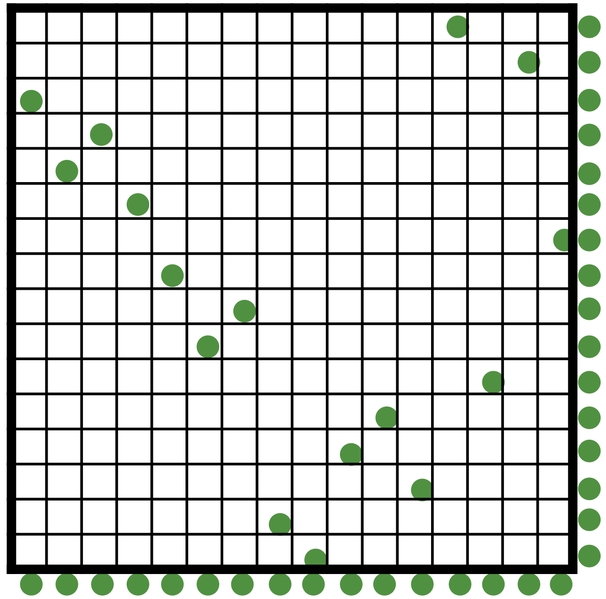


Jitter

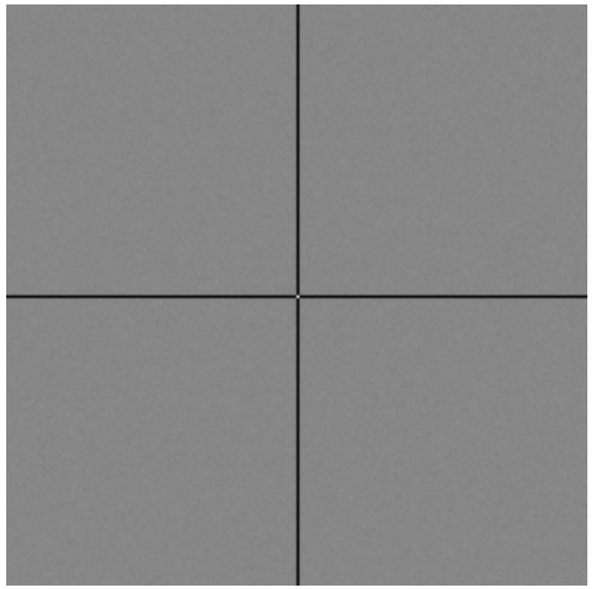


Jitter
Spectrum

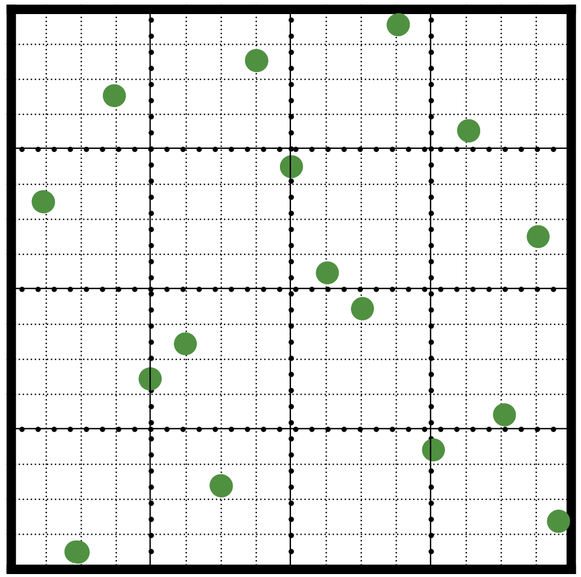
Anisotropic Sampling Power Spectra



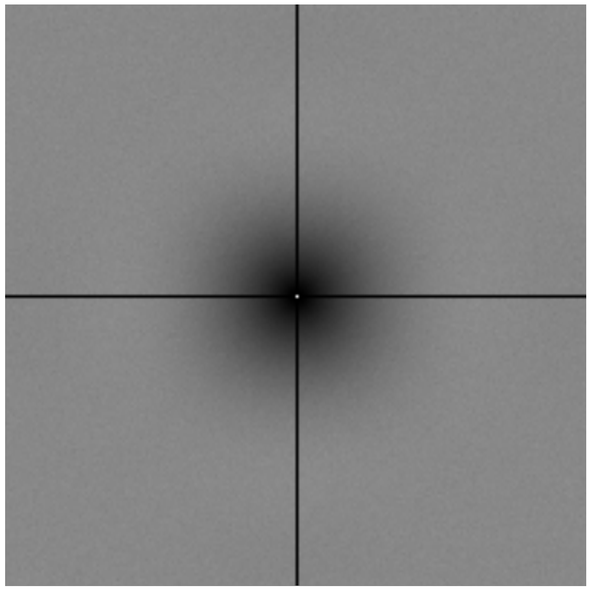
N-rooks /
Latin Hypercube



N-rooks
Spectrum



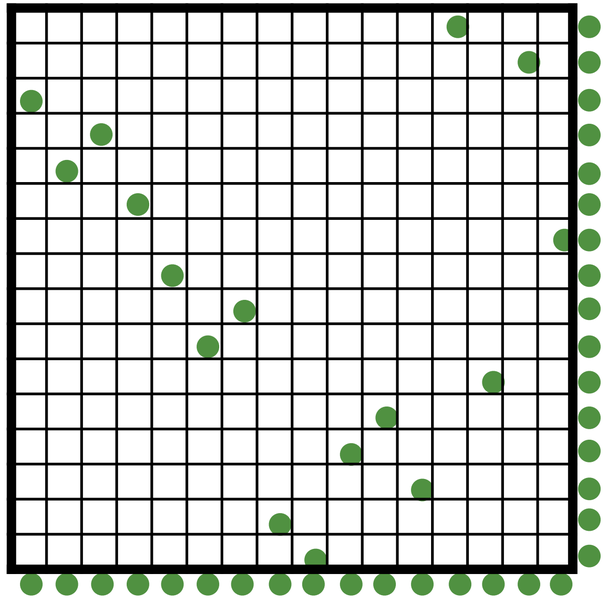
Multi-Jitter



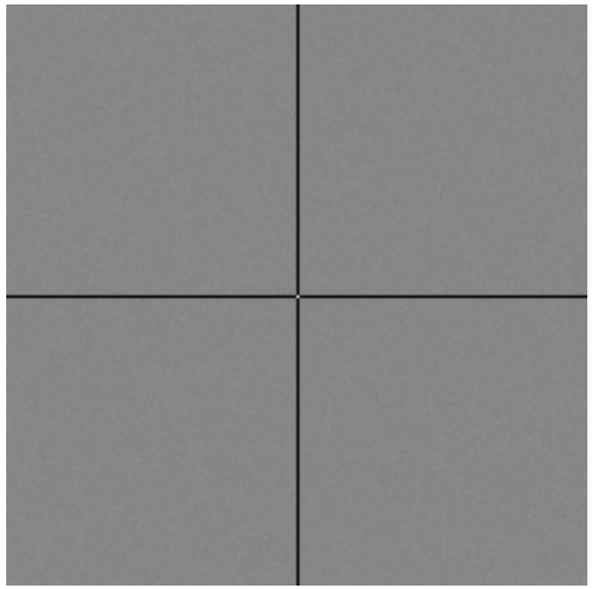
Multi-Jitter
Spectrum

Chiu et al. [1993]

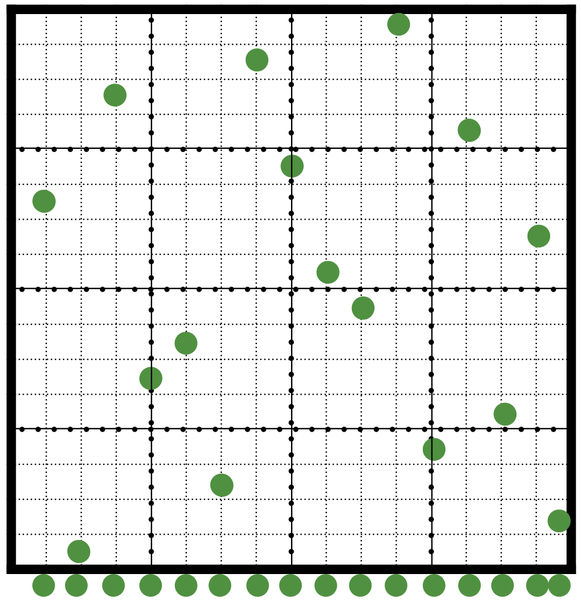
Anisotropic Sampling Power Spectra



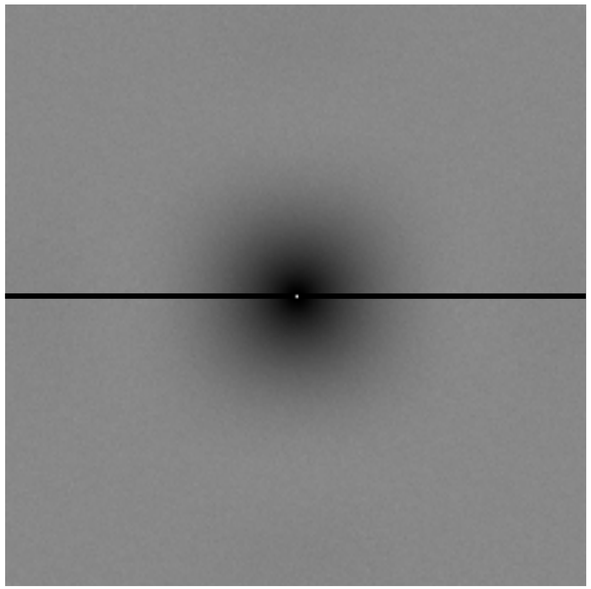
N-rooks /
Latin Hypercube



N-rooks
Spectrum



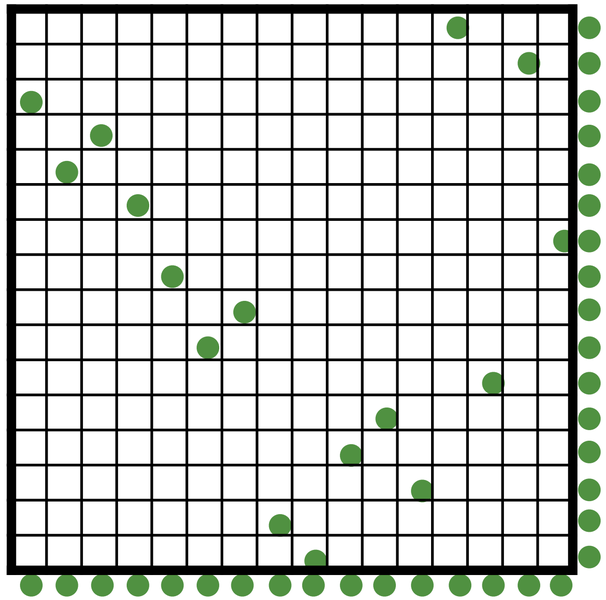
Multi-jitter



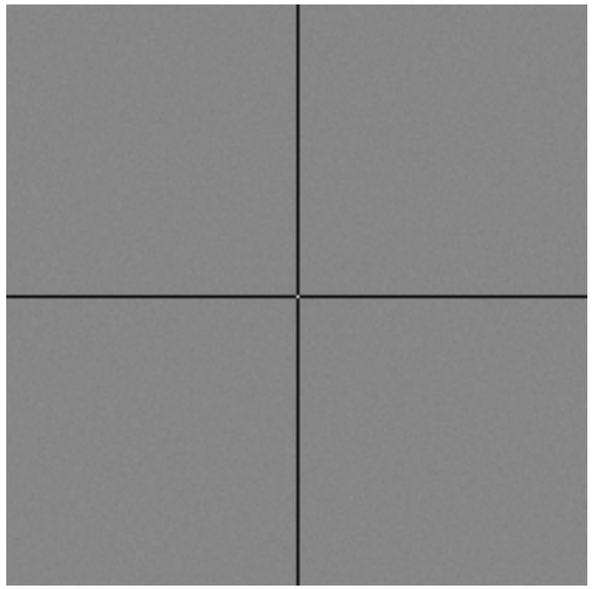
Multi-Jitter
Spectrum

Chiu et al. [1993]

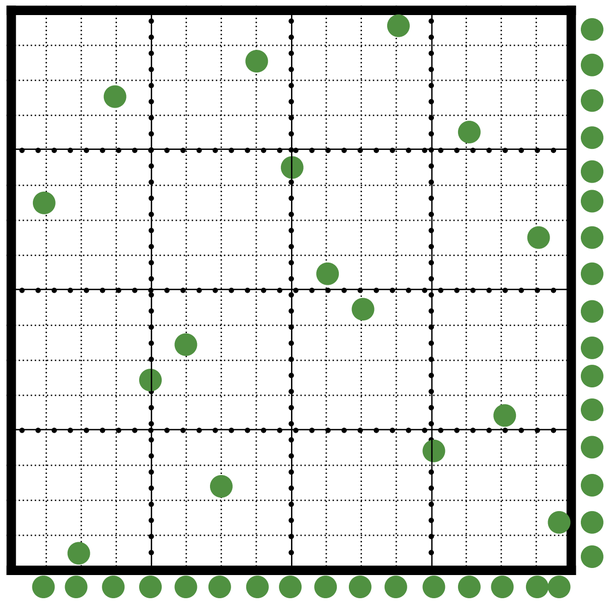
Anisotropic Sampling Power Spectra



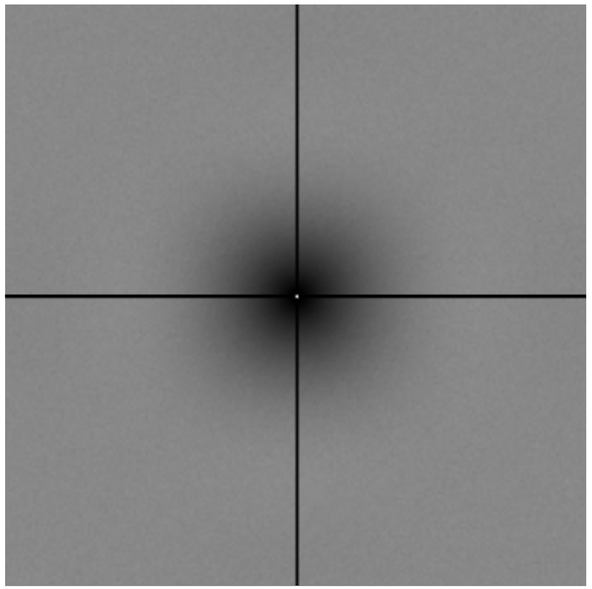
N-rooks /
Latin Hypercube



N-rooks
Spectrum



Multi-jitter

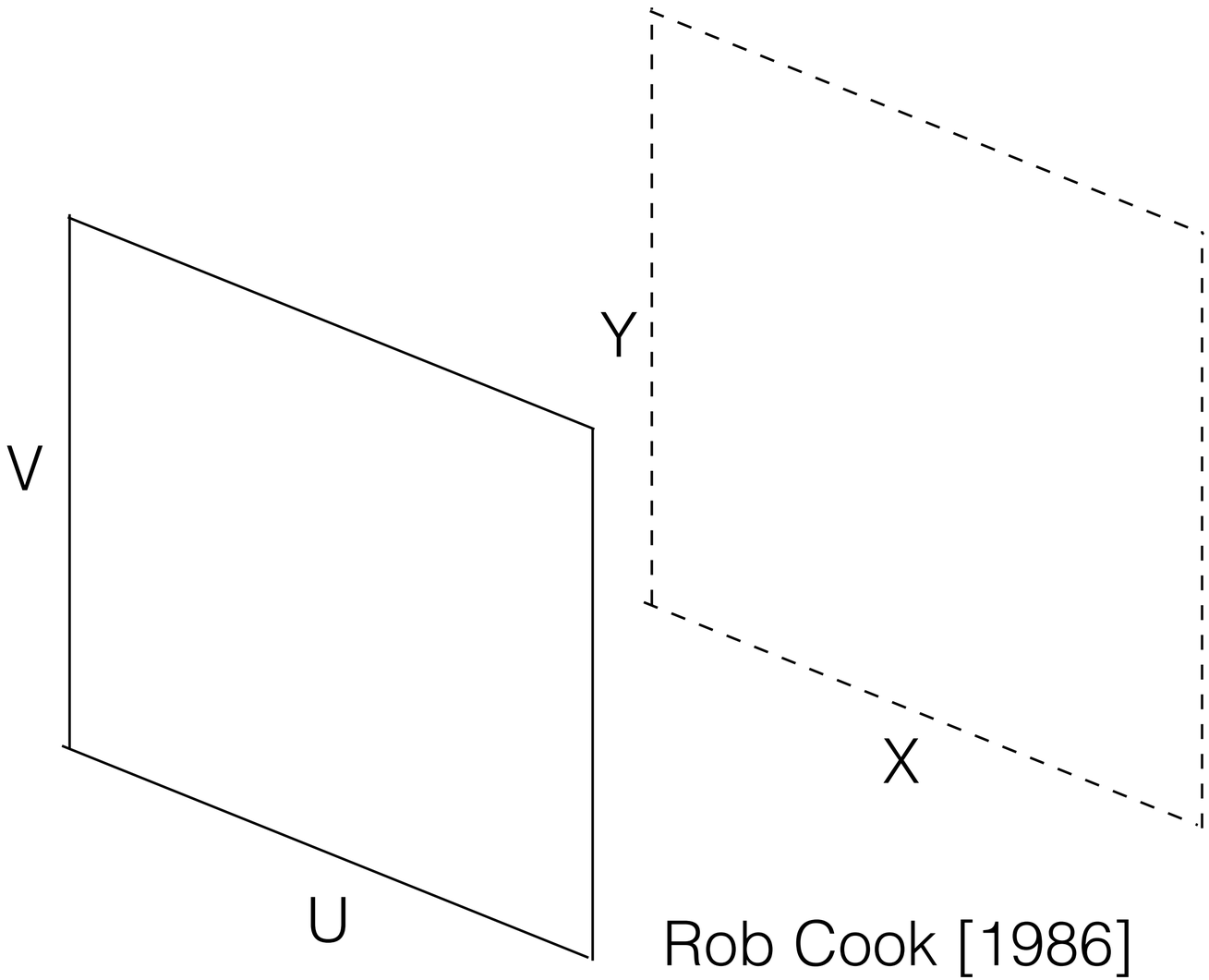


Multi-Jitter
Spectrum

Chiu et al. [1993]

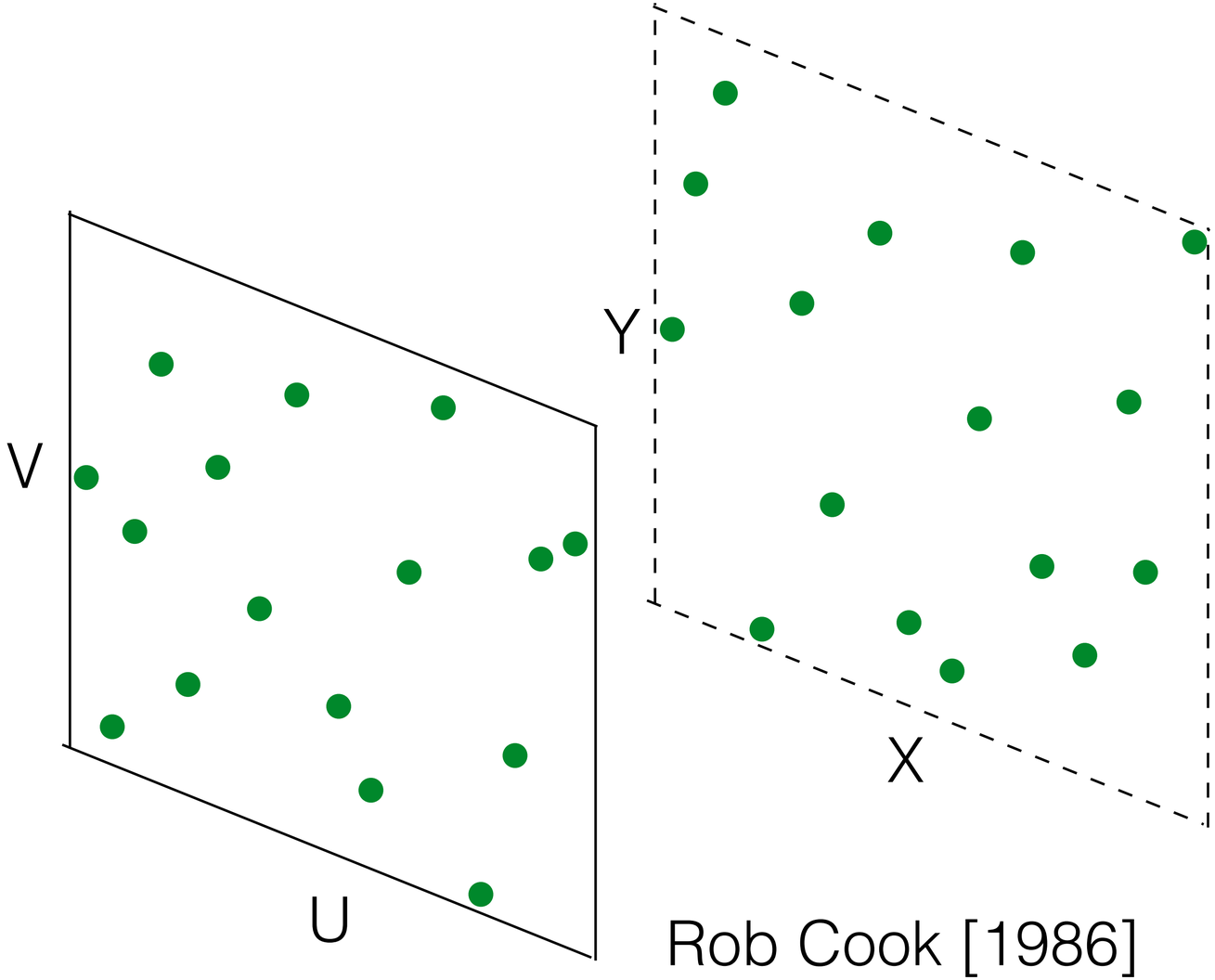
Sampling in Higher Dimensions

4D Sampling



Rob Cook [1986]

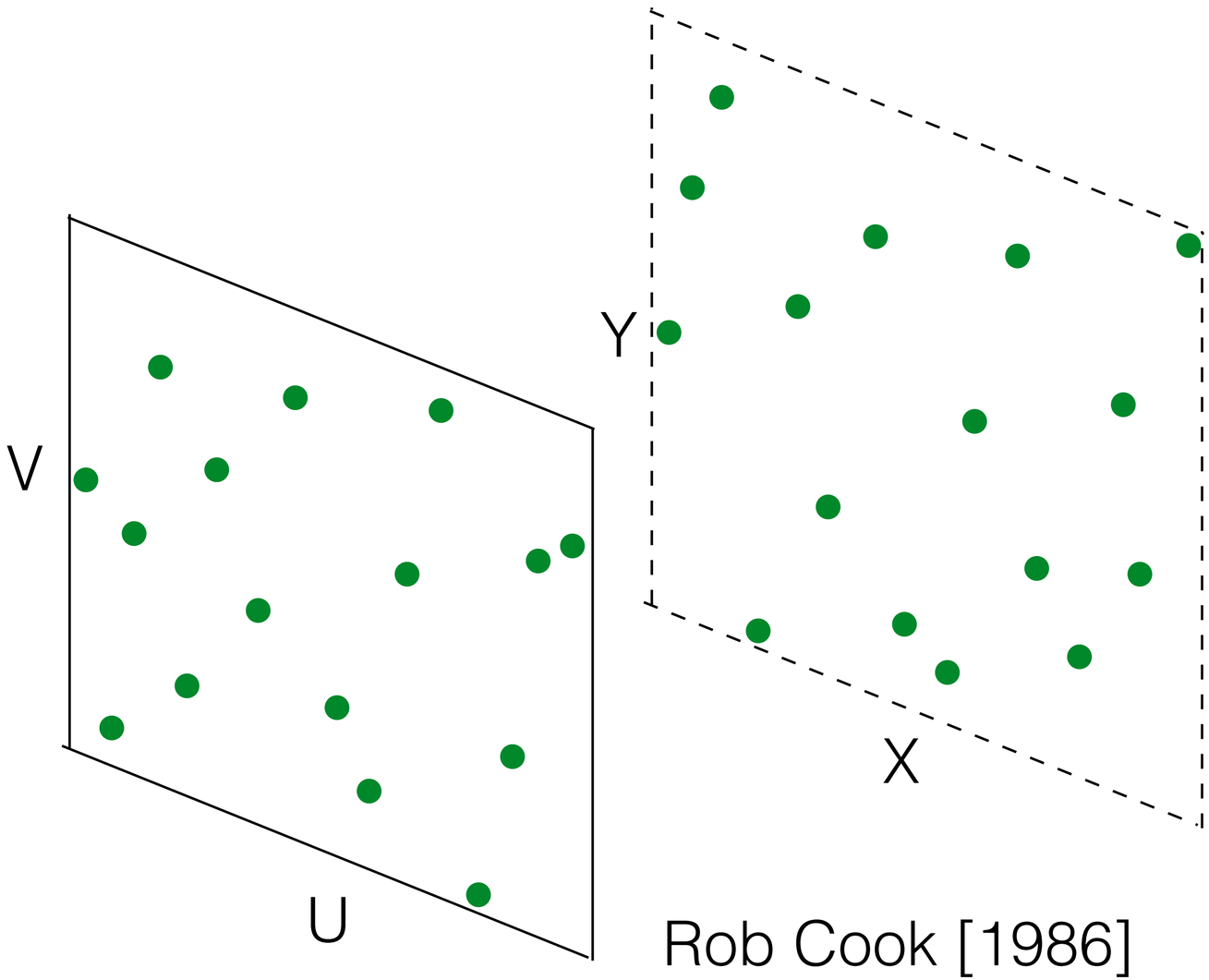
4D Sampling



2D	2D
(x_1, y_1)	(u_1, v_1)
(x_2, y_2)	(u_2, v_2)
(x_3, y_3)	(u_3, v_3)
(x_4, y_4)	(u_4, v_4)
$\vdots$	$\vdots$

Rob Cook [1986]

4D Sampling



Rob Cook [1986]

2D

(x_1, y_1)
(x_2, y_2)
(x_3, y_3)
(x_4, y_4)
$\vdots$

2D

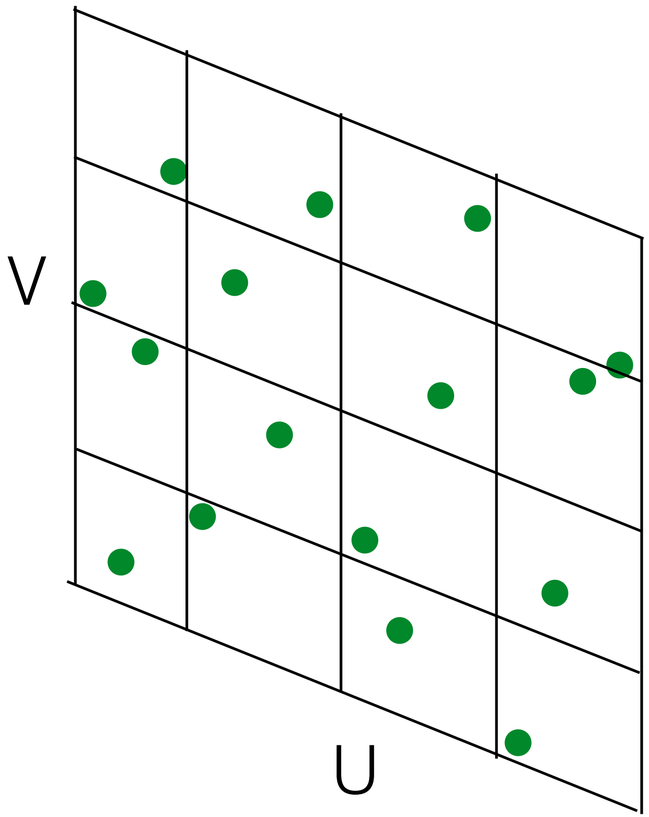
(u_1, v_1)
(u_2, v_2)
(u_3, v_3)
(u_4, v_4)
$\vdots$

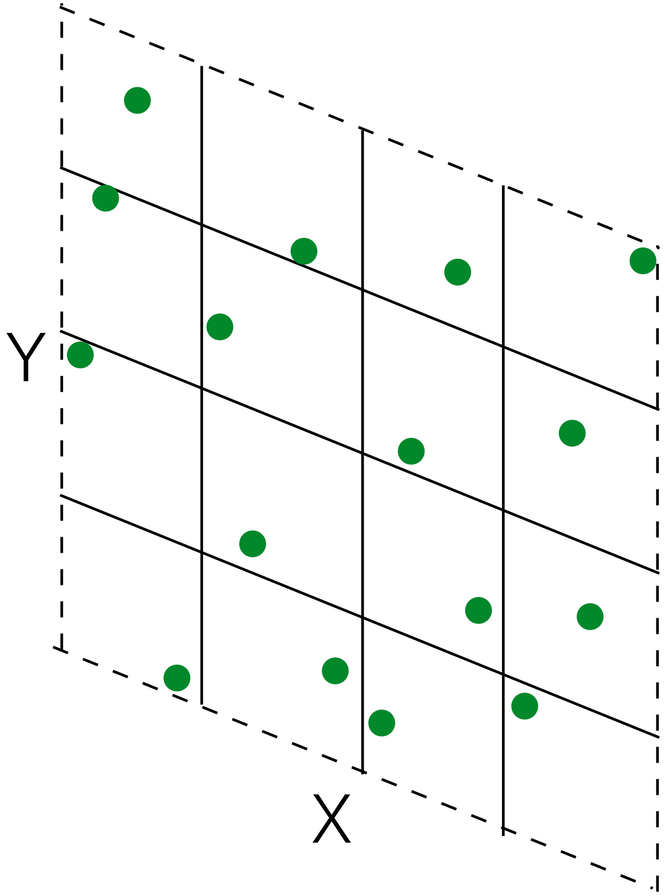
4D

(x_1, y_1, u_3, v_3)
(x_2, y_2, u_1, v_1)
(x_3, y_3, u_4, v_4)
(x_4, y_4, u_2, v_2)
$\vdots$

4D Sampling

Uncorrelated Jitter





Rob Cook [1986]

2D

(x_1, y_1)
(x_2, y_2)
(x_3, y_3)
(x_4, y_4)
$\vdots$

2D

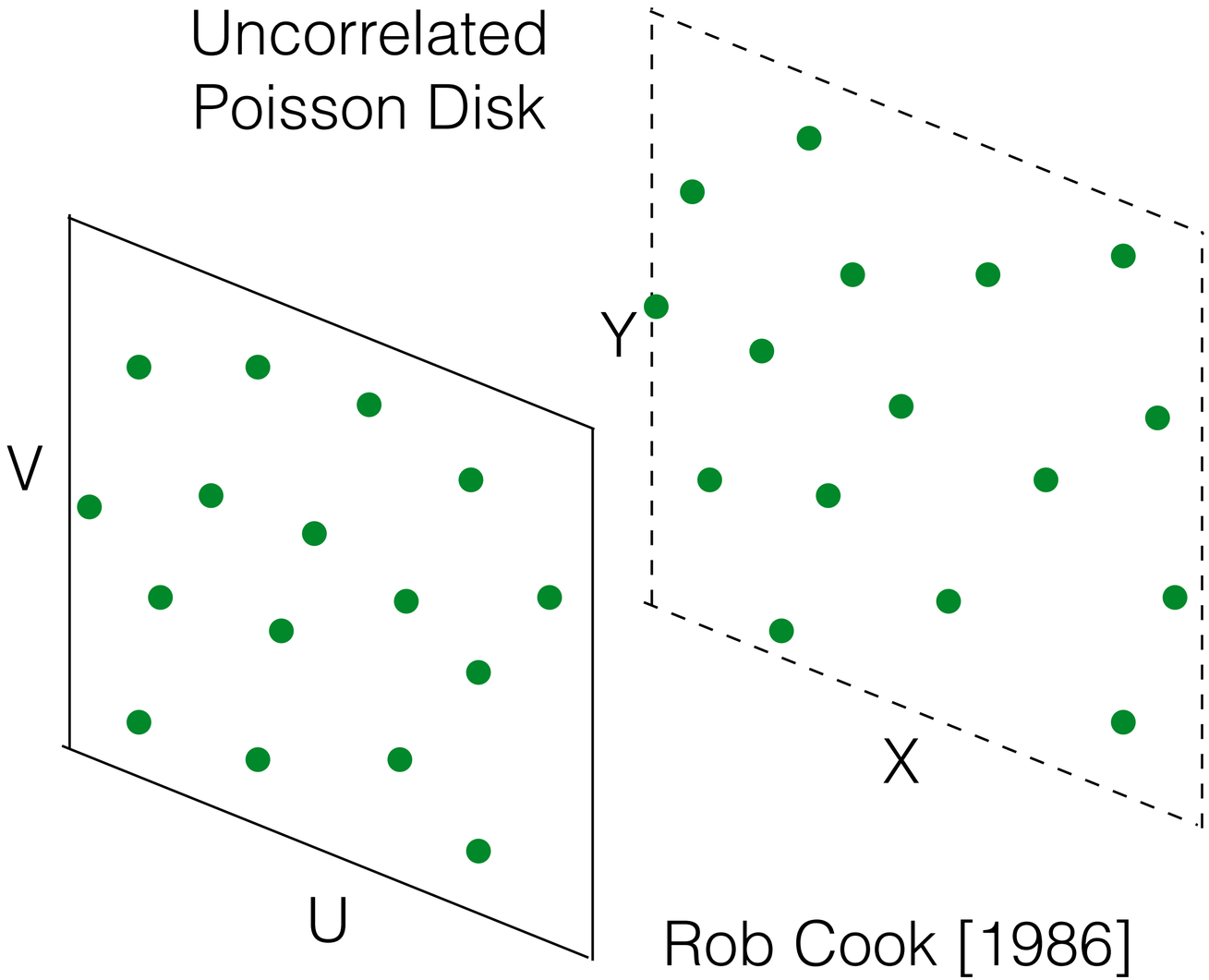
(u_1, v_1)
(u_2, v_2)
(u_3, v_3)
(u_4, v_4)
$\vdots$

4D

(x_1, y_1, u_3, v_3)
(x_2, y_2, u_1, v_1)
(x_3, y_3, u_4, v_4)
(x_4, y_4, u_2, v_2)
$\vdots$

4D Sampling

Uncorrelated
Poisson Disk



Rob Cook [1986]

2D

(x_1, y_1)
(x_2, y_2)
(x_3, y_3)
(x_4, y_4)
$\vdots$

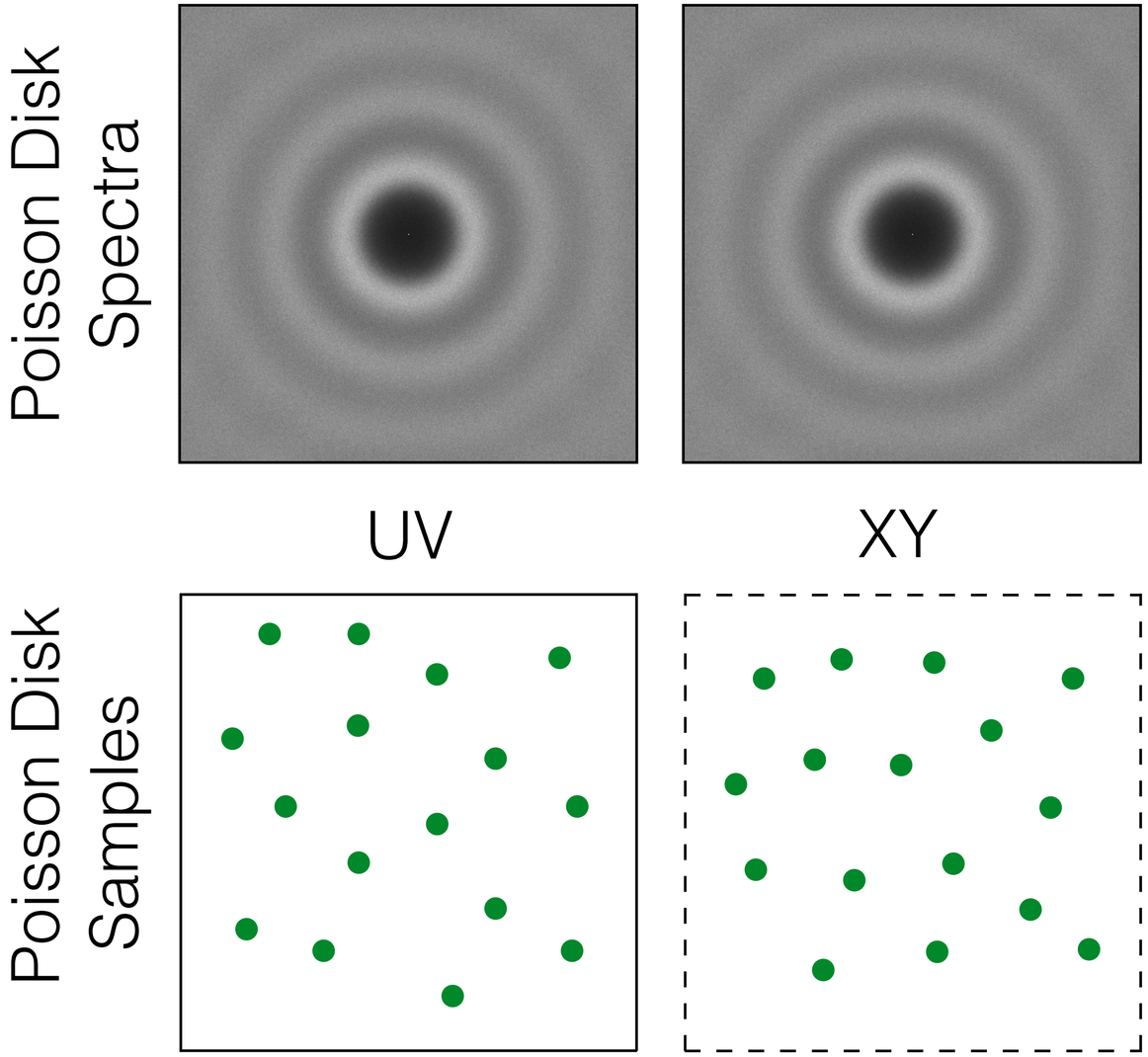
2D

(u_1, v_1)
(u_2, v_2)
(u_3, v_3)
(u_4, v_4)
$\vdots$

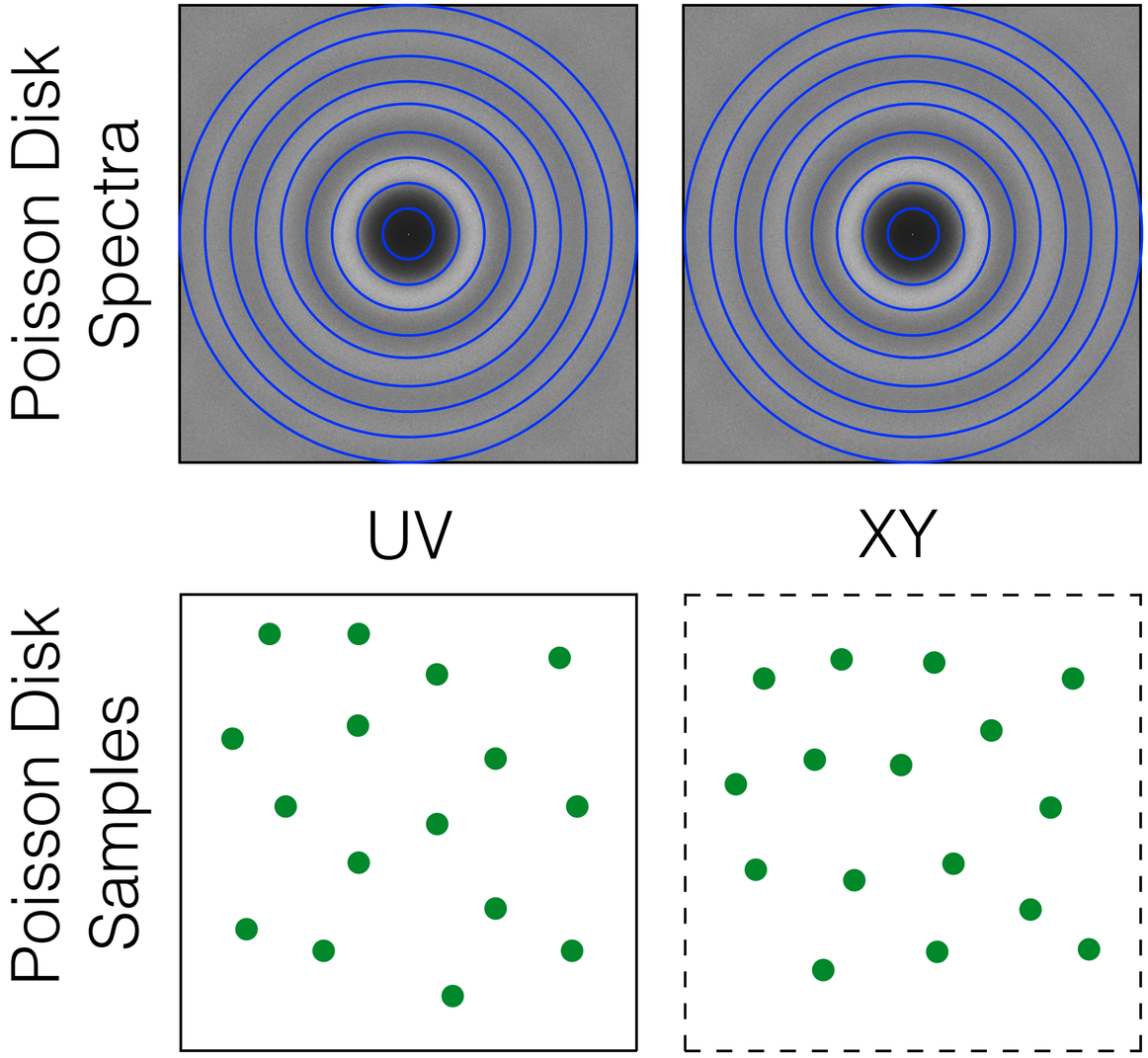
4D

(x_1, y_1, u_3, v_3)
(x_2, y_2, u_1, v_1)
(x_3, y_3, u_4, v_4)
(x_4, y_4, u_2, v_2)
$\vdots$

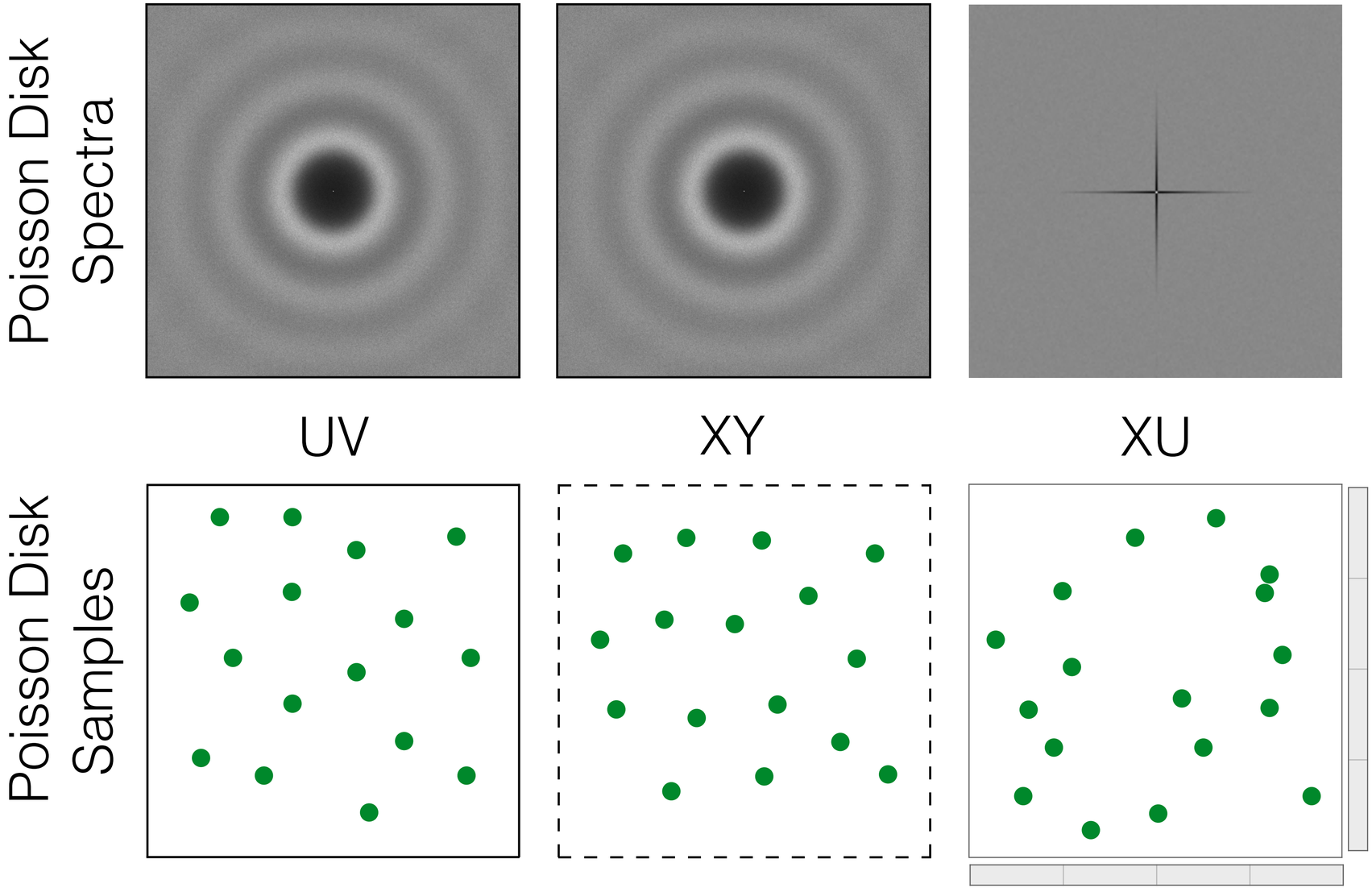
4D Sampling Spectra along Projections



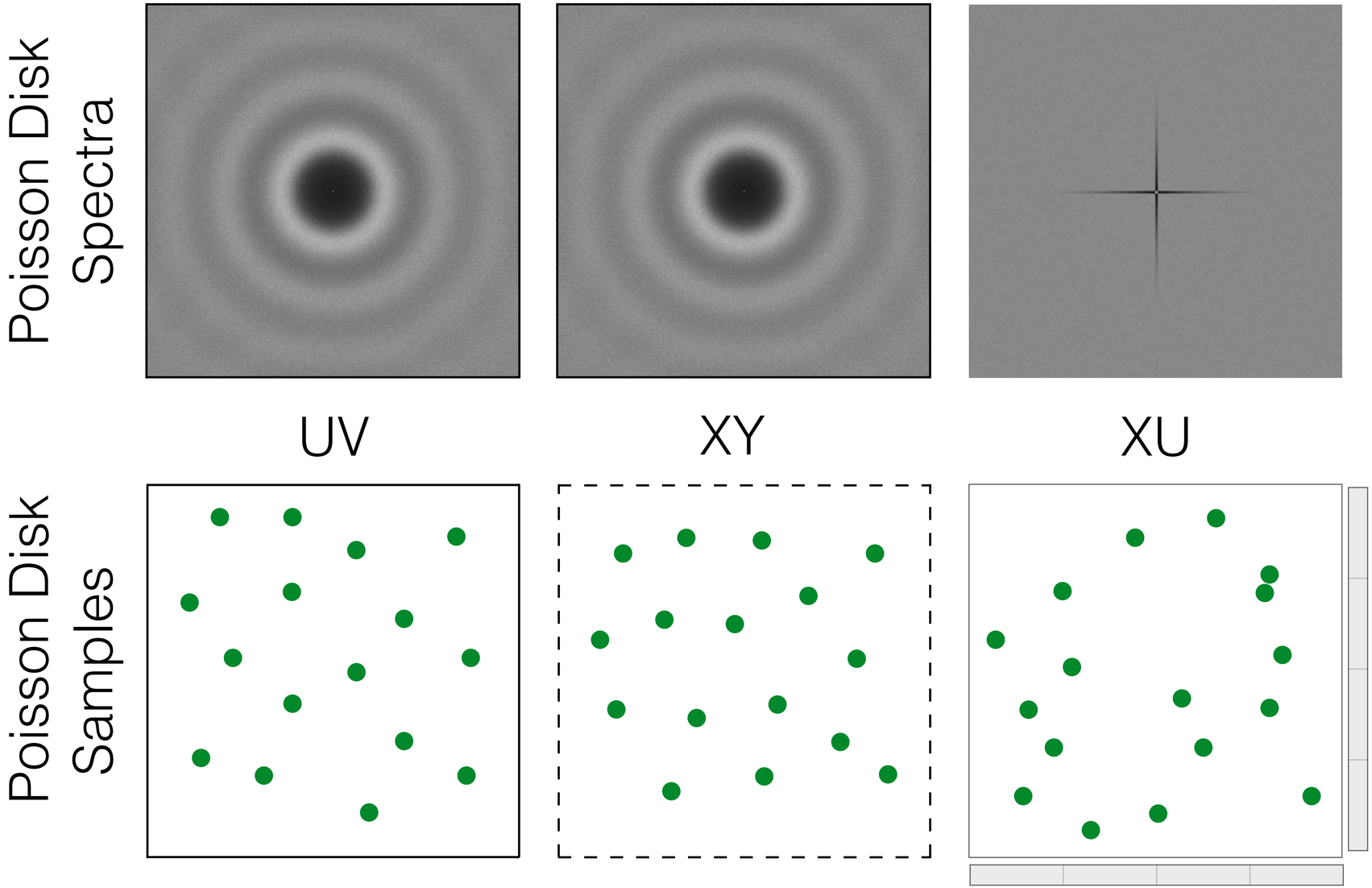
4D Sampling Spectra along Projections



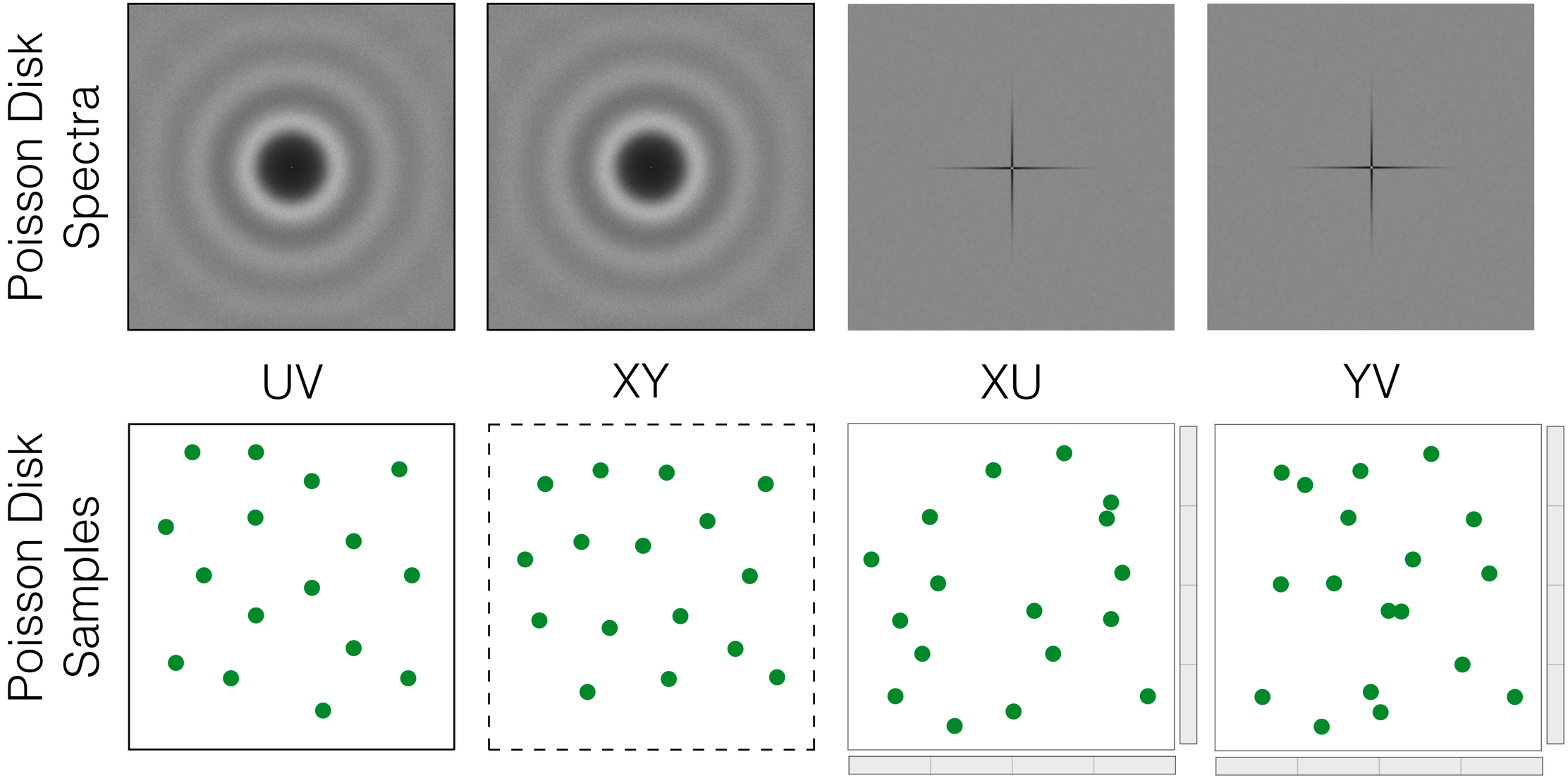
4D Sampling Spectra along Projections



4D Sampling Spectra along Projections



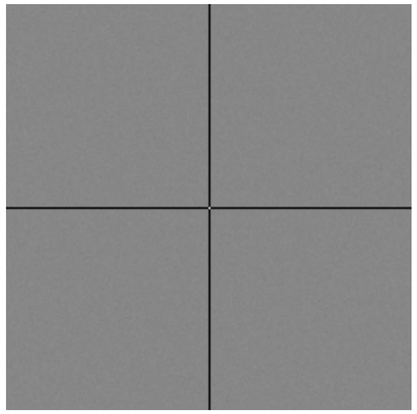
4D Sampling Spectra along Projections



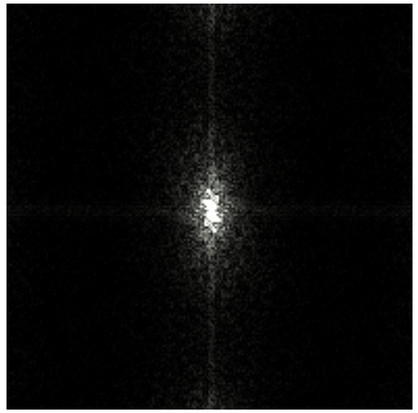
How can we perform Convergence Analysis for Anisotropic Sampling Spectra ?

Variance Formulation for Anisotropic Sampling Spectra

$$\mathrm{Var}(I_N) = \int_\Omega \langle \mathcal{P}_{S_N}(\nu) \rangle \times \mathcal{P}_f(\nu)\, d\nu$$



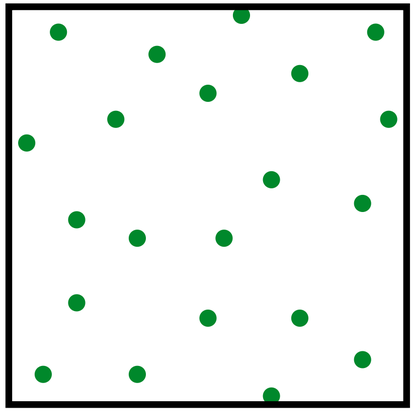
N-rooks spectrum



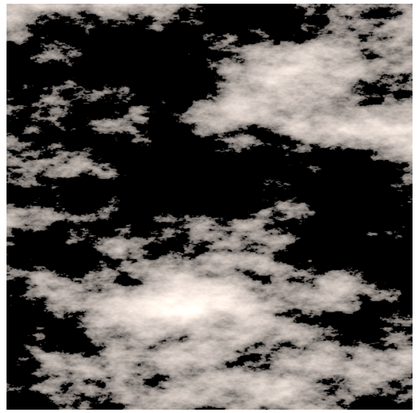
Integrand spectrum

$S_N(\vec{x})$

N-rooks

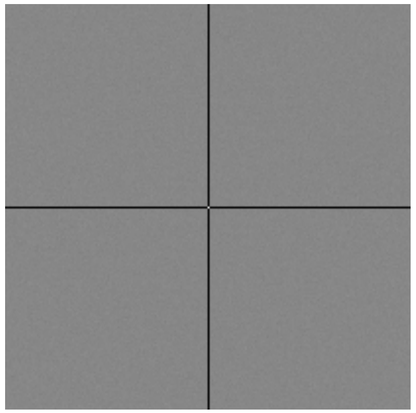


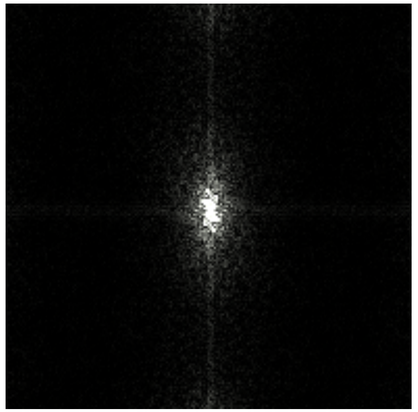
$f(\vec{x})$



Variance Formulation for Anisotropic Sampling Spectra

$$\mathrm{Var}(I_N) = \int_0^\infty \rho^{d-1} \int_{\mathcal{S}^{d-1}} \Big\langle \mathcal{P}_{S_N}(\rho\mathbf{n}) \Big\rangle \times \mathcal{P}_f(\rho\mathbf{n}) \, d\mathbf{n} \, d\rho$$

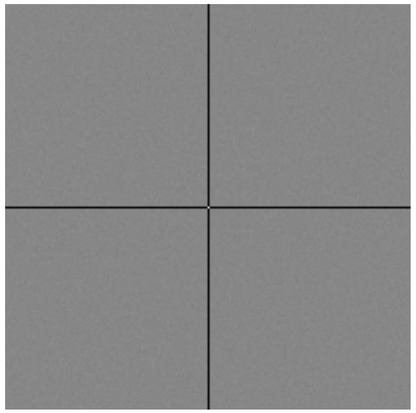




Variance Formulation for Anisotropic Sampling Spectra

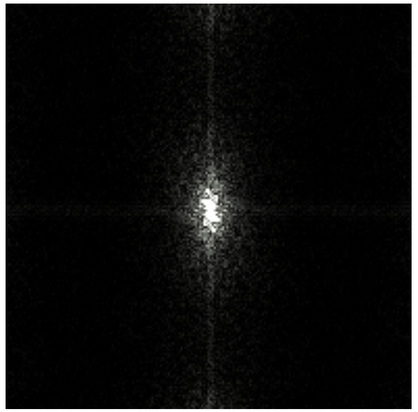
$$\mathrm{Var}(I_N) = \int_{\mathcal{S}^{d-1}} \int_0^\infty \rho^{d-1}$$

$\left\langle \mathcal{P}_{S_N}(\rho\mathbf{n}) \right\rangle$



$\times$

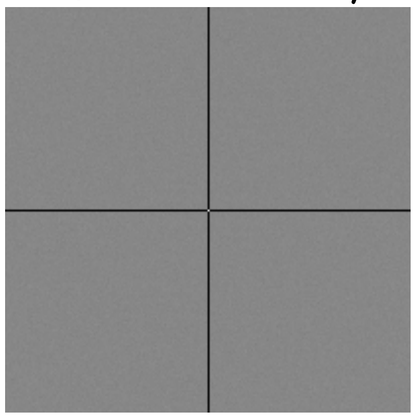
$\mathcal{P}_f(\rho\mathbf{n})$

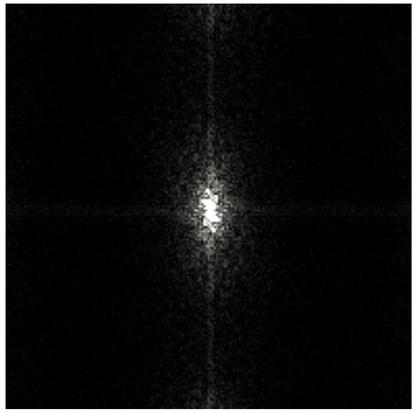


$$d\rho\, d\mathbf{n}$$

Variance Formulation for Anisotropic Sampling Spectra

$$\mathrm{Var}(I_N) = \int_{\mathcal{S}^{d-1}} \int_0^{\infty} \rho^{d-1} \left\langle \mathcal{P}_{S_N}(\rho_k \mathbf{n_k}) \right\rangle \times \mathcal{P}_f(\rho_k \mathbf{n}_k) \, d\rho \, d\mathbf{n}$$

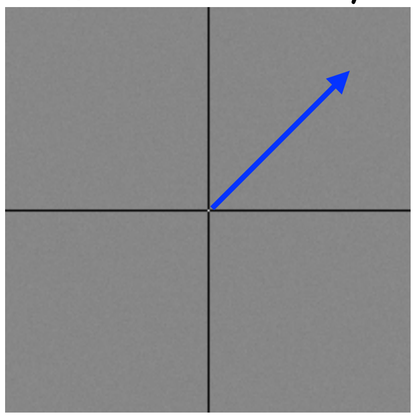




Variance Formulation for Anisotropic Sampling Spectra

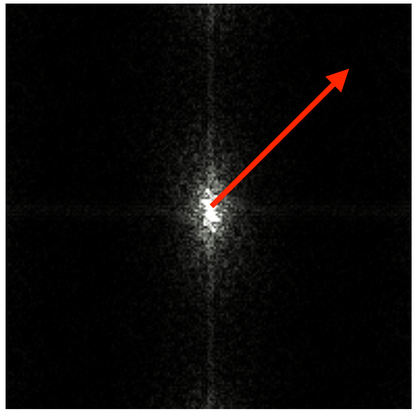
$$\mathrm{Var}(I_N) = \int_{\mathcal{S}^{d-1}} \int_0^\infty \rho^{d-1}$$

$\left\langle \mathcal{P}_{S_N}(\rho_k \mathbf{n_k}) \right\rangle$



$\times$

$\mathcal{P}_f(\rho_k \mathbf{n}_k)$

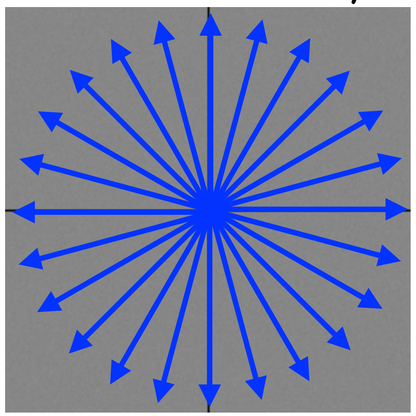


$$d\rho\, d\mathbf{n}$$

Variance Formulation for Anisotropic Sampling Spectra

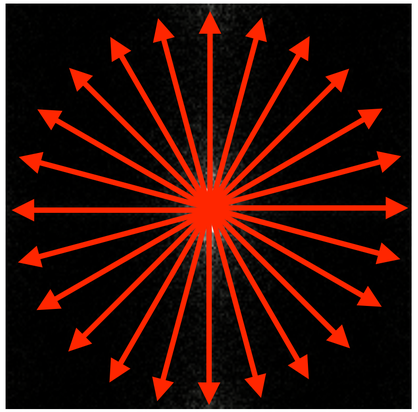
$$\mathrm{Var}(I_N) = \int_{\mathcal{S}^{d-1}} \int_0^{\infty} \rho^{d-1}$$

$\left\langle \mathcal{P}_{S_N}(\rho_k \mathbf{n_k}) \right\rangle$



$\times$

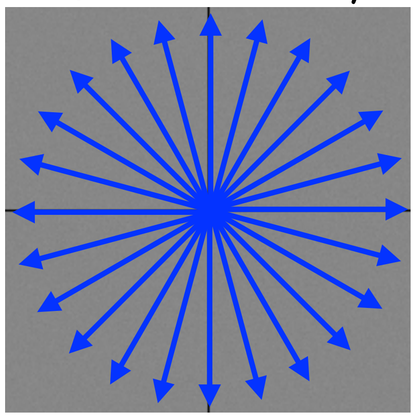
$\mathcal{P}_f(\rho_k \mathbf{n}_k)$

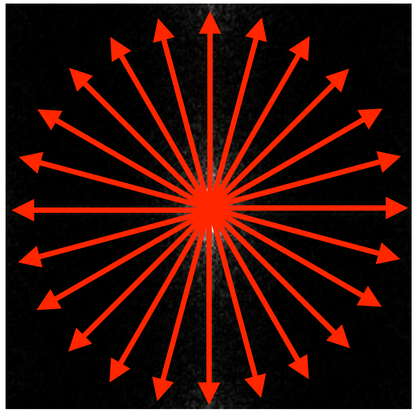


$d\rho\, d\mathbf{n}$

Variance Formulation for Anisotropic Sampling Spectra

$$\mathrm{Var}(I_N) = \lim_{m\to\infty} \sum_{k=1}^{m} \int_0^\infty \rho^{d-1} \left\langle \mathcal{P}_{S_N}(\rho_k \mathbf{n_k}) \right\rangle \times \mathcal{P}_f(\rho_k \mathbf{n}_k)\, d\rho\, \Delta\mathbf{n}_k$$





Variance Formulation for Anisotropic Sampling Spectra

$$\mathrm{Var}(I_N) = \lim_{m\to\infty} \sum_{k=1}^{m} \int_0^\infty \rho^{d-1} \quad \left\langle \mathcal{P}_{S_N}(\rho_k \mathbf{n_k}) \right\rangle \quad \times \quad \mathcal{P}_f(\rho_k \mathbf{n}_k) \quad d\rho \, \Delta \mathbf{n}_k$$

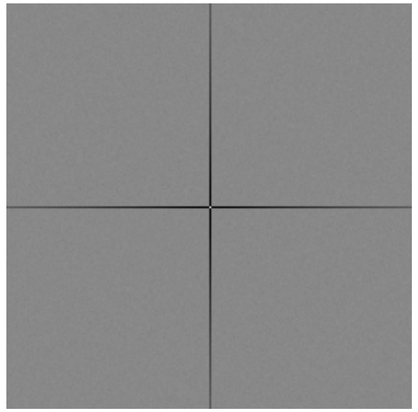
Variance Formulation for Anisotropic Sampling Spectra

$$\mathrm{Var}(I_N) = \lim_{m\to\infty} \sum_{k=1}^{m} \int_0^\infty \rho^{d-1} \left\langle \mathcal{P}_{S_N}(\rho_k \mathbf{n_k}) \right\rangle \mathcal{P}_f(\rho_k \mathbf{n}_k)\, d\rho\, \Delta \mathbf{n}_k$$

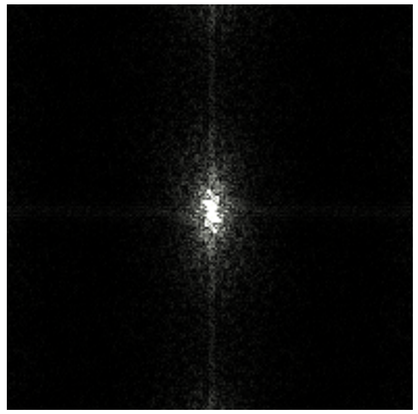
Variance Formulation for Anisotropic Sampling Spectra

$$\mathrm{Var}(I_N) = \lim_{m\to\infty} \sum_{k=1}^{m} \boxed{\int_0^\infty \rho^{d-1} \left\langle \mathcal{P}_{S_N}(\rho_k \mathbf{n_k}) \right\rangle \mathcal{P}_f(\rho_k \mathbf{n}_k)\, d\rho} \Delta \mathbf{n}_k$$

$\left\langle \mathcal{P}_{S_N}(\rho_k \mathbf{n_k}) \right\rangle$



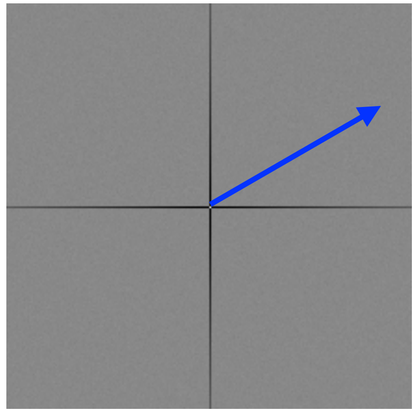
$\mathcal{P}_f(\rho_k \mathbf{n}_k)$



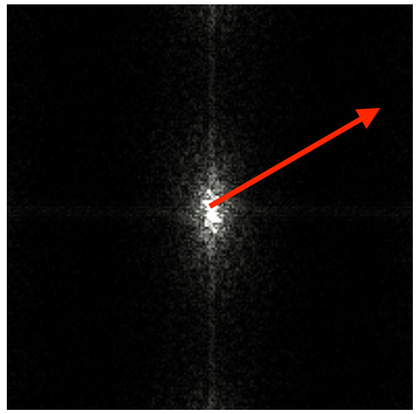
Variance Formulation for Anisotropic Sampling Spectra

$$\mathrm{Var}(I_N) = \lim_{m\to\infty} \sum_{k=1}^{m} \boxed{\int_0^\infty \rho^{d-1} \left\langle \mathcal{P}_{S_N}(\rho_k \mathbf{n_k}) \right\rangle \mathcal{P}_f(\rho_k \mathbf{n}_k)\, d\rho} \Delta \mathbf{n}_k$$

$\left\langle \mathcal{P}_{S_N}(\rho_k \mathbf{n_k}) \right\rangle$



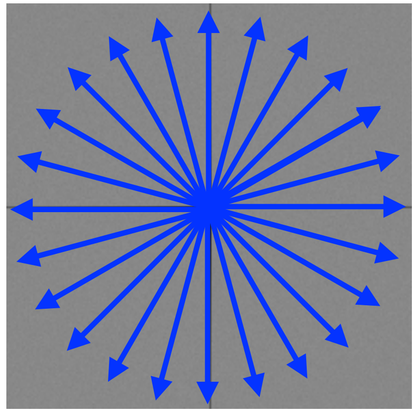
$\mathcal{P}_f(\rho_k \mathbf{n}_k)$



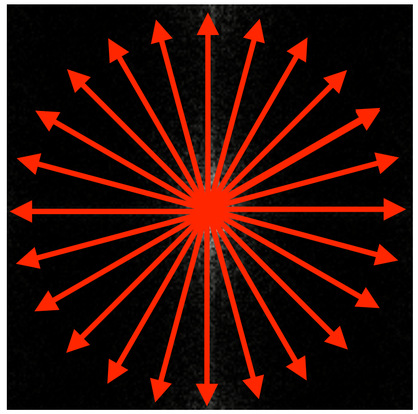
Variance Formulation for Anisotropic Sampling Spectra

$$\mathrm{Var}(I_N) = \lim_{m\to\infty} \sum_{k=1}^{m} \boxed{\int_0^\infty \rho^{d-1} \left\langle \mathcal{P}_{S_N}(\rho_k \mathbf{n_k}) \right\rangle \mathcal{P}_f(\rho_k \mathbf{n}_k)\, d\rho} \Delta \mathbf{n}_k$$

$\left\langle \mathcal{P}_{S_N}(\rho_k \mathbf{n_k}) \right\rangle$



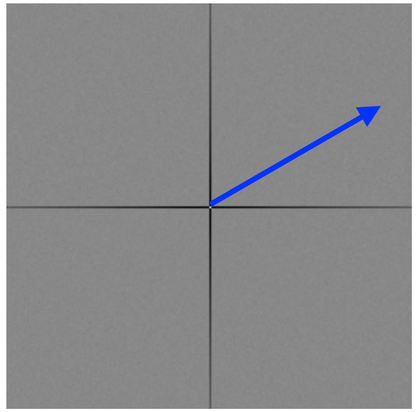
$\mathcal{P}_f(\rho_k \mathbf{n}_k)$



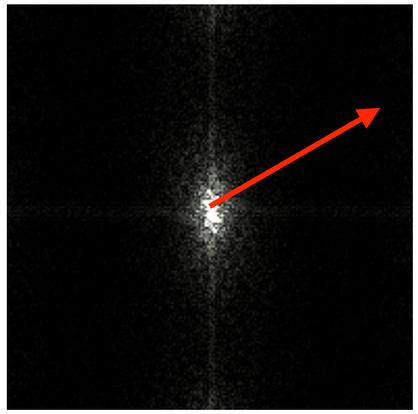
Variance Formulation for Anisotropic Sampling Spectra

$$\mathrm{Var}(I_N) = \lim_{m\to\infty} \sum_{k=1}^{m} \boxed{\int_0^\infty \rho^{d-1} \left\langle \mathcal{P}_{S_N}(\rho_k \mathbf{n_k}) \right\rangle \mathcal{P}_f(\rho_k \mathbf{n}_k)\, d\rho} \Delta \mathbf{n}_k$$

$\left\langle \mathcal{P}_{S_N}(\rho_k \mathbf{n_k}) \right\rangle$

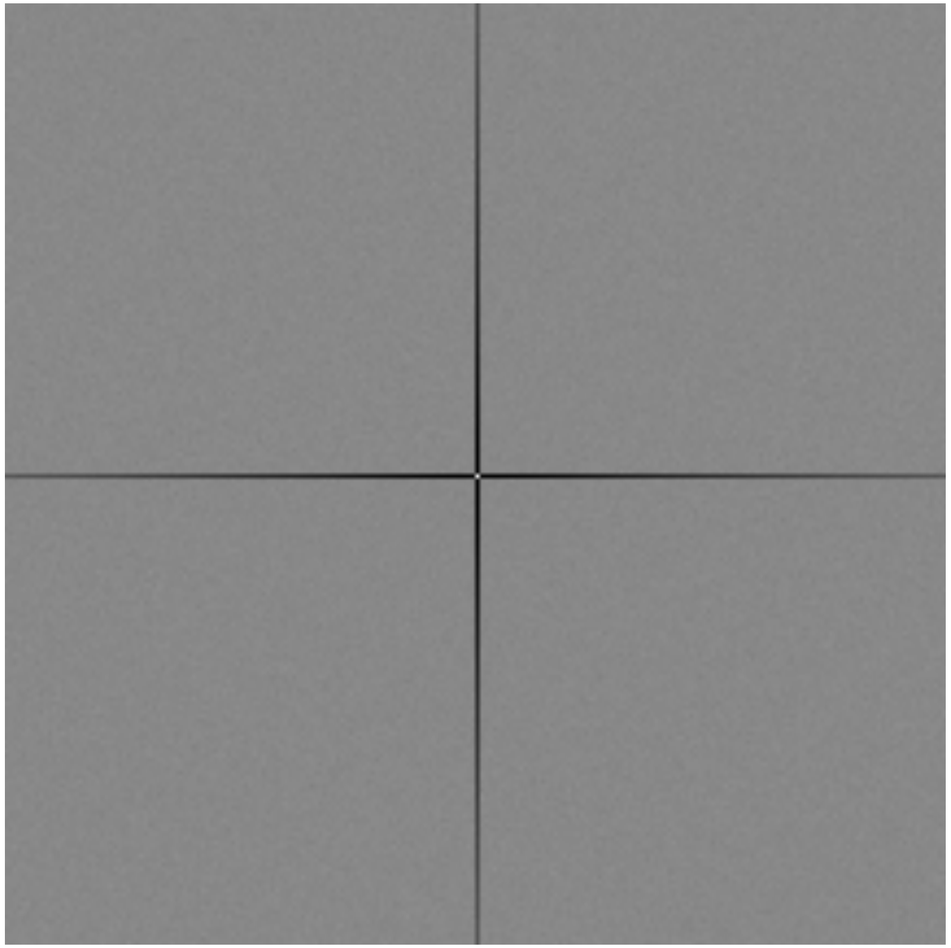


$\mathcal{P}_f(\rho_k \mathbf{n}_k)$



Convergence Analysis for Anisotropic Sampling Spectra

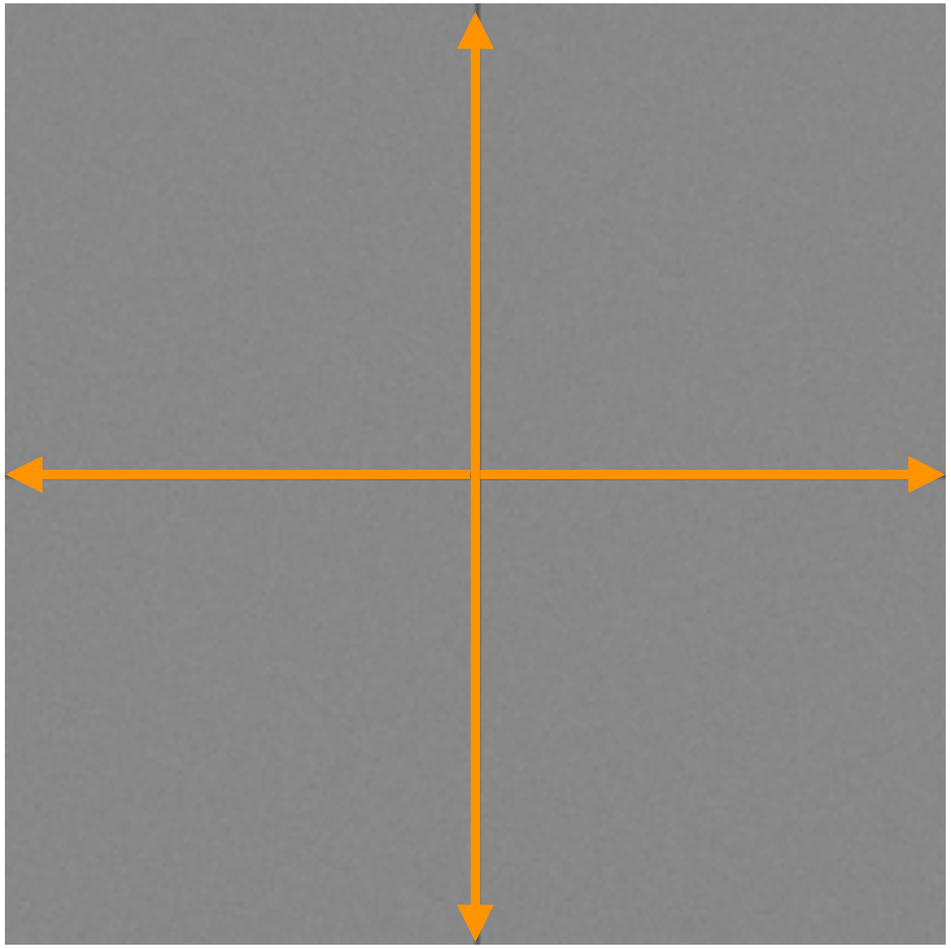
Power Spectrum



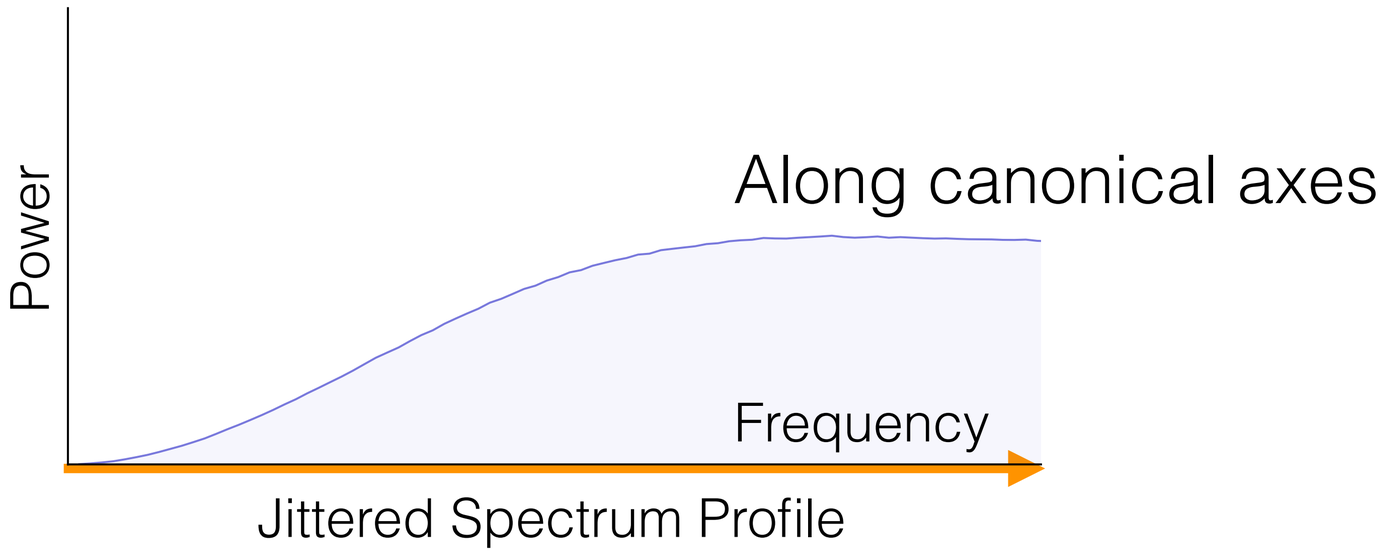
Radial Power Spectrum

Convergence Analysis for Anisotropic Sampling Spectra

Power Spectrum

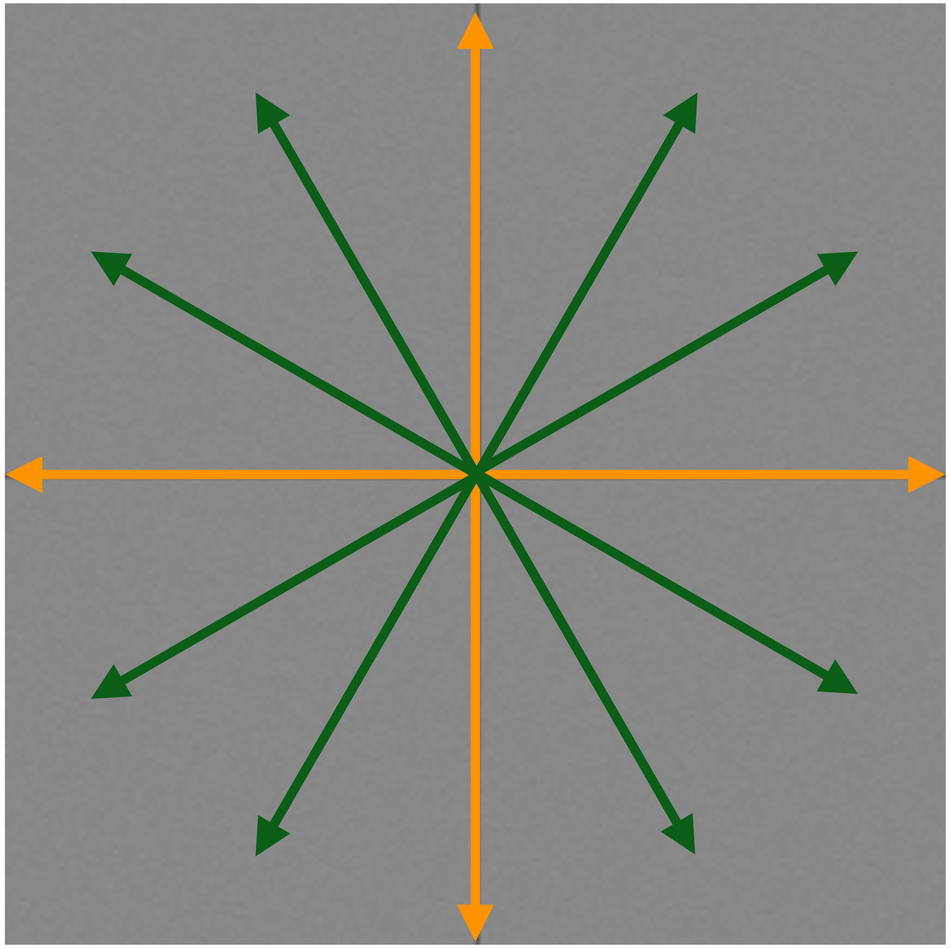


Radial Power Spectrum

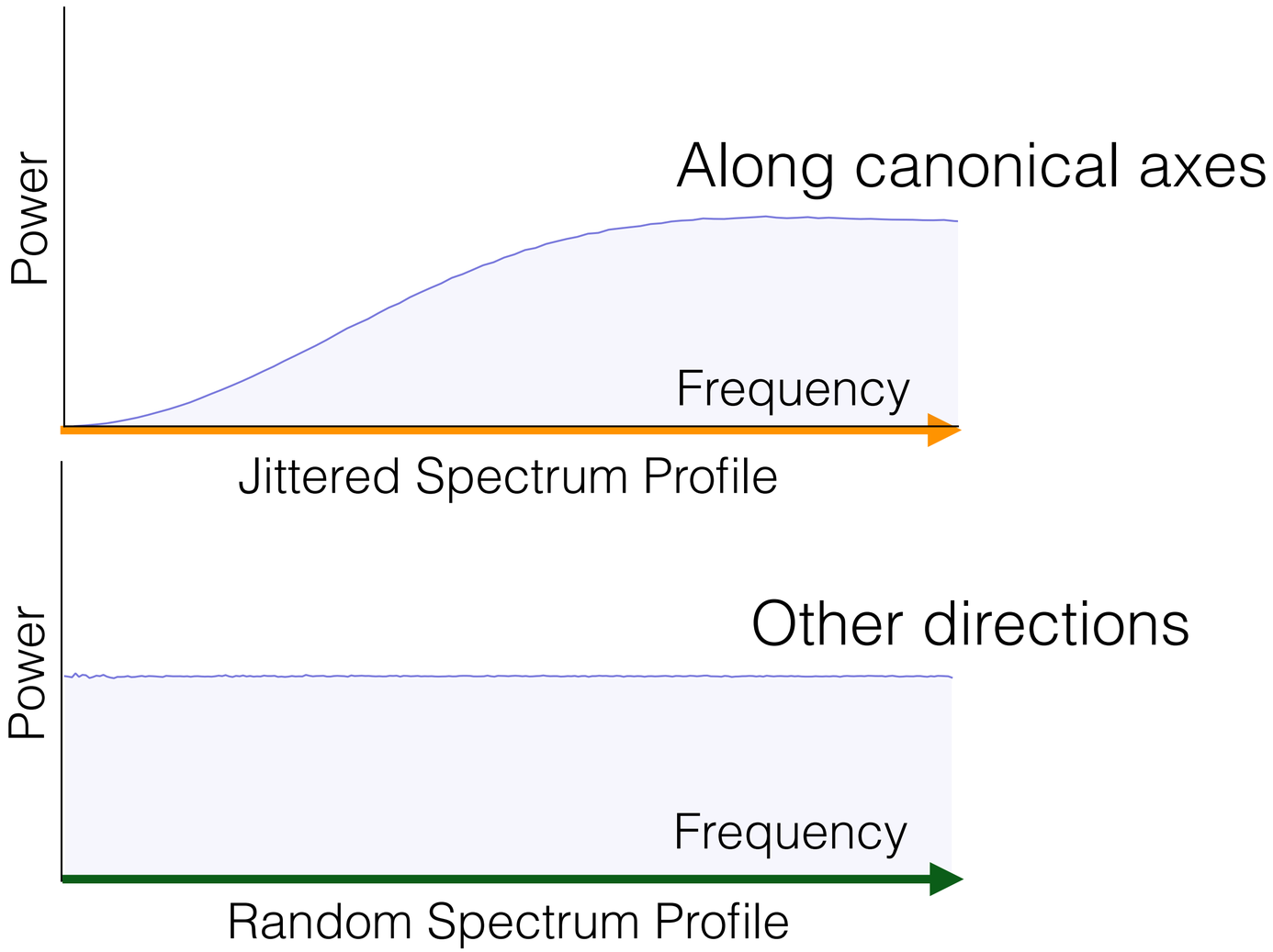


Convergence Analysis for Anisotropic Sampling Spectra

Power Spectrum

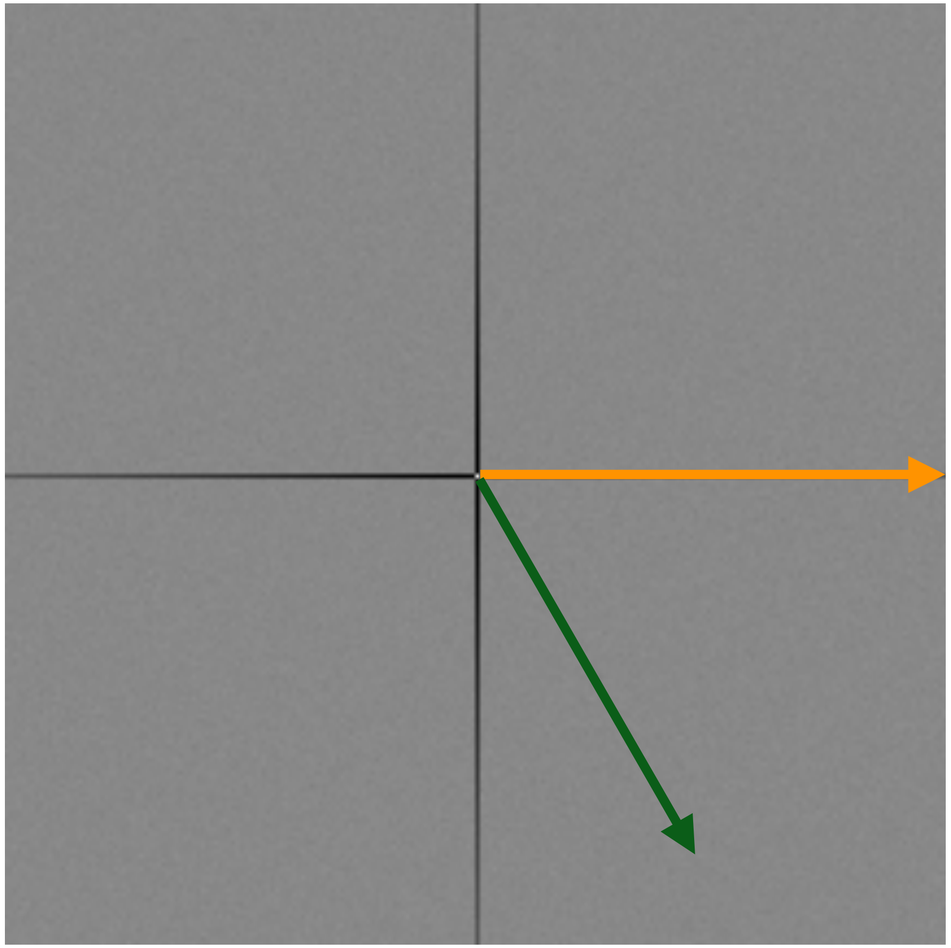


Radial Power Spectrum

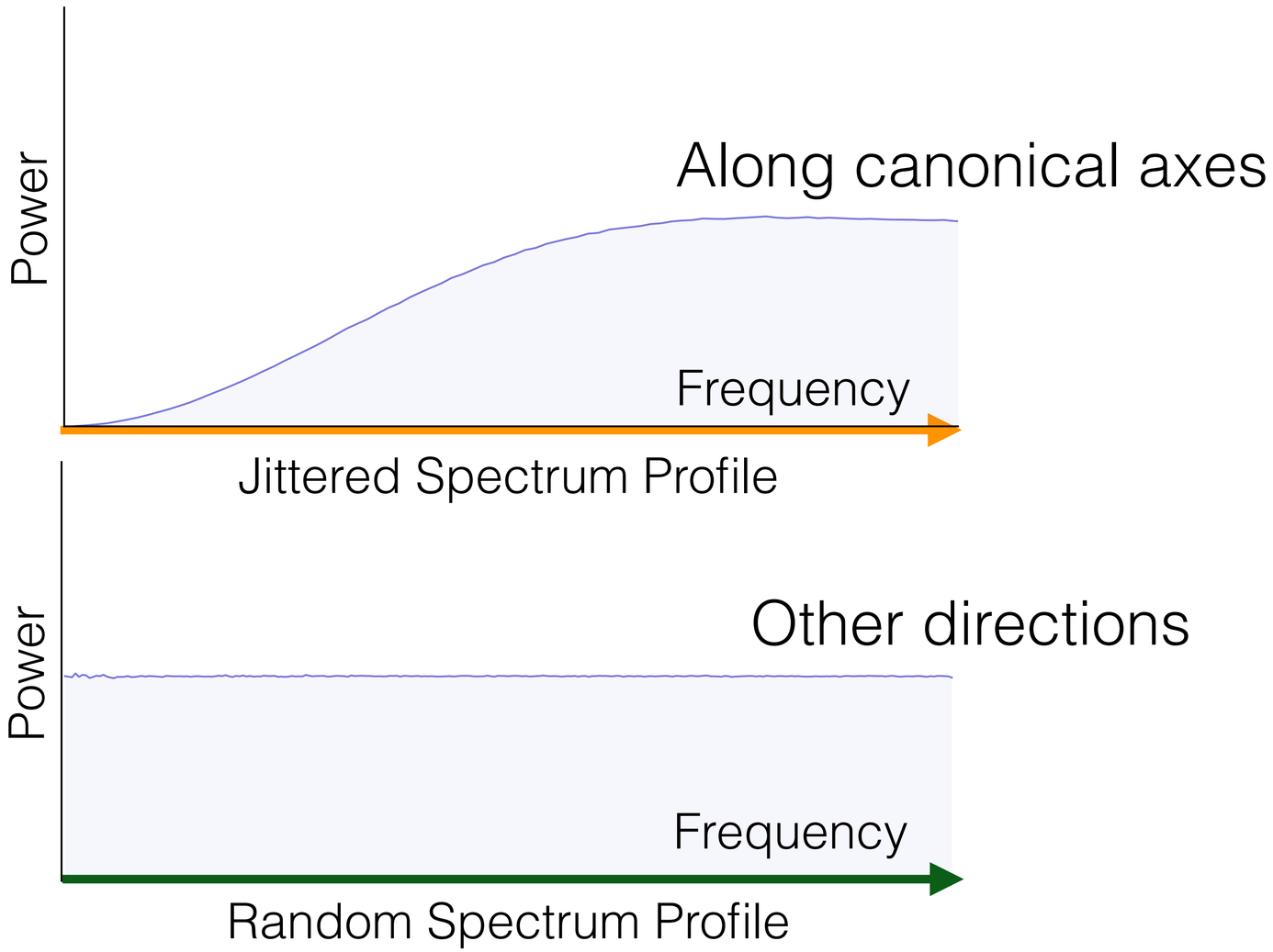


Convergence Analysis for Anisotropic Sampling Spectra

Power Spectrum

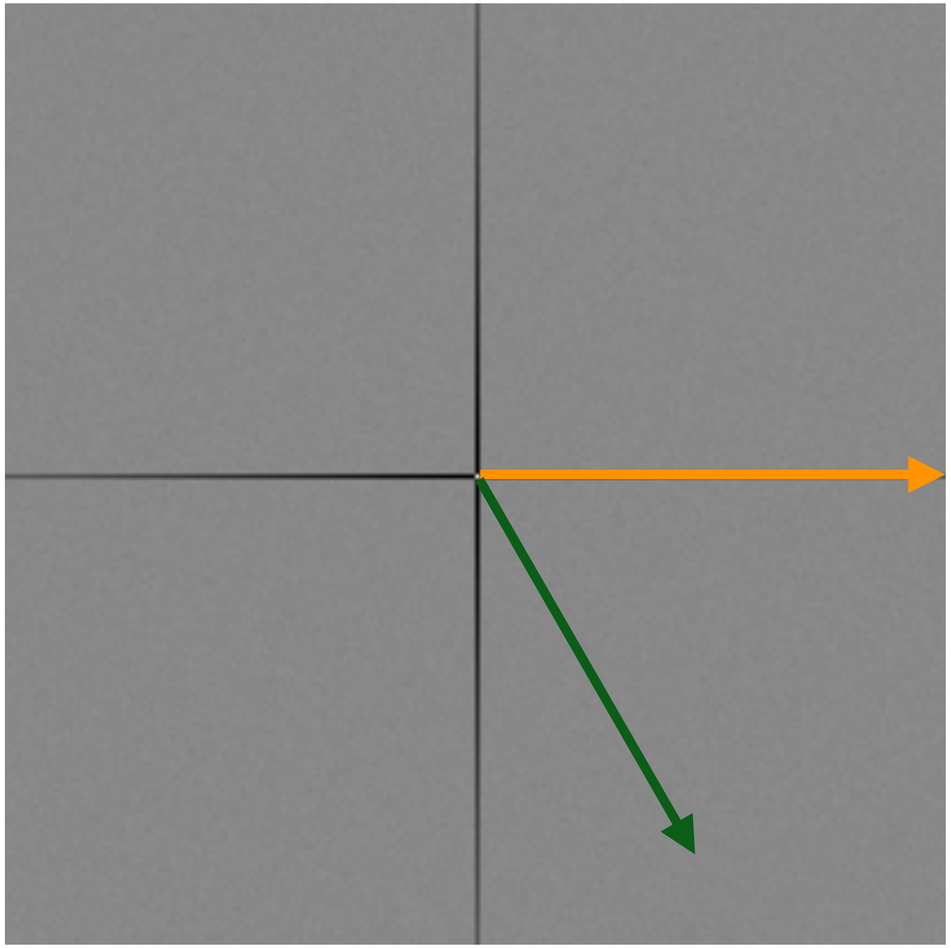


Radial Power Spectrum

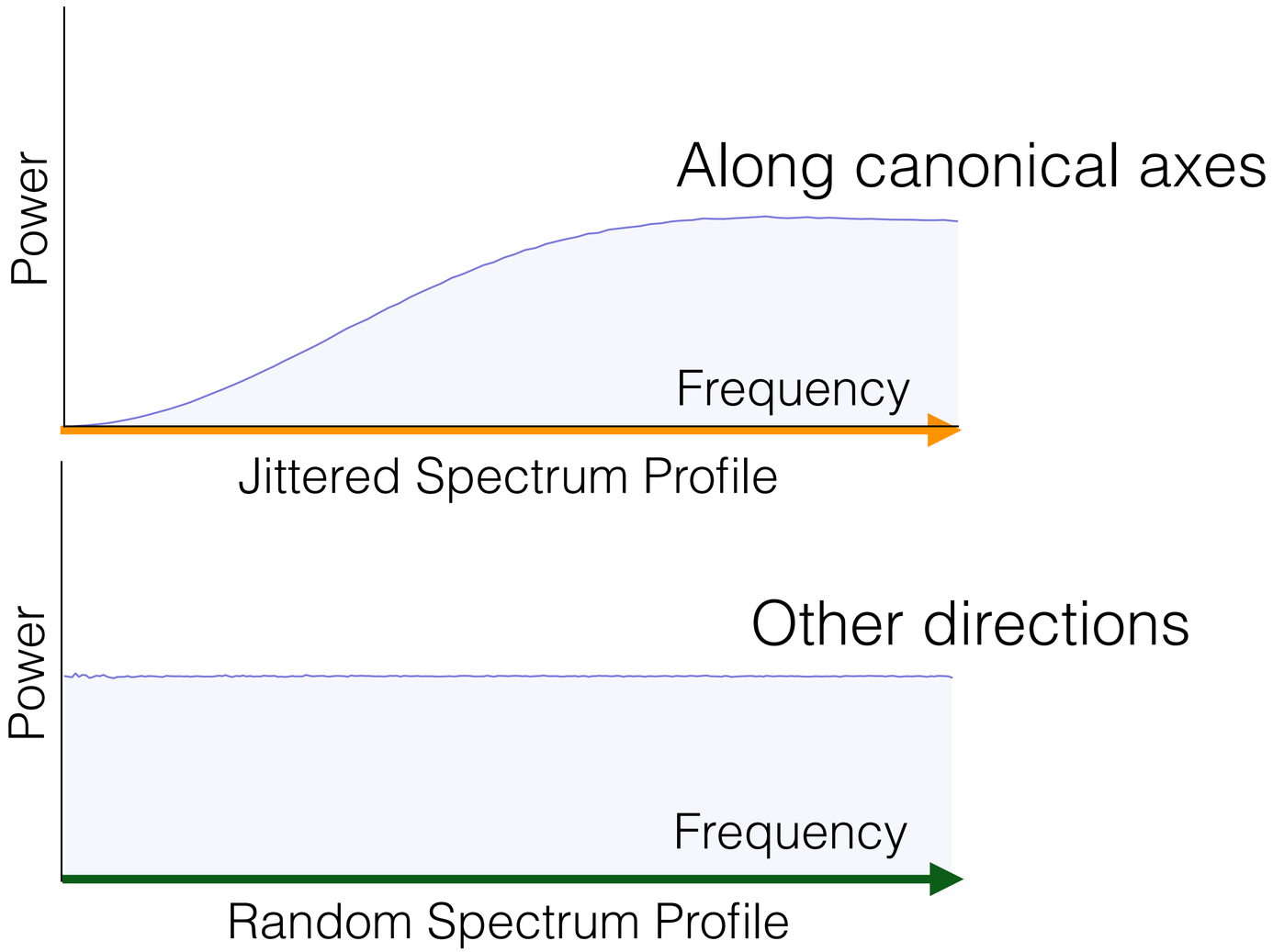


Convergence Analysis for Anisotropic Sampling Spectra

Power Spectrum

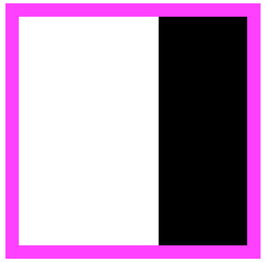


Radial Power Spectrum



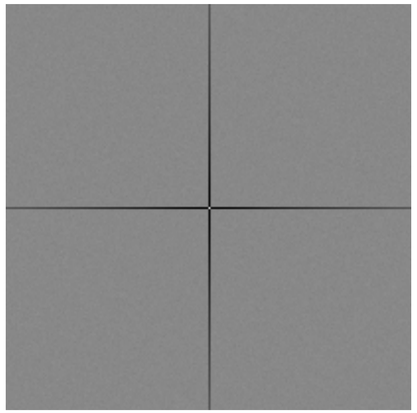
Variance due to N-rooks Sampler

$f(\vec{x})$



$$\mathrm{Var}(I_N) = \int_\Omega$$

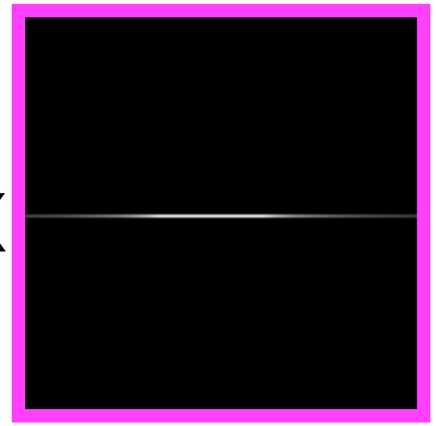
$\langle \mathcal{P}_{S_N}(\nu) \rangle$



N-rooks spectrum

X

$\mathcal{P}_f(\nu)$

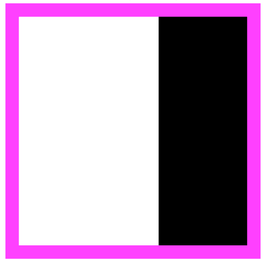


Integrand spectrum

$d\nu$

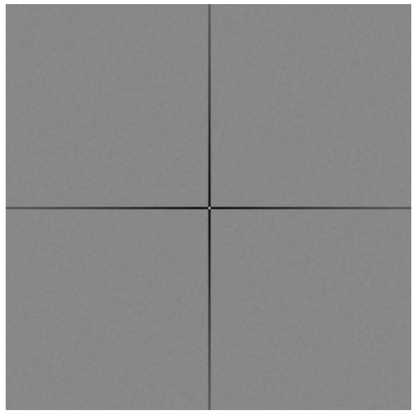
Variance due to N-rooks Sampler

$f(\vec{x})$



$$\mathrm{Var}(I_N) = \int_\Omega$$

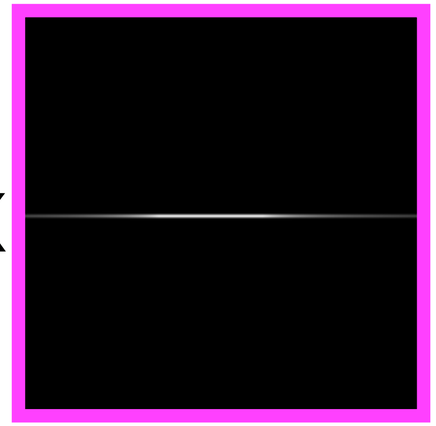
$\langle \mathcal{P}_{S_N}(\nu) \rangle$



N-rooks spectrum

X

$\mathcal{P}_f(\nu)$

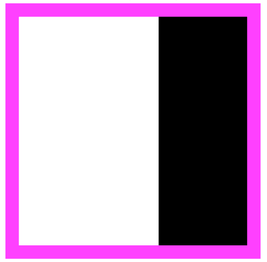


Integrand spectrum

$d\nu$

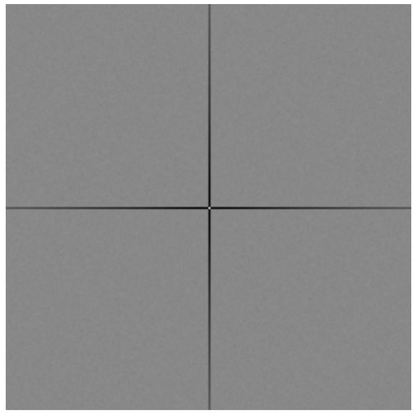
Variance due to N-rooks Sampler

$f(\vec{x})$



$$\mathrm{Var}(I_N) = \int_\Omega$$

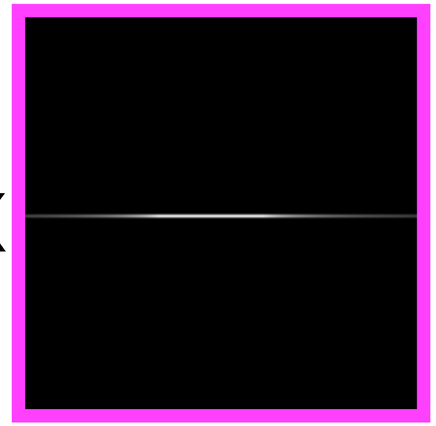
$\langle \mathcal{P}_{S_N}(\nu) \rangle$



N-rooks spectrum

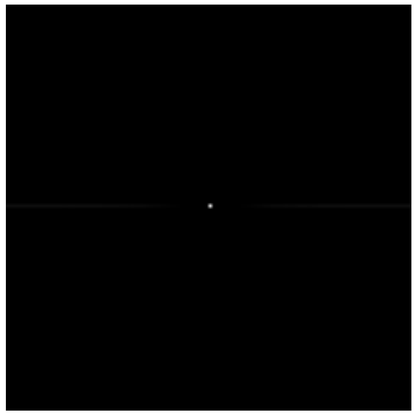
X

$\mathcal{P}_f(\nu)$



Integrand spectrum

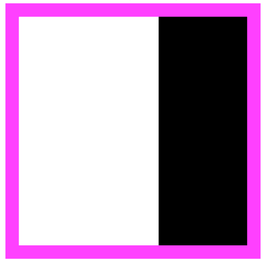
$$d\nu = \int_\Omega$$



$$d\nu$$

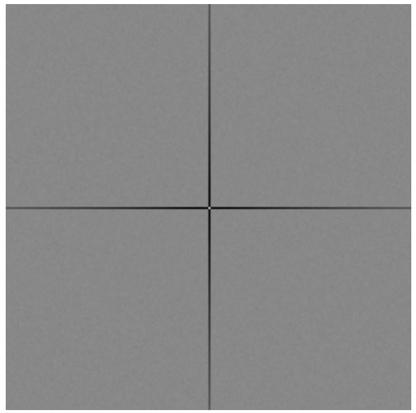
Variance due to N-rooks Sampler

$f(\vec{x})$



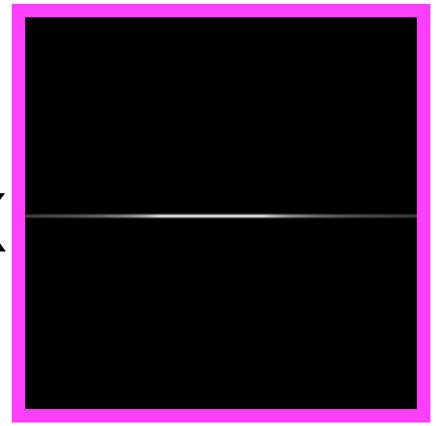
$\langle \mathcal{P}_{S_N}(\nu) \rangle$ $\mathcal{P}_f(\nu)$

$$\mathrm{Var}(I_N) = \int_\Omega$$



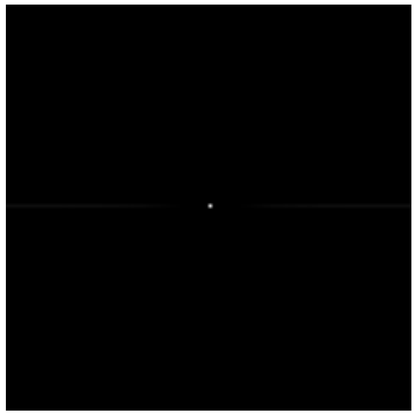
N-rooks spectrum

X



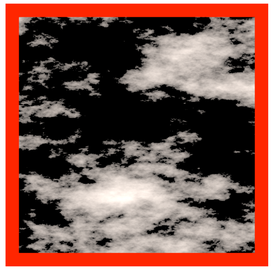
Integrand spectrum

$$d\nu = \int_\Omega$$

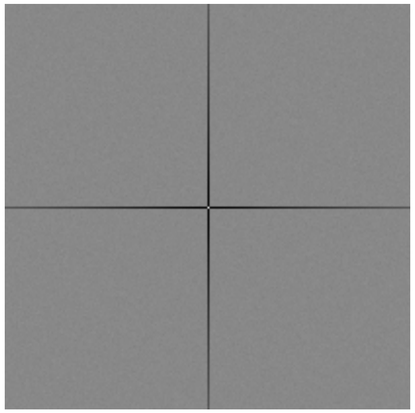


$$d\nu$$

$f(\vec{x})$

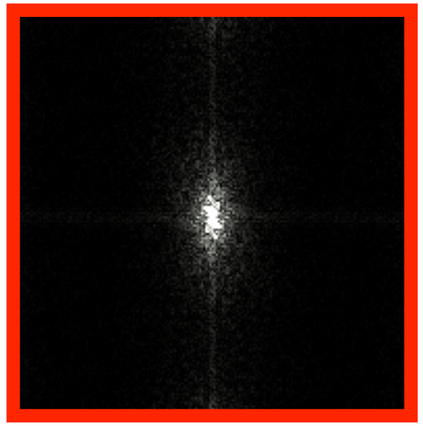


$$\mathrm{Var}(I_N) = \int_\Omega$$



N-rooks spectrum

X

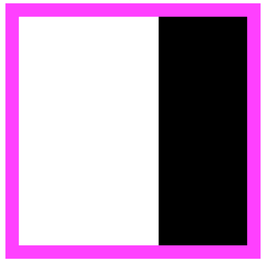


Integrand spectrum

$$d\nu$$

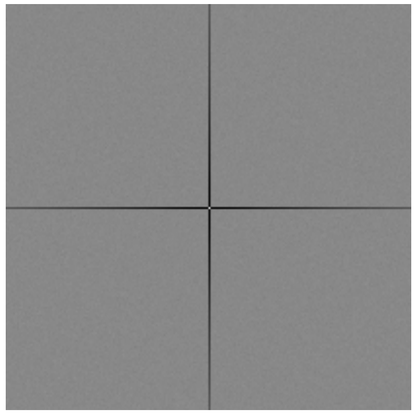
Variance due to N-rooks Sampler

$f(\vec{x})$



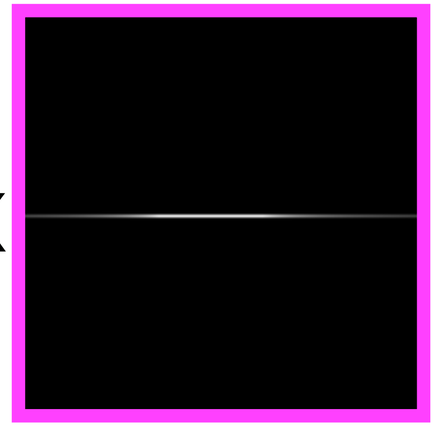
$\langle \mathcal{P}_{S_N}(\nu) \rangle$ $\mathcal{P}_f(\nu)$

$$\mathrm{Var}(I_N) = \int_\Omega$$



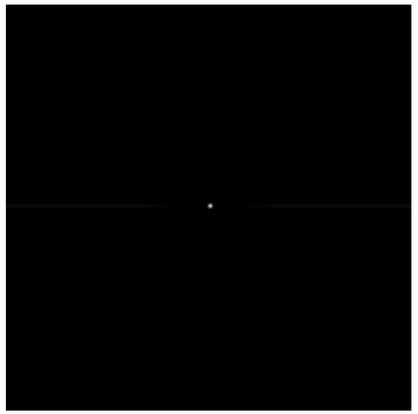
N-rooks spectrum

X



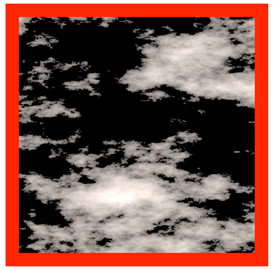
Integrand spectrum

$$d\nu = \int_\Omega$$

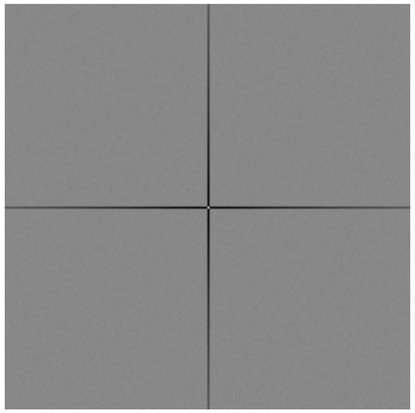


$$d\nu$$

$f(\vec{x})$

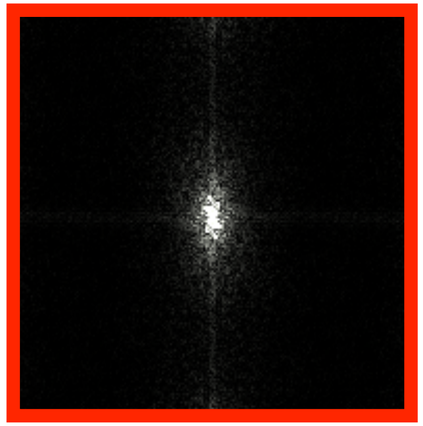


$$\mathrm{Var}(I_N) = \int_\Omega$$



N-rooks spectrum

X

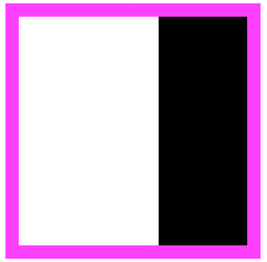


Integrand spectrum

$$d\nu$$

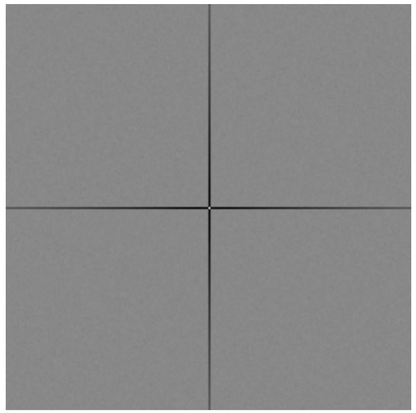
Variance due to N-rooks Sampler

$f(\vec{x})$



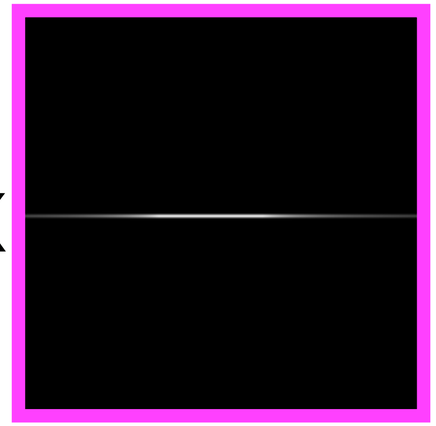
$\langle \mathcal{P}_{S_N}(\nu) \rangle$ $\mathcal{P}_f(\nu)$

$$\mathrm{Var}(I_N) = \int_\Omega$$



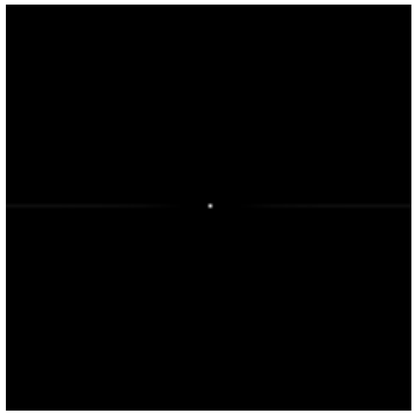
N-rooks spectrum

X



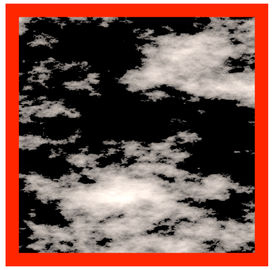
Integrand spectrum

$$d\nu = \int_\Omega$$

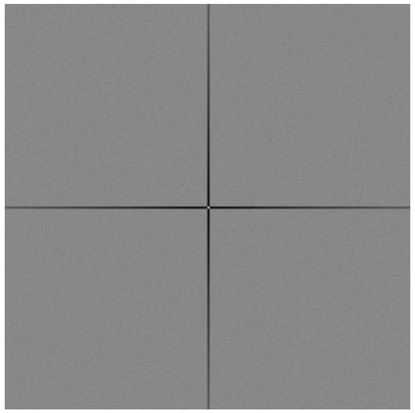


$$d\nu$$

$f(\vec{x})$

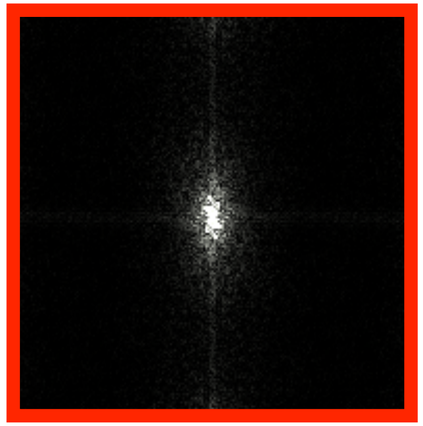


$$\mathrm{Var}(I_N) = \int_\Omega$$



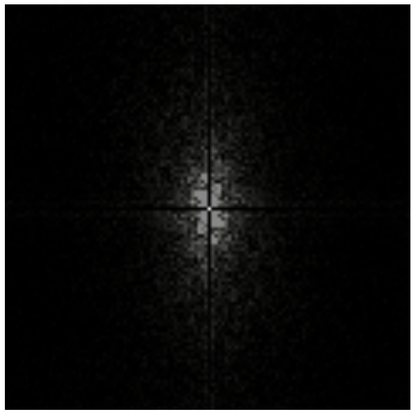
N-rooks spectrum

X



Integrand spectrum

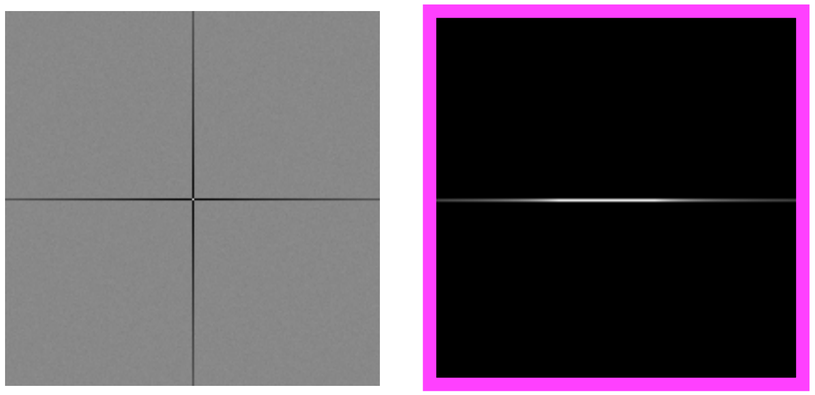
$$d\nu = \int_\Omega$$



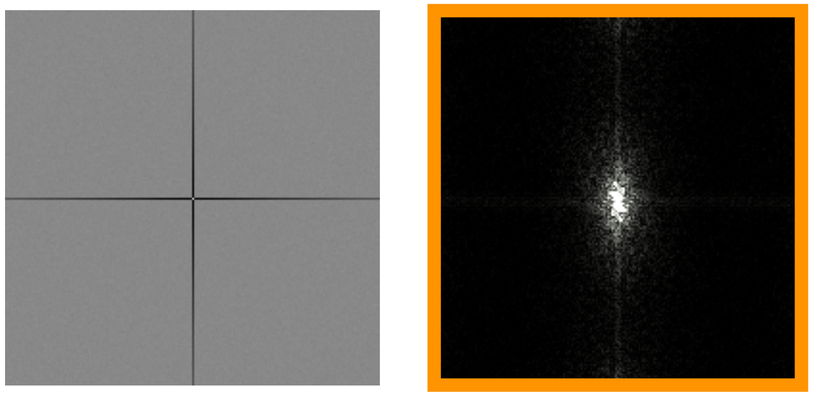
$$d\nu$$

Variance Convergence of Latin Hypercube (N-rooks)

$\langle \mathcal{P}_{S_N}(\nu) \rangle$ $\mathcal{P}_f(\nu)$



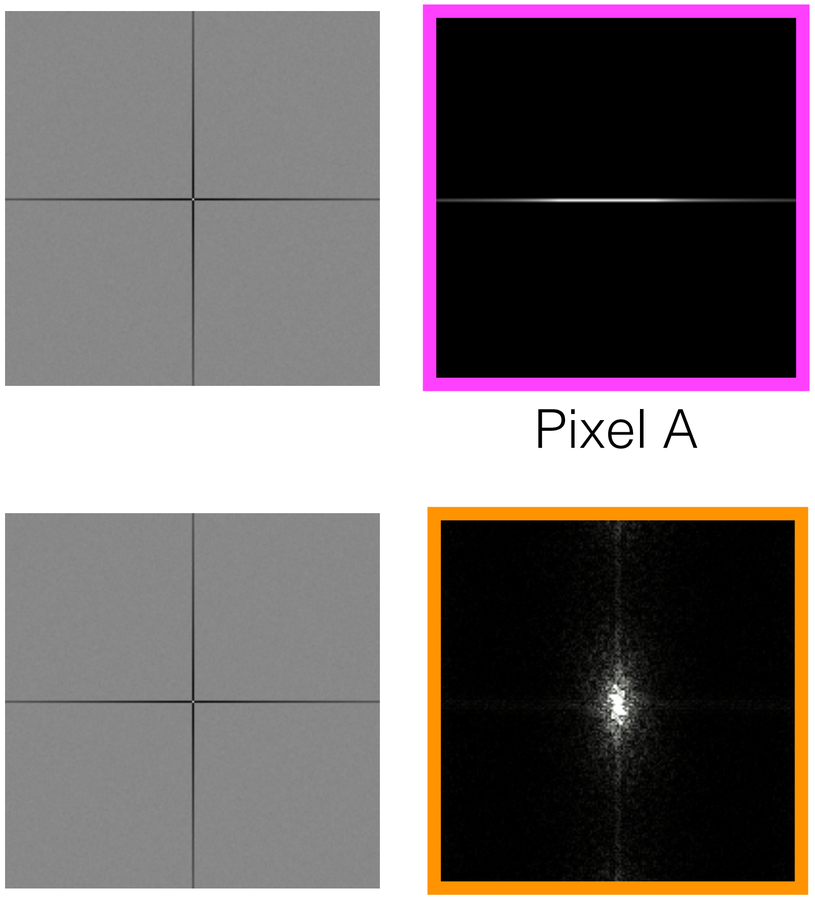
Pixel A

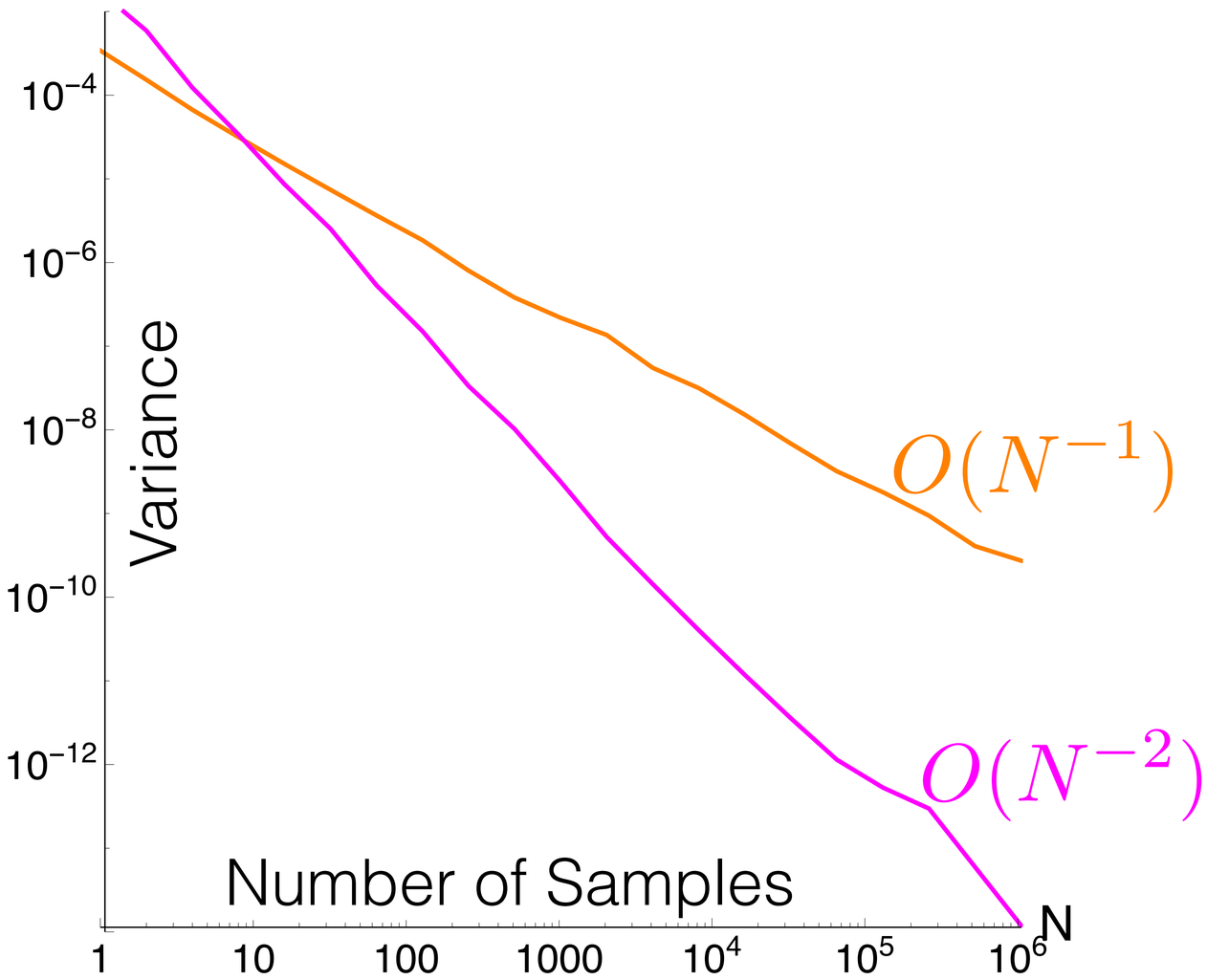


Pixel B

Variance Convergence of Latin Hypercube (N-rooks)

$\langle \mathcal{P}_{S_N}(\nu) \rangle$ $\mathcal{P}_f(\nu)$



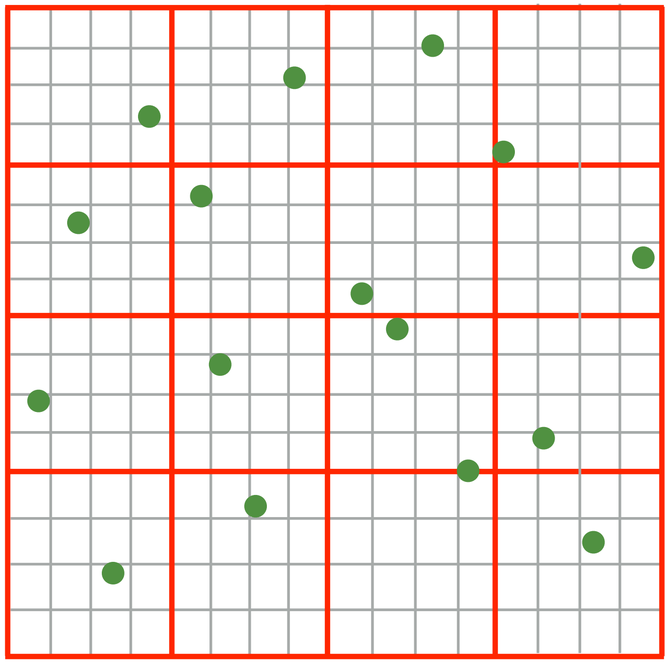


Non-Axis Aligned Integrand Spectra

$\mathcal{P}_f(\nu)$

Integrand Spectrum

Non-Axis Aligned Integrand Spectra

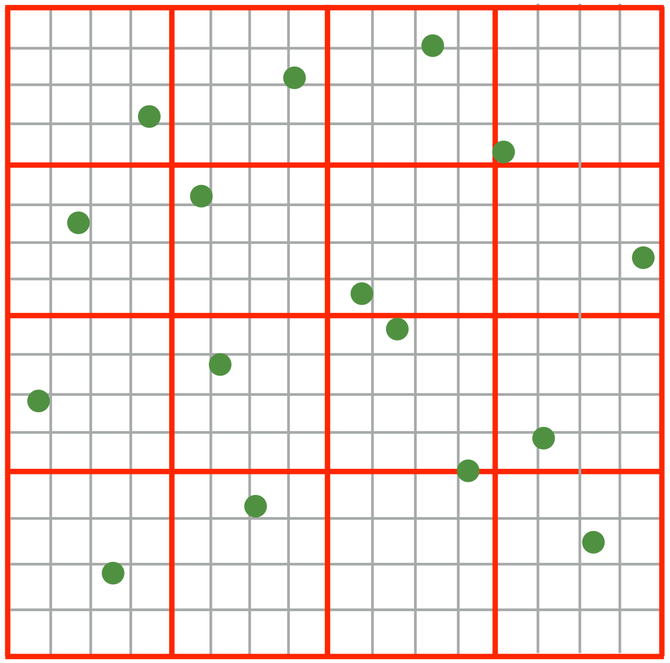


Multi-jittered Samples

$\mathcal{P}_f(\nu)$

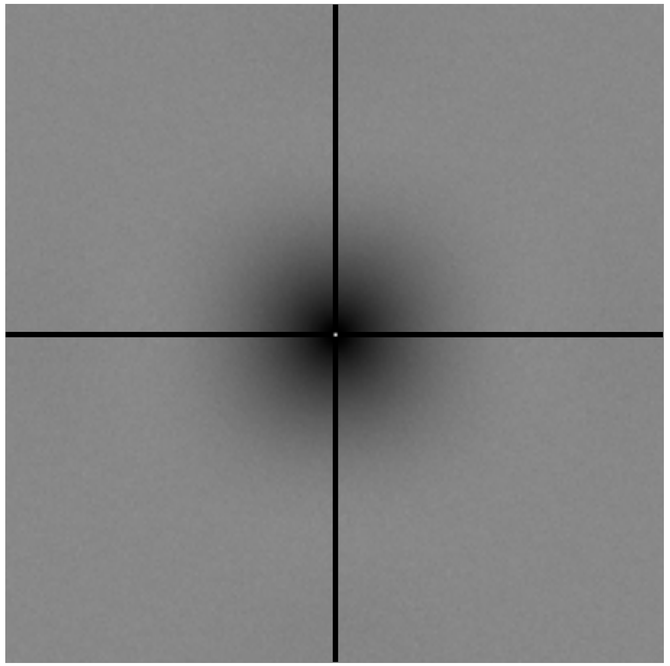
Integrand Spectrum

Non-Axis Aligned Integrand Spectra



Multi-jittered Samples

$\langle \mathcal{P}_{S_N}(\nu) \rangle$

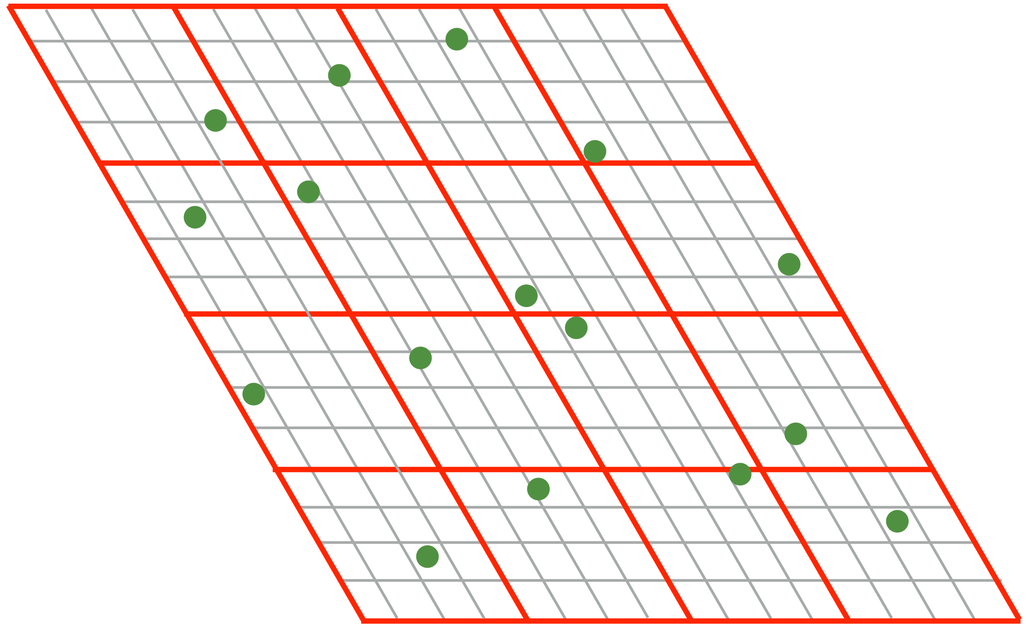


Sampling Spectrum

$\mathcal{P}_f(\nu)$

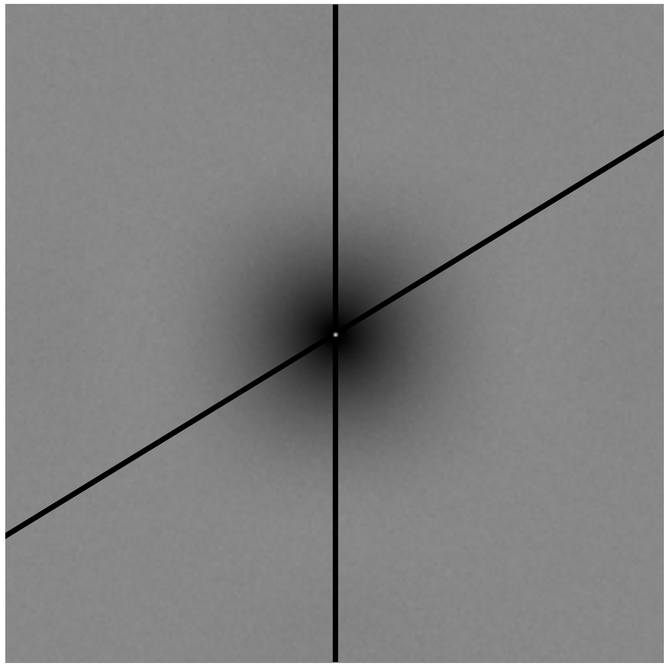
Integrand Spectrum

Shearing Multi-Jittered Samples



Sheared Samples

$\langle \mathcal{P}_{S_N}(\nu) \rangle$

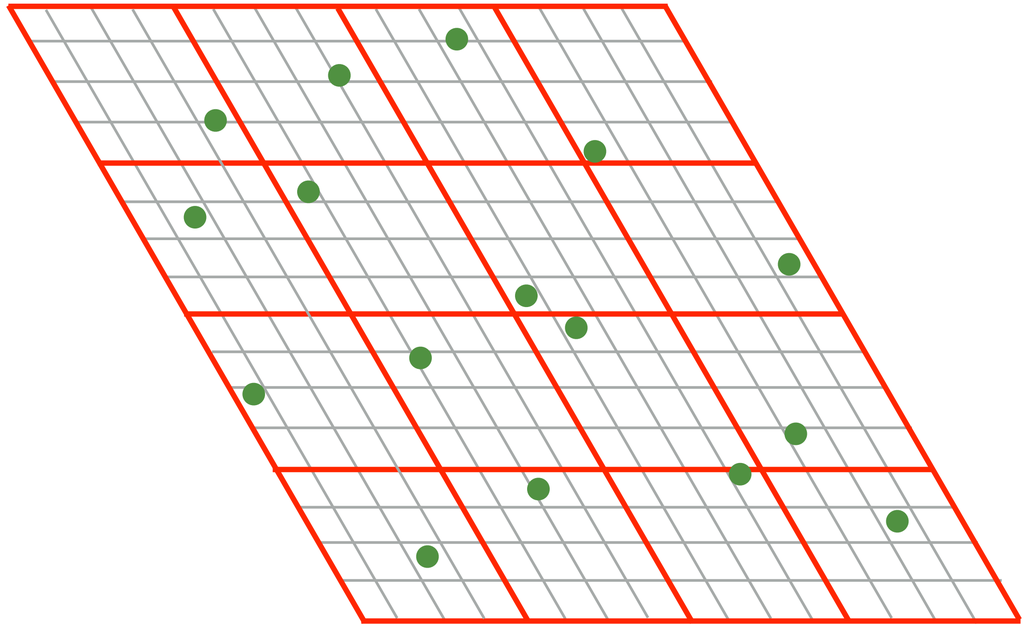


Sheared Spectrum

$\mathcal{P}_f(\nu)$

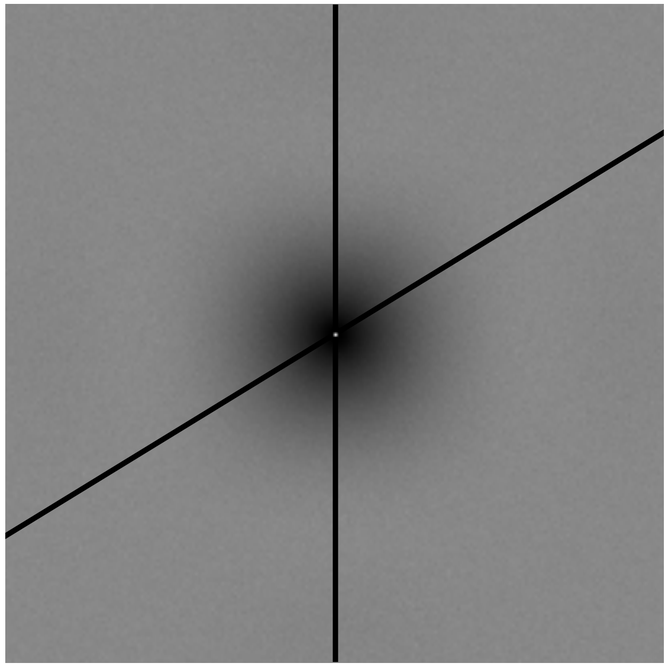
Integrand Spectrum

How can we determine the sample shearing parameters ?



Sheared Samples

$\langle \mathcal{P}_{S_N}(\nu) \rangle$



Sheared Spectrum

$\mathcal{P}_f(\nu)$

Integrand Spectrum

Spatio-temporal Sampling for Reconstructing Distribution Effects

Philipp Slusallek *Karol Myszkowski*
Gurprit Singh

Multi-dimensional adaptive sampling of distribution effects

Fourier Analysis of Light Transport

Temporal reconstruction of distribution effects

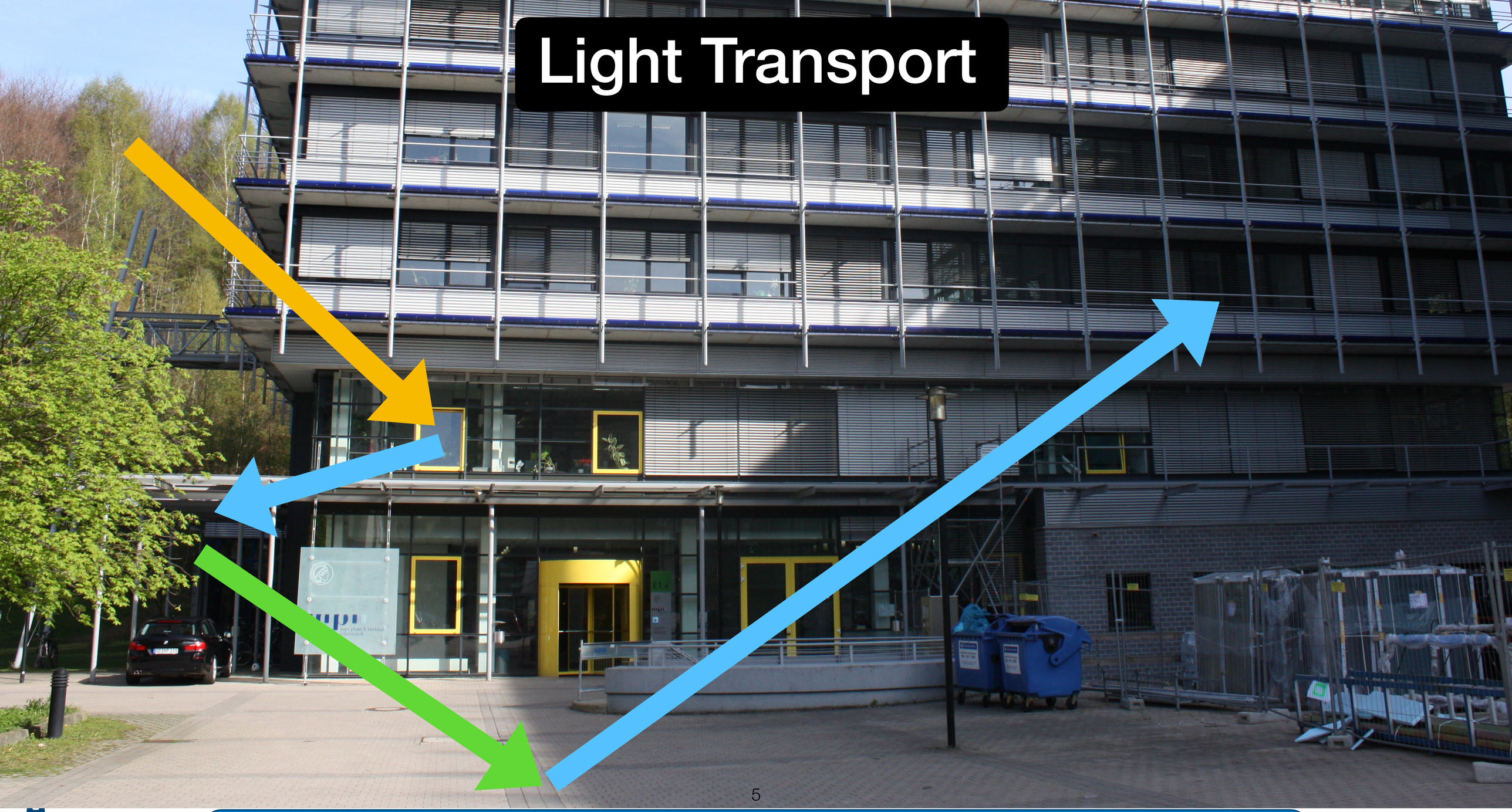
Multi-dimensional adaptive spatio-temporal sampling

Hachisuka et al. [2008]

Multi-dimensional adaptive sampling of distribution effects

Fourier Analysis of Light Transport

Light Transport

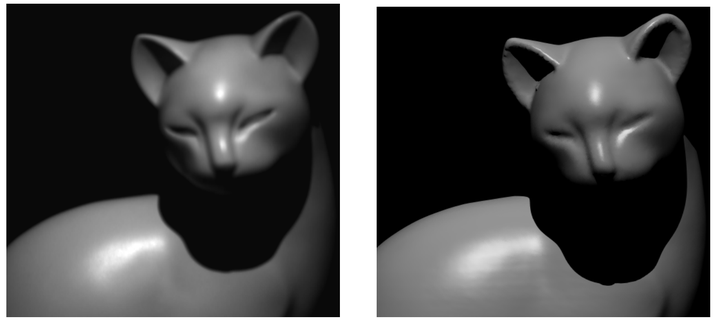


Understanding, manipulating and computing signals

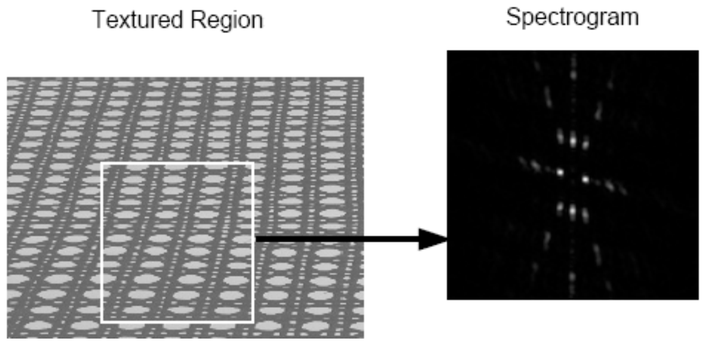
- Discontinuites
  - where things change
- Gradients
  - Useful for interpoloation
- **Frequency content (today's main course)**
  - Useful for sampling
  - Useful for inverse problems
    - Sometimes useful as basis functions
  - Statistics

And all these capture perceptual properties

Frequency contents matter in vision



Inverse lighting

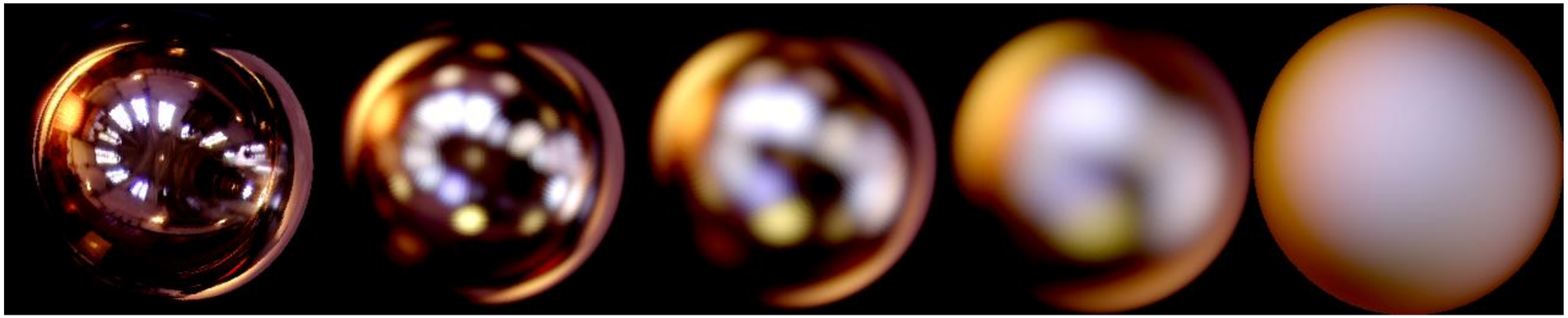


Shape from texture

Shape from (de)focus

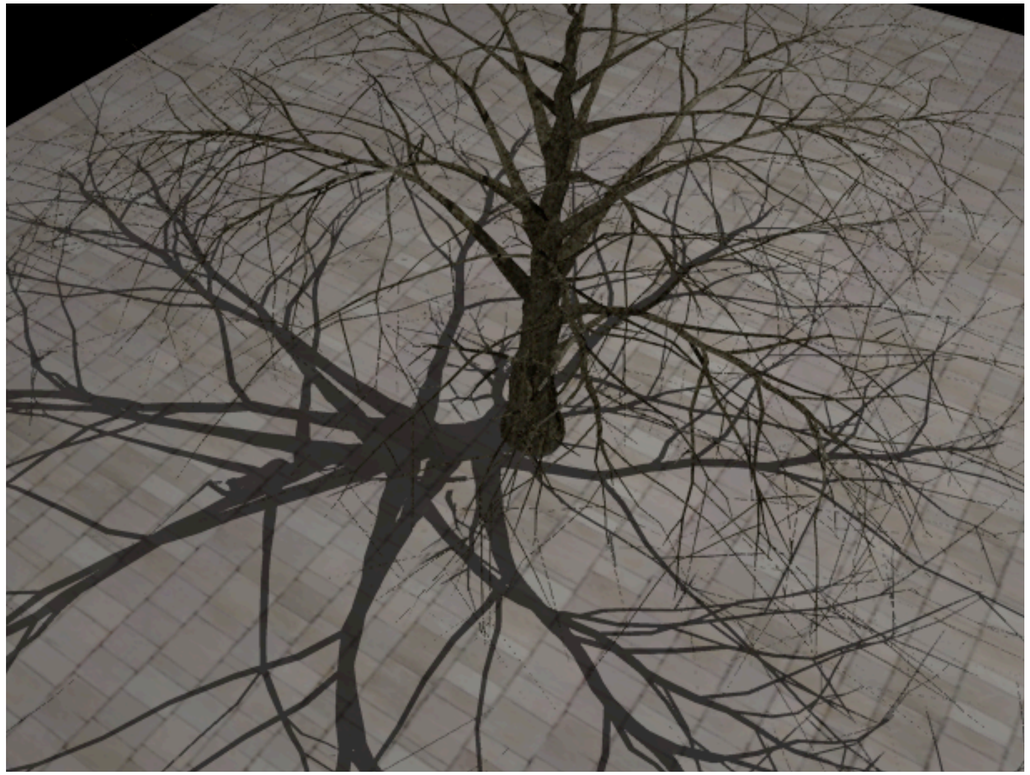
Illumination effects

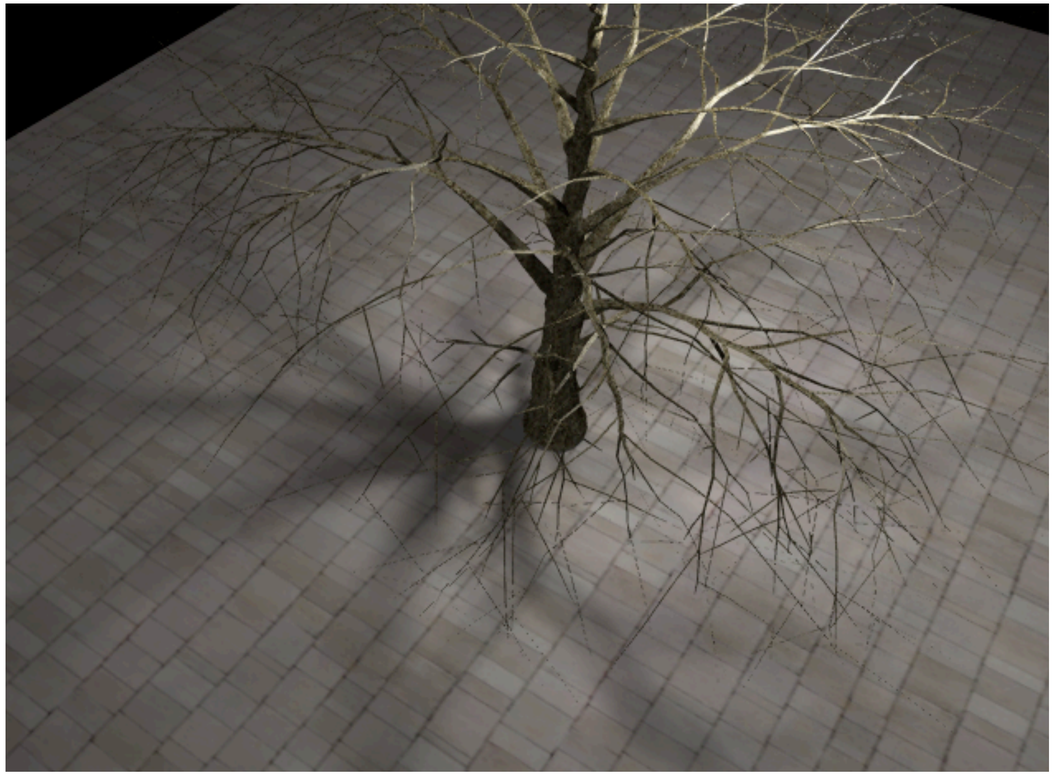
Blurry reflections



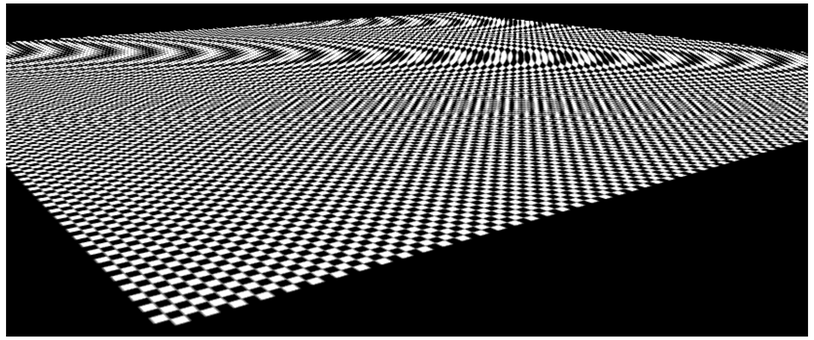
Illumination effects

Shadow boundaries:





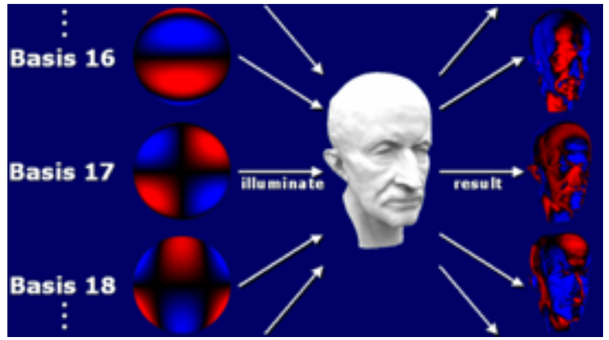
Frequency contents matter in graphics



Sampling, antialiasing

Texture filtering

Light Field Sampling

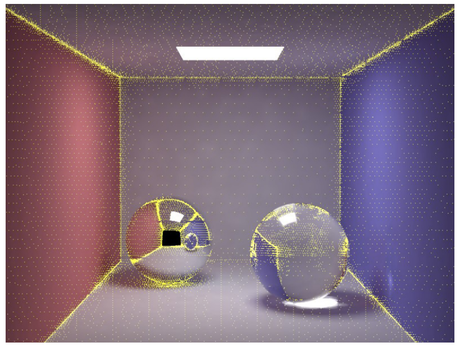


Fourier-like basis

Precomputed radiance transfer

Wavelet radiosity

Spherical harmonics



Low frequency assumption

Irradiance caching

How does light interactions in a scene explain the frequency content?

How does light interactions in a scene explain the frequency content?

Theoretical framework:

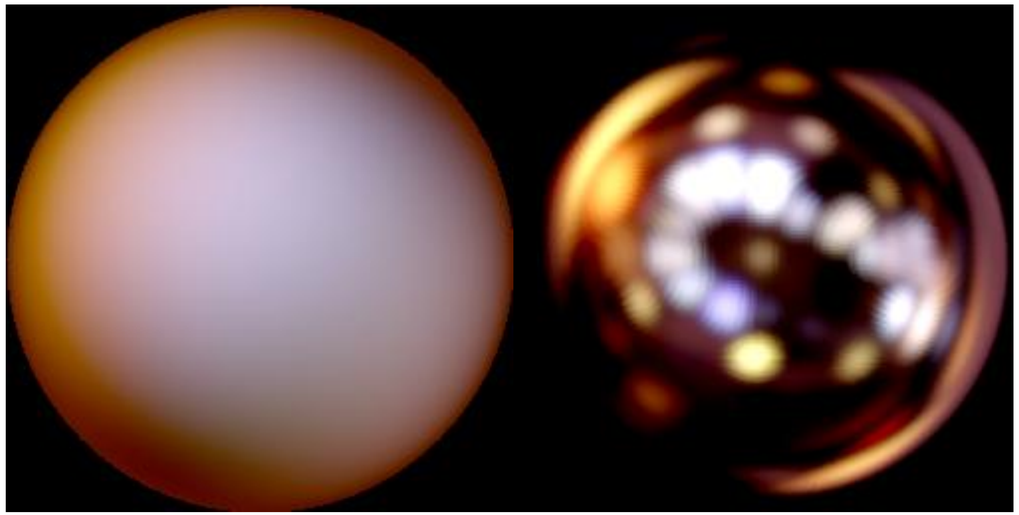
Understanding the frequency content of the radiance function

Mathematical equations of the light transport → Fourier spectrum of the Illumination in the scene

Spatial and Angular frequency

Spatial frequency

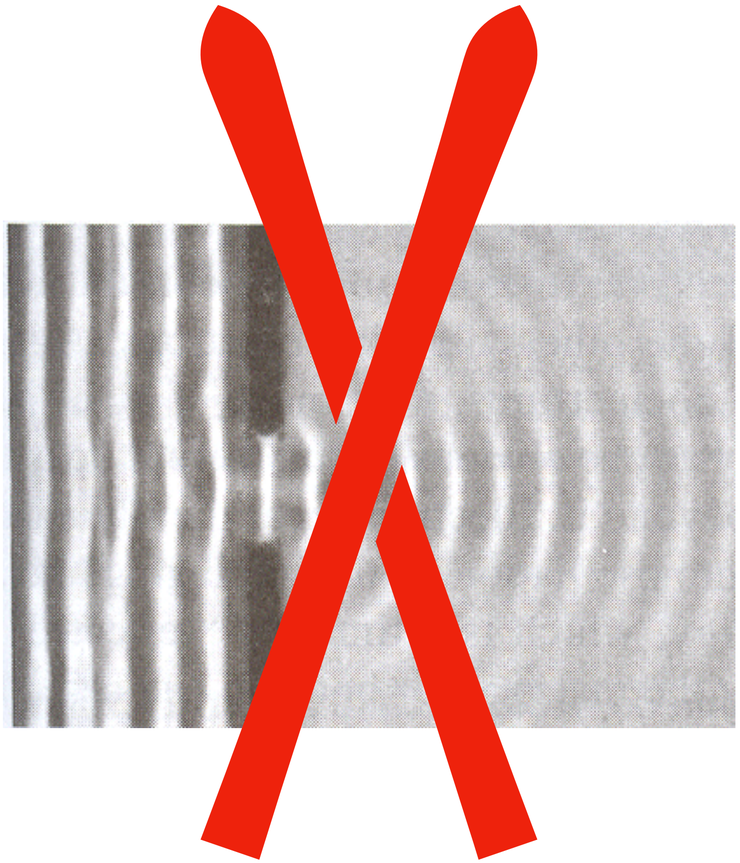
(e.g., shadows, textures)

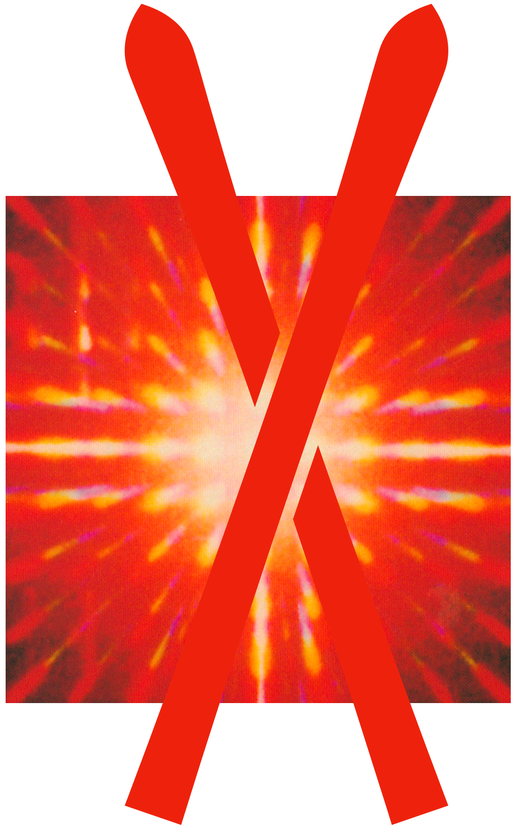


Angular frequency

(e.g., blurry highlights)

Disclaimer: no Fourier optics



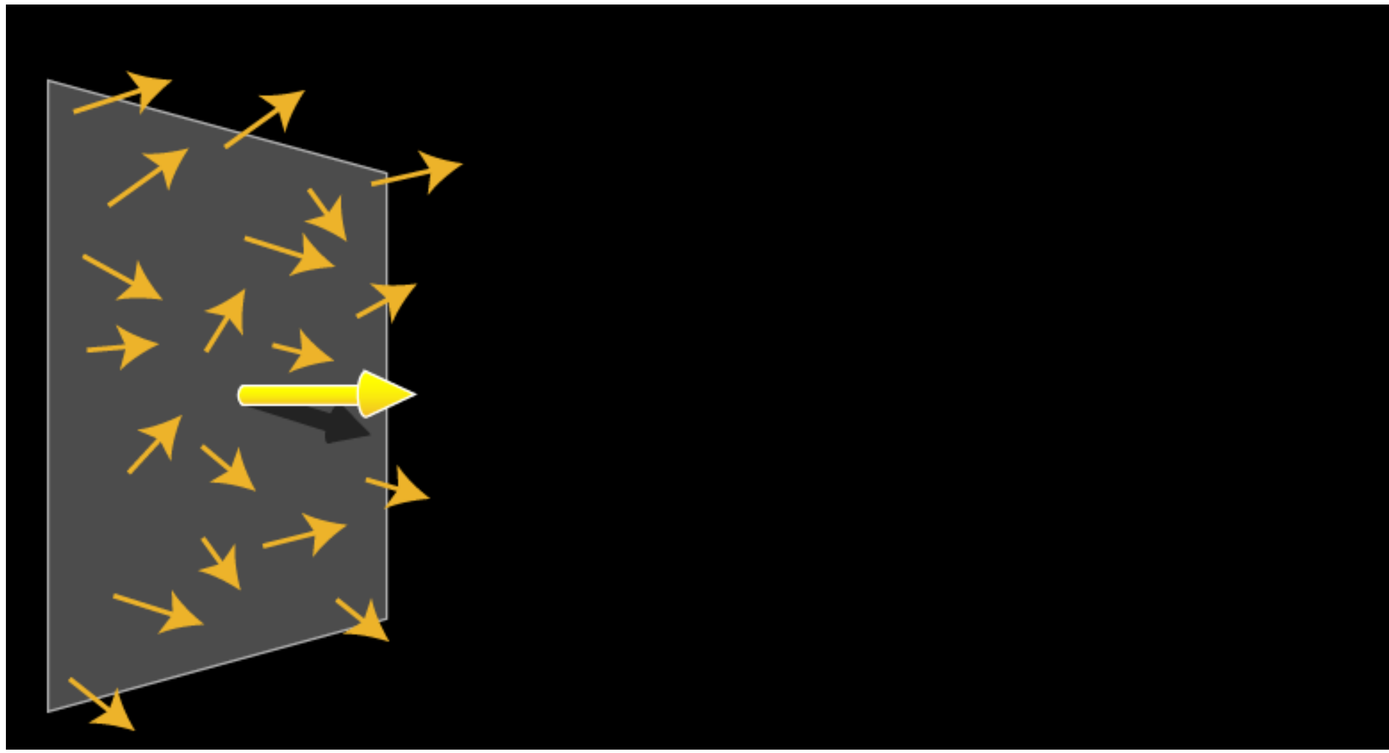


Only geometrical optics

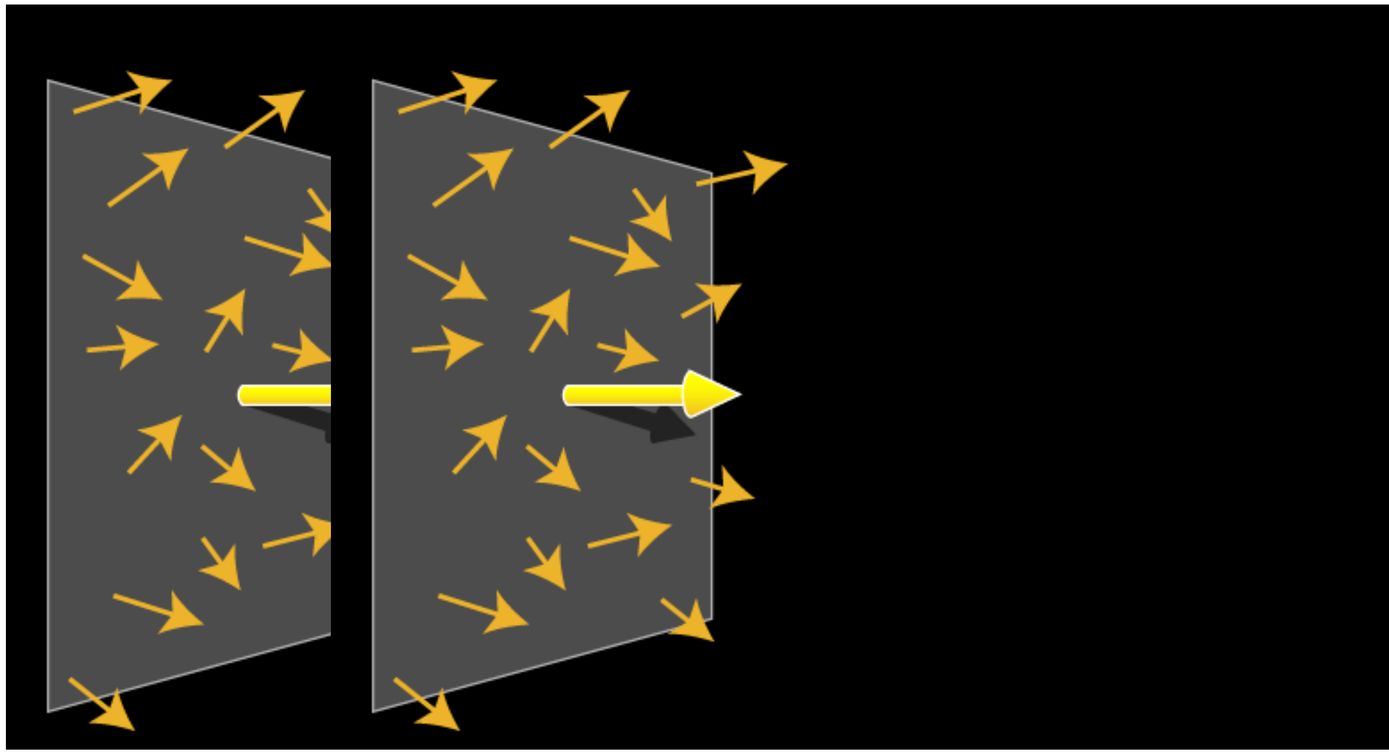
Light transport in a scene

Light transport in a scene

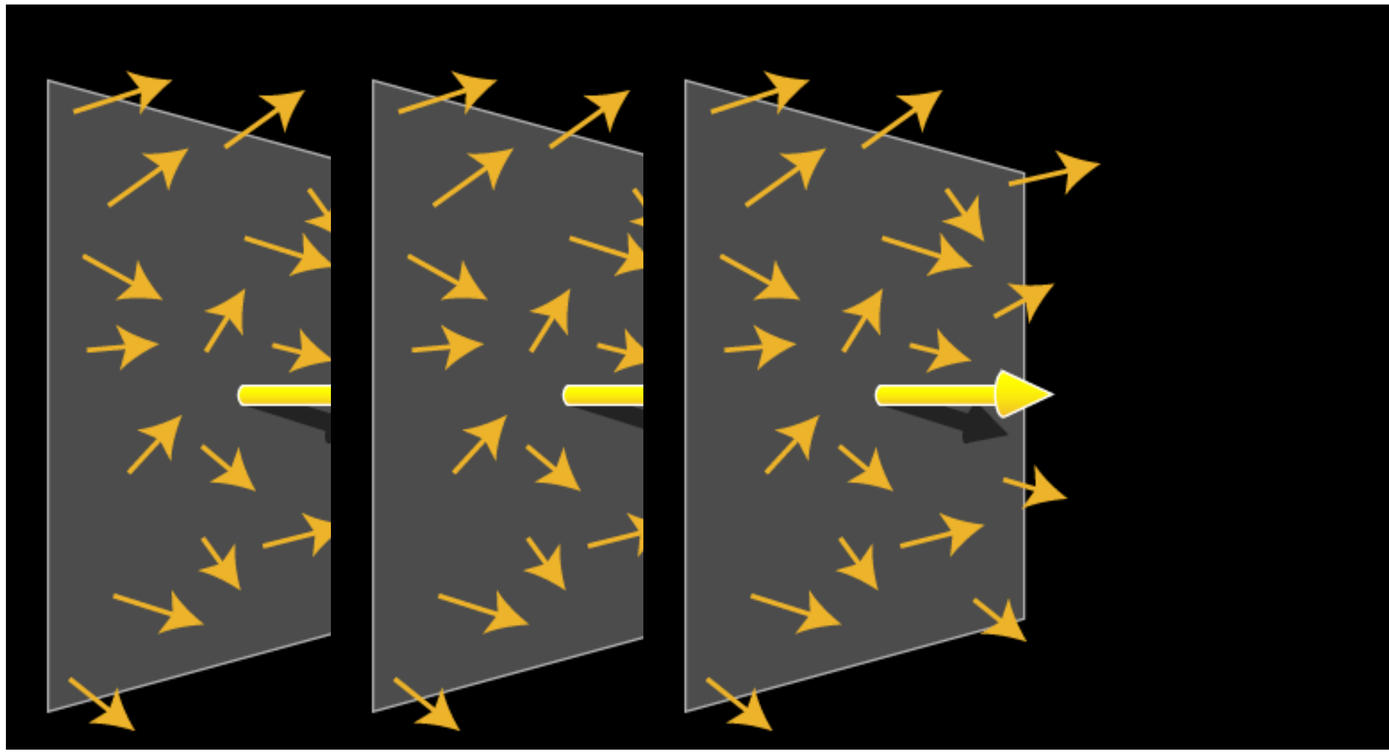
Flatland: Light transport in a scene



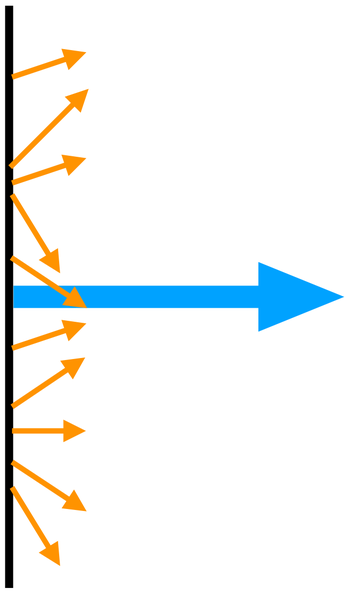
Flatland: Light transport in a scene



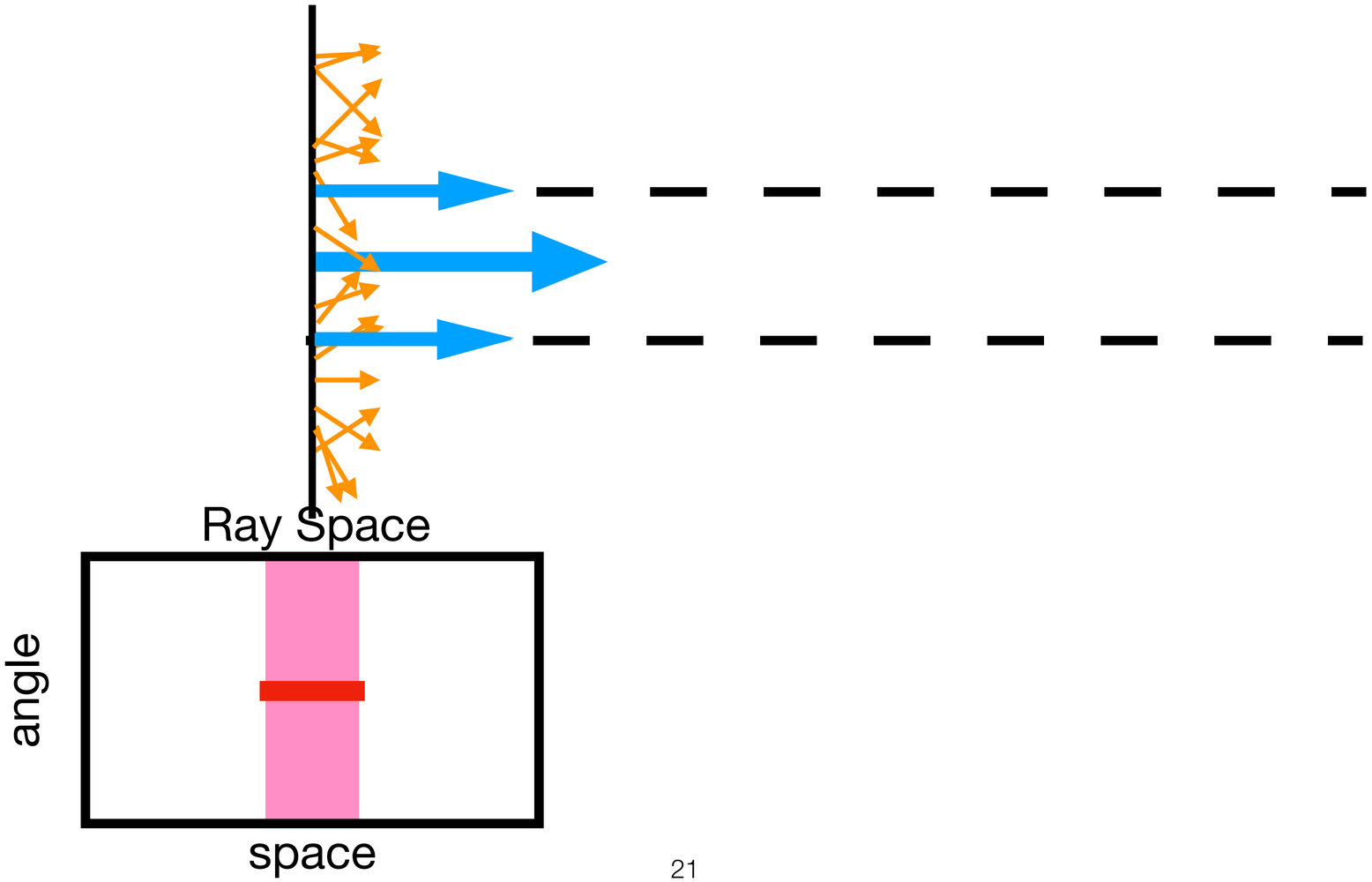
Flatland: Light transport in a scene



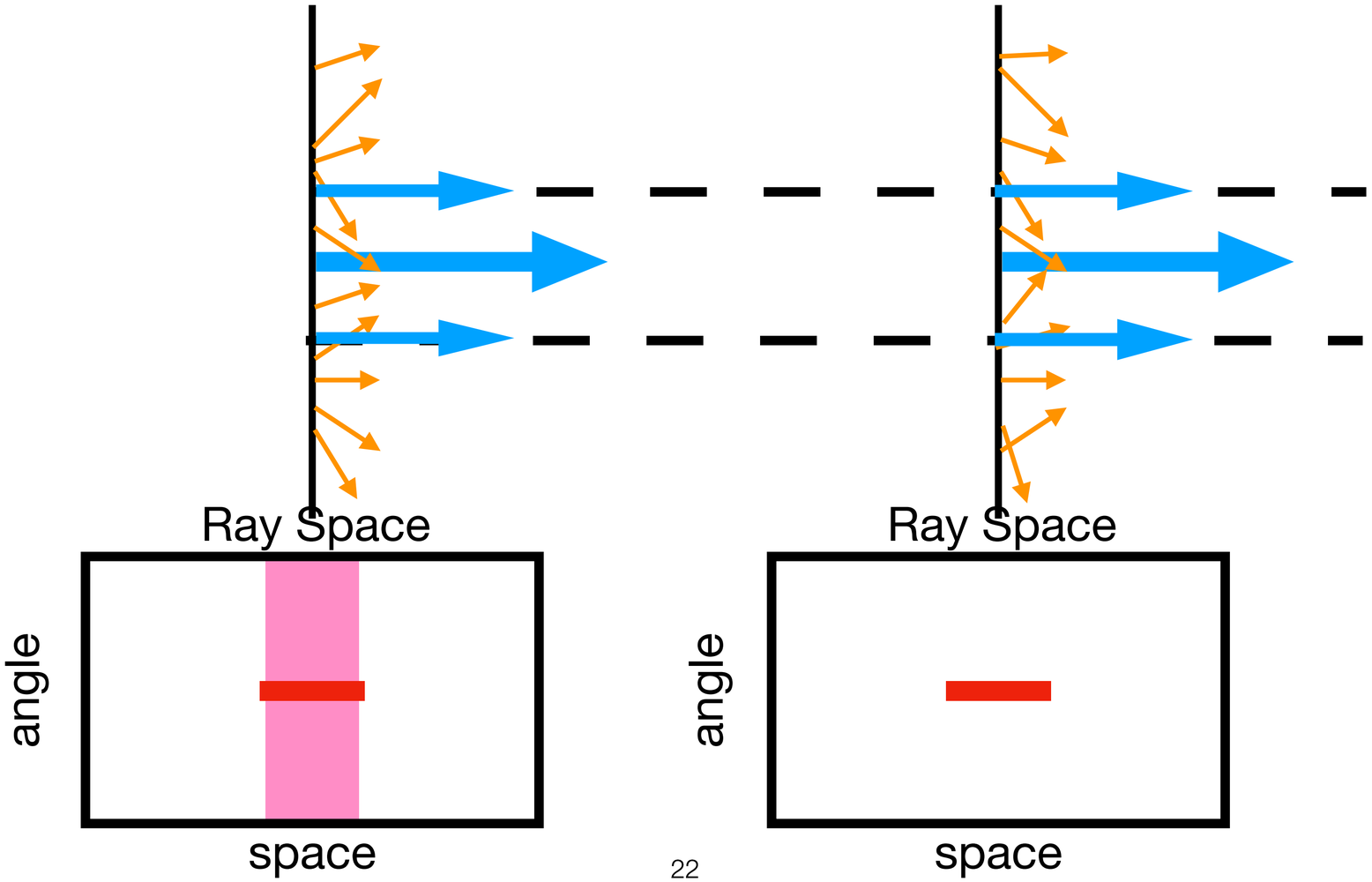
Flatland: Light transport



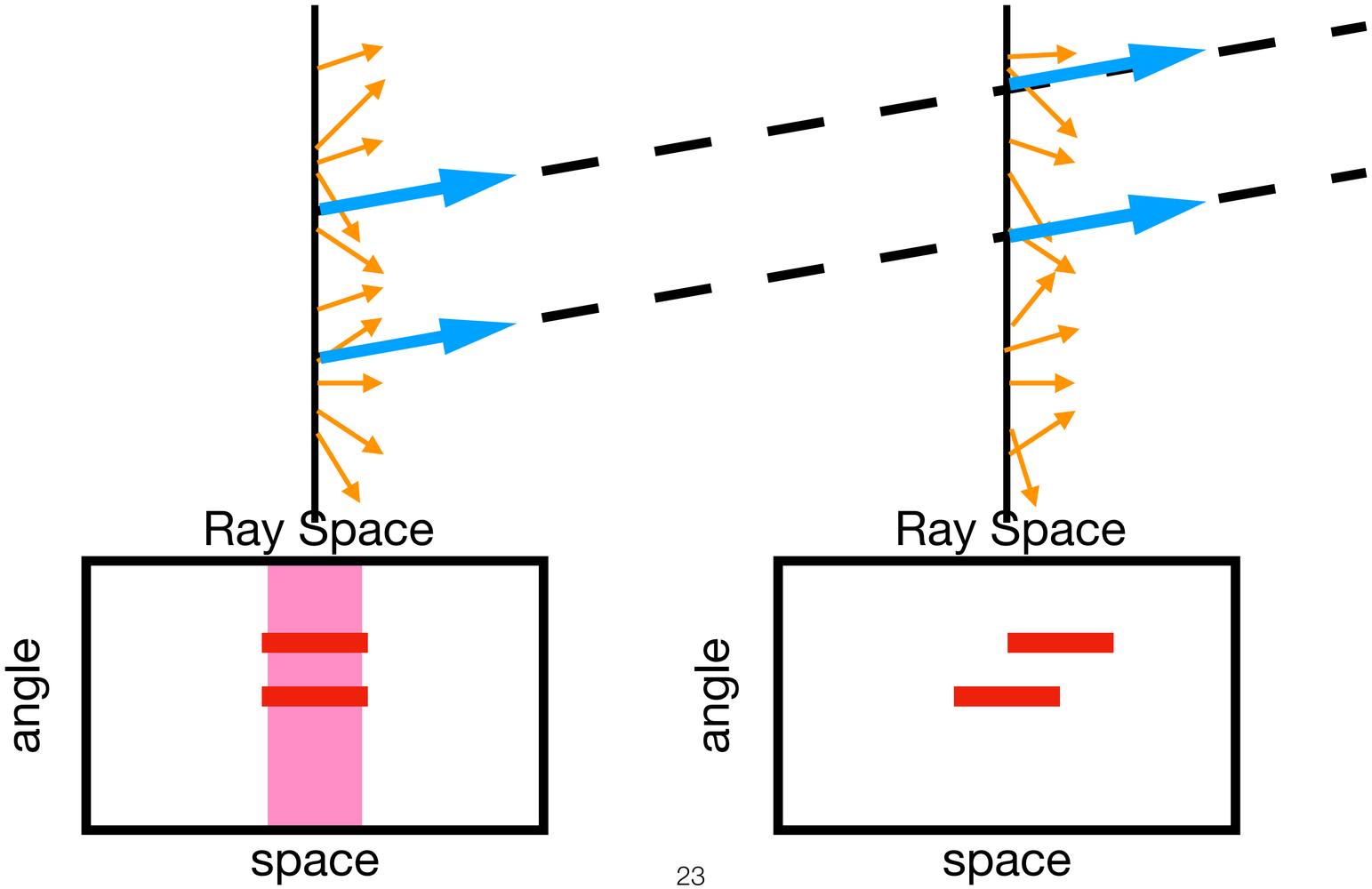
Flatland: Light transport



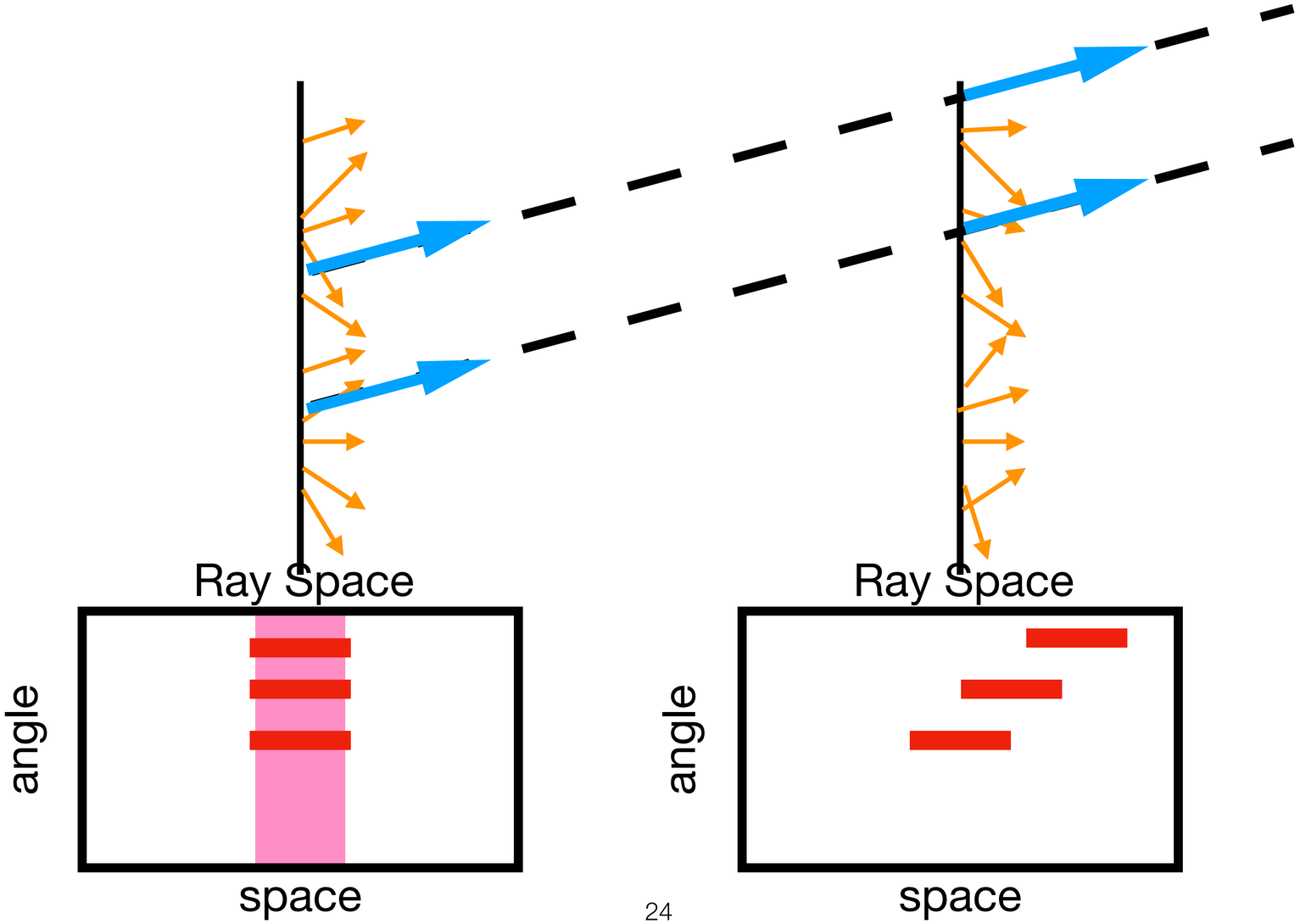
Flatland: Light transport



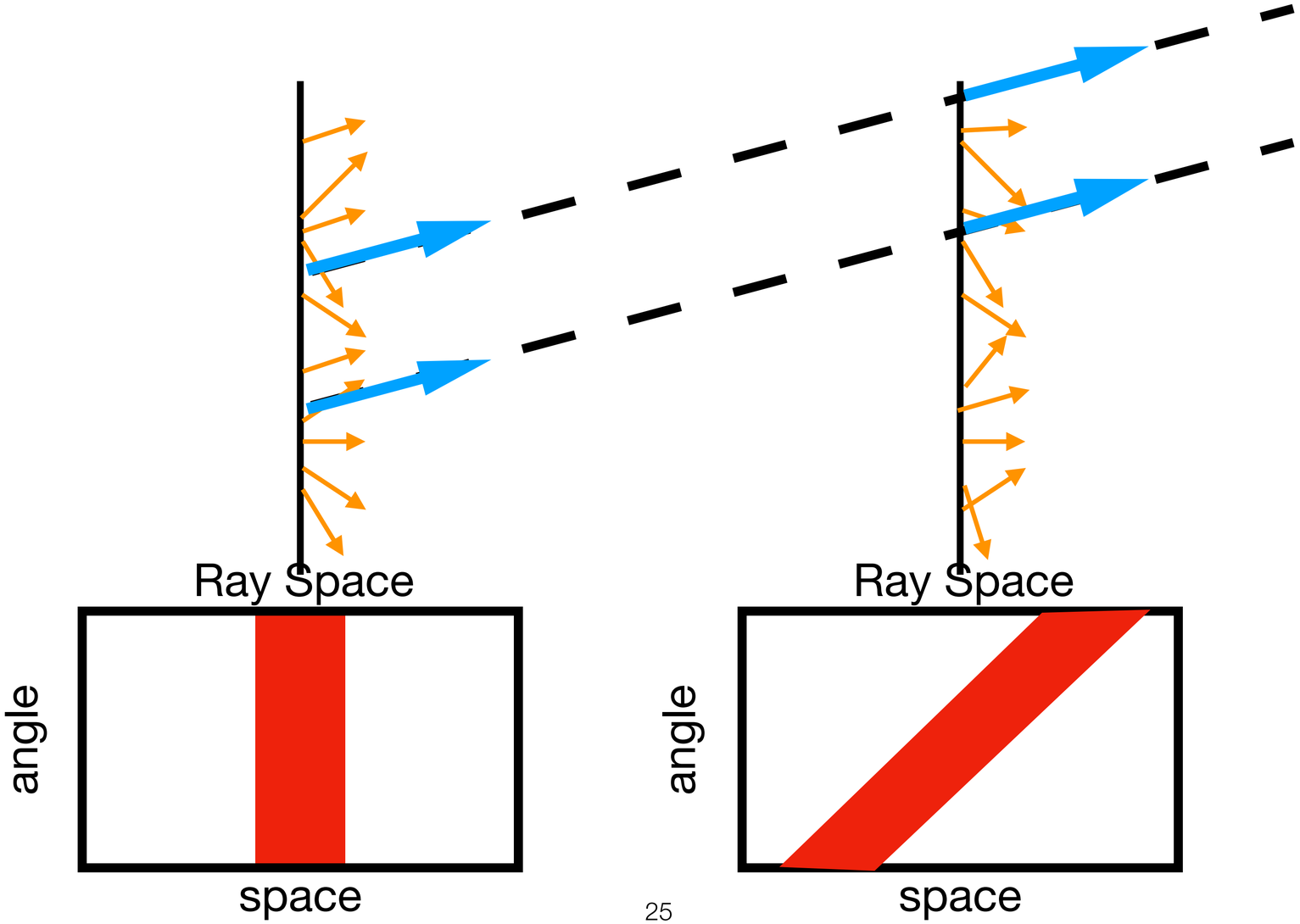
Flatland: Light transport



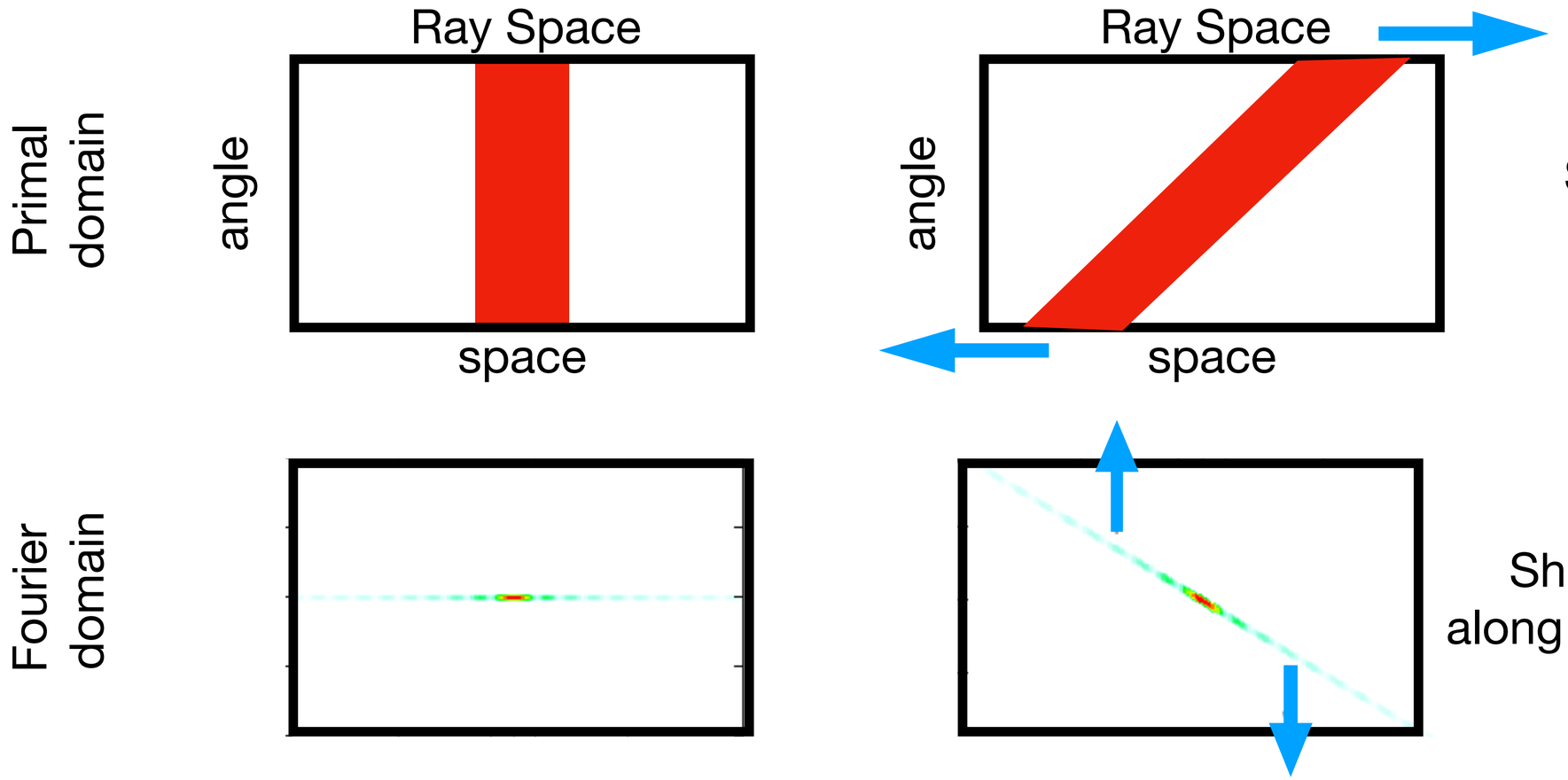
Flatland: Light transport



Flatland: Light transport



Flatland: Light transport

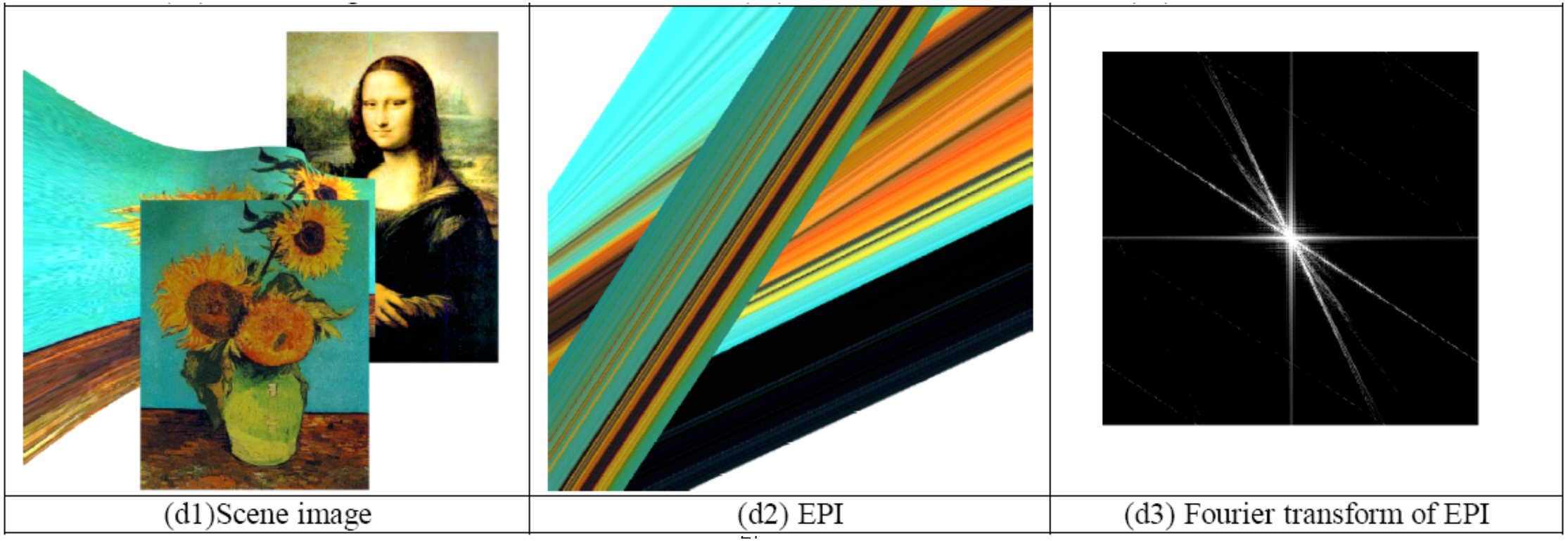


Shear in primal

Shear in Fourier, but along the other dimension

Transport --> Shear

Consistent with literature [see Plenoptic Sampling by Chai et al. 2000]

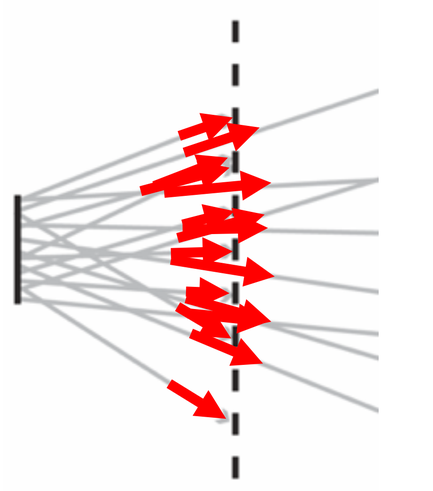


Occlusions

Consider planar occluders

Multiplication by binary function

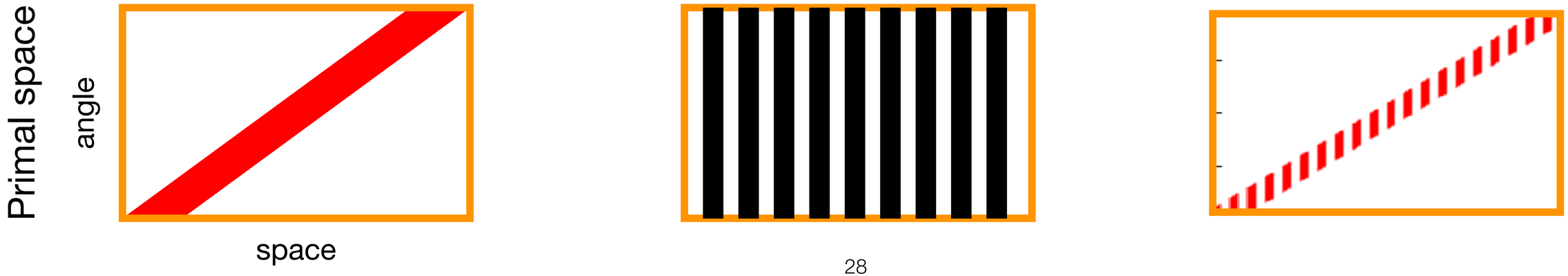
- mostly in space



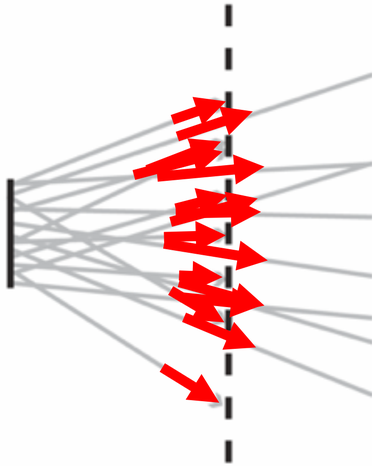
Before Occlusion

Blocker function

After occlusion



Occlusions



Multiplication in Primal domain is Convolution in Fourier domain

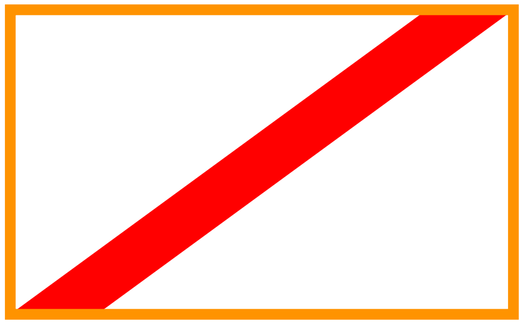
Before Occlusion

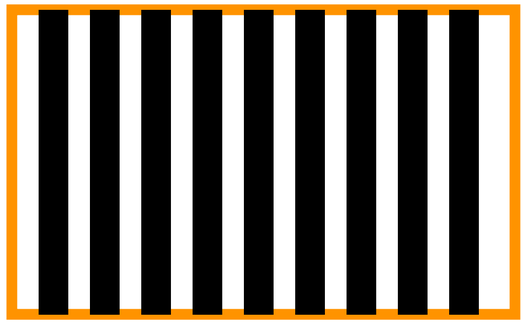
Blocker function

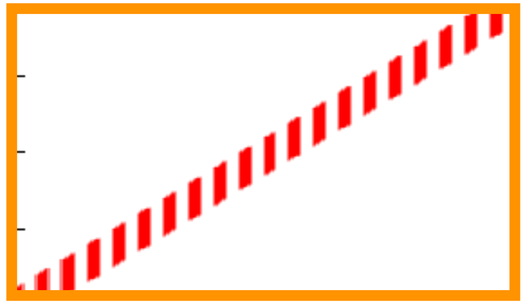
After occlusion

Primal

angle





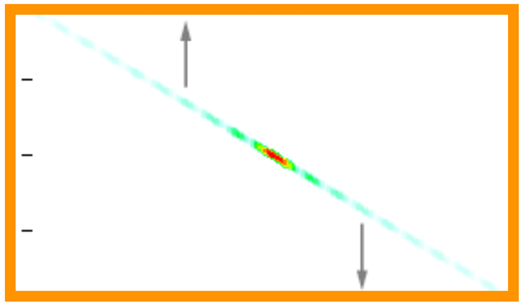


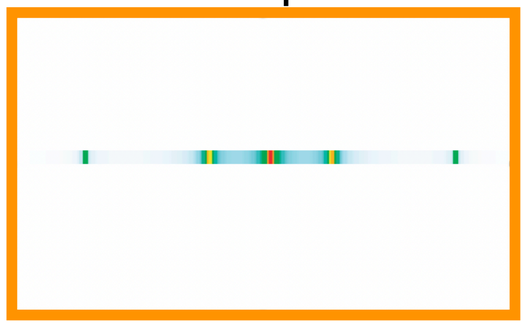
space

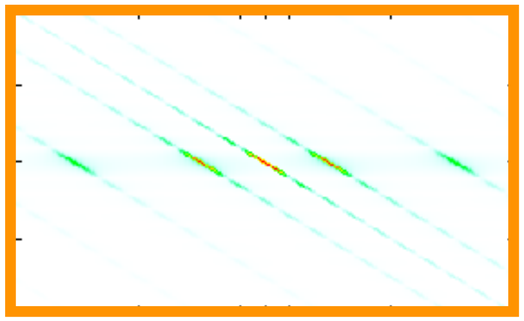
Blocker Spectrum

Fourier

$\Omega_v(\text{angle})$







$\Omega_x(\text{space})$

Main Transforms: Summary

	Transformations	Effects
Transport	Shear	
Occlusion	Convolution/Multiplication	Adds spatial frequencies
BRDF	Multiplication/Convolution	Removes angular frequencies
Curvature	Shear	

Frequency analysis of light transport [Durand et al. 2005]

Reconstructing Motion Blur

Motion blur

Objects move while the camera shutter is open

Image is "blurred" over time

Expensive for special effects

Necessary to remove "strobing" in animation

Garfield: A tale of two kitties

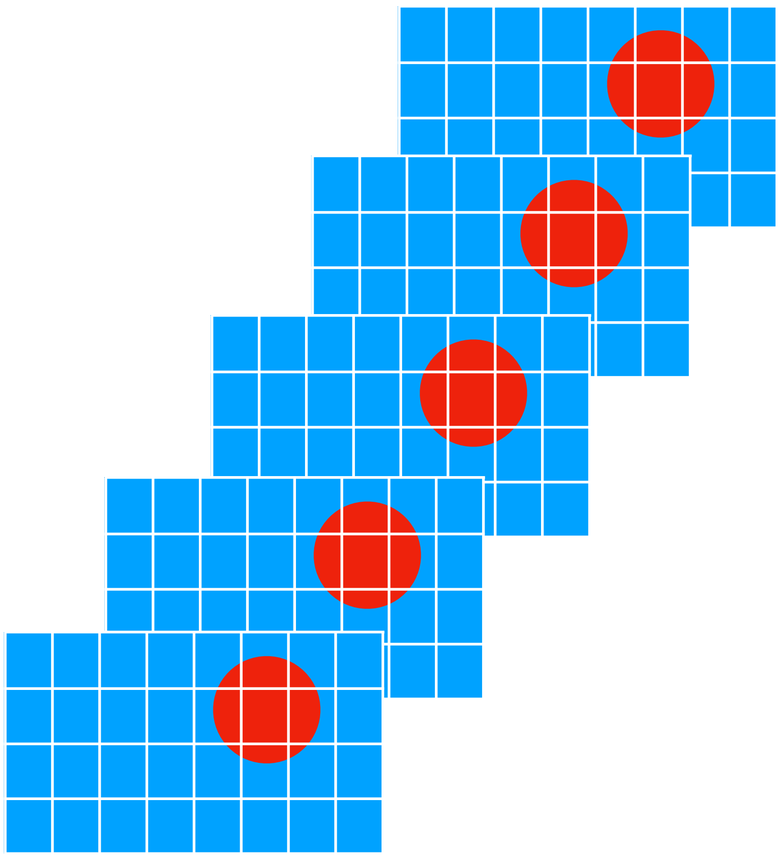
Rhythm & Hues Studios

The Incredibles

Pixar Animation Studios

Walt Disney Pictures

Motion blur: Simple approach



$t = 0.1$

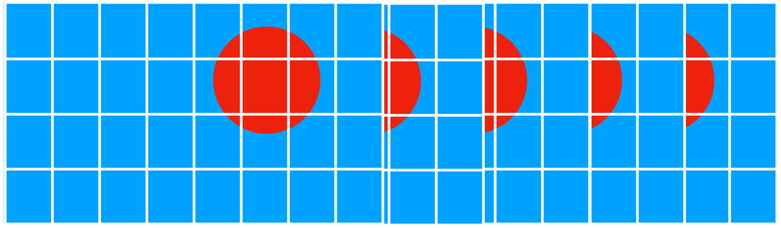
$t = 0.3$

$t = 0.5$

$t = 0.7$

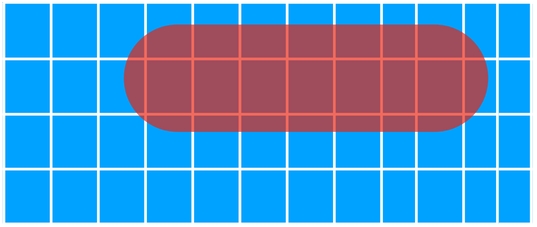
$t = 0.9$

Motion blur: Simple approach



$t = 0.5$

Motion blur: Simple approach



$$t \in [0, 1]$$

Simple approach

The simple approach is expensive

Can we do better?

Observation

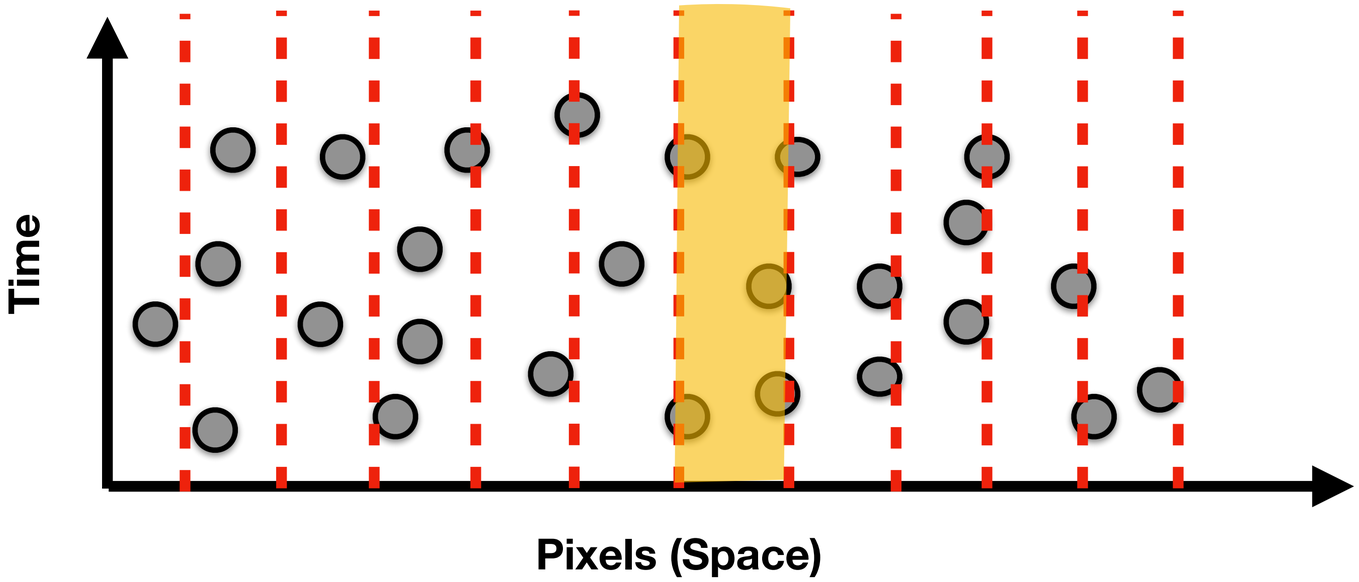
Motion blur is expensive

Motion blur removes spatial complexity

Standard Method

Use axis-aligned pixel filters at each pixel

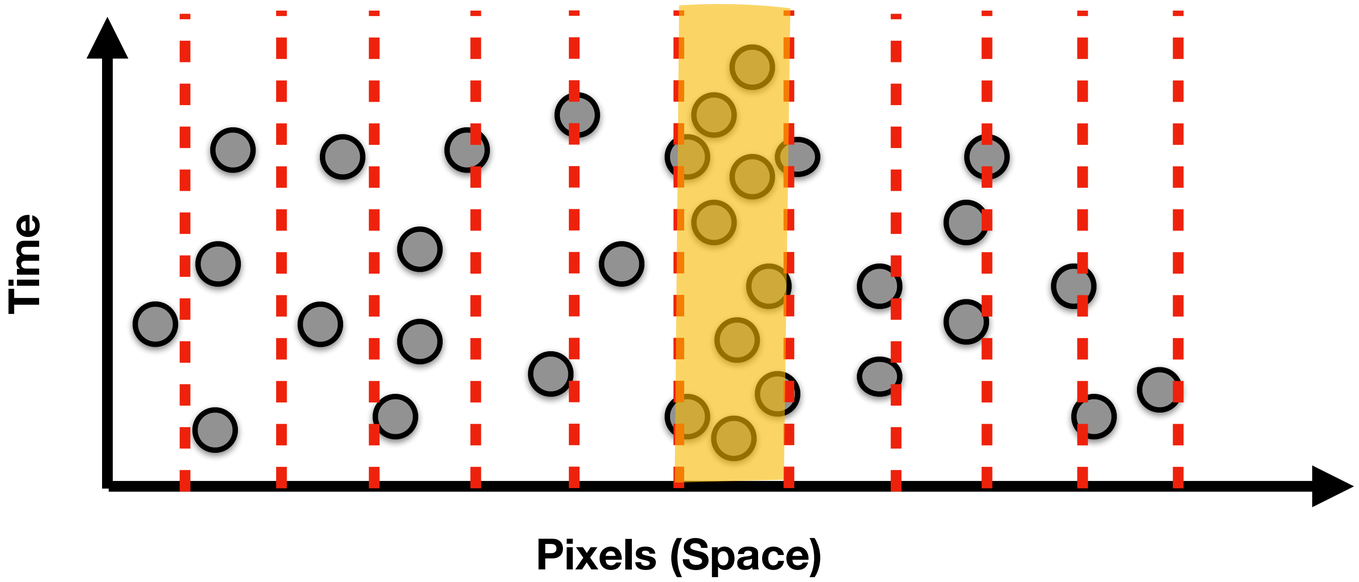
Requires many samples



Standard Method

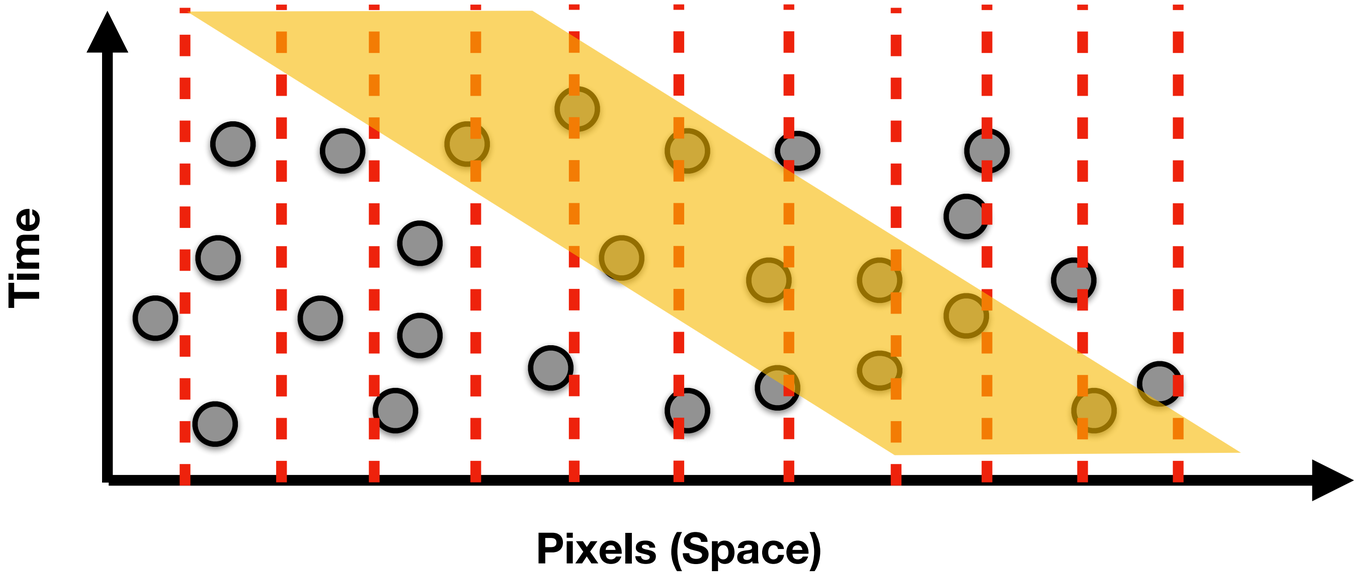
Use axis-aligned pixel filters at each pixel

Requires many samples



Filter shearing based on frequency analysis of light transport

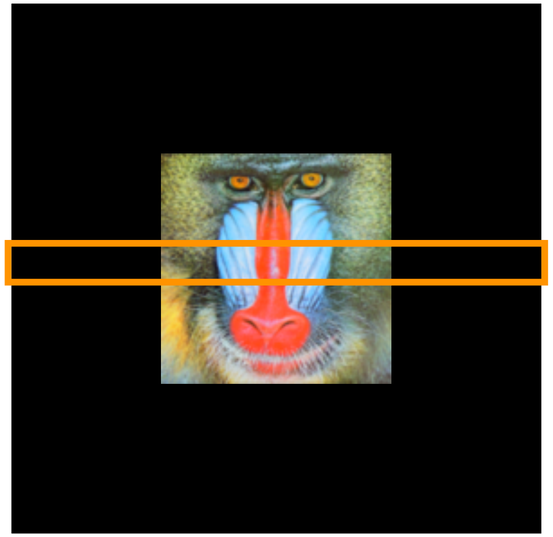
We will look at how to reuse nearby pixel samples to reconstruct using filters derived using the frequency analysis of light transport



Basic Example

No velocity: static scene $t \in [0, 1)$

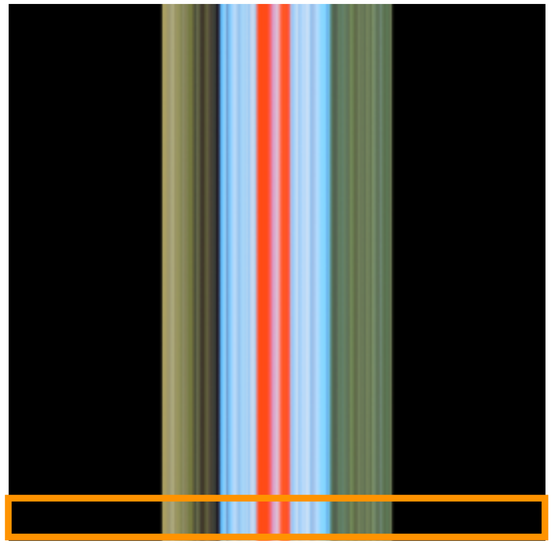
f(x,y)



y

x

f(x,t)



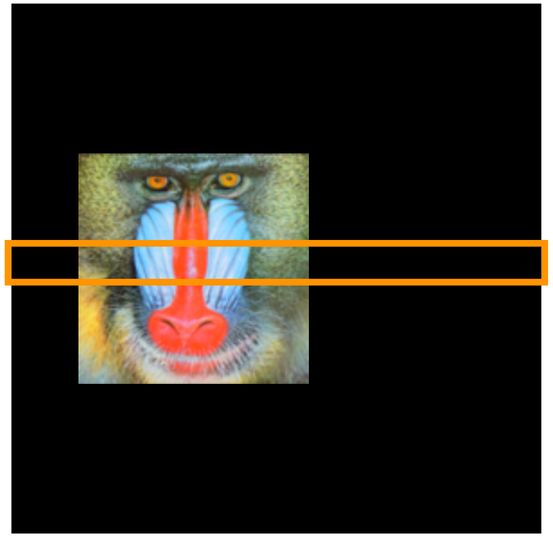
t

x

Basic Example

Low velocity $t \in [0, 1)$

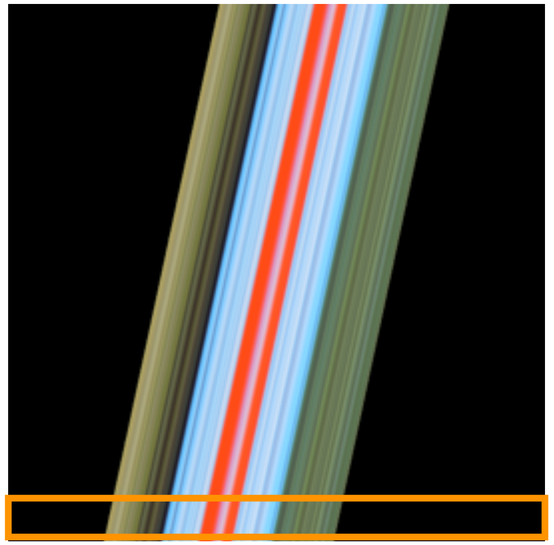
f(x,y)



y

x

f(x,t)



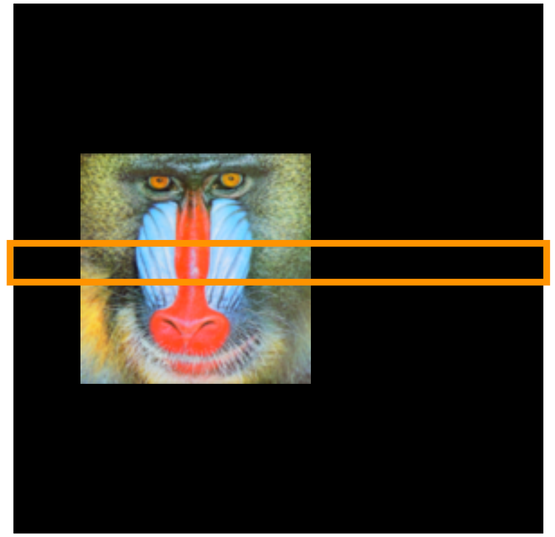
t

x

Basic Example

Low velocity $t \in [0, 1)$

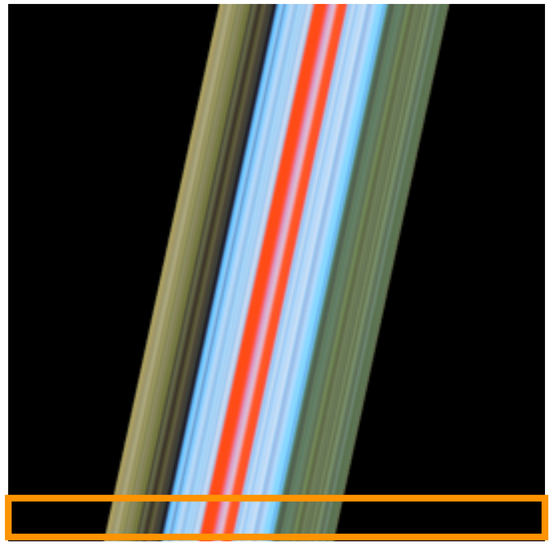
f(x,y)



y

x

f(x,t)



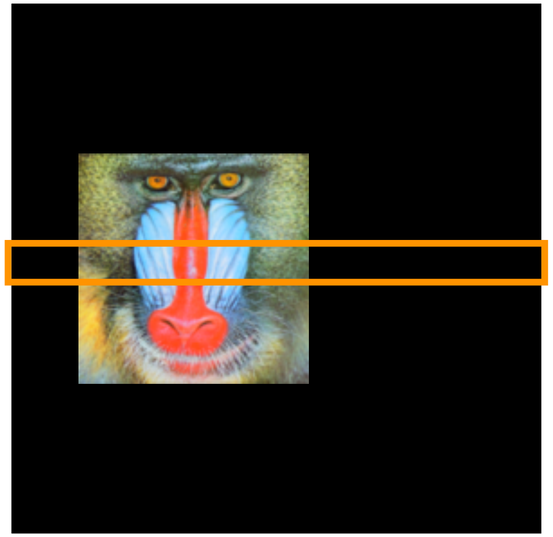
t

x

Basic Example

Low velocity $t \in [0, 1)$

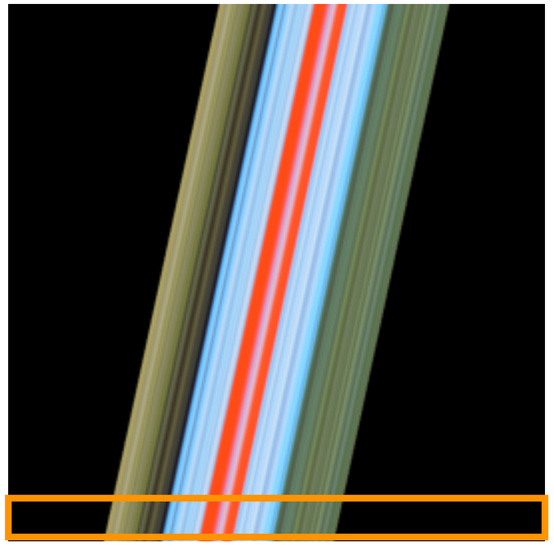
f(x,y)



y

x

f(x,t)



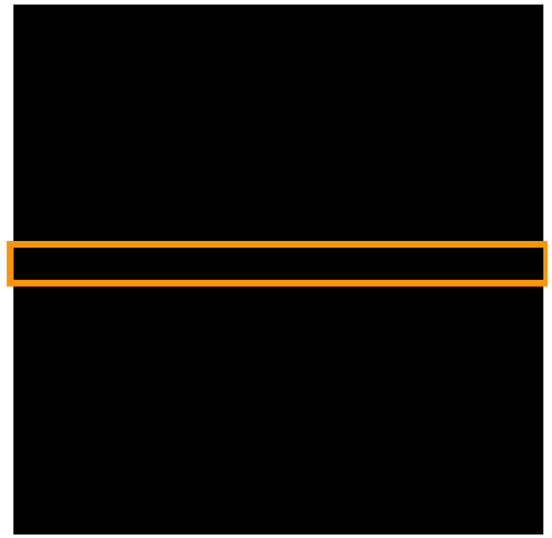
t

x

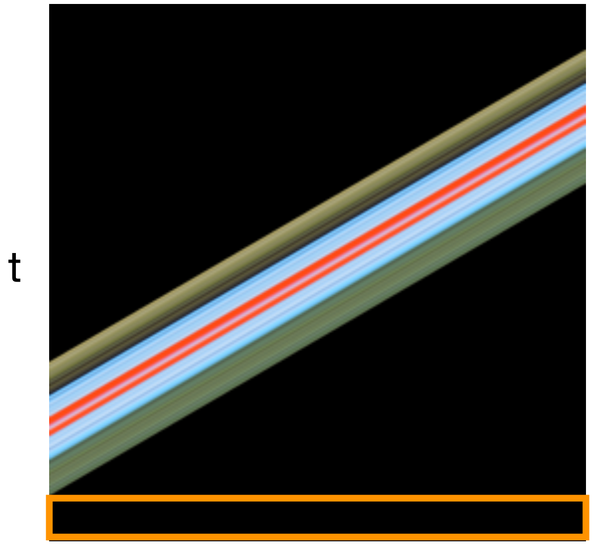
Basic Example

High velocity $t \in [0, 1)$

f(x,y)



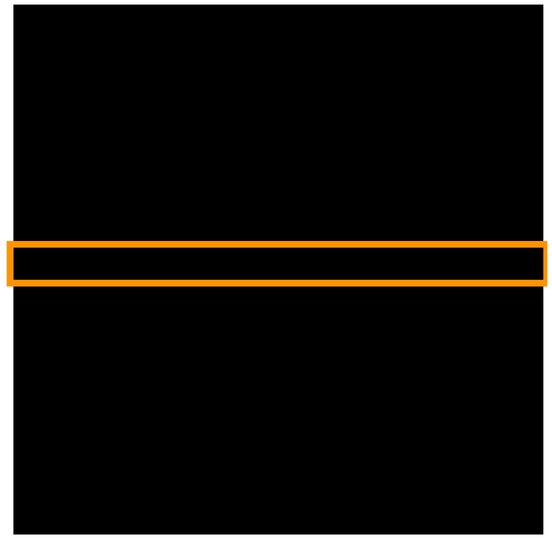
f(x,t)



Basic Example

High velocity $t \in [0, 1)$

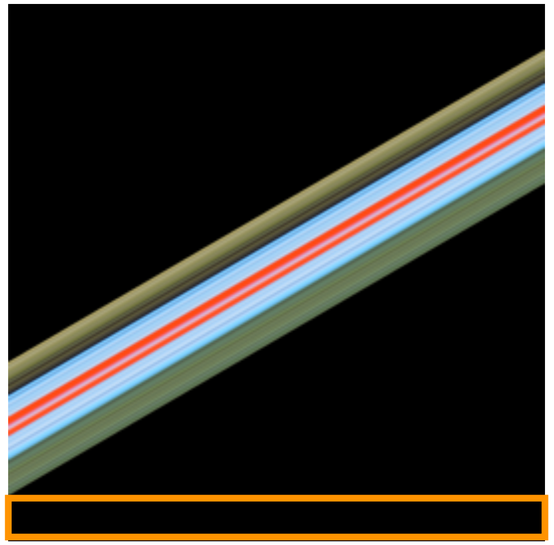
f(x,y)



y

x

f(x,t)

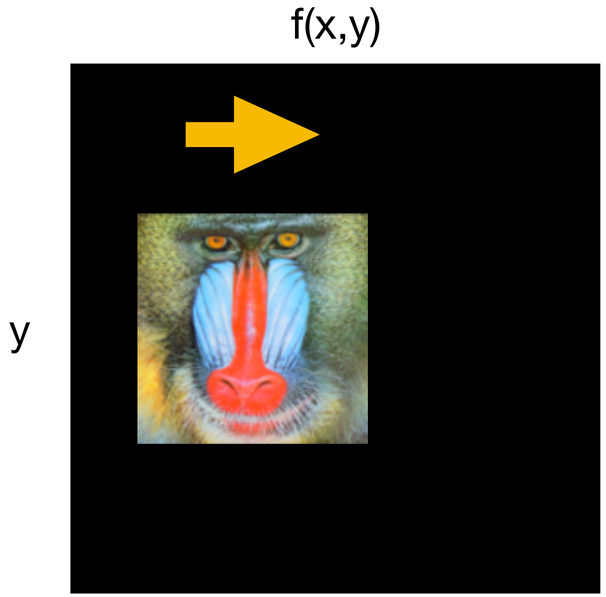


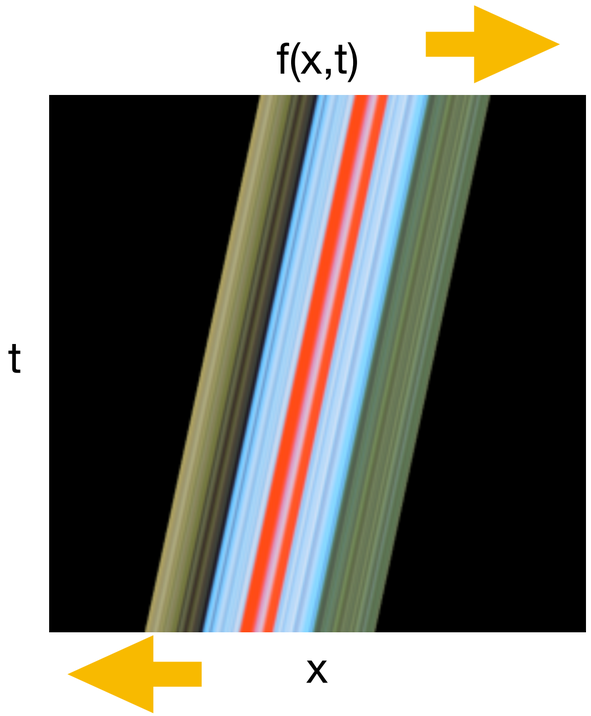
t

x

Shear in space-time

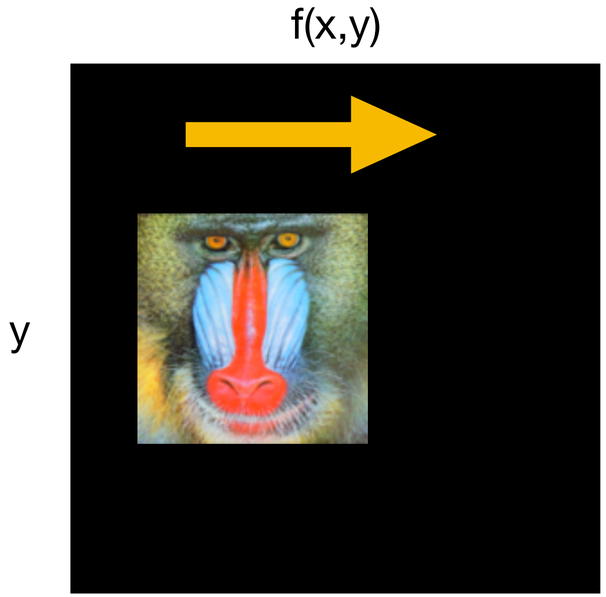
Object moving with low velocity $t \in [0, 1)$

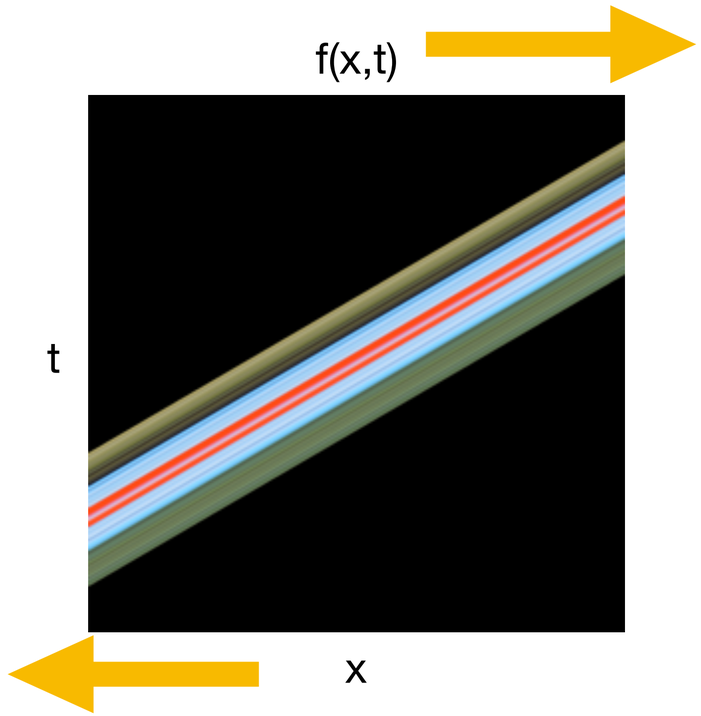




Large shear in space-time

Object moving with high velocity $t \in [0, 1)$





Shear in space-time

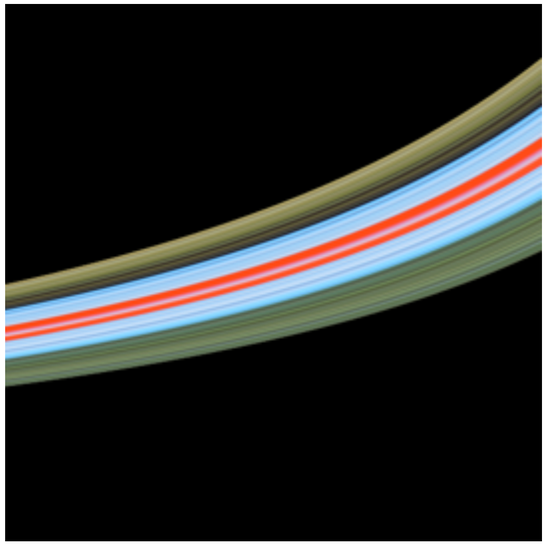
Object moving away from the camera $t \in [0, 1)$

f(x,y)

y

x

f(x,t)



t

x

Camera shutter filter

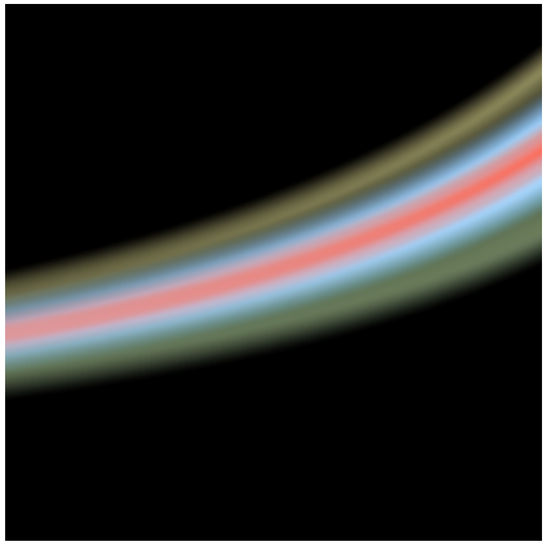
Applying shutter blur across time $t \in [0, 1)$

f(x,y)

y

x

f(x,t)

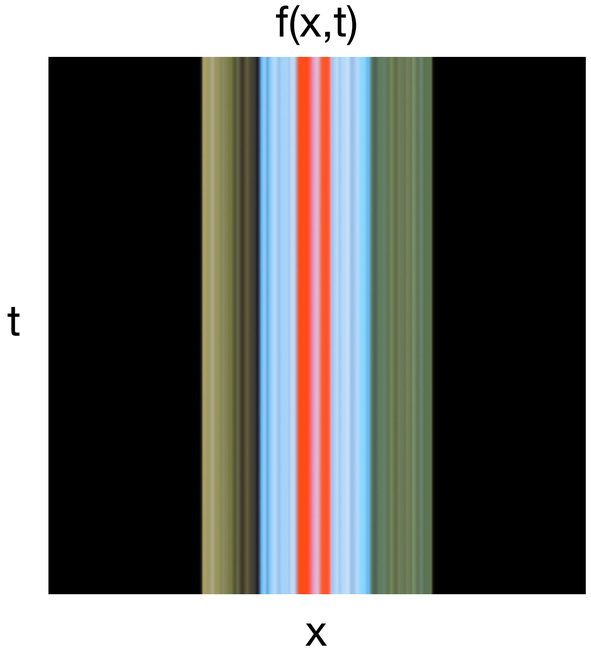


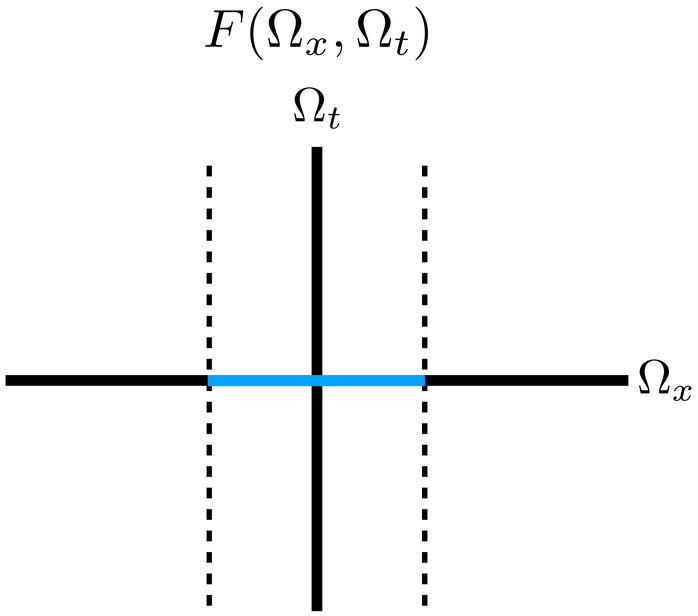
t

x

Basic example: Fourier domain

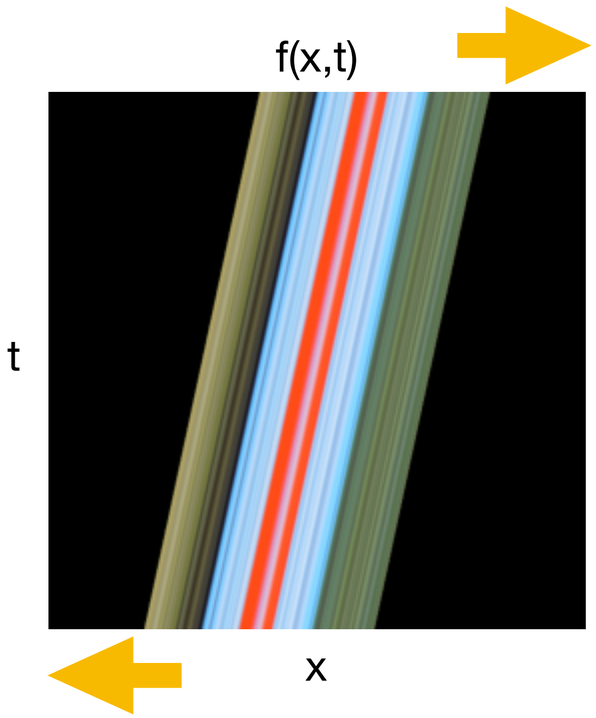
Fourier spectrum, zero velocity $t \in [0, 1)$



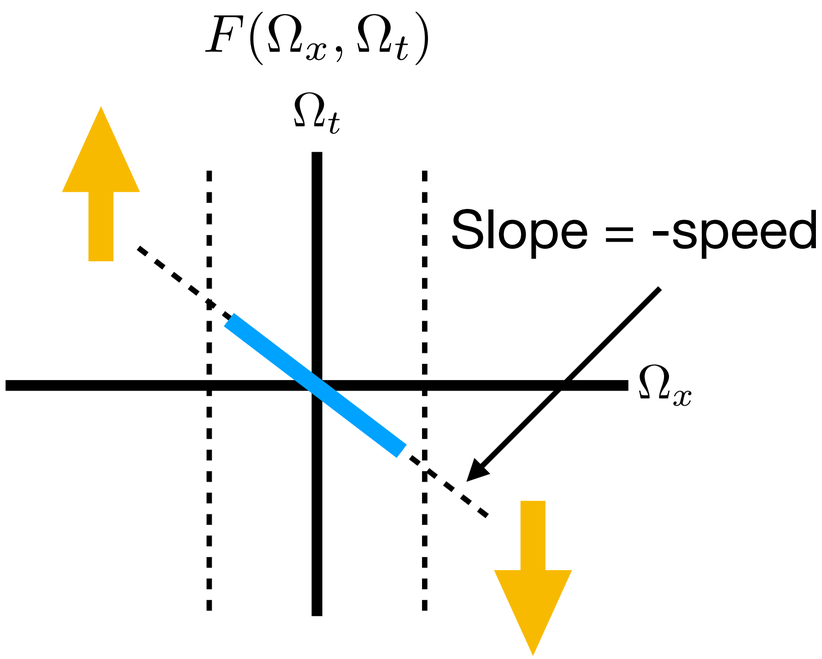


Basic example: Fourier domain

Low velocity, small shear in both domains

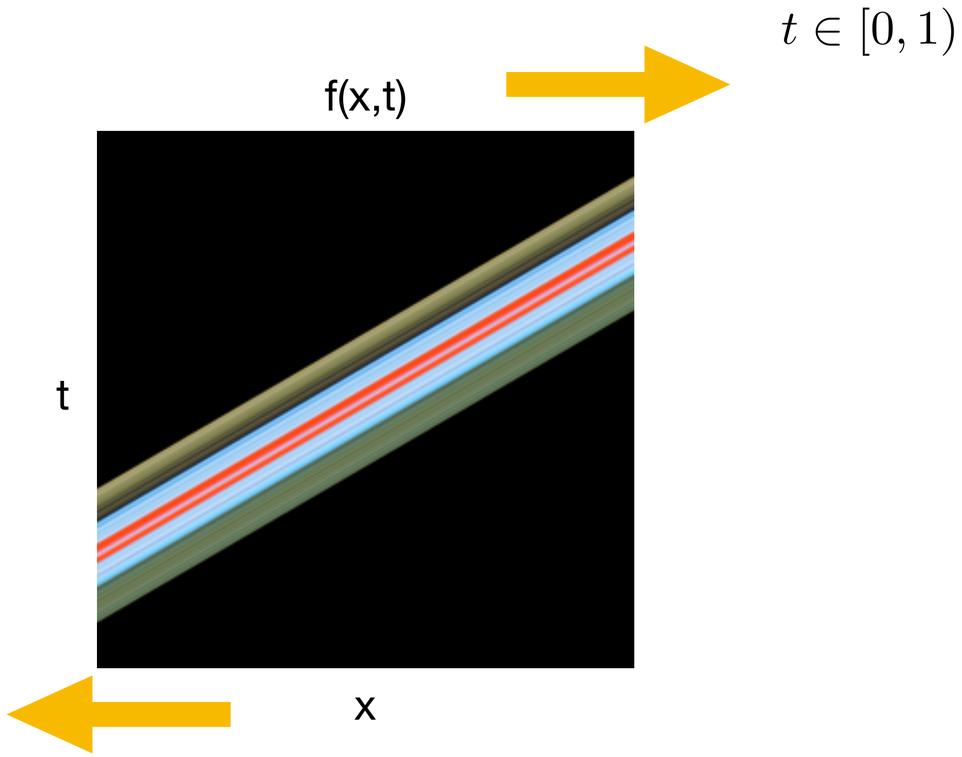


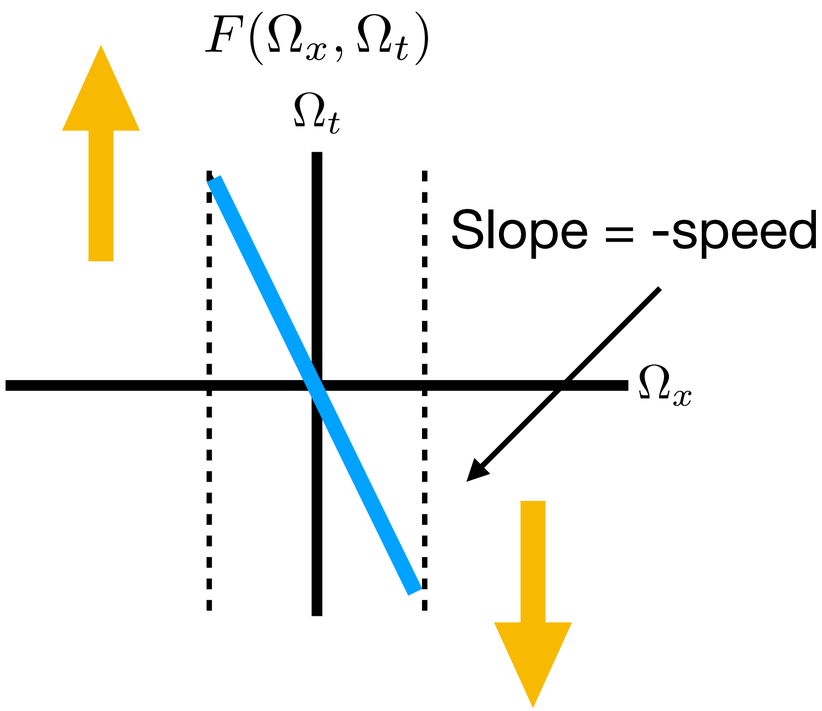
$t \in [0, 1)$



Basic example: Fourier domain

Low velocity, small shear in both domains



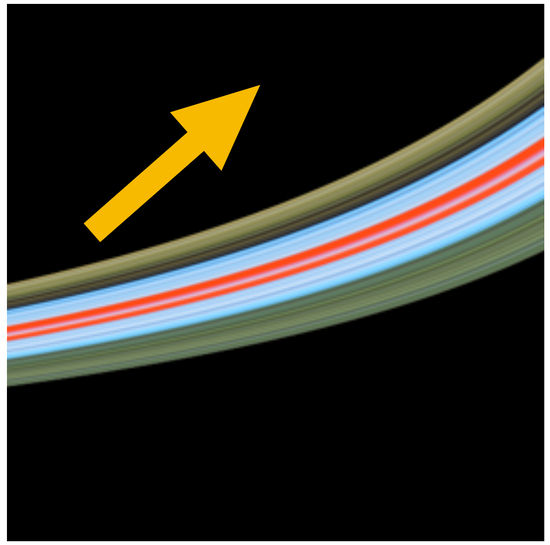


Basic example: Fourier domain

Due to camera motion, slopes are varying

$t \in [0, 1)$

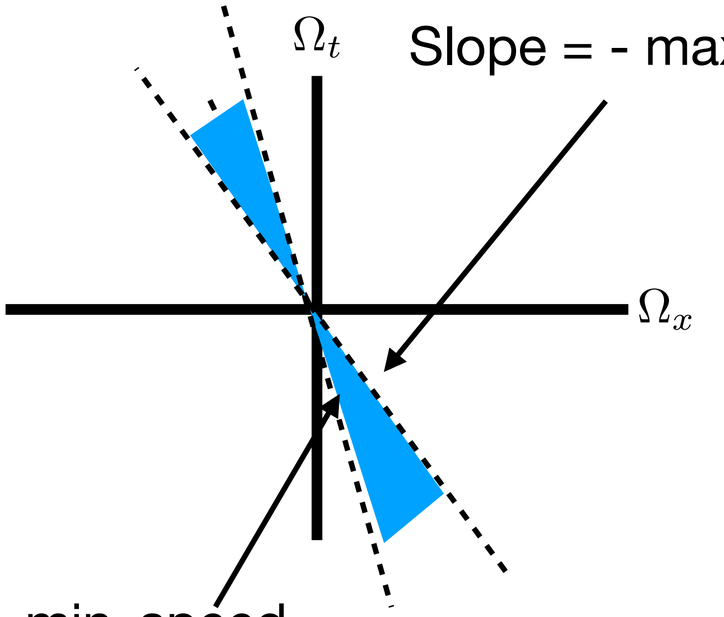
f(x,t)



t

x

$F(\Omega_x, \Omega_t)$



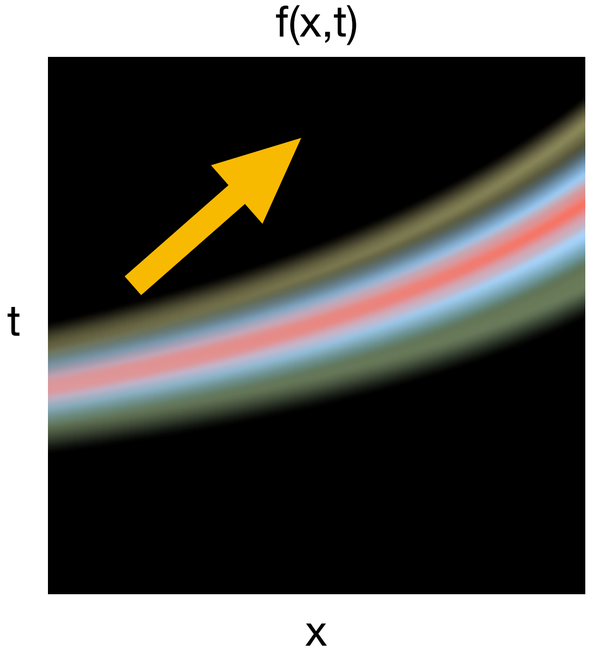
Slope = - max_speed

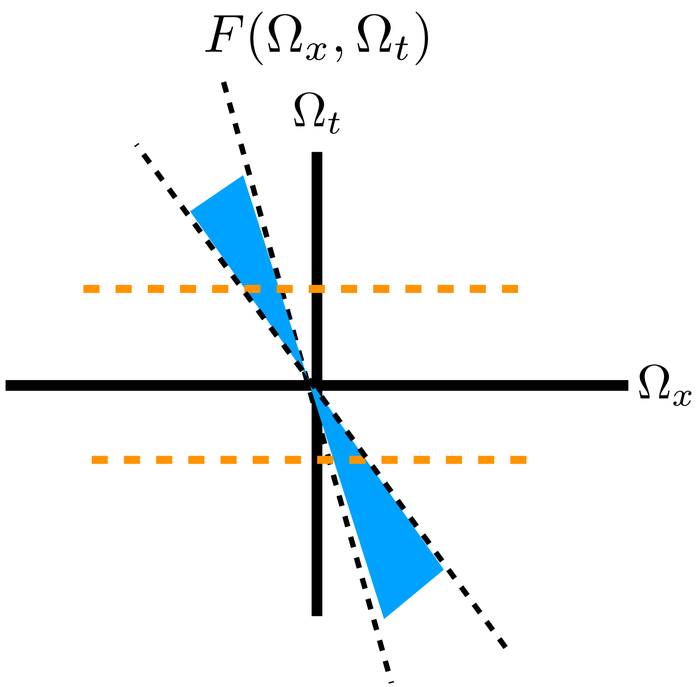
Slope = - min_speed

Basic example: Fourier domain

When shutter blur is applied, only low frequencies matter

$t \in [0, 1)$

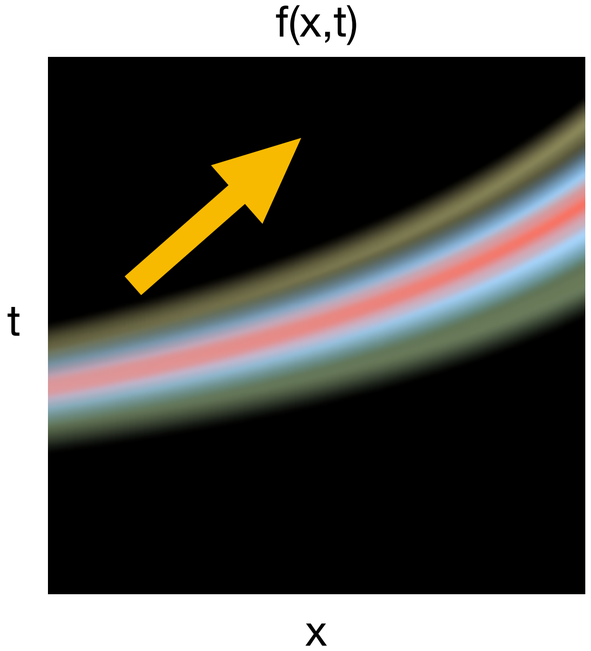


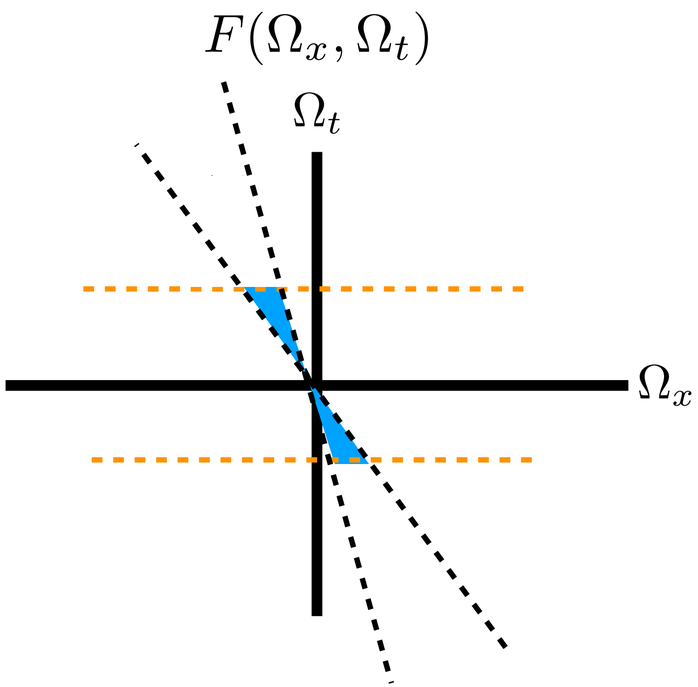


Basic example: Fourier domain

When shutter blur is applied, only low frequencies matter

$t \in [0, 1)$

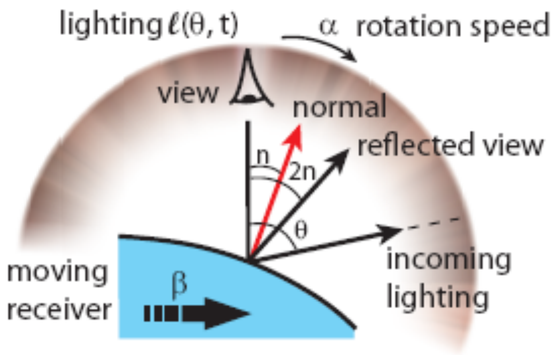


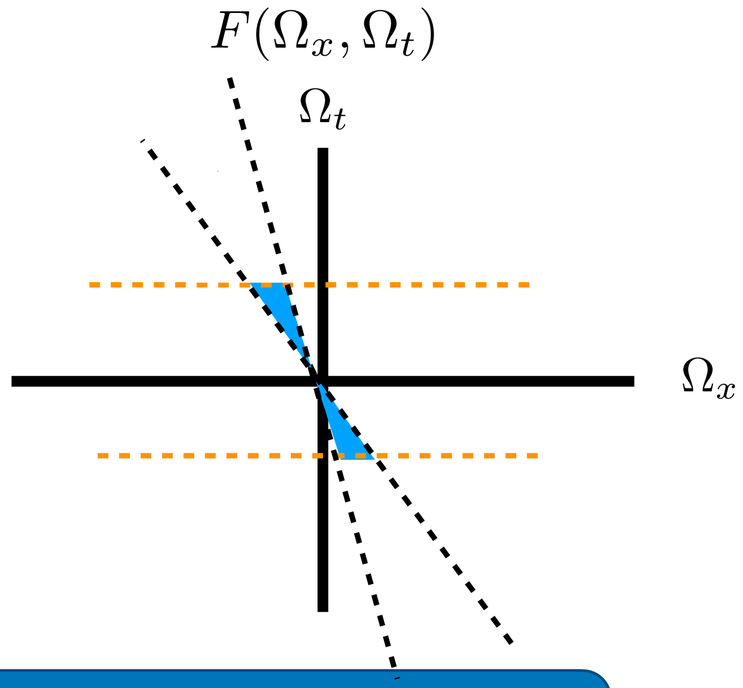


Main Insights

Common case = double wedge spectra

Shutter indirectly removes spatial frequencies





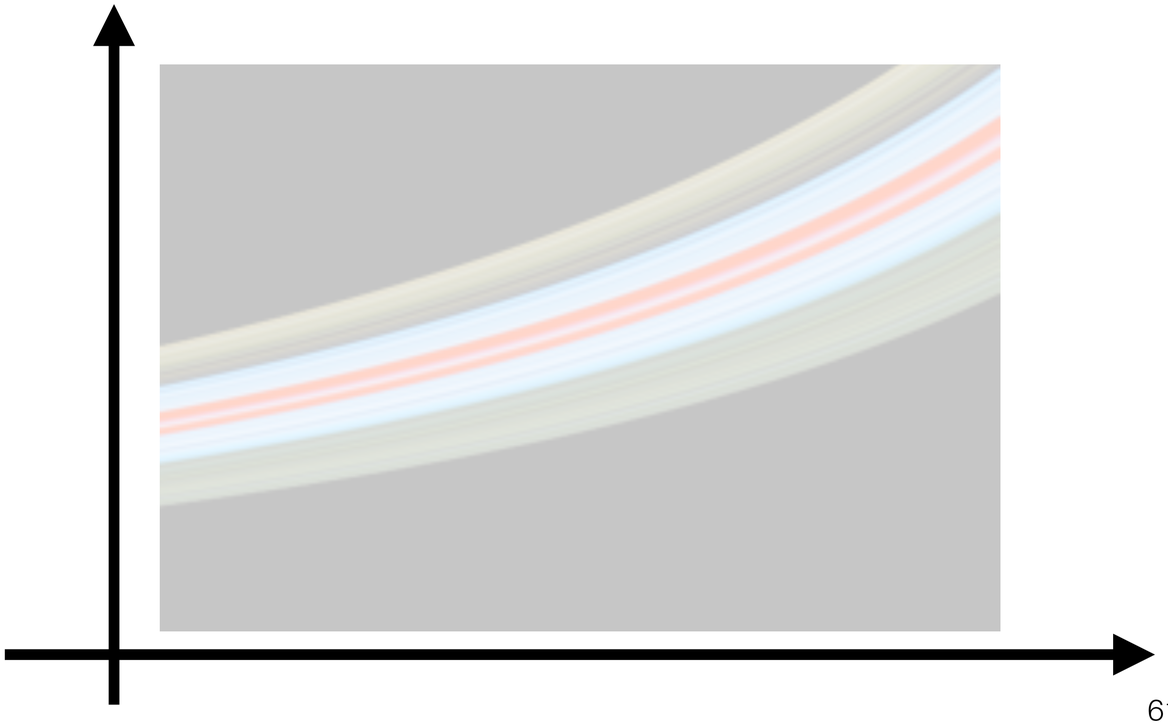
Sampling and Filtering Goals

Minimal sampling rate to prevent aliasing

Derive shape of the reconstruction filters

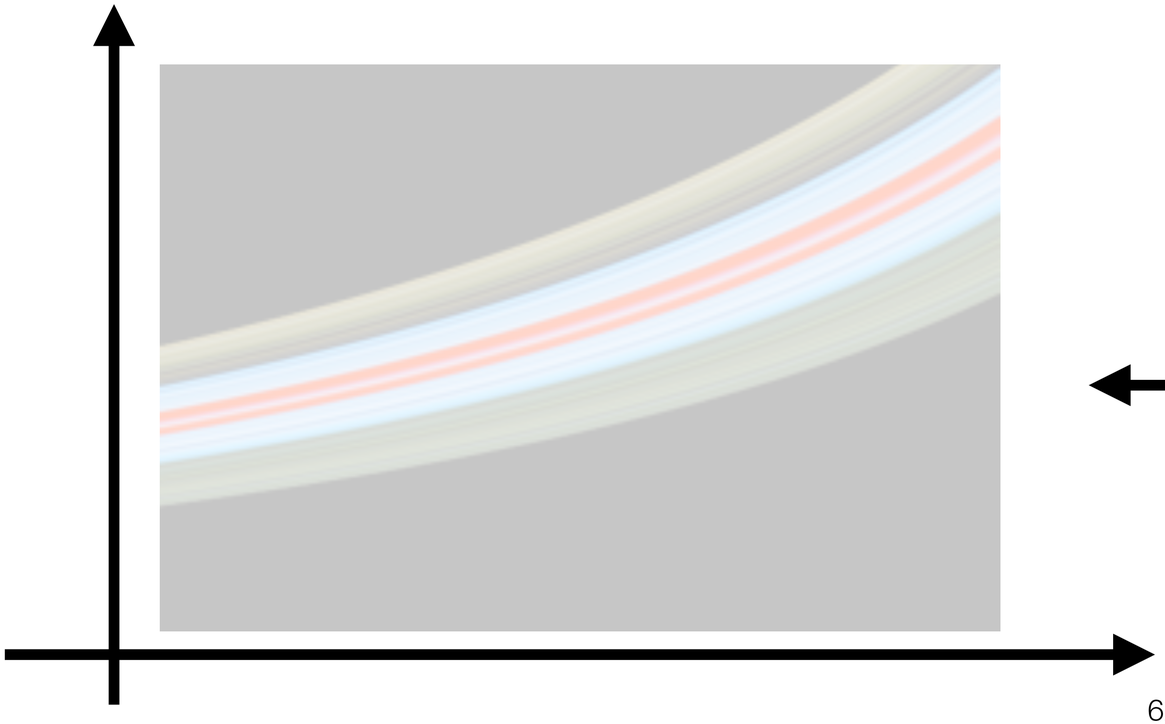
Sampling in Fourier Domain

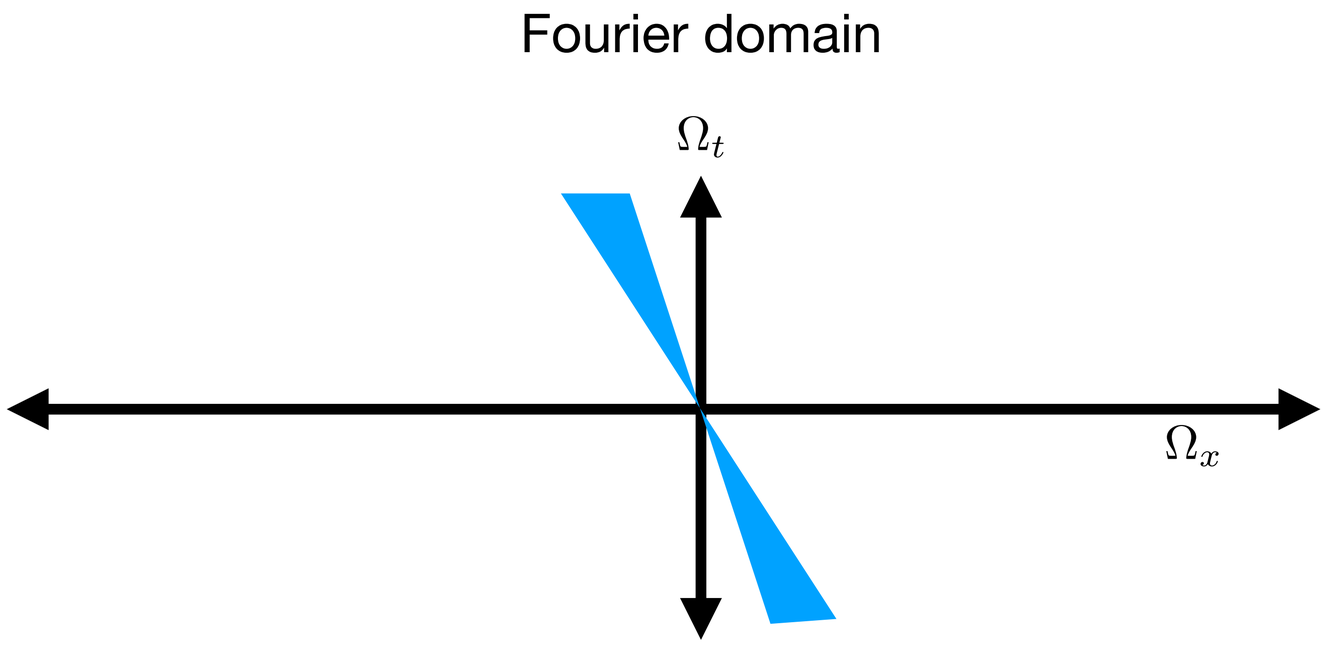
Sampling produces replicas in the Fourier domain



Sampling in Fourier Domain

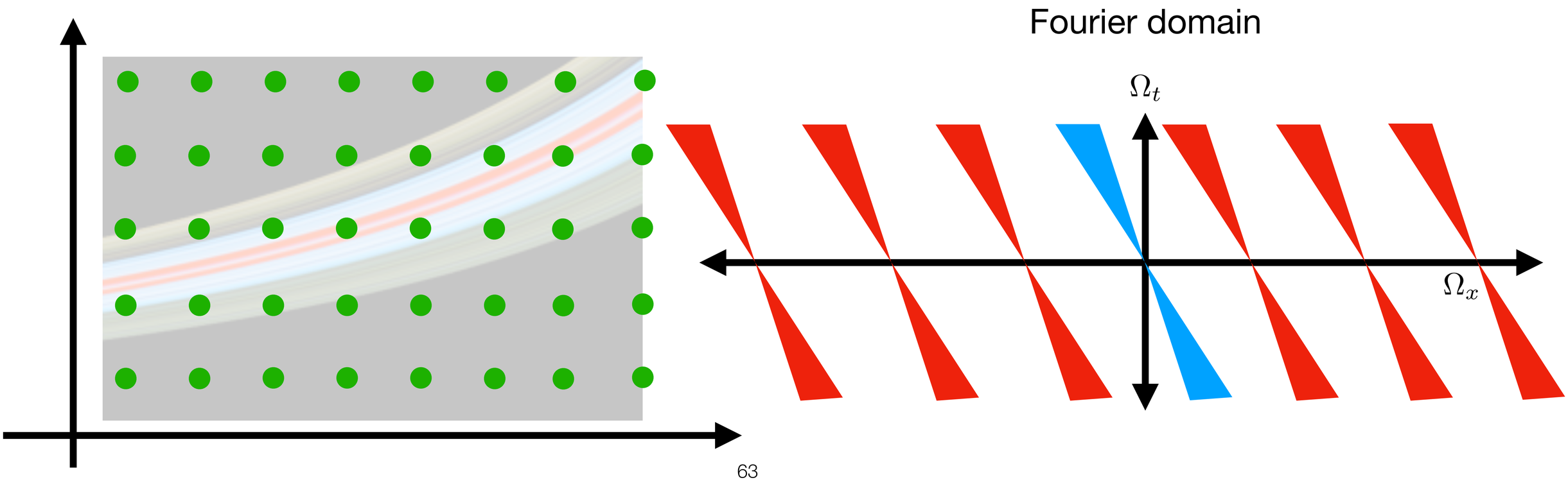
Let's say the corresponding image has a Fourier spectrum as shown on the right side





Sampling in Fourier Domain

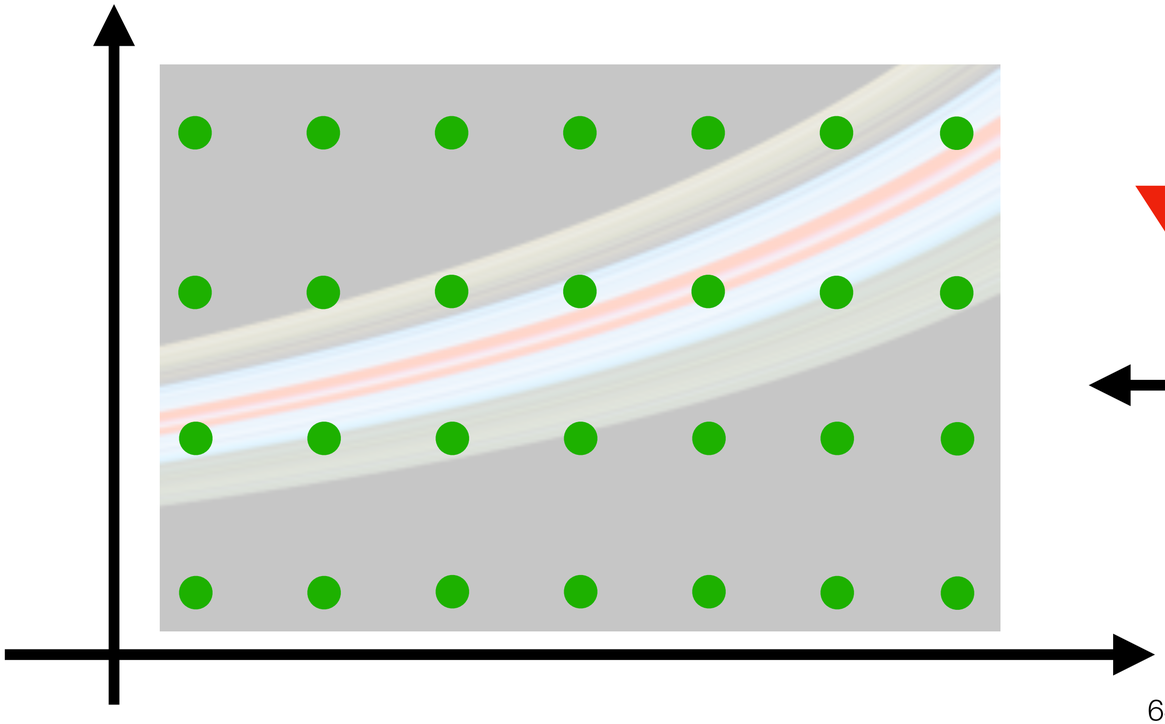
Sampling produces replicas in the Fourier domain



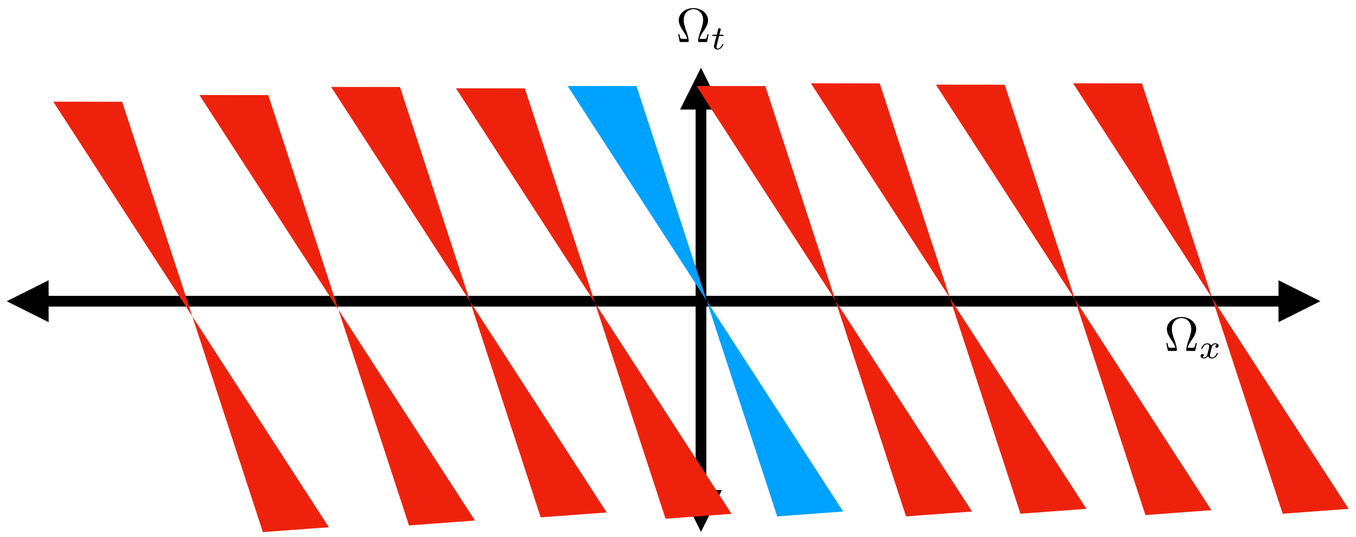
Sampling in Fourier Domain

Sampling produces replicas in the Fourier domain

Sparse sampling produces denser replicas

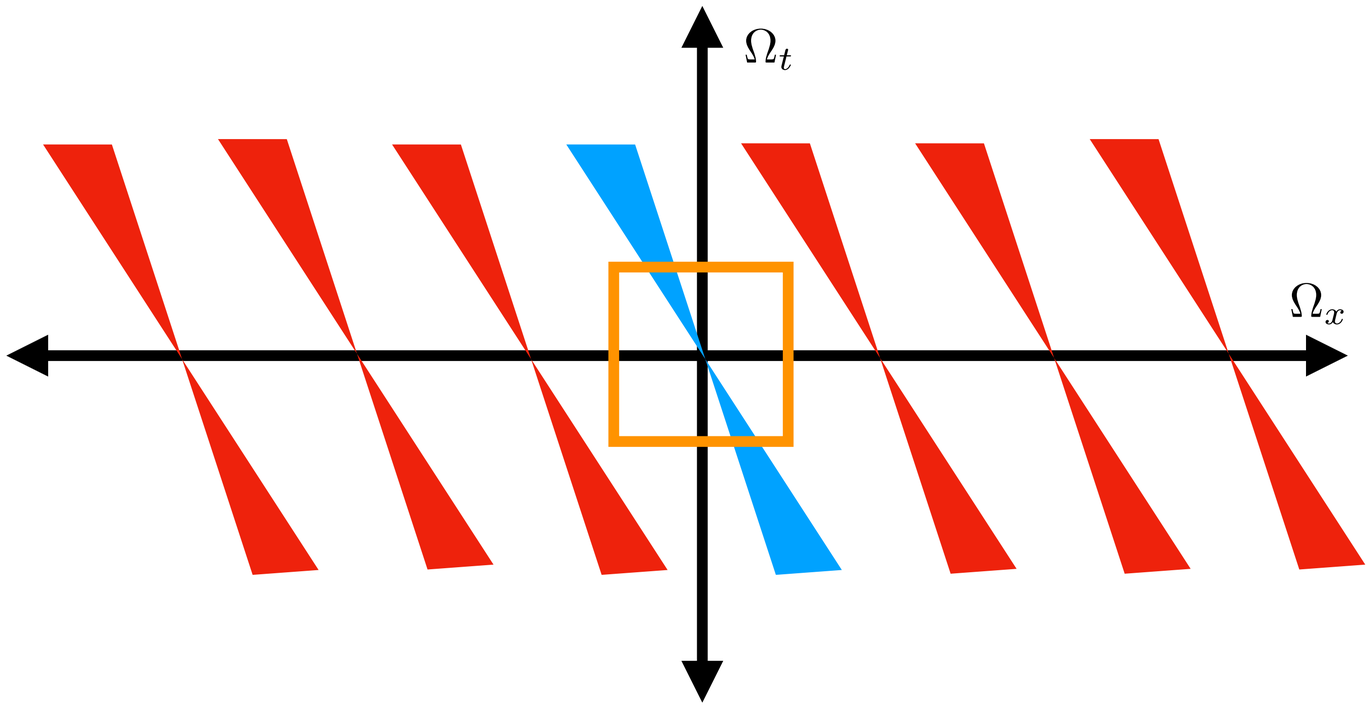


Fourier domain



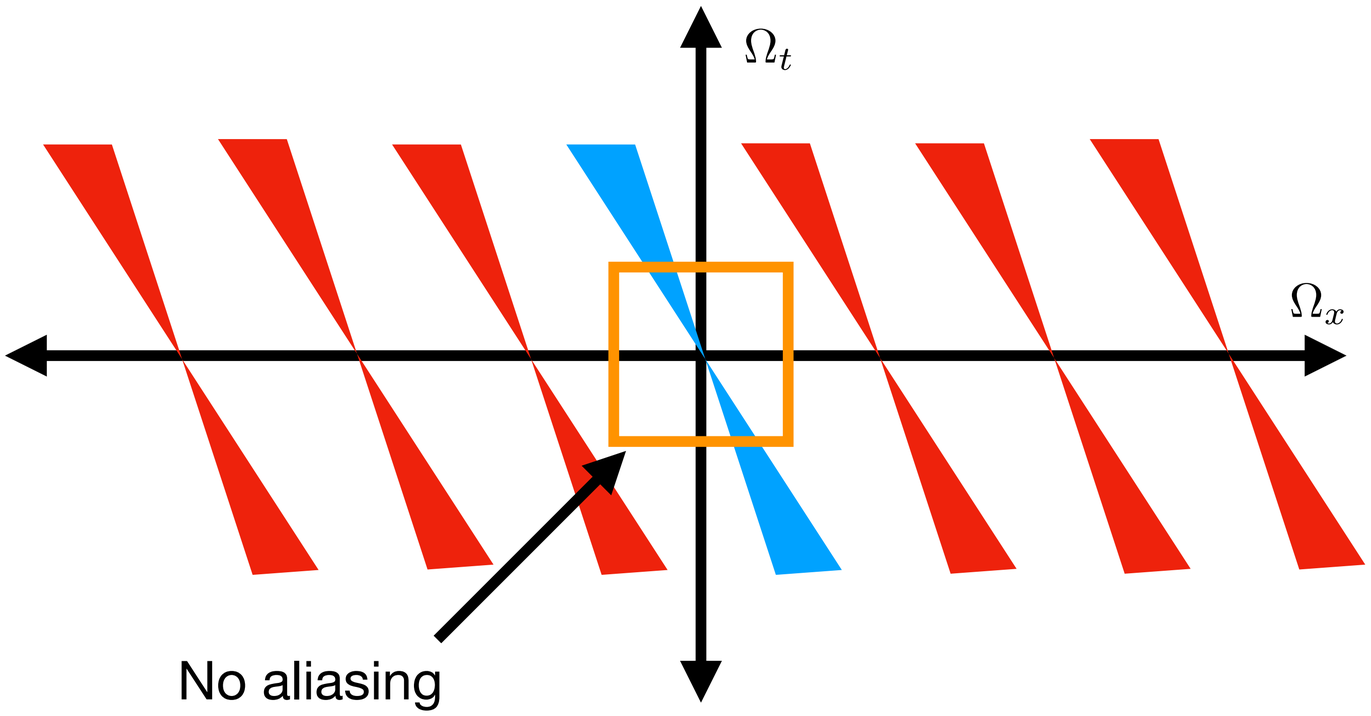
Standard Reconstruction Filtering

Standard filer, dense sampling, slow



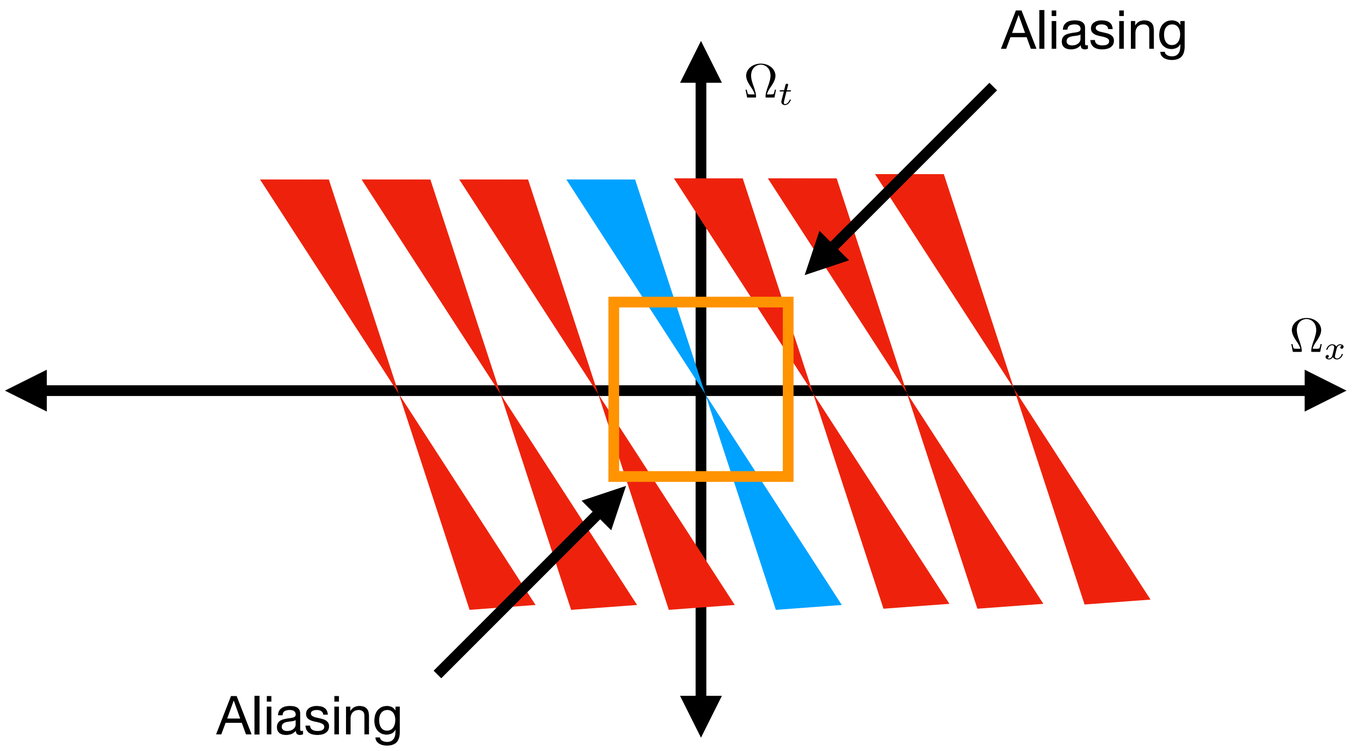
Standard Reconstruction Filtering

Standard filer, dense sampling, slow



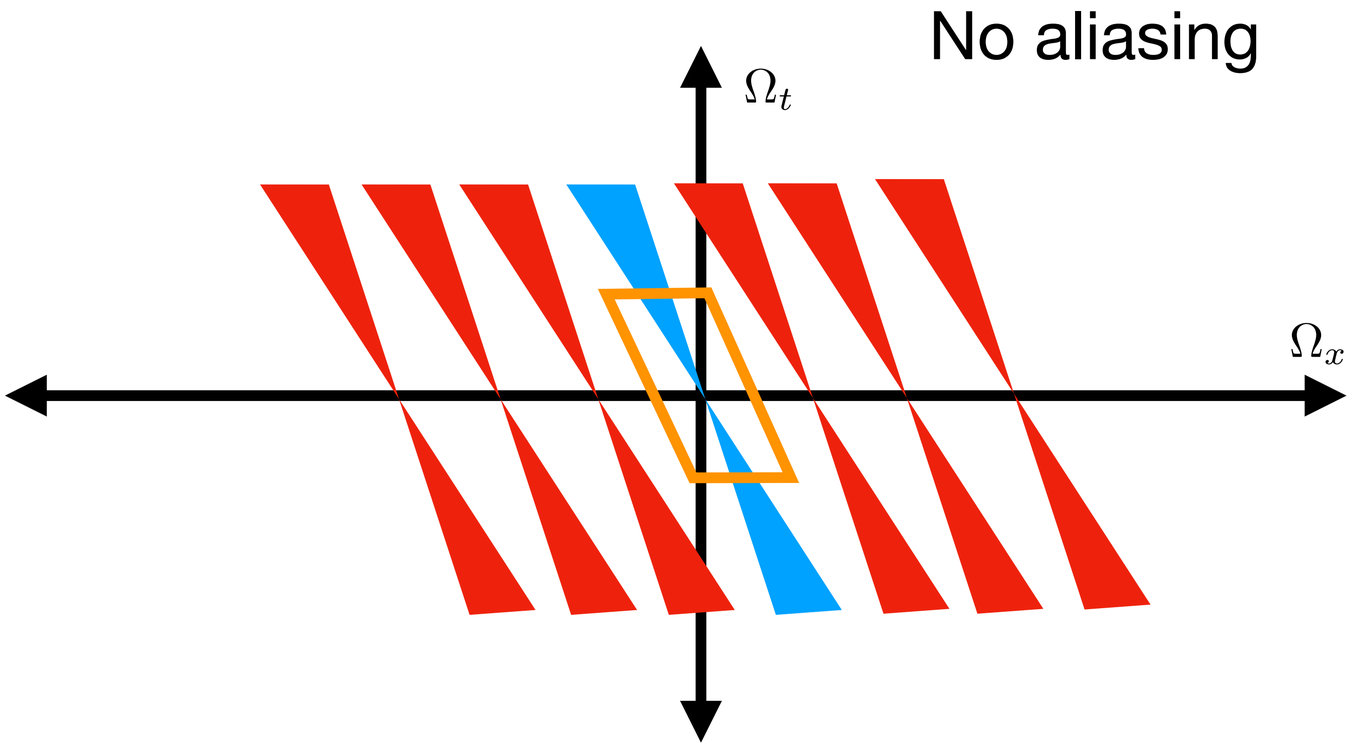
Standard Reconstruction Filtering

Standard filer, sparse sampling, fast



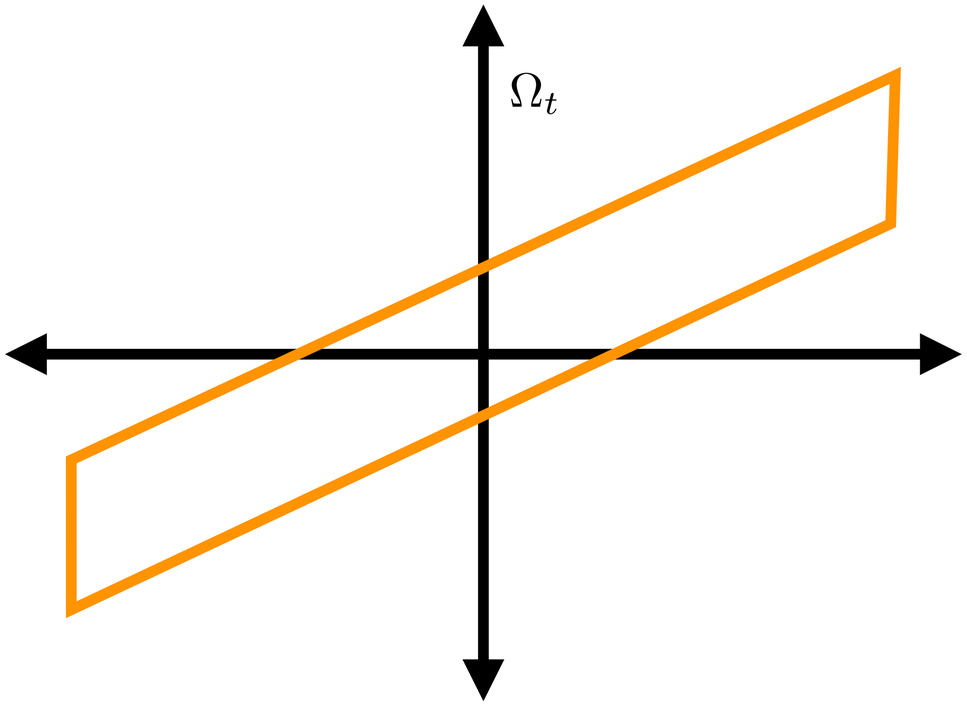
Sheared Reconstruction Filter

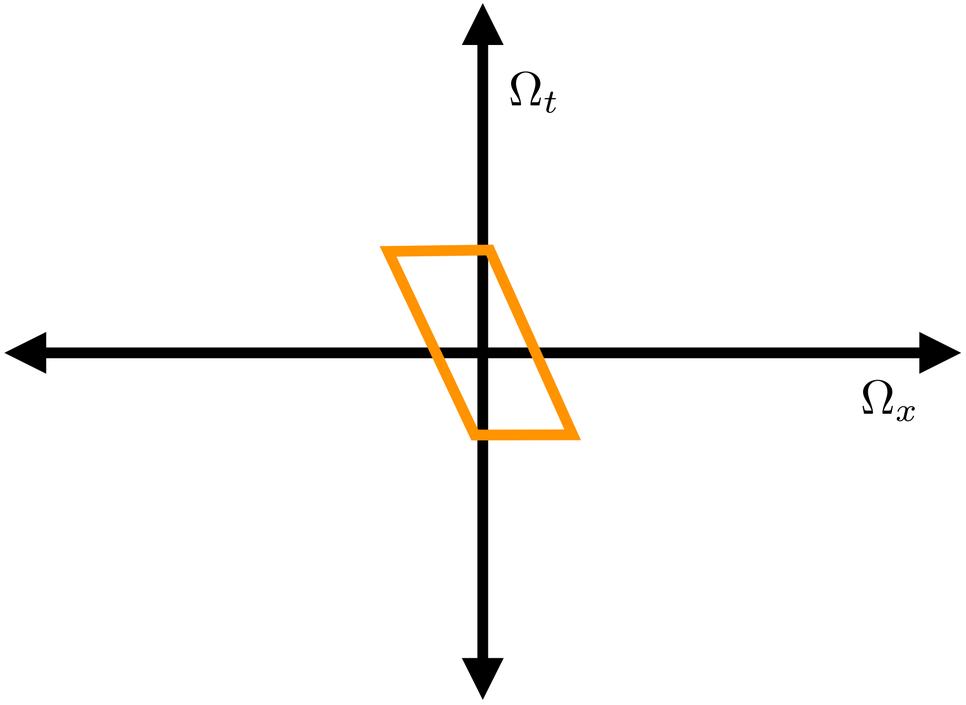
Standard filer, sparse sampling, fast



Sheared Reconstruction Filter

Compact shape in Fourier = wide space-time





Main Insights

Sheared filter allows for many fewer samples

Filters in action: Car example

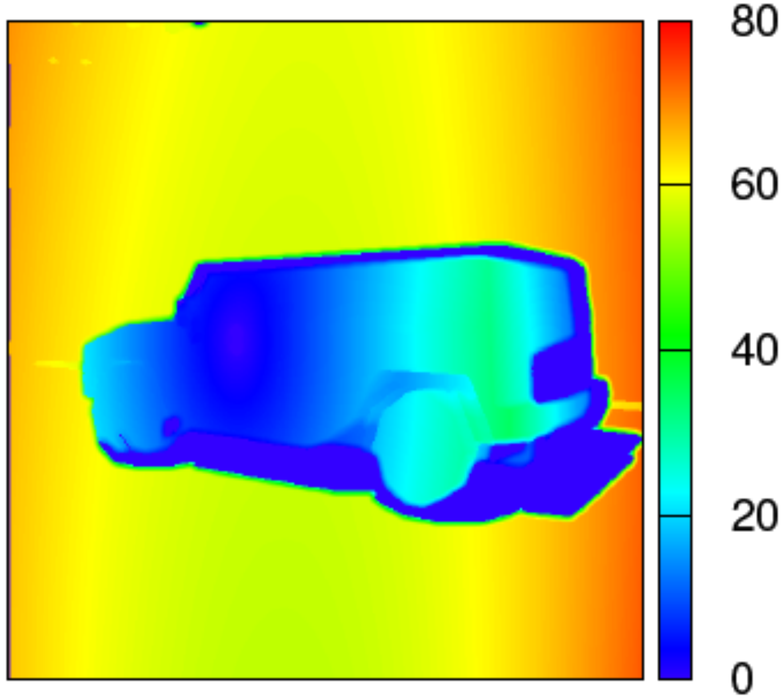
Static

Motion blurred

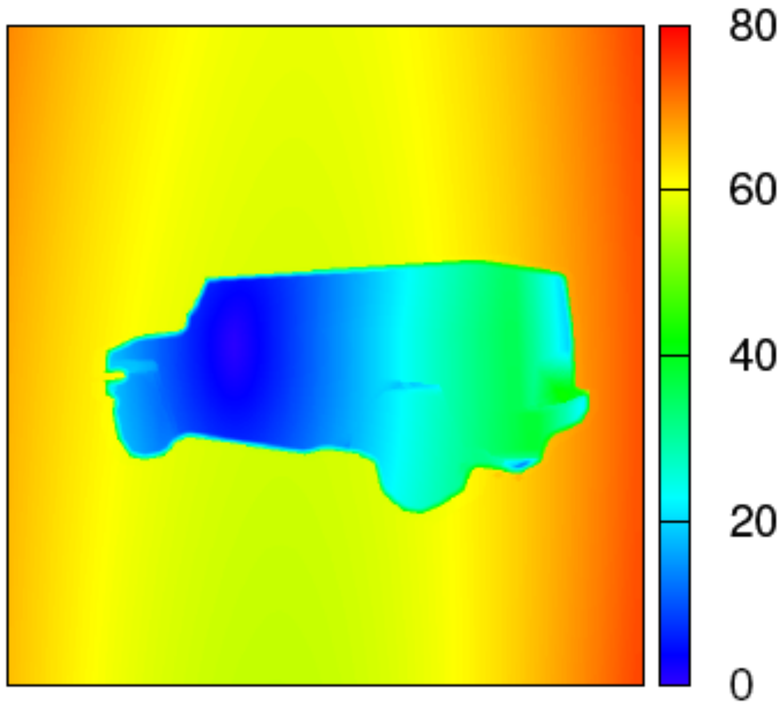
Implementation: stage 1

Sparse sampling to compute velocity bounds

min speed



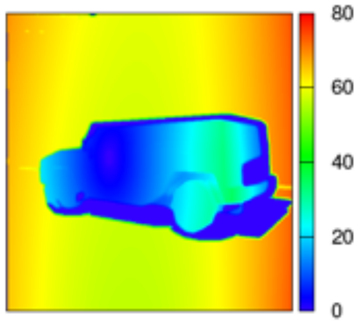
max speed



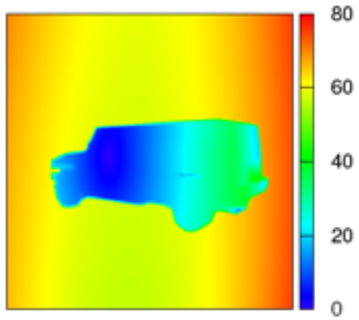
Implementation: stage 1

Calculate filter widths and sampling rates

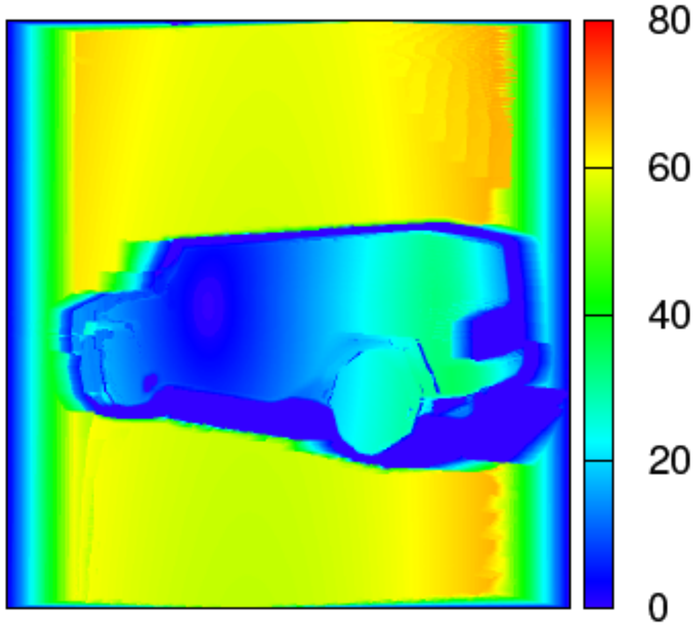
min speed



max speed



filter width

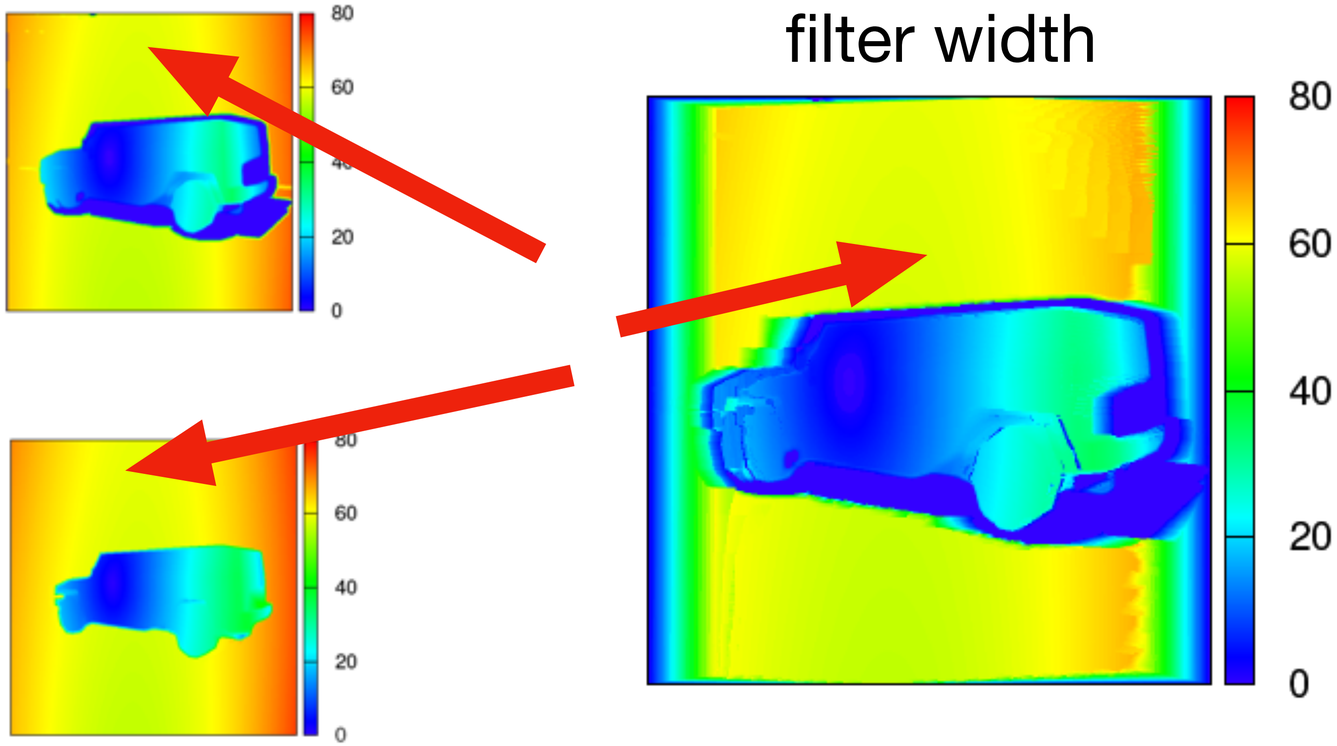


Implementation: stage 1

Uniform velocities, wide filter, low samples

min speed

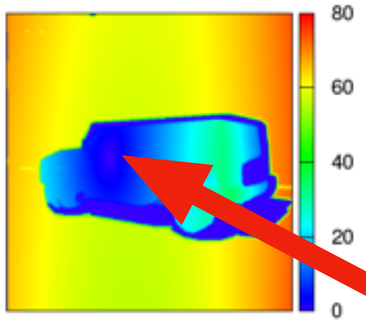
max speed



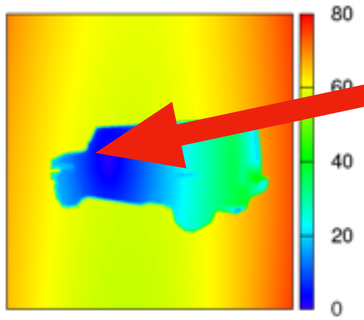
Implementation: stage 1

Static surface, small filter, low samples

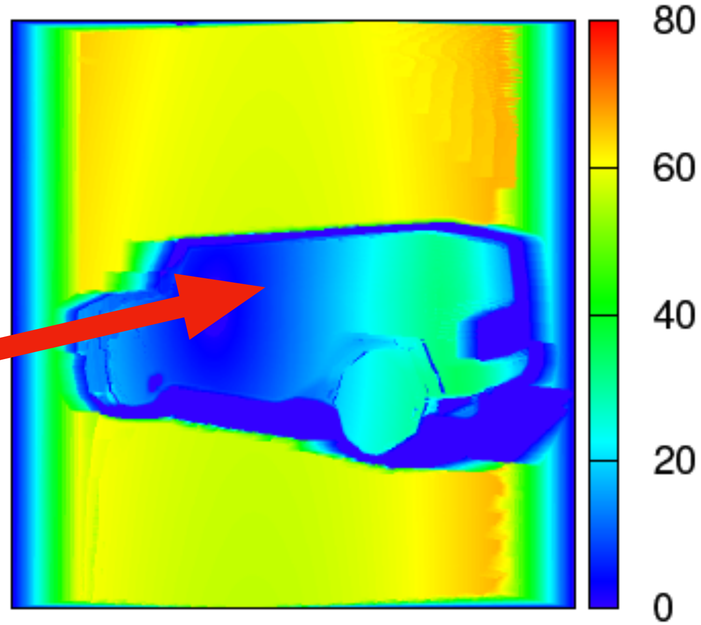
min speed



max speed



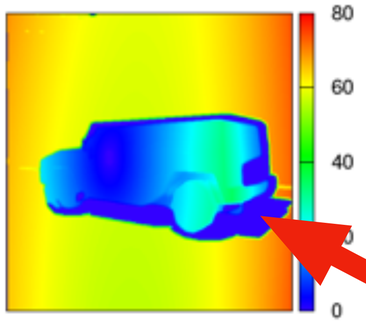
filter width



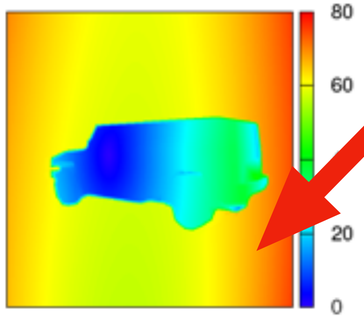
Implementation: stage 1

Varying velocities, small filter, high samples

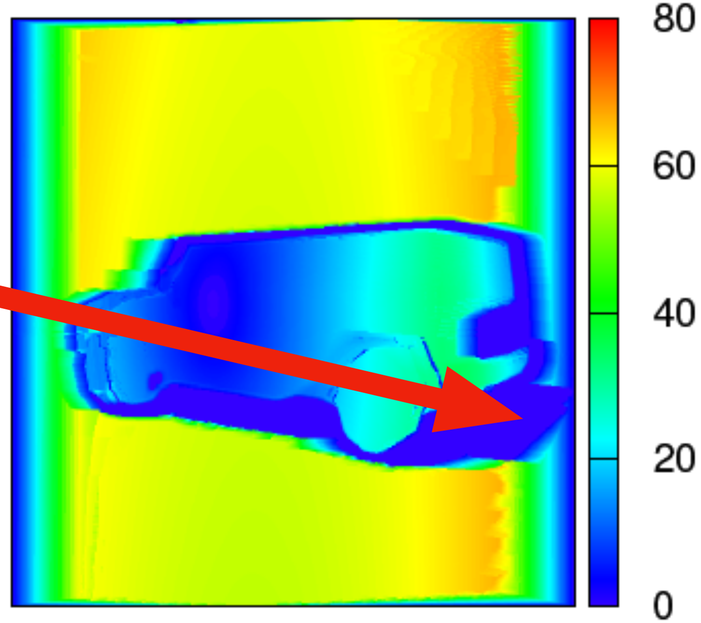
min speed



max speed



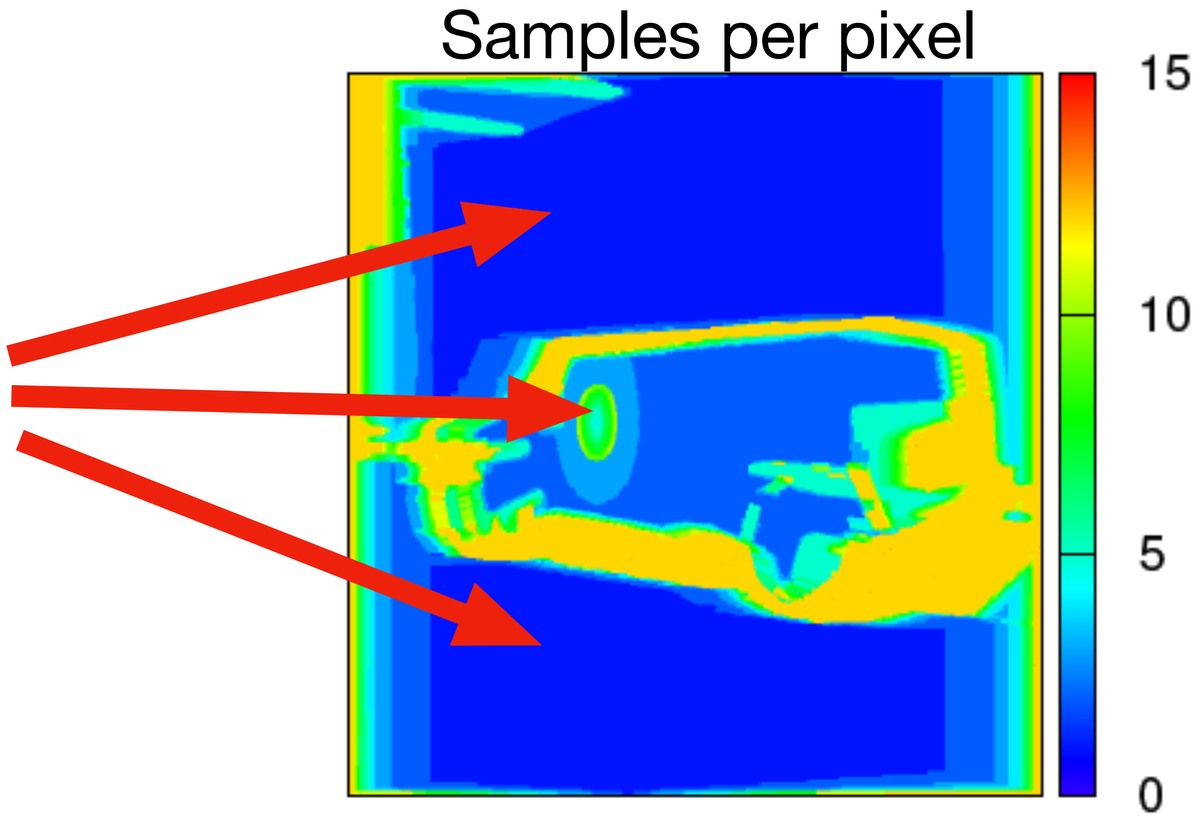
filter width



Implementation: stage 2

Then, compute sampling densities

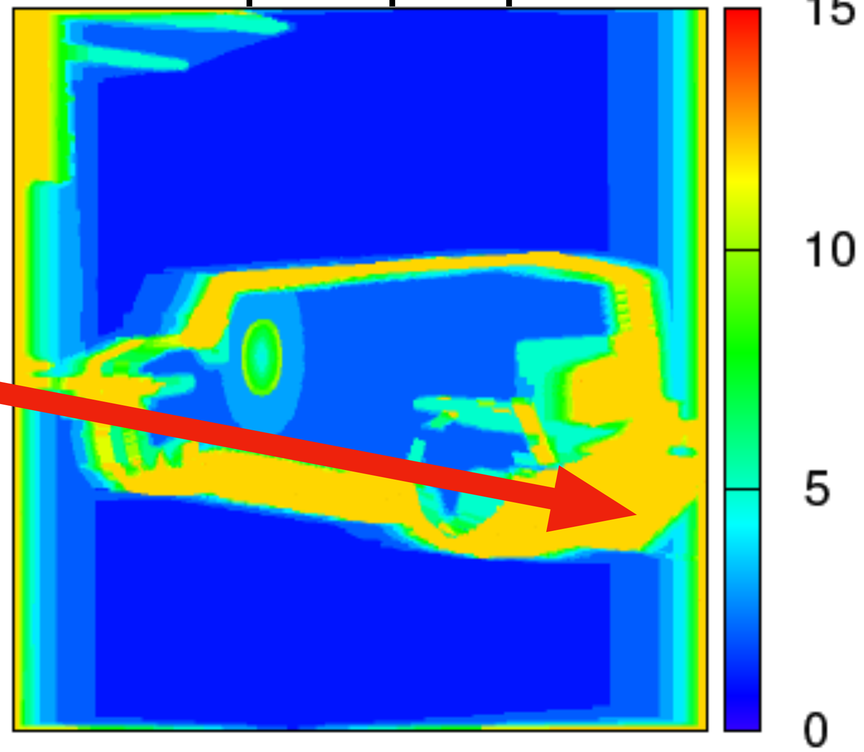
Uniform velocities = low sample count



Implementation: stage 2

Then, compute sampling densities

Samples per pixel

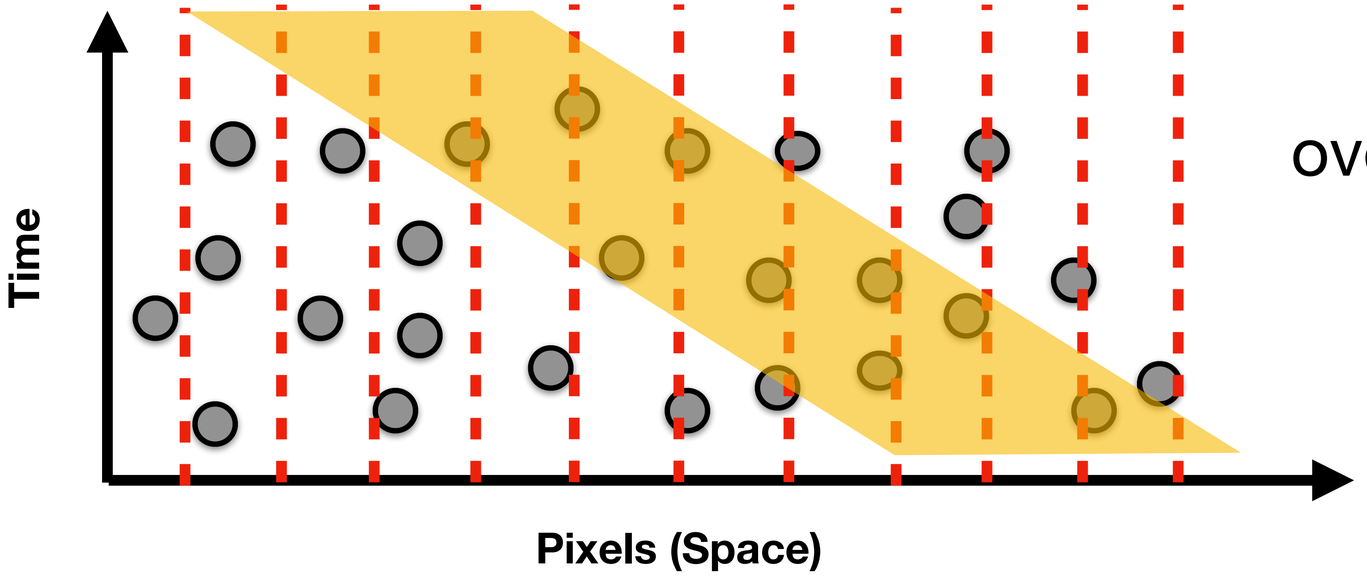


Varying velocities = high sample count

Implementation: stage 3

Render sample locations in space-time

Apply sheared filters to nearby samples



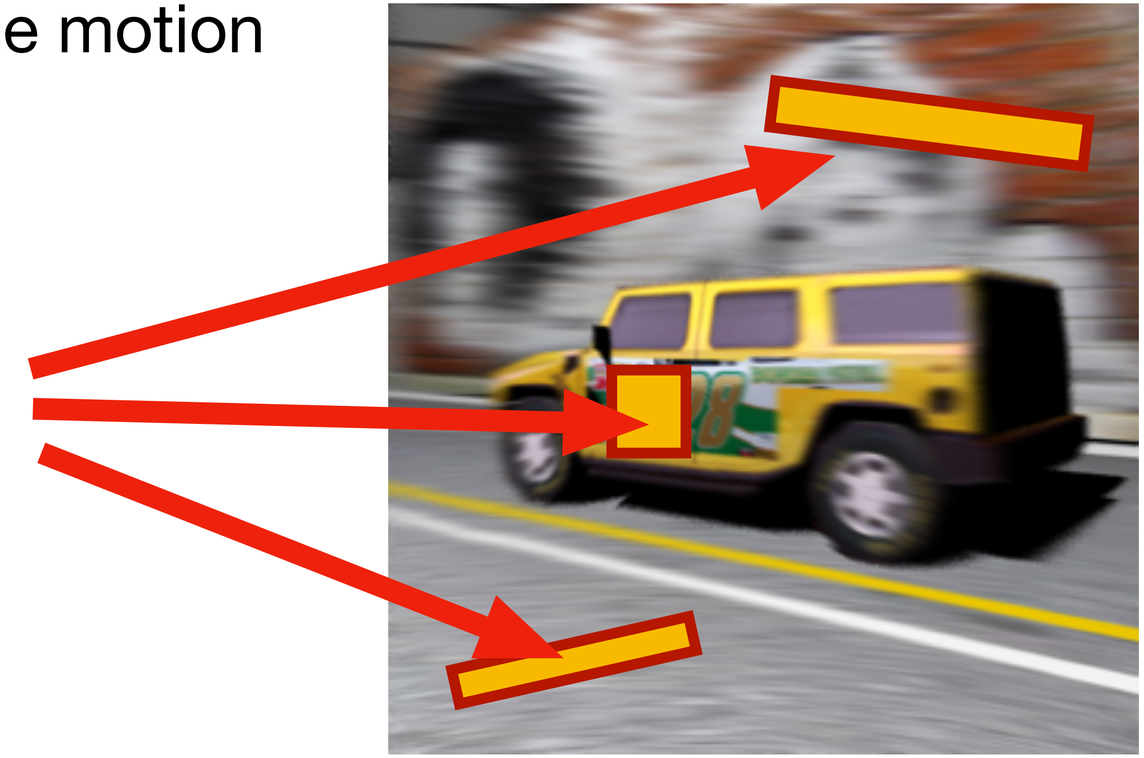
Sheared filters overlaps samples across multiple pixels

Implementation: stage 3

Filters stretched along the direction of motion

Preseve frequencies orthogonal to the motion

Filter shapes



Results

Sheared Filters
4spp

Stratified Sampling
4spp

Multi-dimensional adaptive sampling of distribution effects

Fourier Analysis of Light Transport

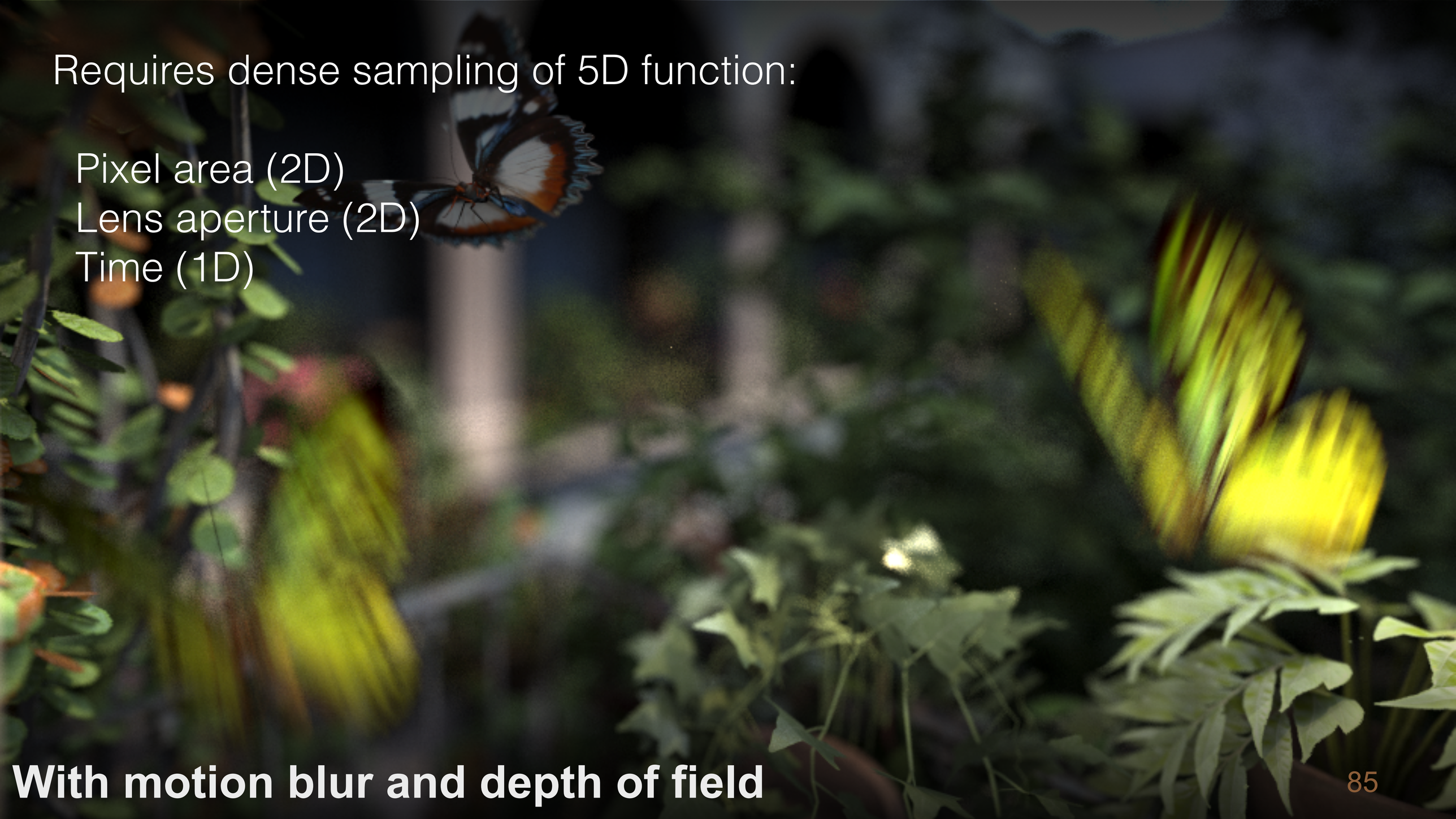
Temporal reconstruction of distribution effects

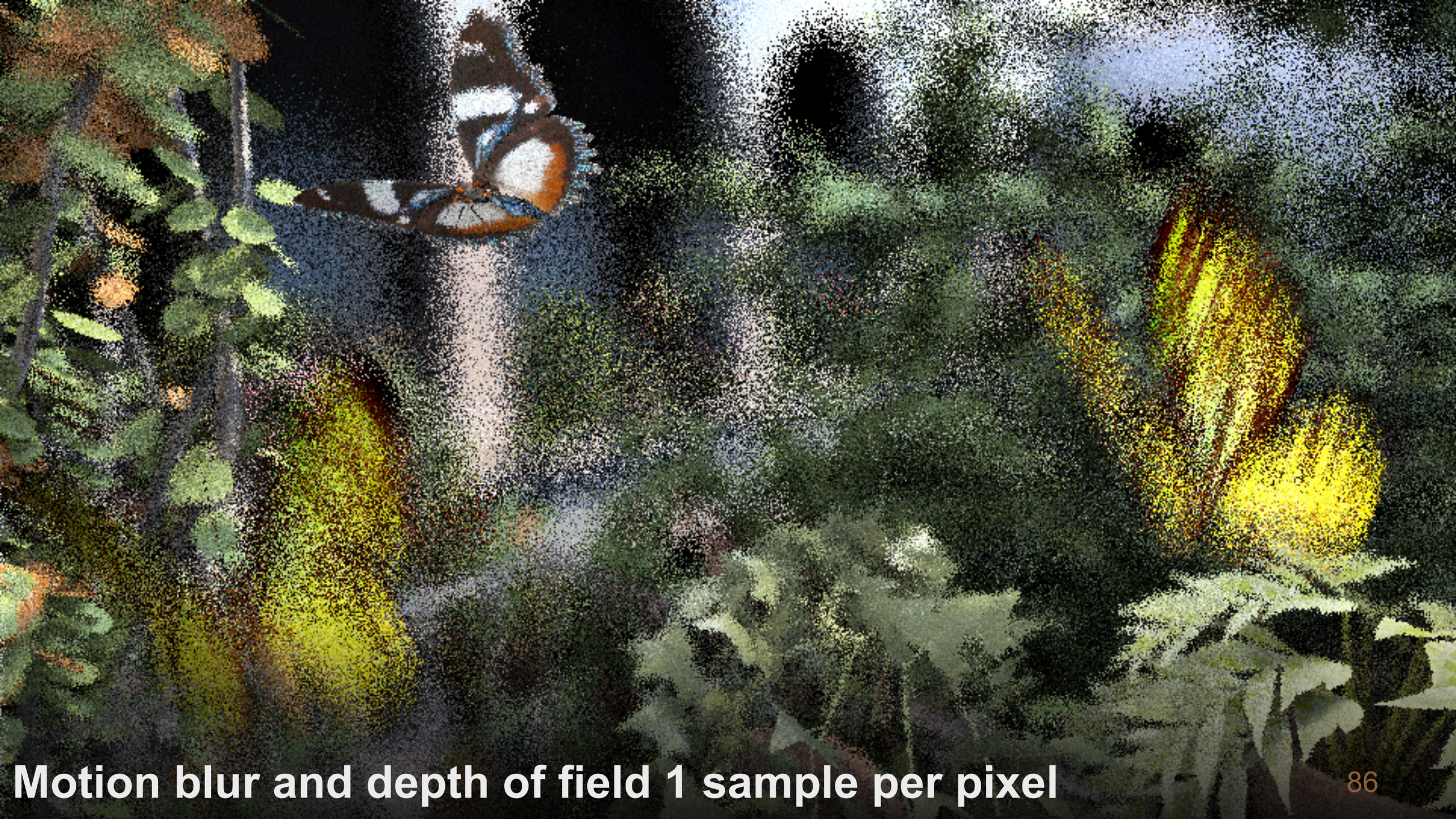
Temporal Light-Field Reconstruction for Rendering Distribution Effects

Lehtinen et al. [2011]

Slides courtesy: Jakko Lehtinen

Pinhole image

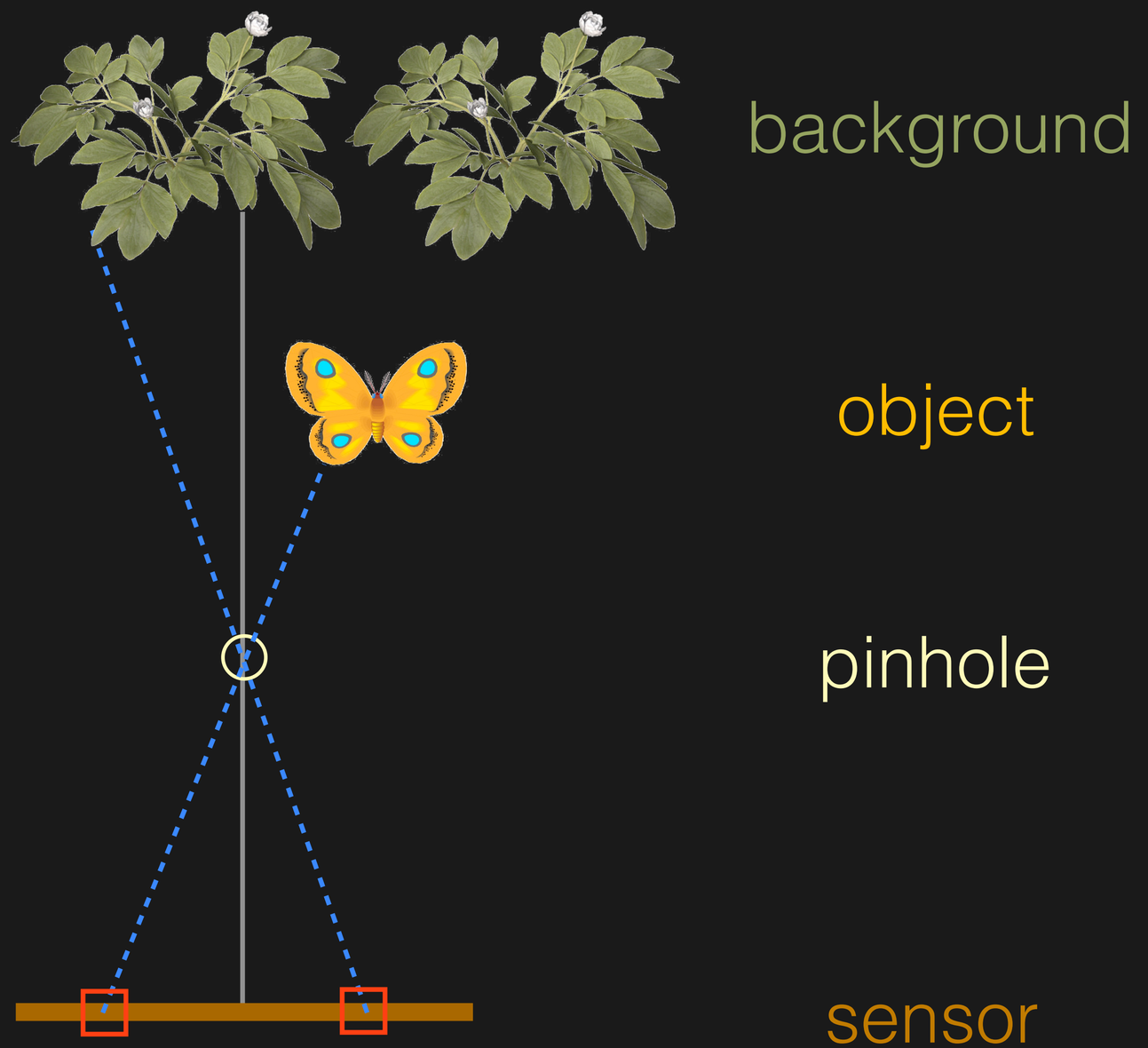

Requires dense sampling of 5D function:
Pixel area (2D)
Lens aperture (2D)
Time (1D)
With motion blur and depth of field



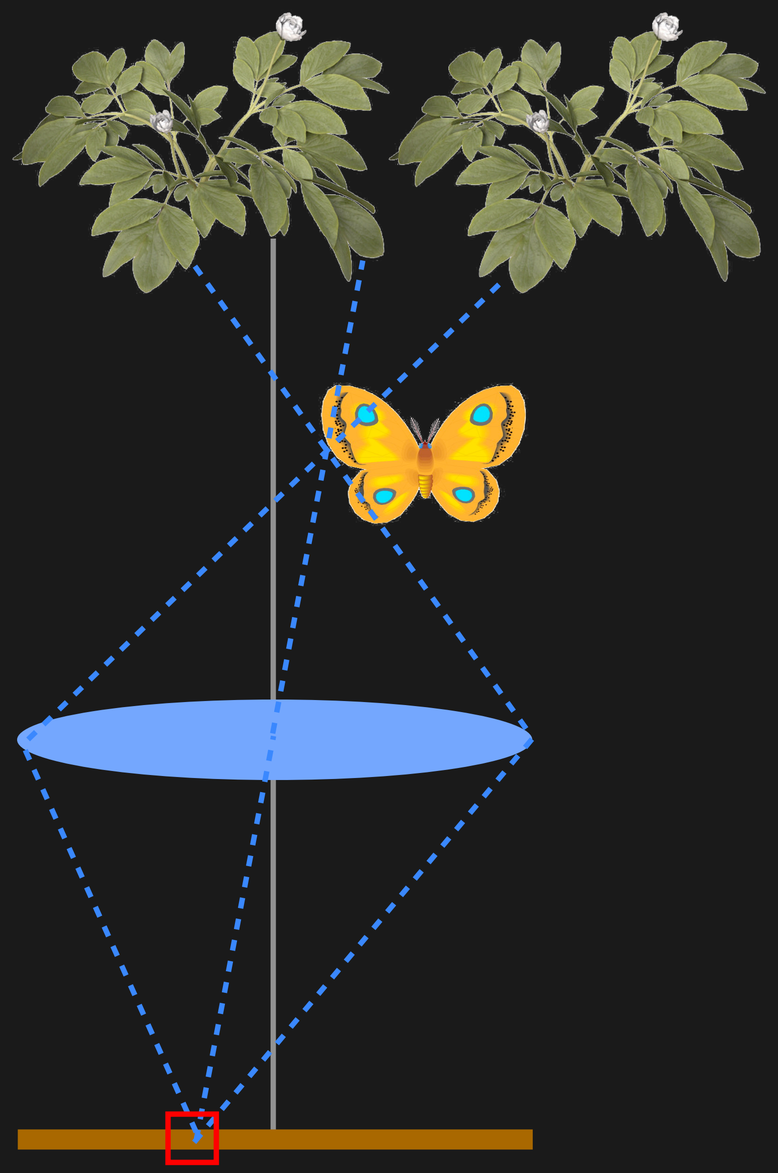
Motion blur and depth of field 1 sample per pixel

Our reconstruction

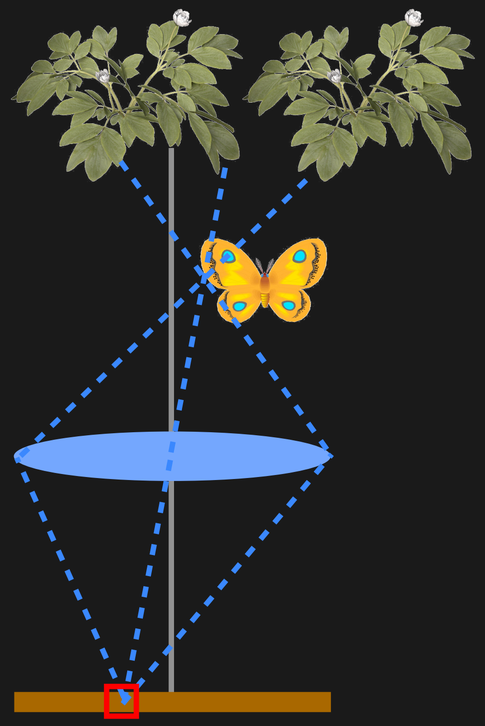
Pinhole camera model



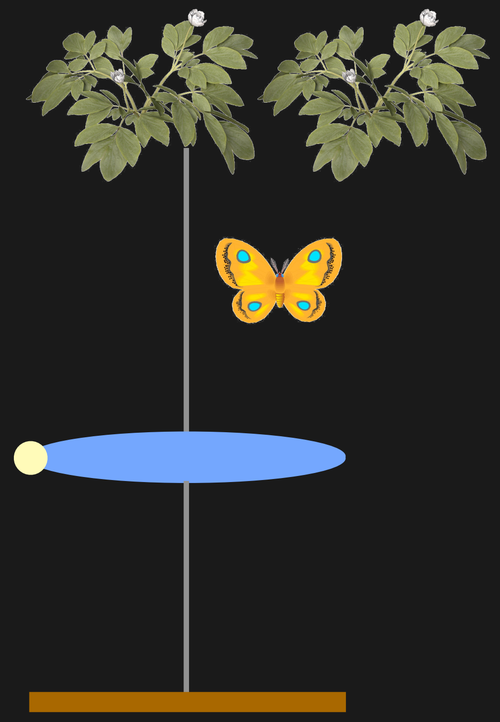
Thin lens camera model

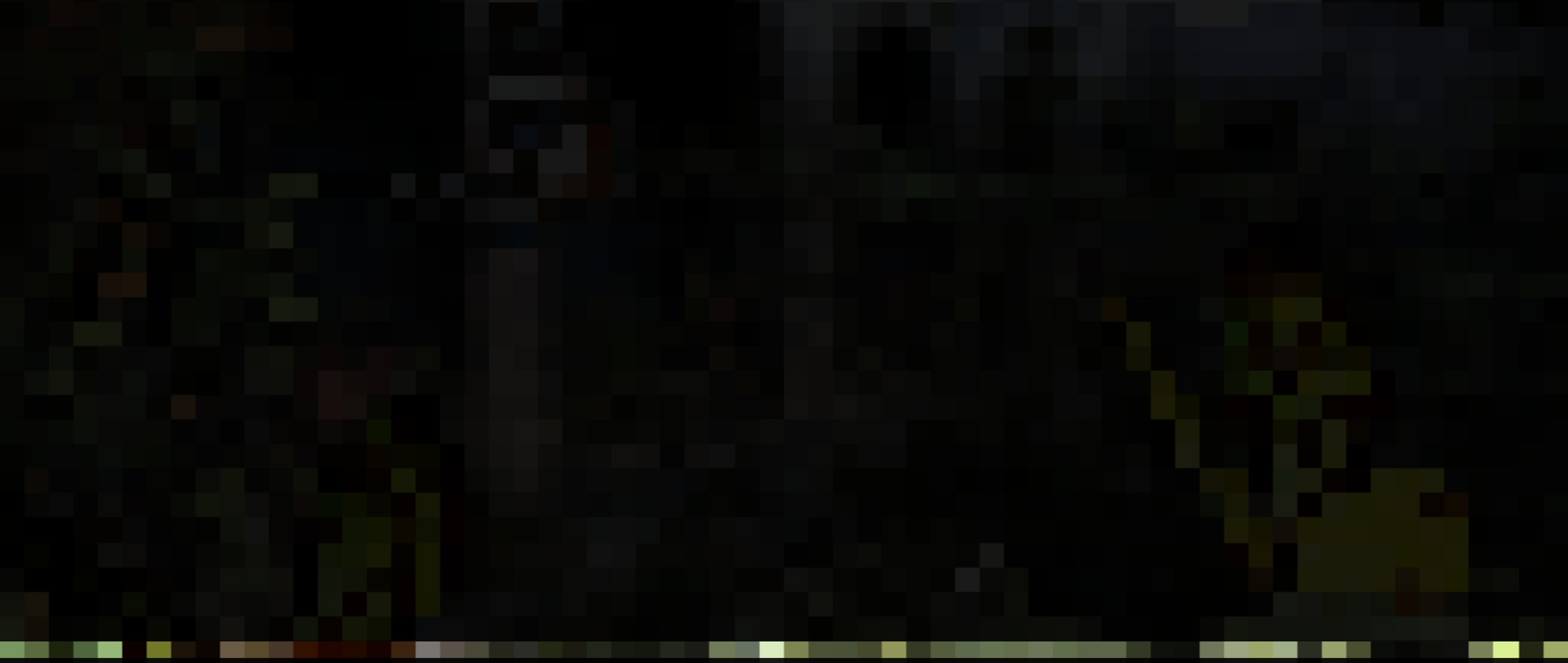


Depth of field

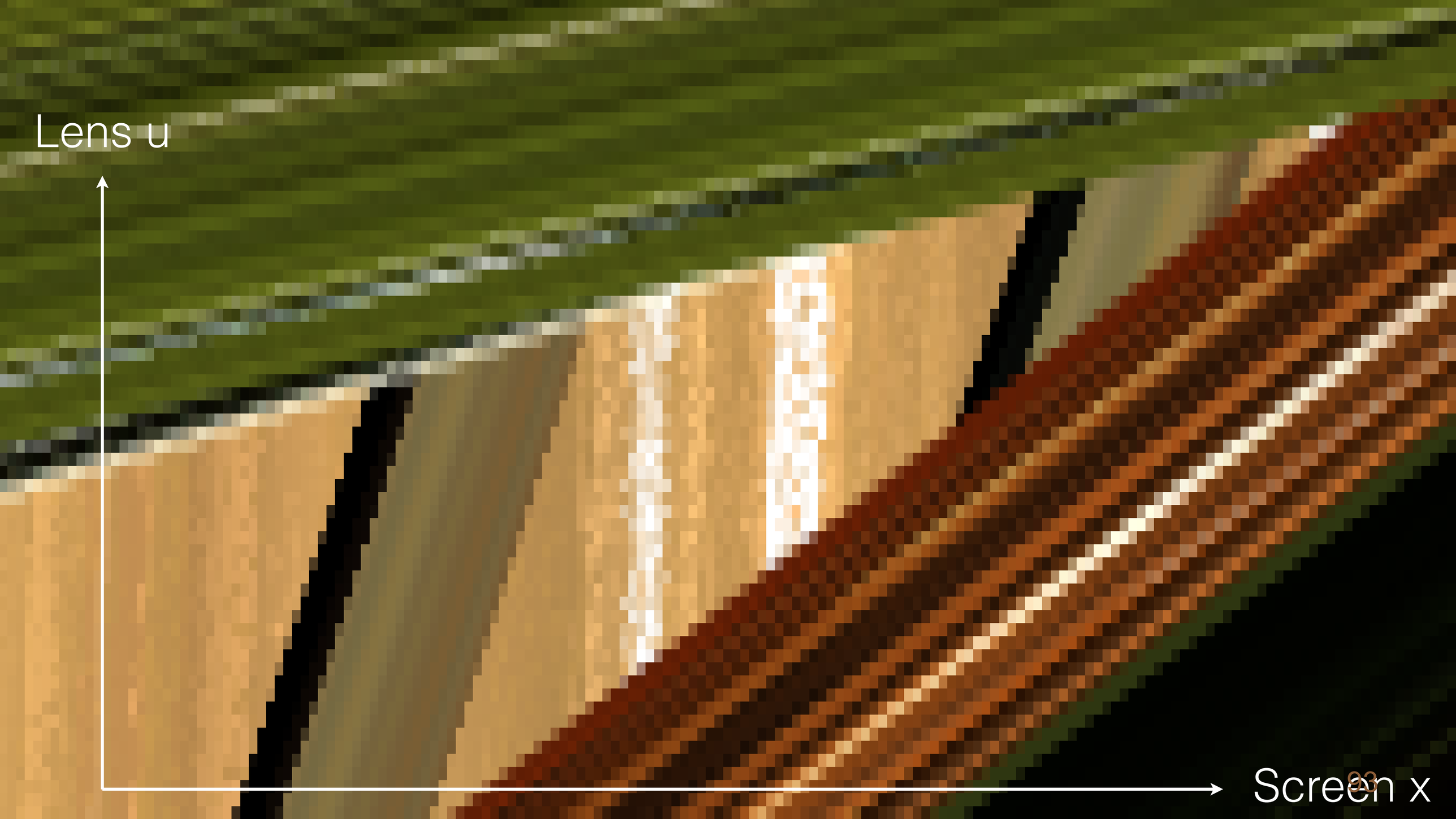


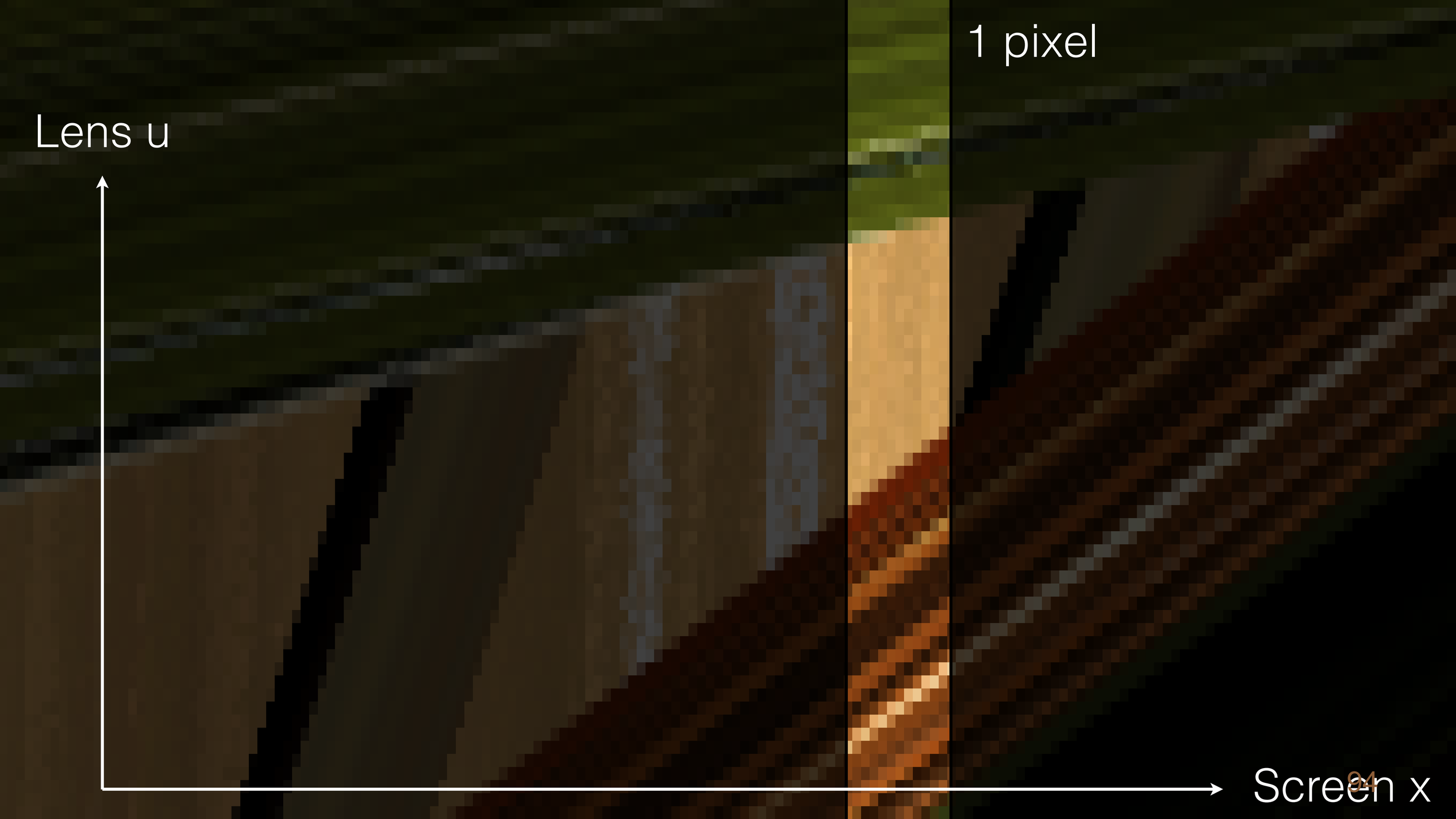
Depth of field





1 scanline

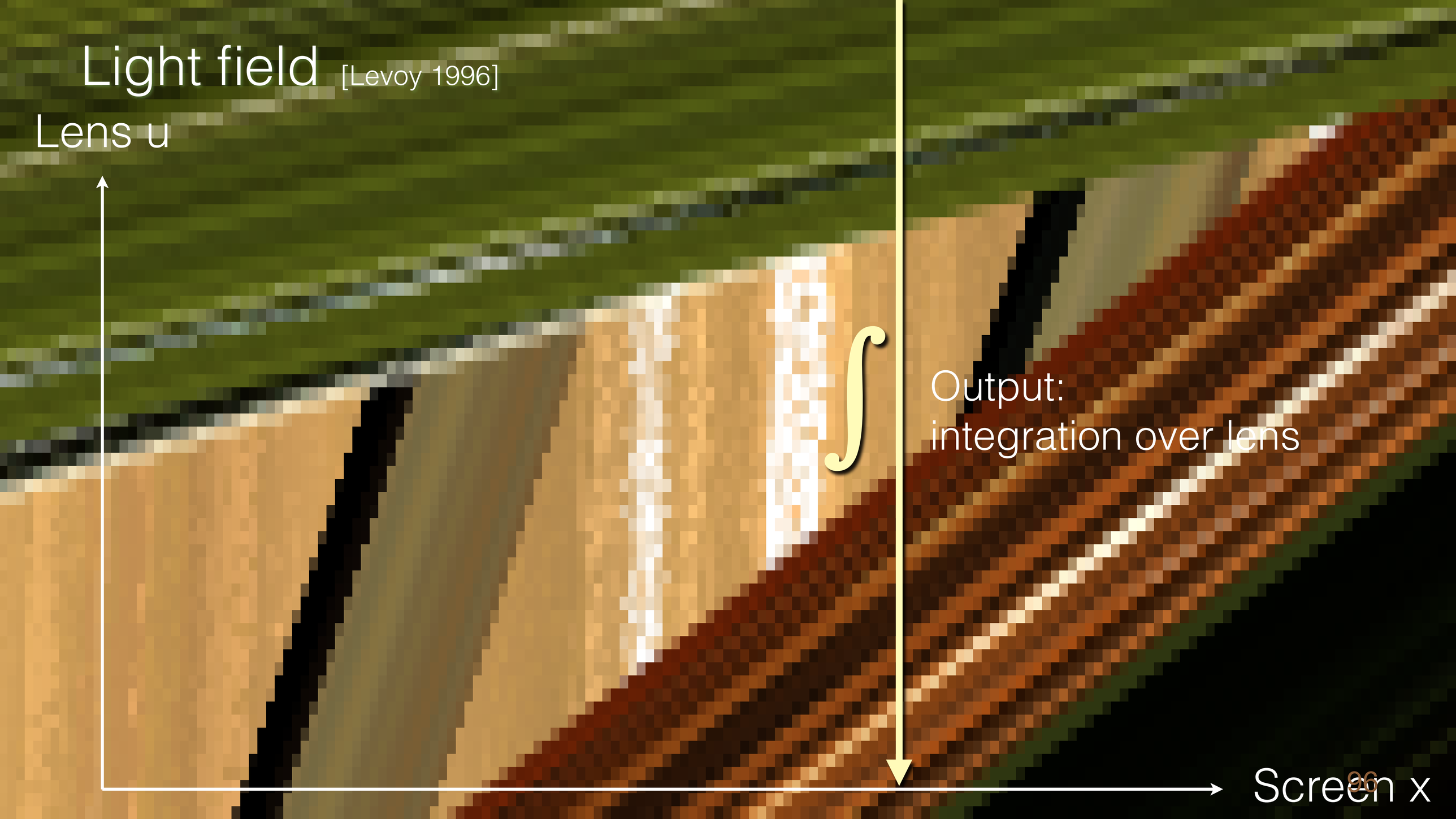

Lens u
Screen x


1 pixel
Lens u
Screen x

Lens u

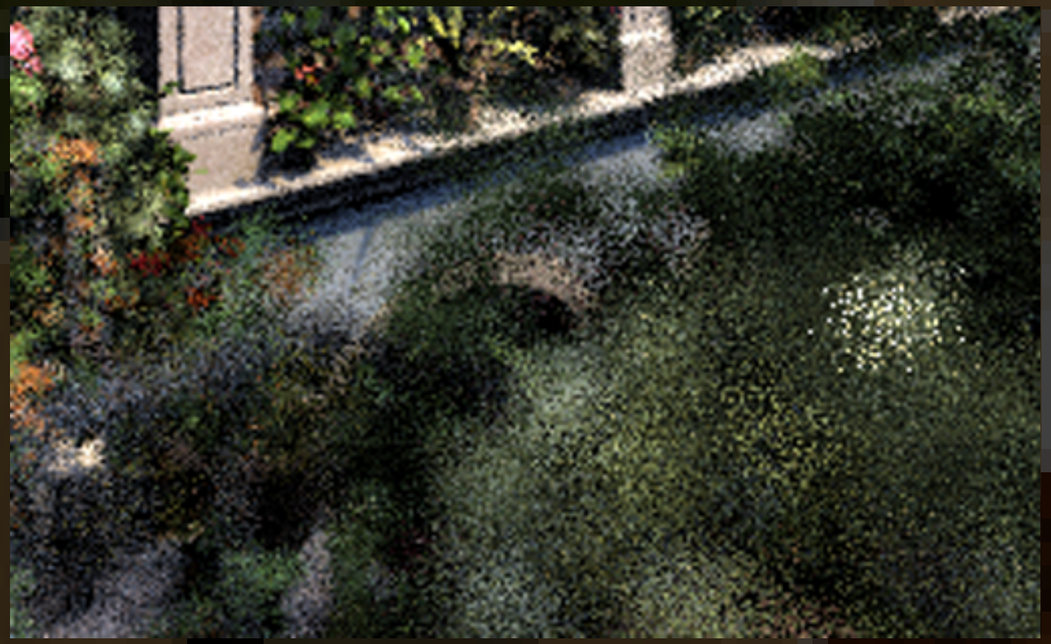
Screen x

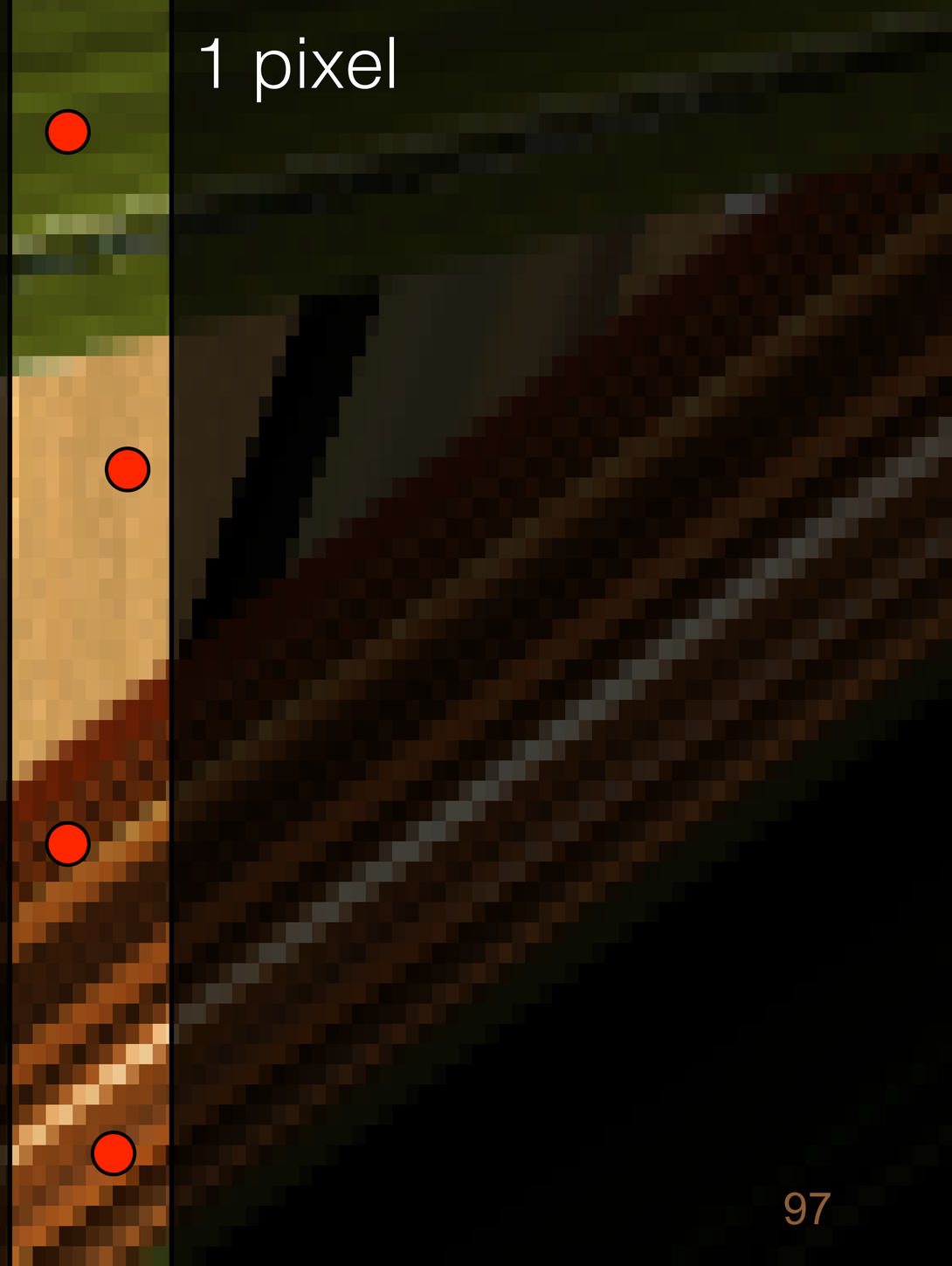
Light field [Levoy 1996]



Monte Carlo sampling

Low sample density leads to noise





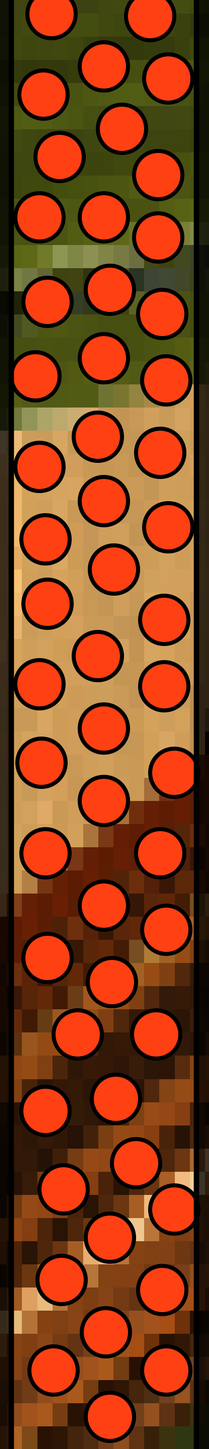
Lens u

Screen x

Monte Carlo sampling

Need many samples to capture the signal:

computationally expensive



Lens u

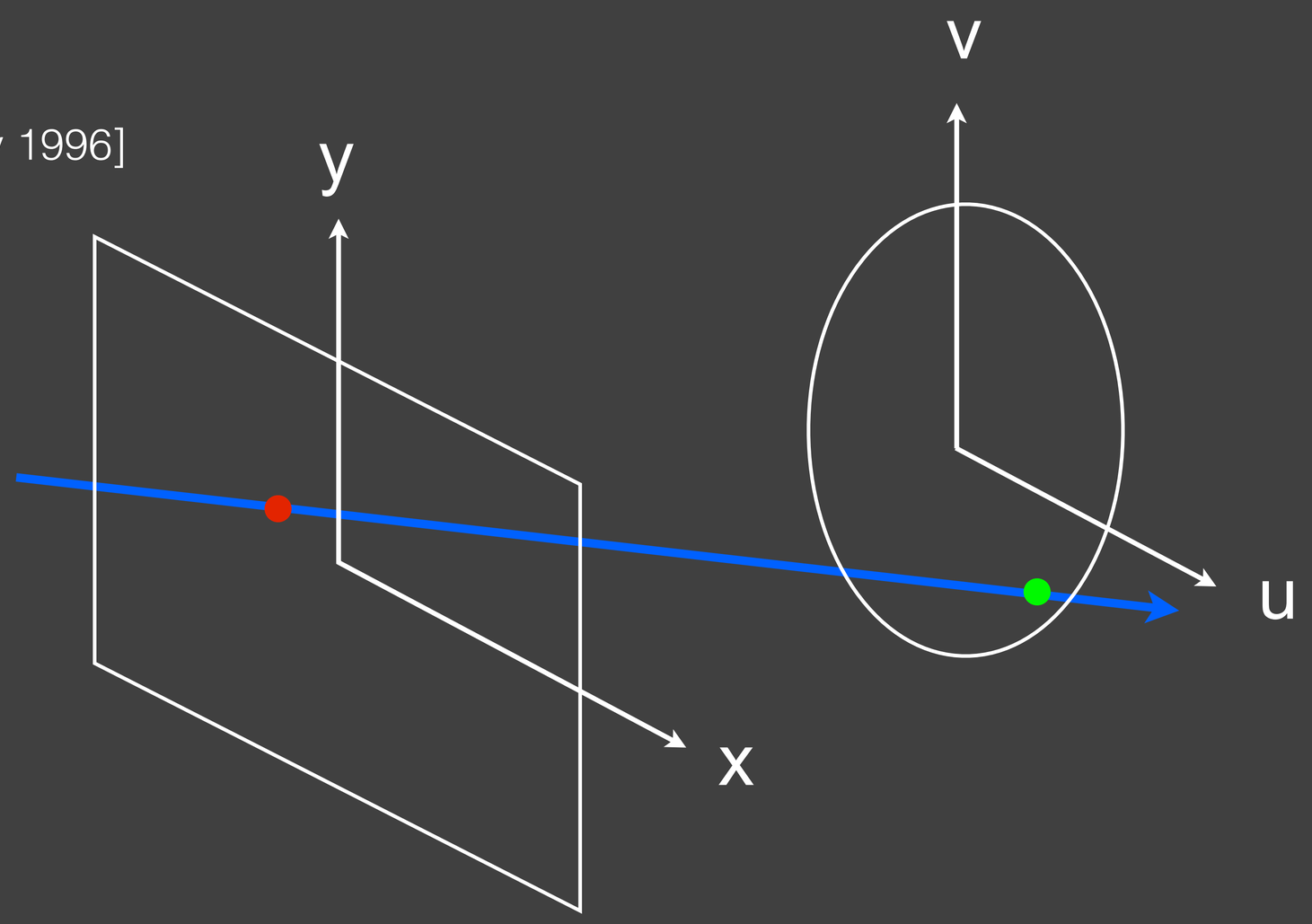
Screen x

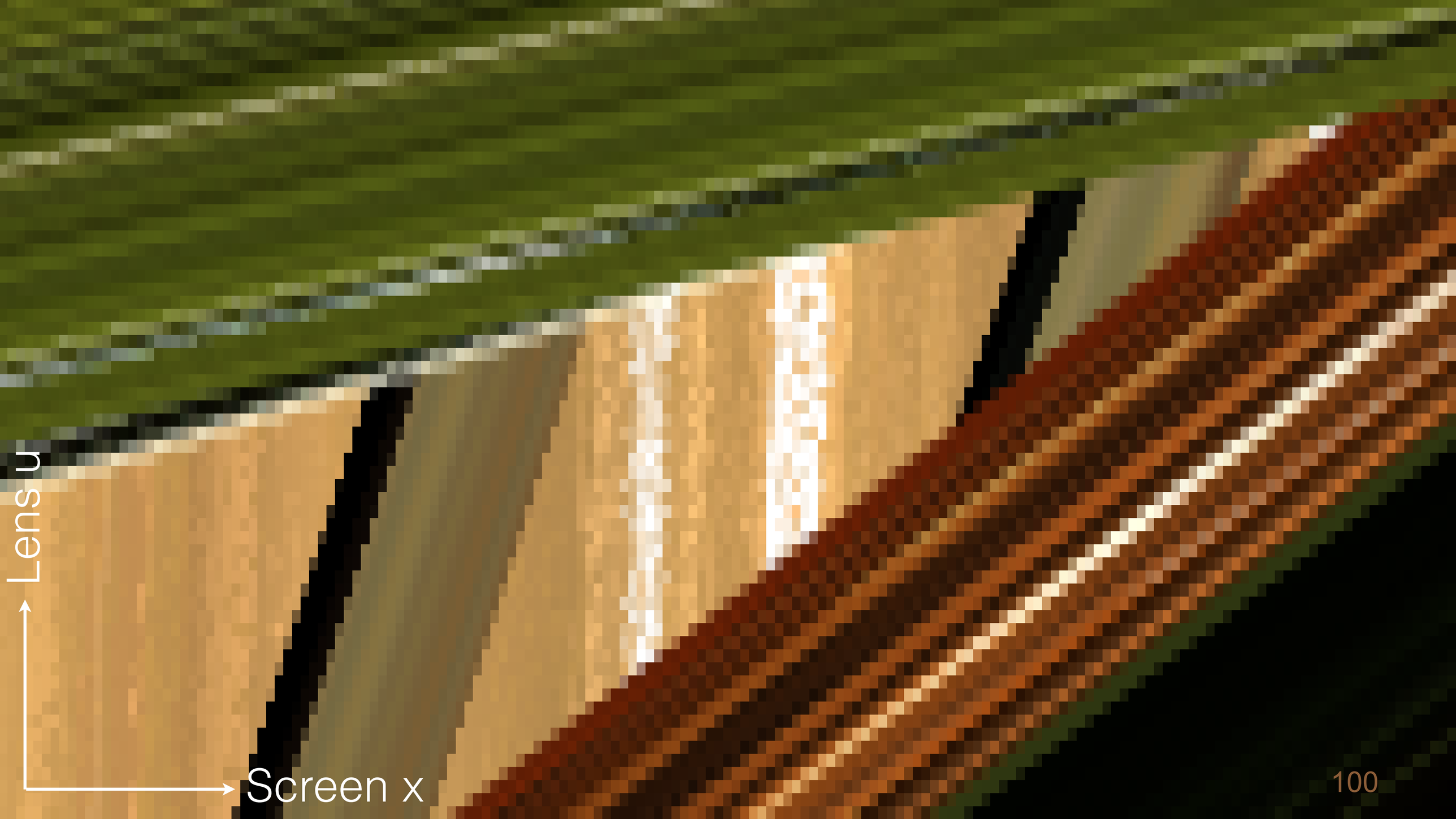
Temporal light fields

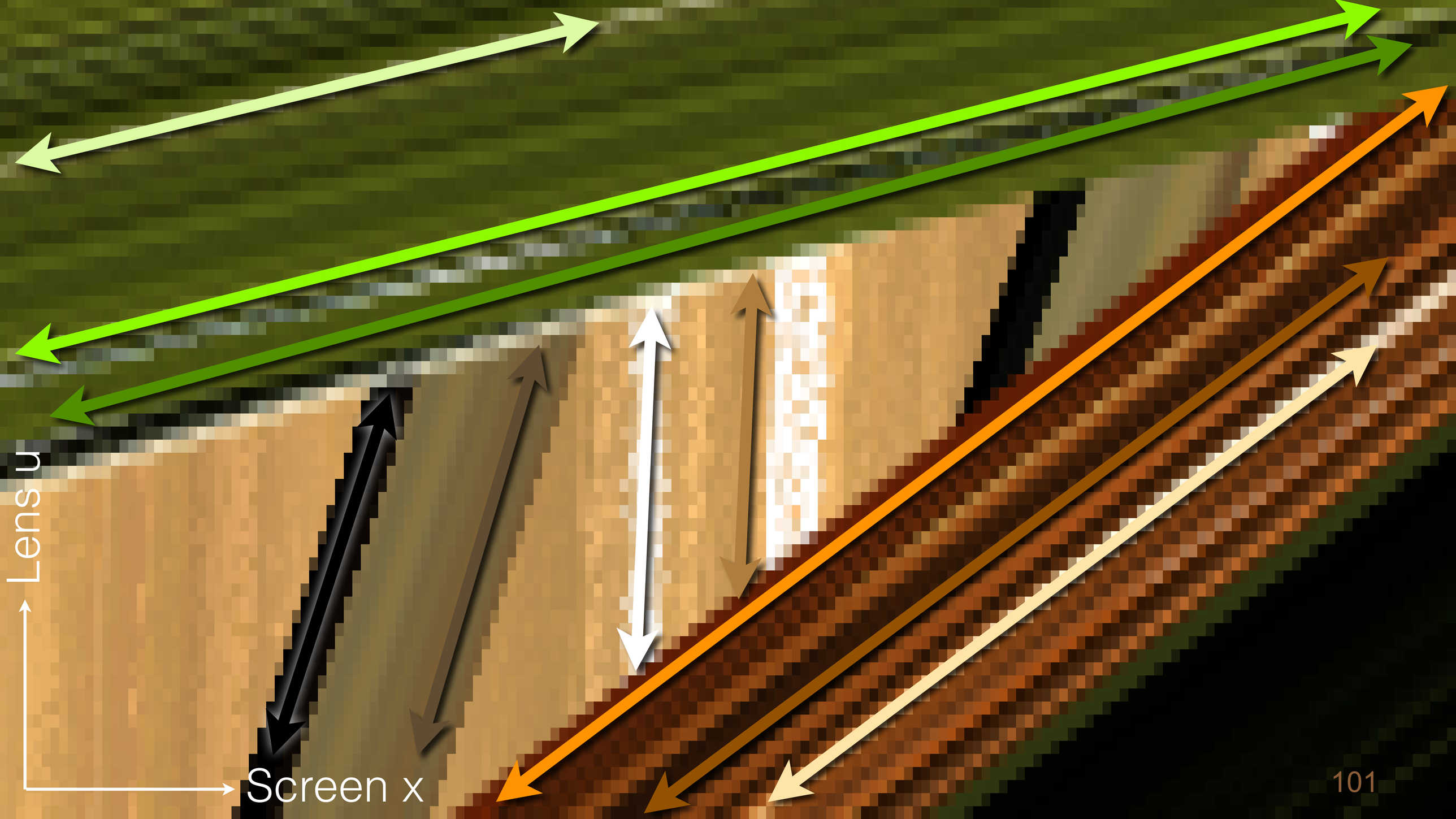
Traditional light field is 4D [Levoy 1996]

x,y over sensor (2D)
u,v over lens (2D)

Add time dimension for moving geometry (5D)

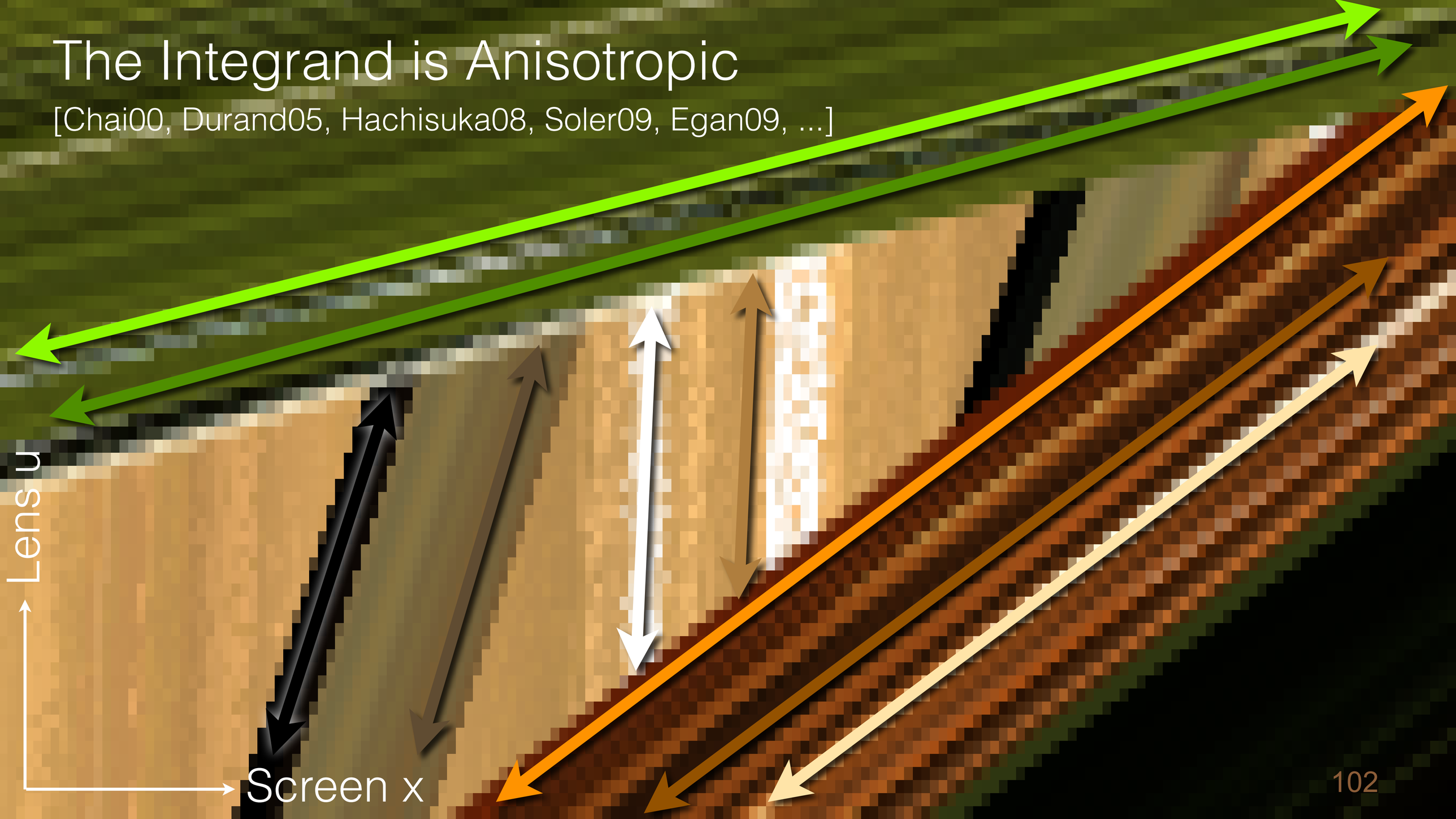



Lens u
Screen x

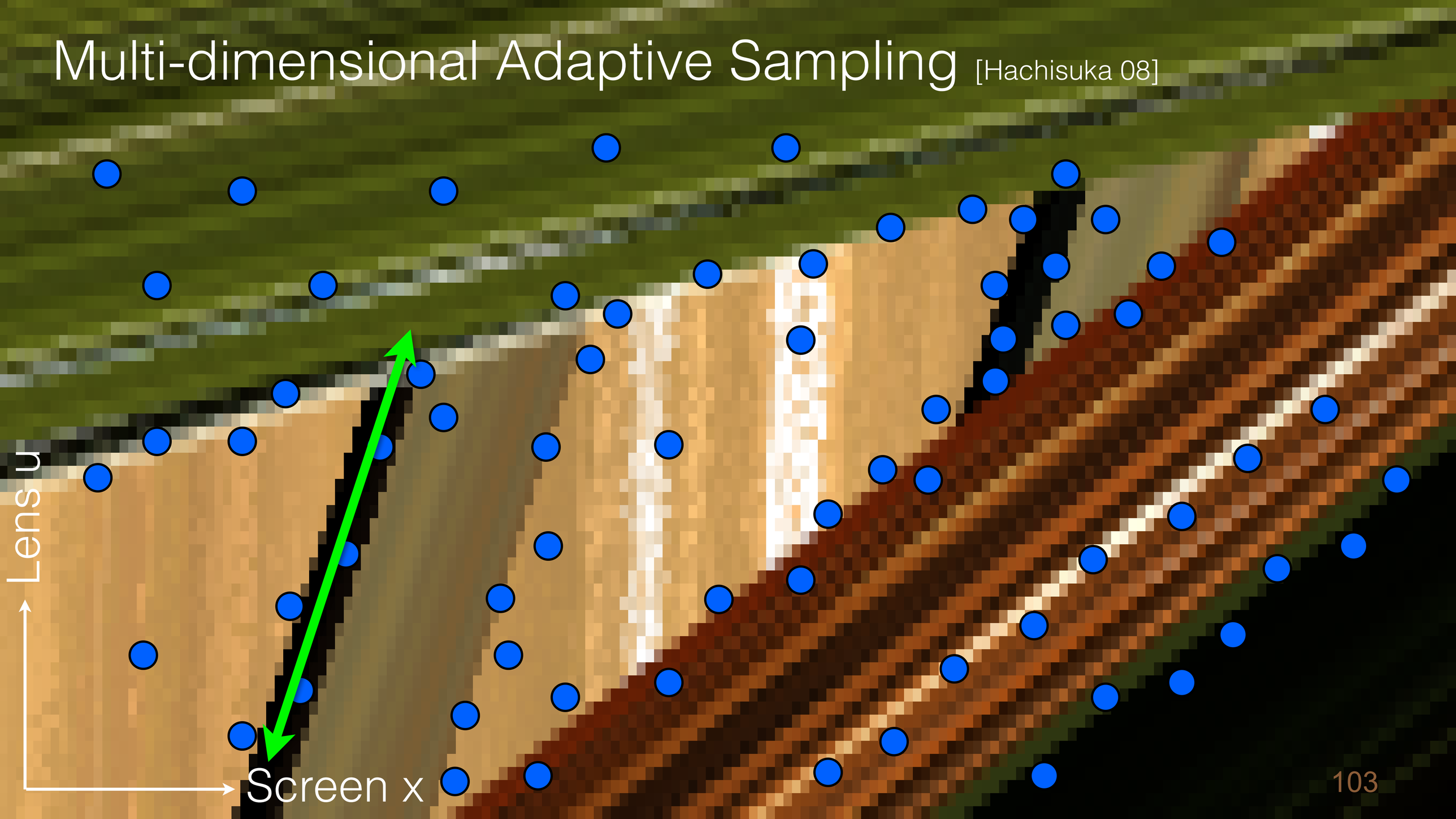

Lens u
Screen x

The Integrand is Anisotropic

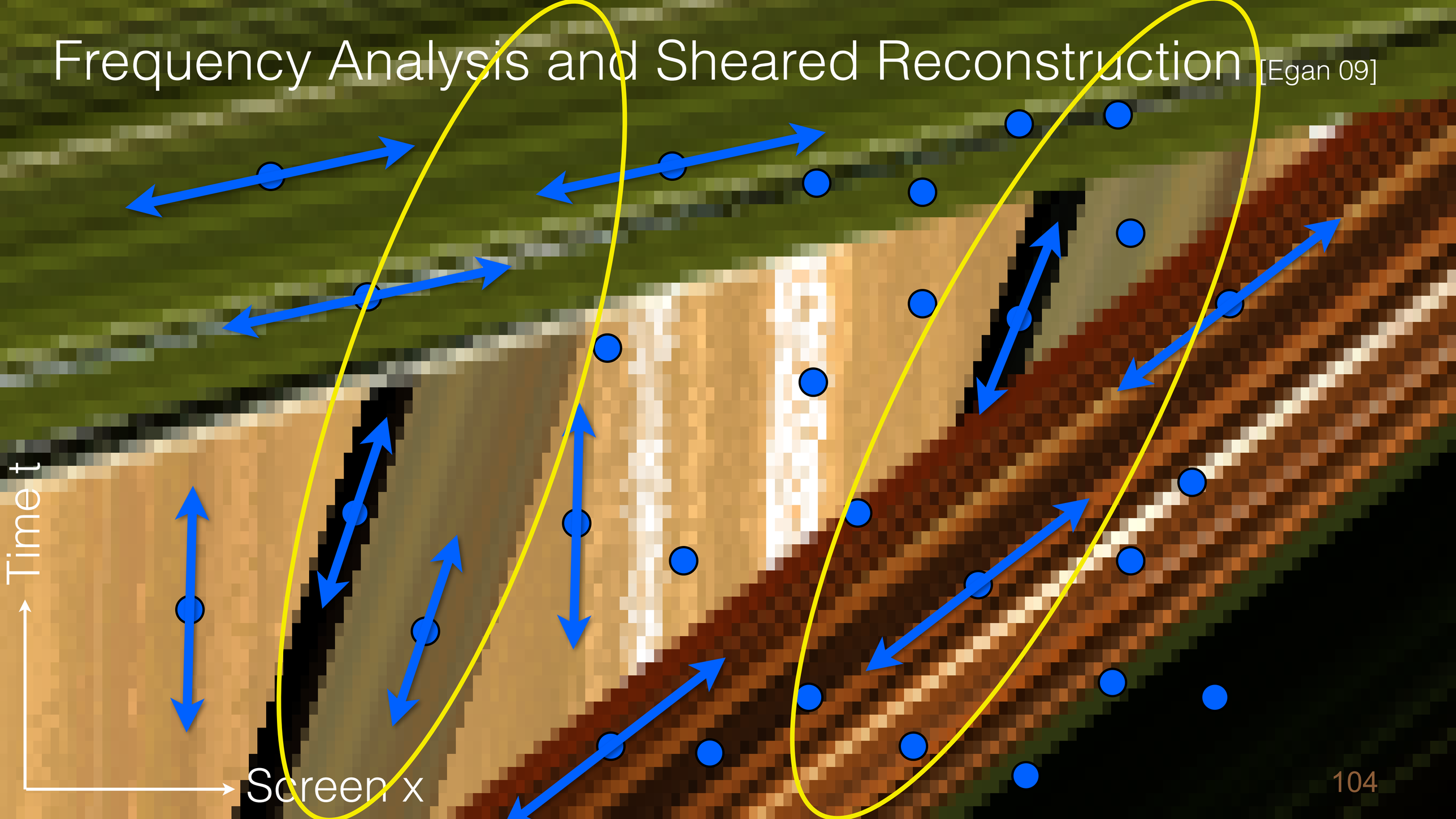
[Chai00, Durand05, Hachisuka08, Soler09, Egan09, ...]



Multi-dimensional Adaptive Sampling [Hachisuka 08]



Frequency Analysis and Sheared Reconstruction [Egan 09]



Our approach

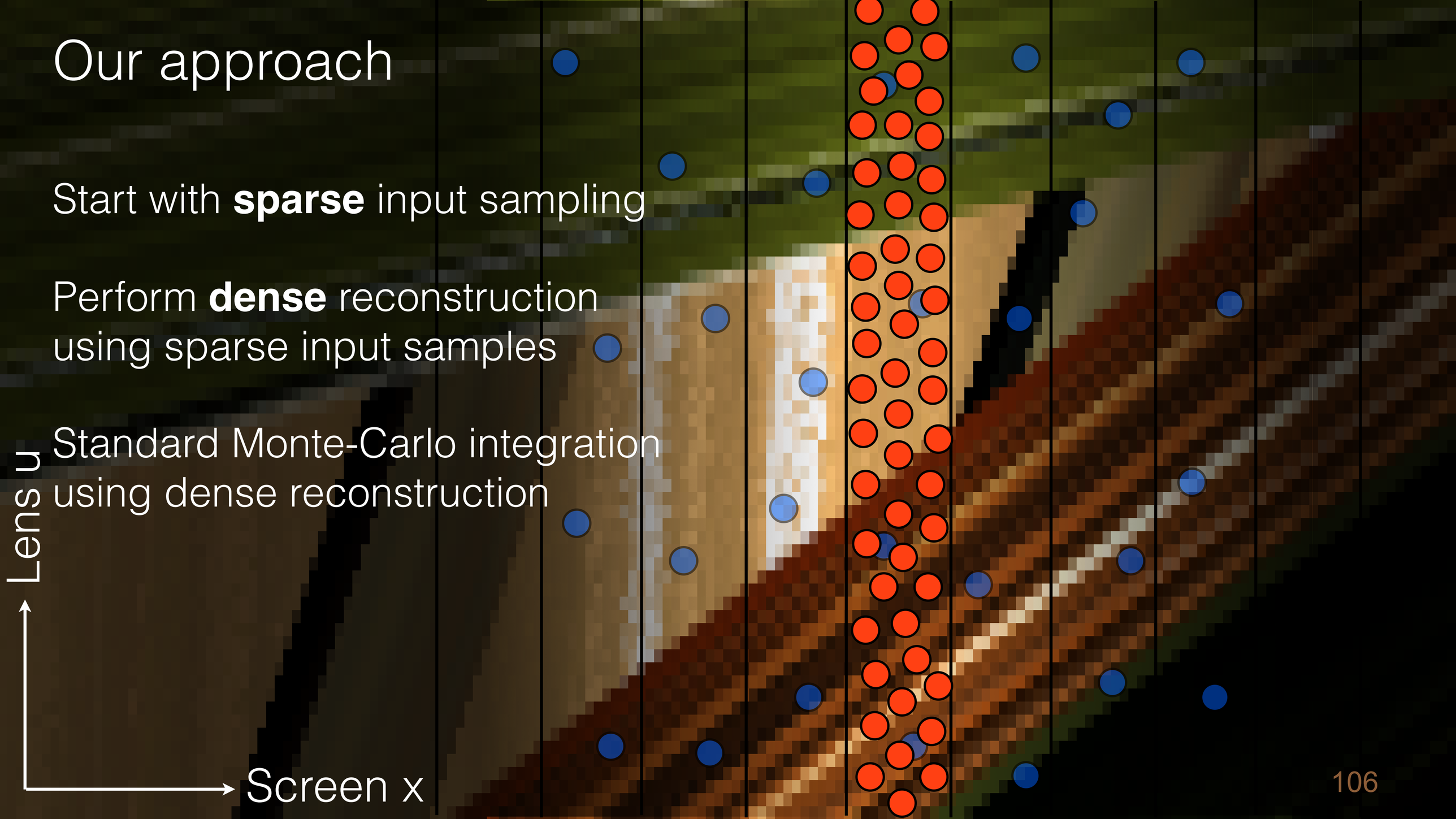
Start with **sparse** input sampling

Our approach

Start with **sparse** input sampling

Perform **dense** reconstruction using sparse input samples

Standard Monte-Carlo integration using dense reconstruction



Our input has **slope information**

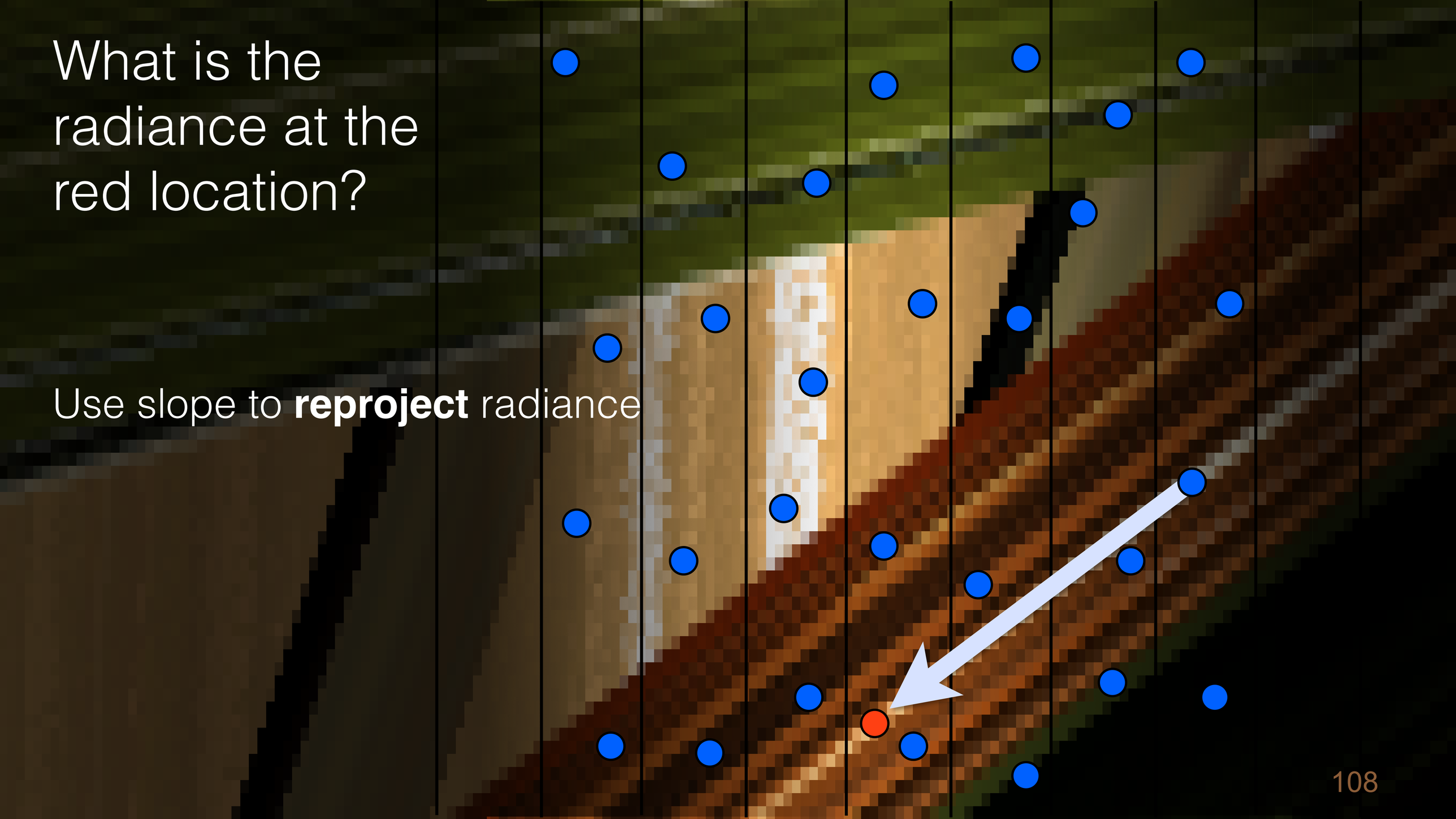
For defocus, proportional to inverse **depth 1/z** [Chai00]

For motion, proportional to inverse **velocity 1/v** [Egan09]

Easy to output from any renderer.

What is the radiance at the red location?

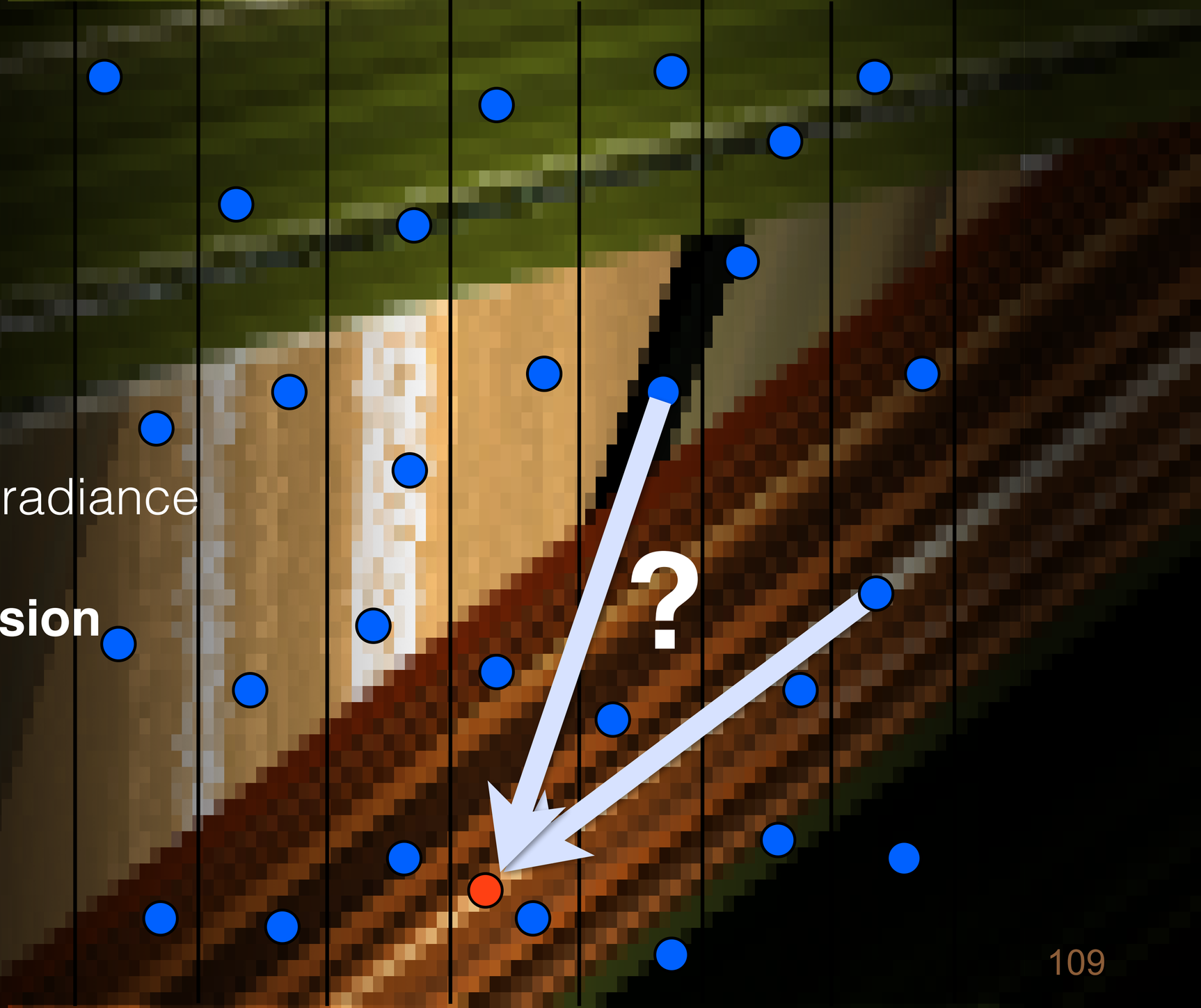
Use slope to **reproject** radiance



What is the radiance at the red location?

Use slope to **reproject** radiance

Must account for **occlusion**



Recap: our approach

Start with **sparse** input sampling

Perform **dense** reconstruction using sparse input samples

- Use **slopes** to reproject
- Account for **visibility**

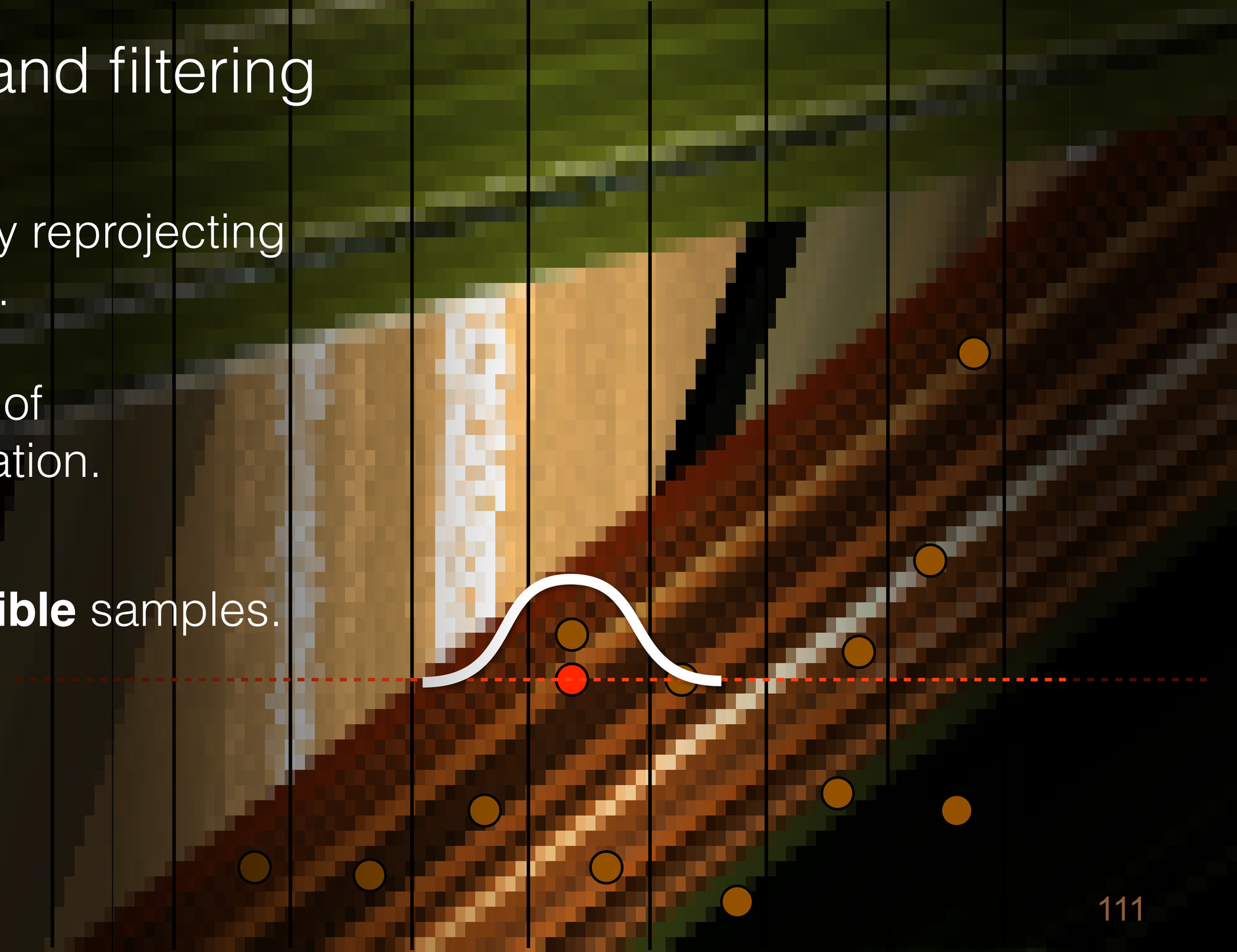
Standard Monte-Carlo integration using dense reconstruction

Reprojection and filtering

Simplify visibility by reprojecting into **screen space**.

Reproject to u, v, t of reconstruction location.

Pixel filter over **visible** samples.

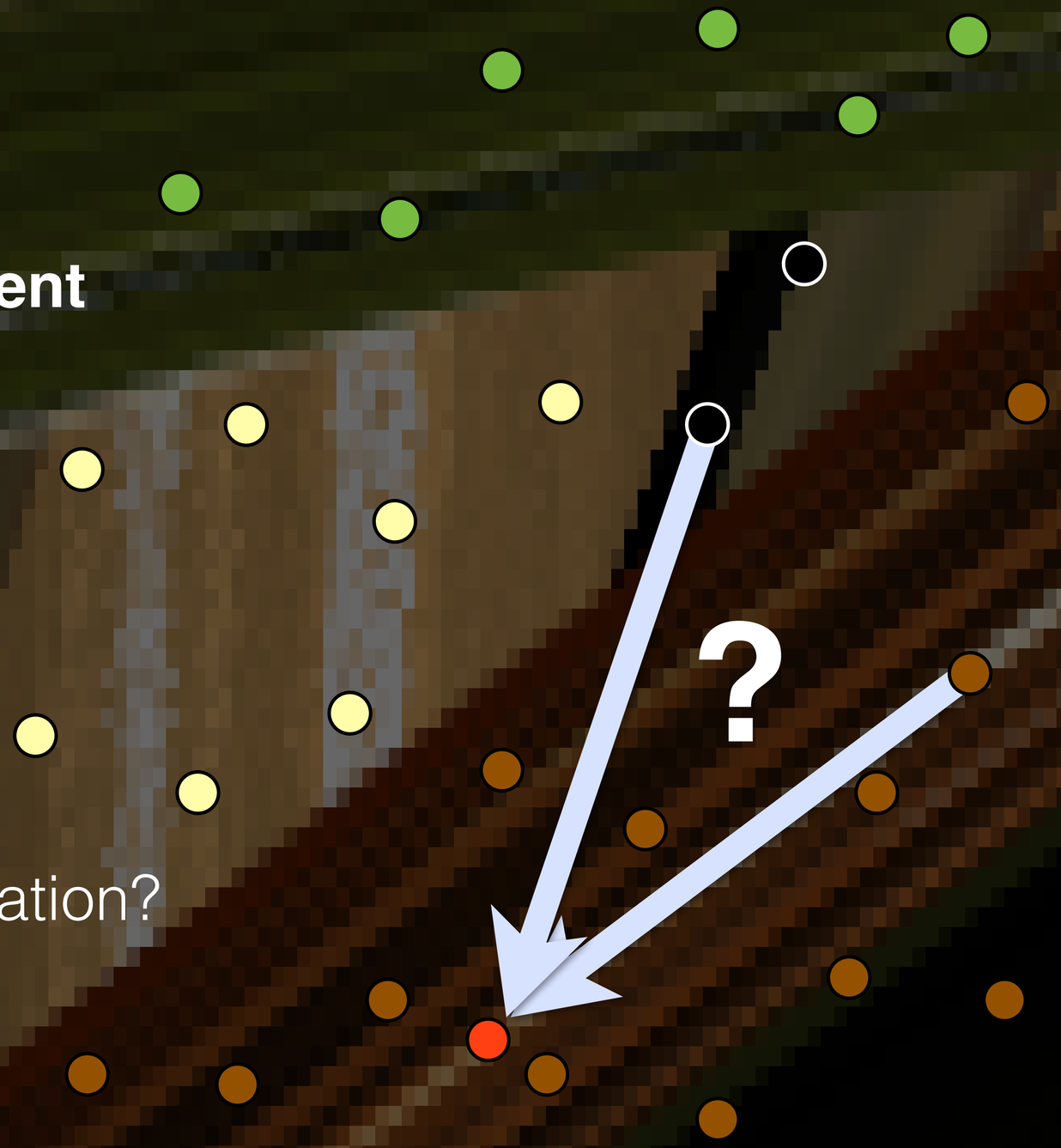


Visibility

Cluster samples into **apparent surfaces** to resolve *z*-order

SameSurface algorithm

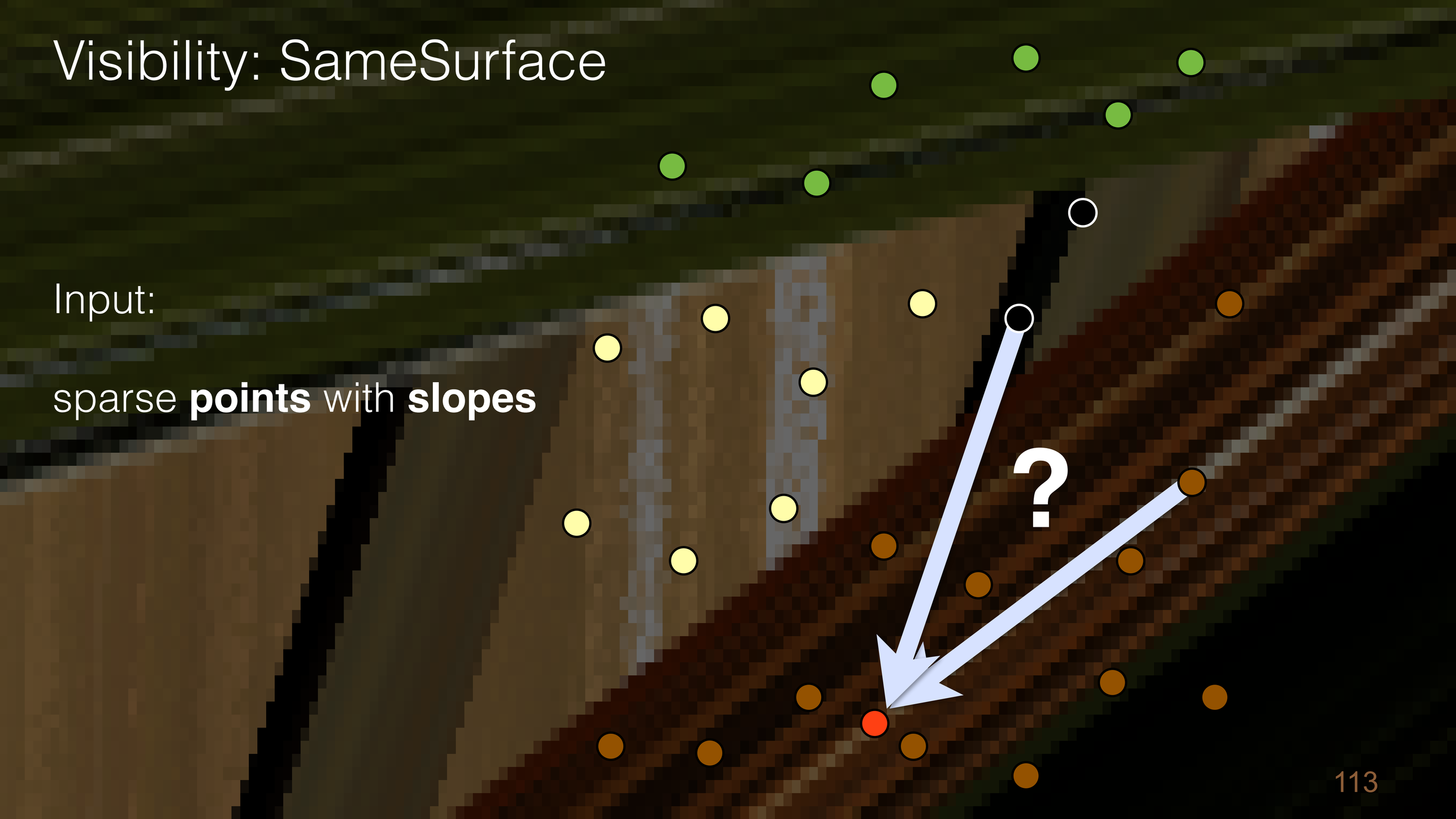
Determining **coverage**:
Does the apparent surface cover my reconstruction location?



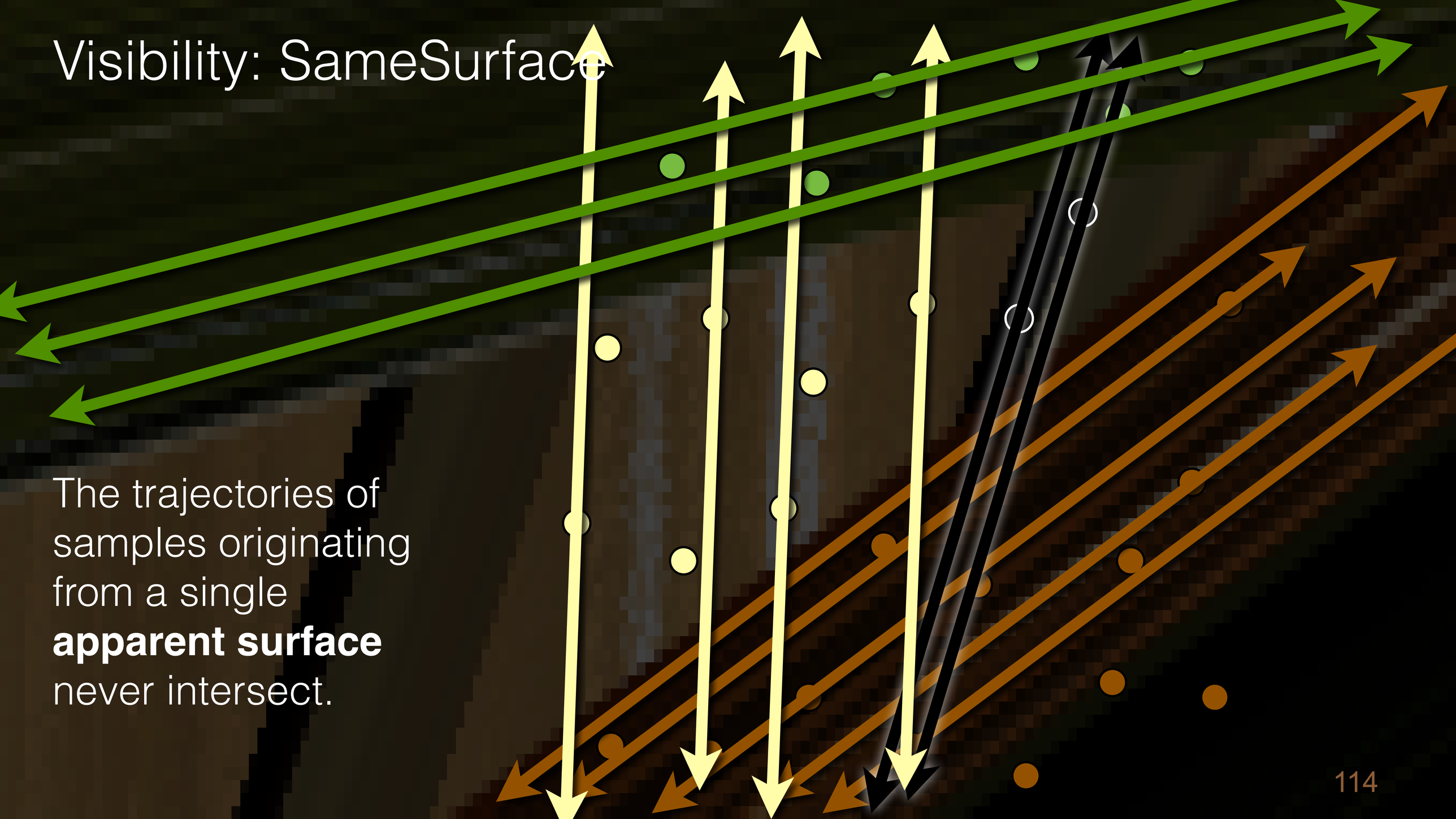
Visibility: SameSurface

Input:

sparse **points** with **slopes**

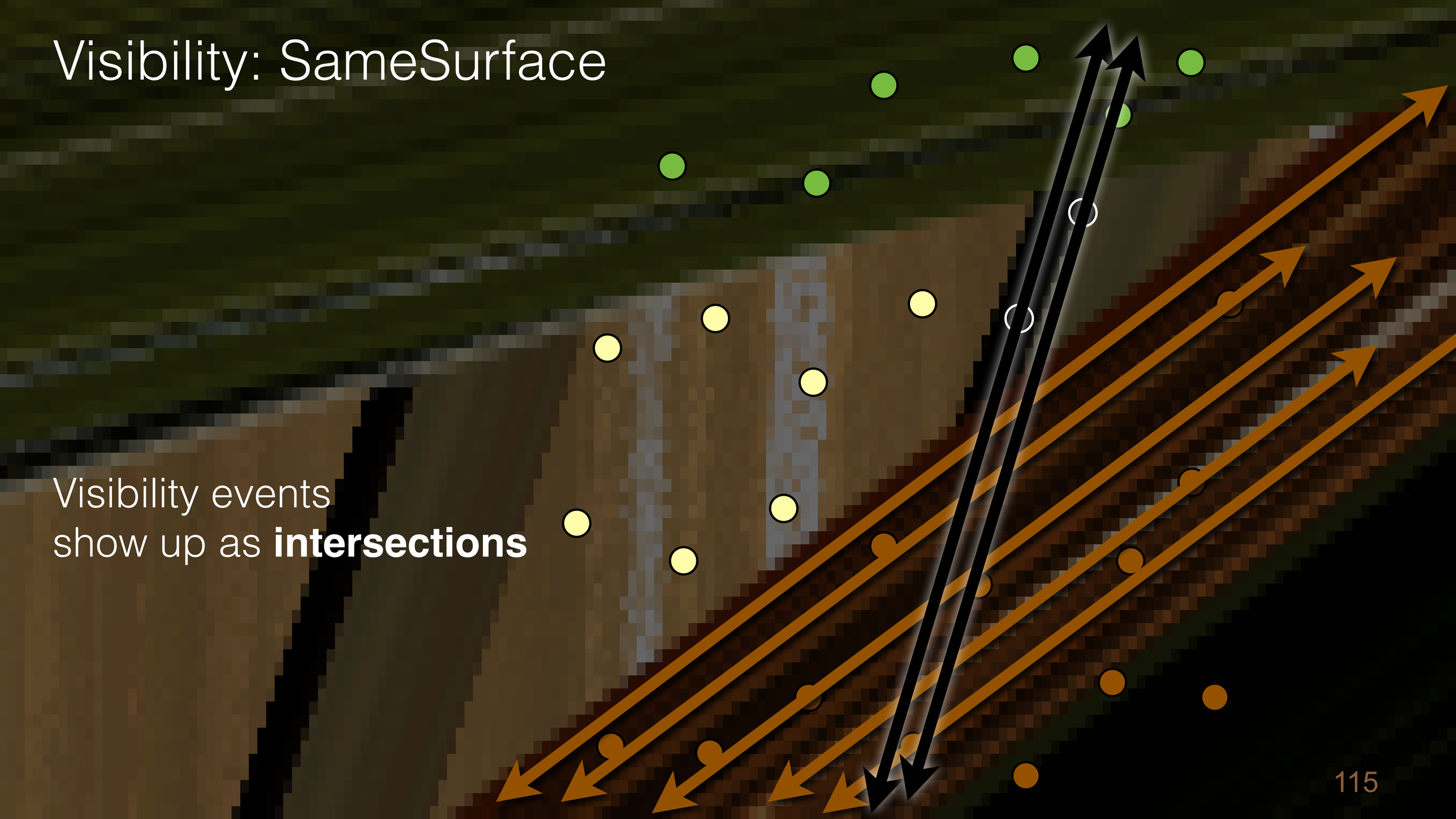


Visibility: SameSurface



The trajectories of samples originating from a single **apparent surface** never intersect.

Visibility: SameSurface



Visibility events
show up as **intersections**

Visibility: Coverage

Does foreground apparent surface cover reconstruction location?

Search foreground samples for spanning triangle.

Recap: our approach

Start with **sparse** input sampling

Perform **dense** reconstruction using sparse input samples

- Use **slopes** to reproject
- Account for **visibility**

Standard Monte-Carlo integration using dense reconstruction

Observations

We only need sample radiance, depth, and velocity (i.e., **slopes**).
Reconstruction is **independent** of the original renderer.

We can **discard** the scene.

Observations

We only need sample radiance, depth, and velocity (i.e., **slopes**).
Reconstruction is **independent** of the original renderer.

We can **discard** the scene.

Need efficient **sample search**:

- Fast motion and large defocus can lead to a single sample contributing to **hundreds** of pixels.

- Build a **hierarchy** over input samples.

Extension to soft shadows

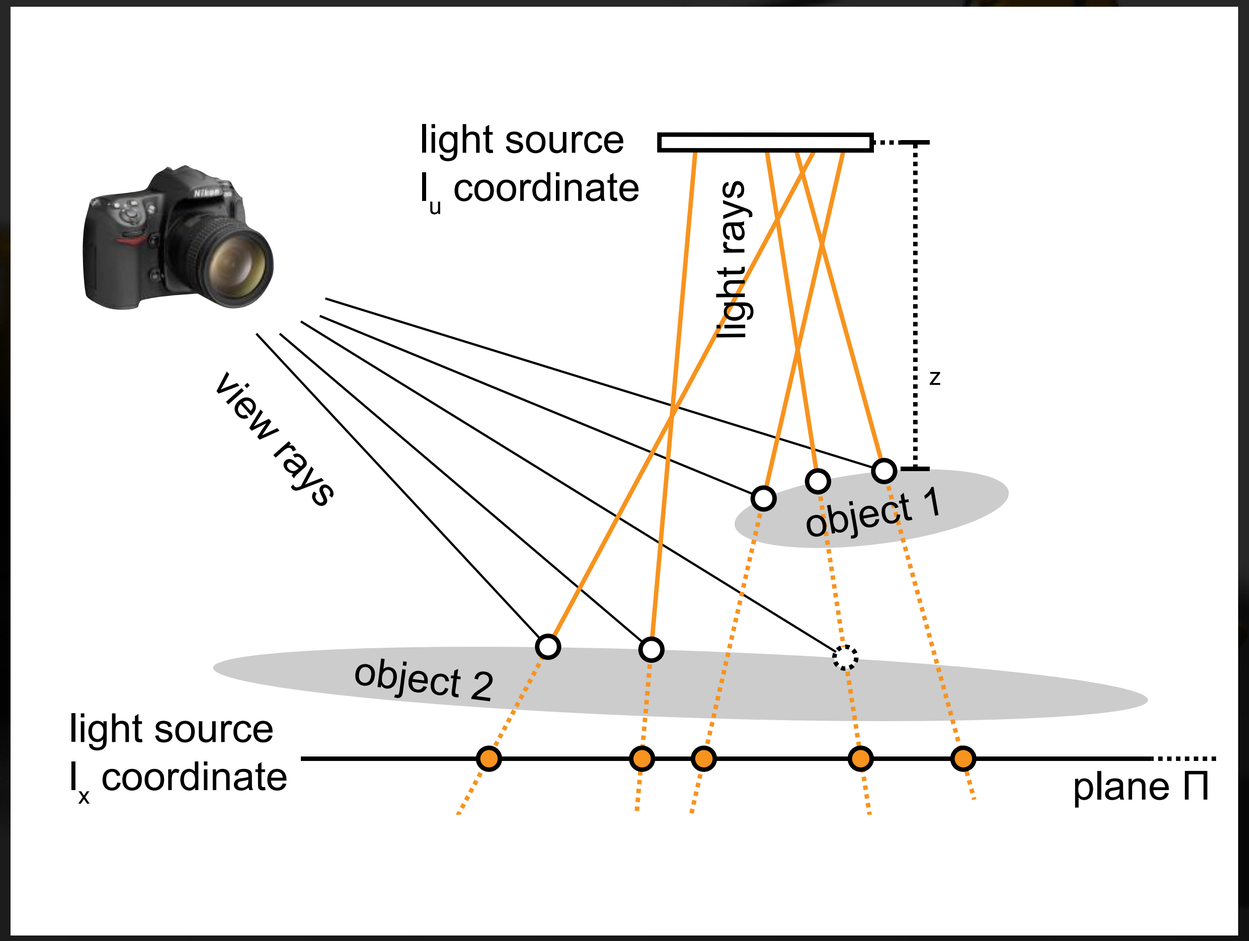
An **area light** is very much like a **lens**.

lens ~ light, sensor ~ virtual plane
Reconstruct **z** instead of radiance

Egan et al. [2010] reconstruct
far field **binary visibility** only.

7D path-tracing style reconstruction
avoiding **combinatorial explosion**

Reconstruct scene point (**5D**)
Reconstruct shadow **z** shade (**2D**)



Results

Implementation

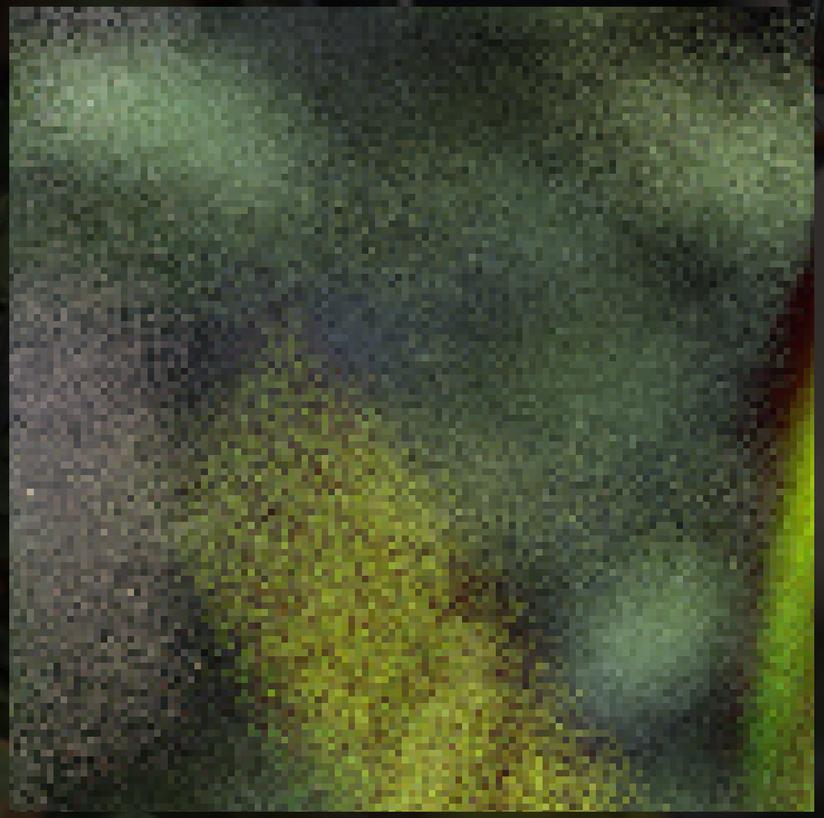
Multithreaded **CPU**
GPU, excluding hierarchy construction

Common sample buffer format accepts outputs from:

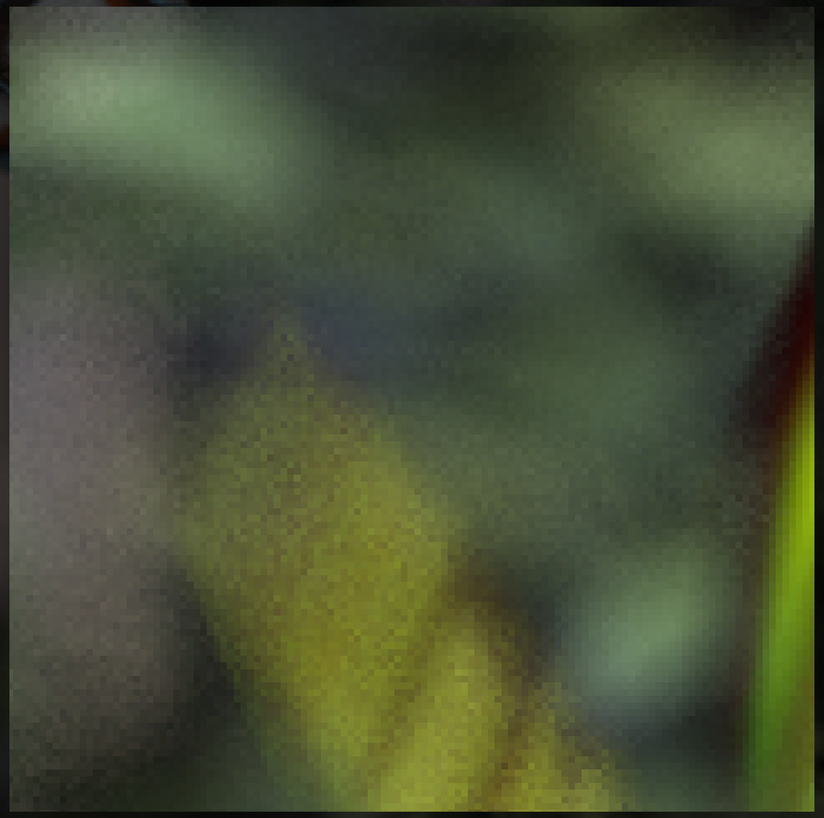
- **PBRT**
- **Pixie** (Open source RenderMan)
- Custom ray tracer


Input: 16 spp
1072 sec (PBRT)

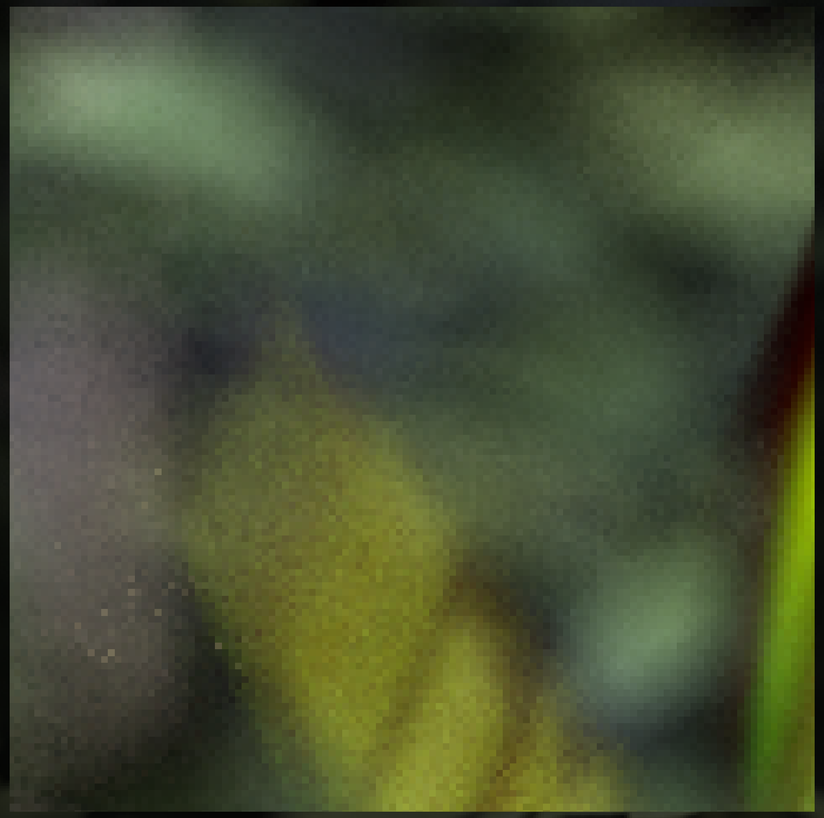
Our result: 16 spp + reconstruction at 128spp
1072 sec (PBRT) + 10 sec (reconstruction)



Input: 16 spp

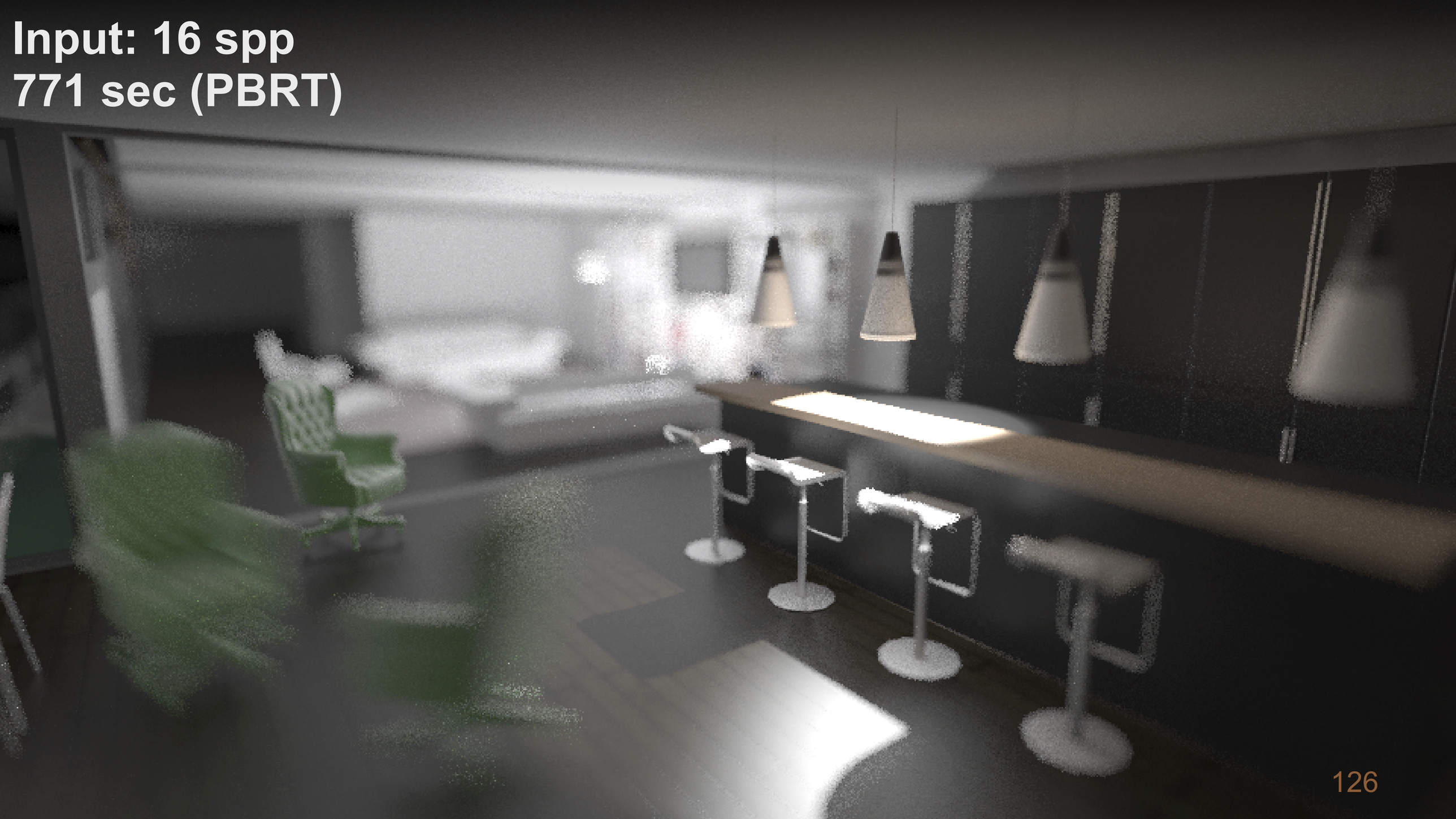


Our result at 128 spp
using same input



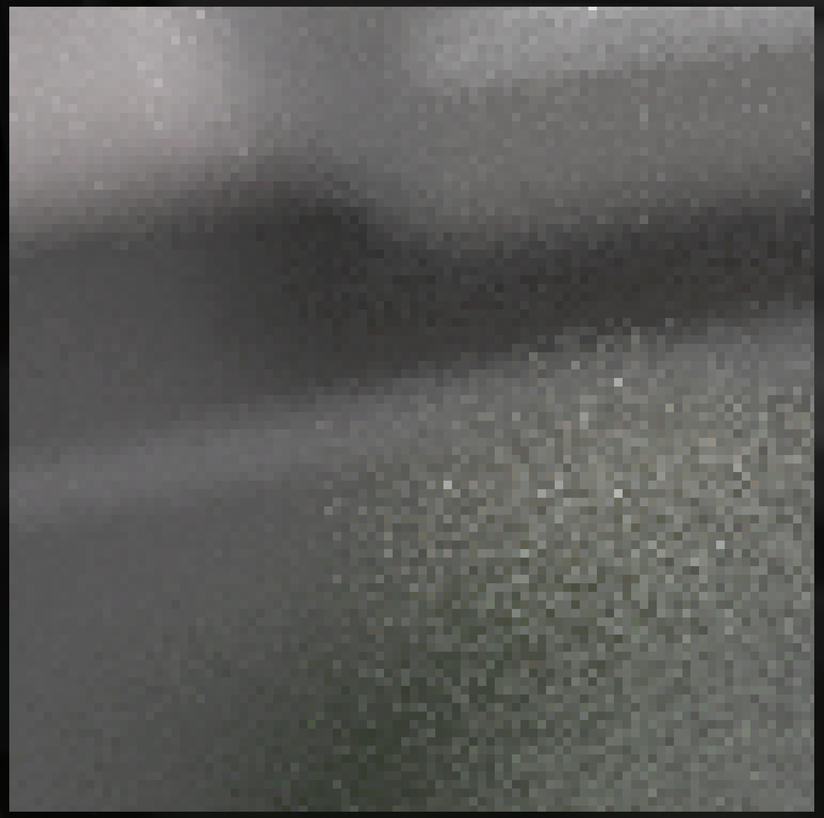
Reference: 256 spp
(16x time)

Our result: 16 spp + reconstruction at 128spp
1072 sec (PBRT) + 10 sec (reconstruction)

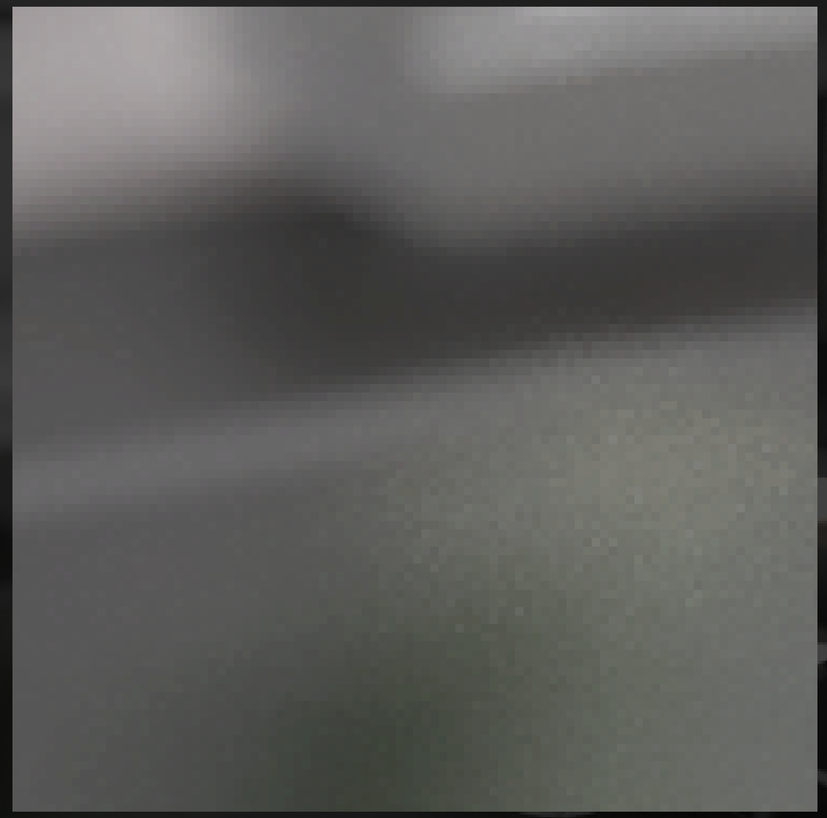

Input: 16 spp
771 sec (PBRT)

Our result: 16 spp + reconstruction at 128spp
771 sec (PBRT) + 10 sec (reconstruction)

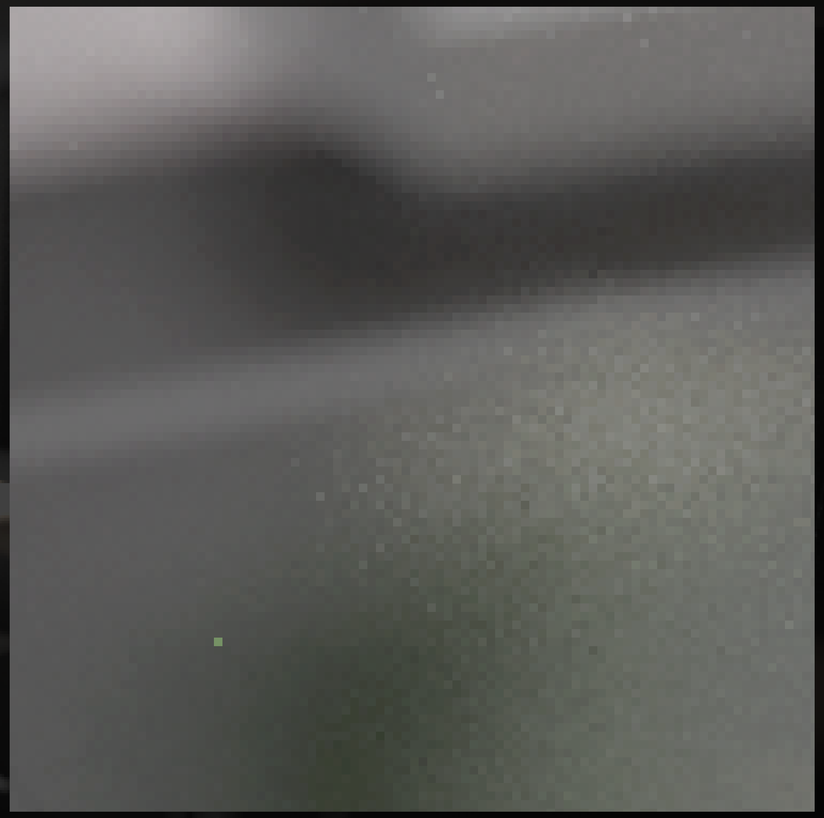
Our result: 16 spp + reconstruction at 128spp
771 sec (PBRT) + 10 sec (reconstruction)



Input: 16 spp

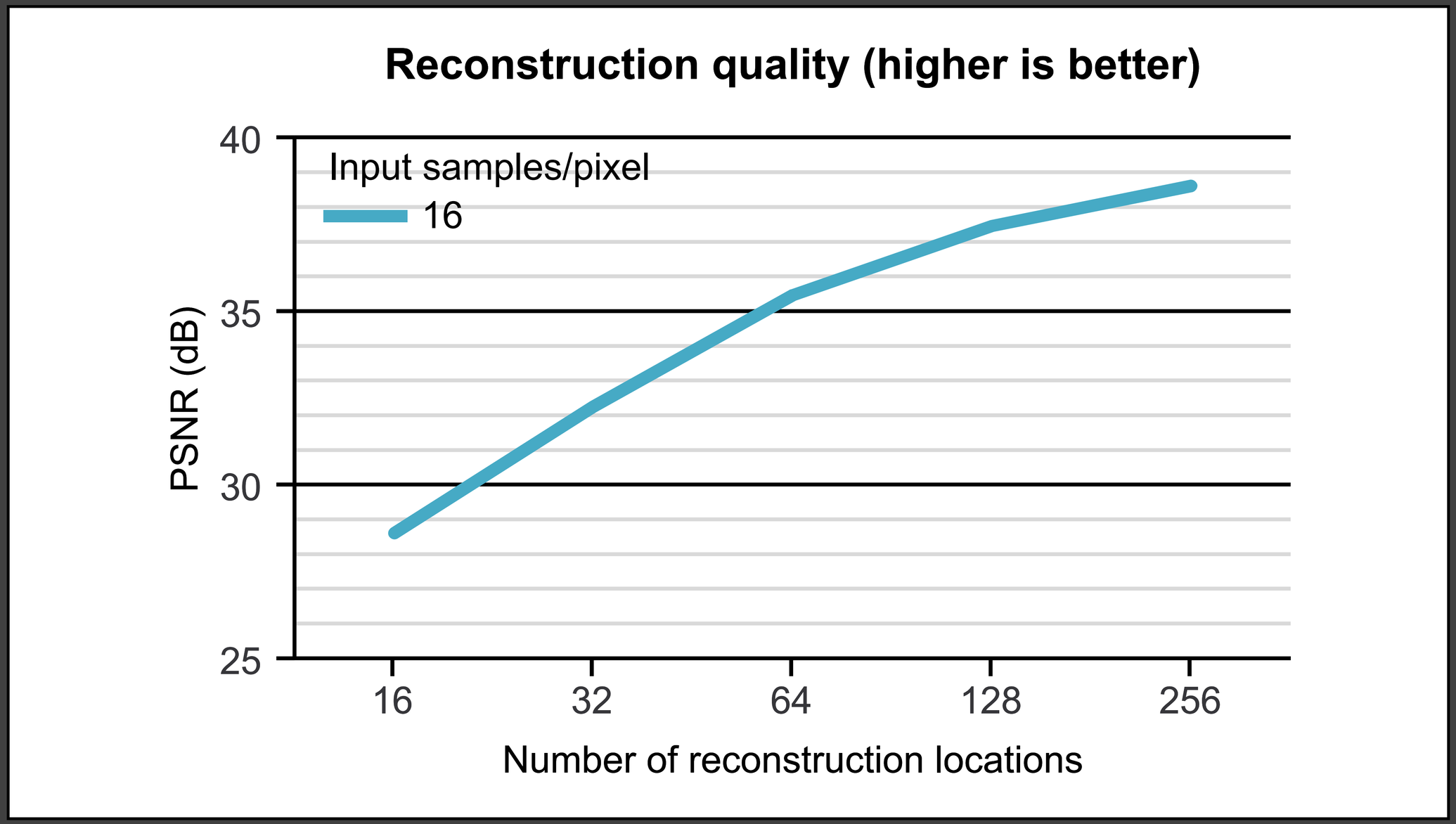


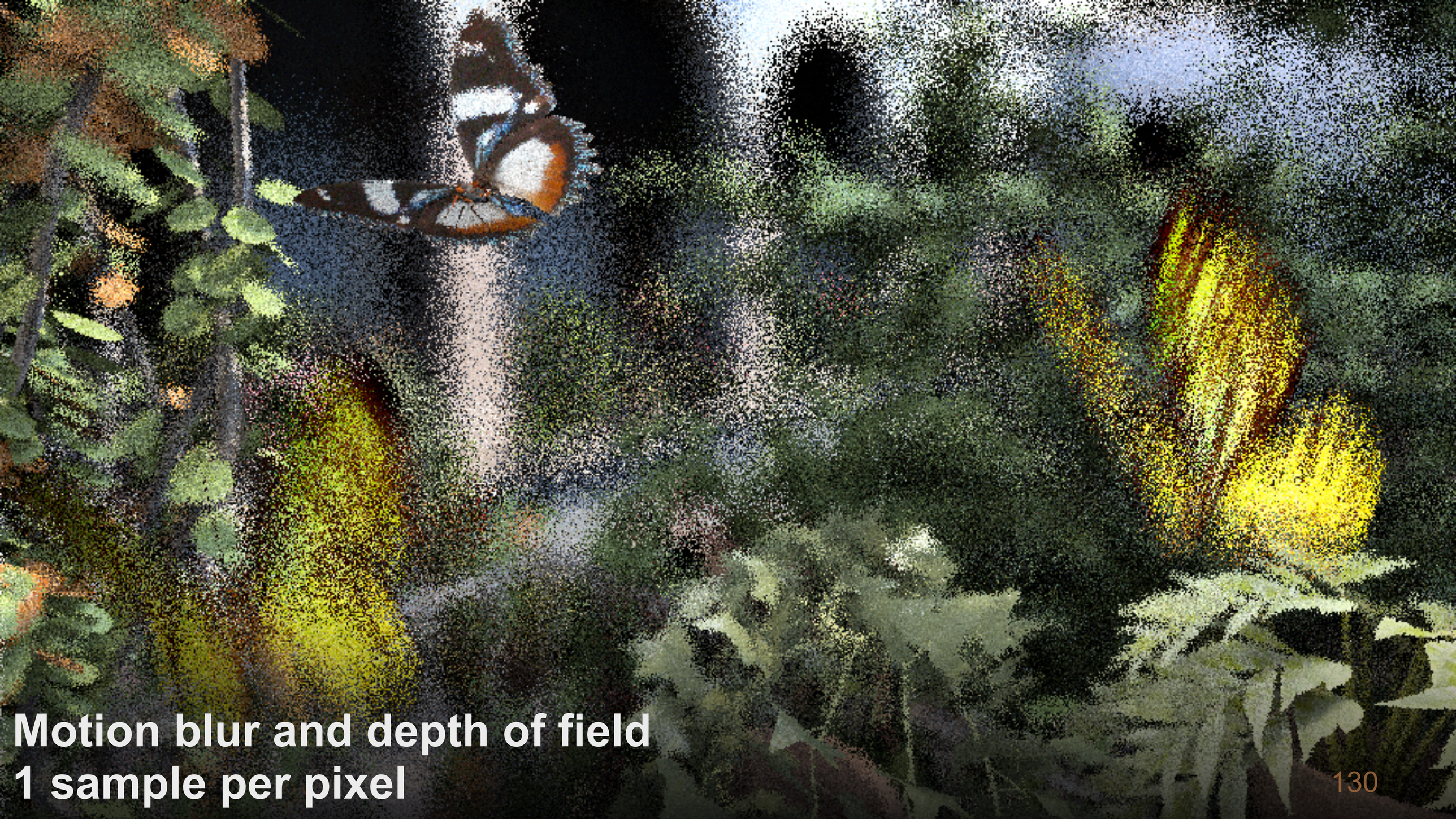
Our result at 128 spp
using same input



Reference: 256 spp
(16x time)

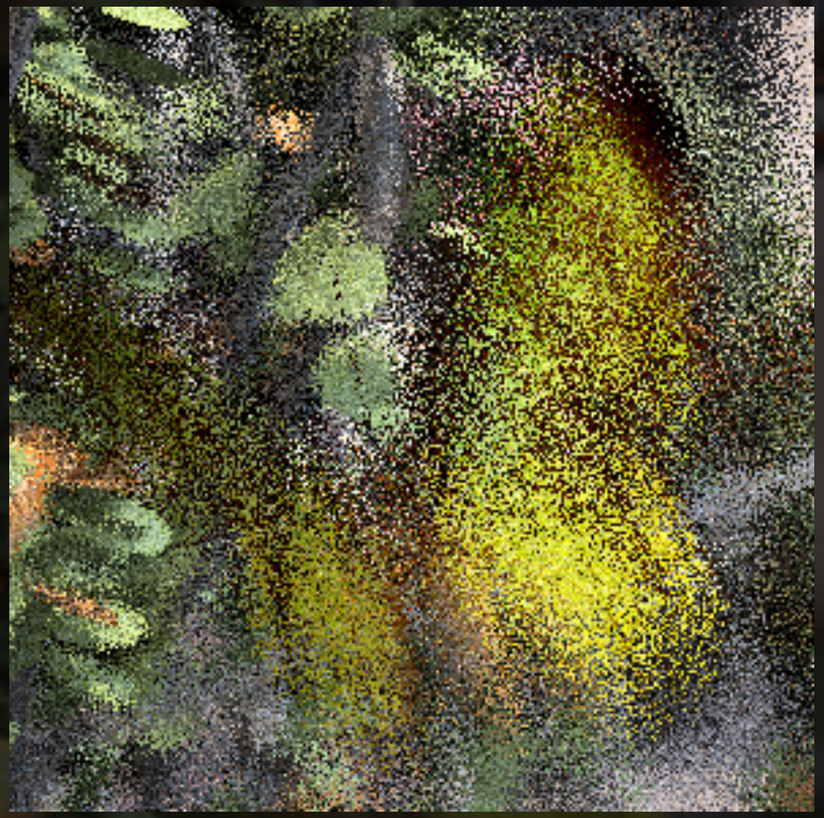
Comparison to reference





Motion blur and depth of field
1 sample per pixel

Proposed reconstruction



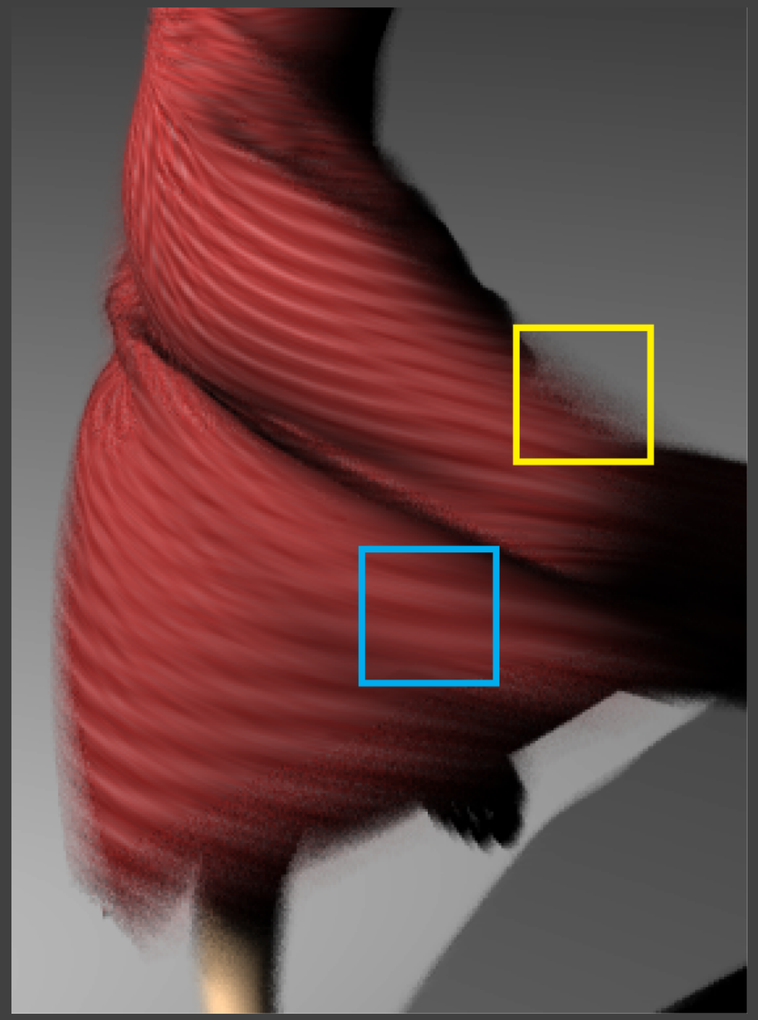
Input: 1 spp

Proposed result:
1 spp -> 128 spp

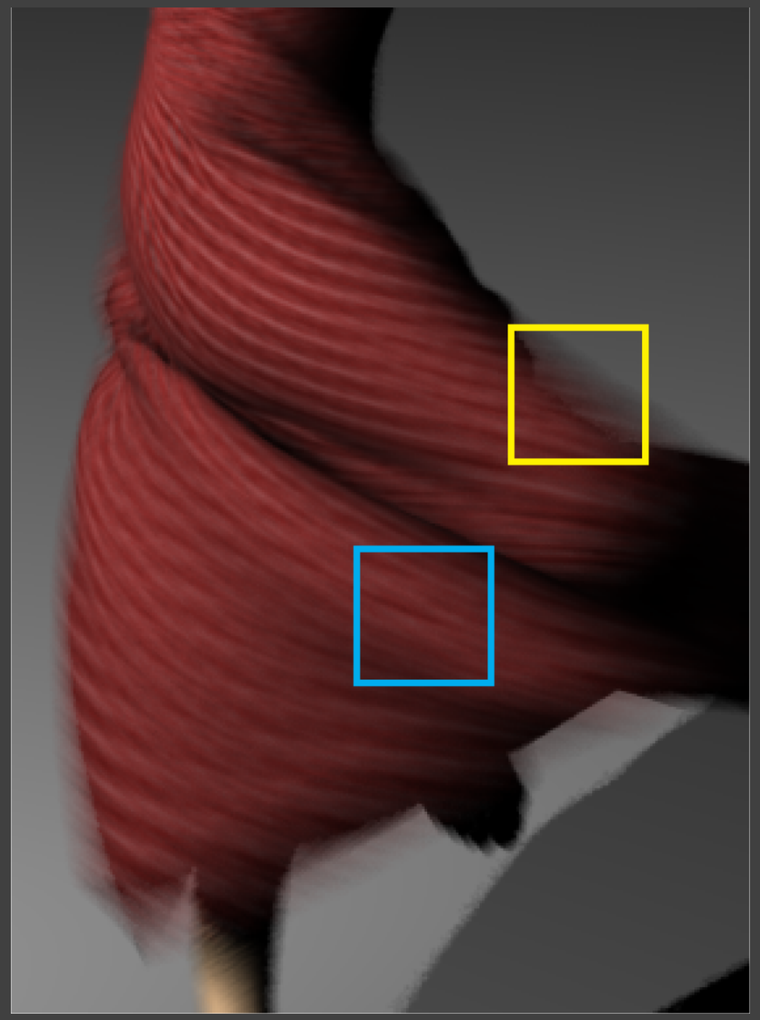
Reference 256 spp
(256x time)

Our reconstruction

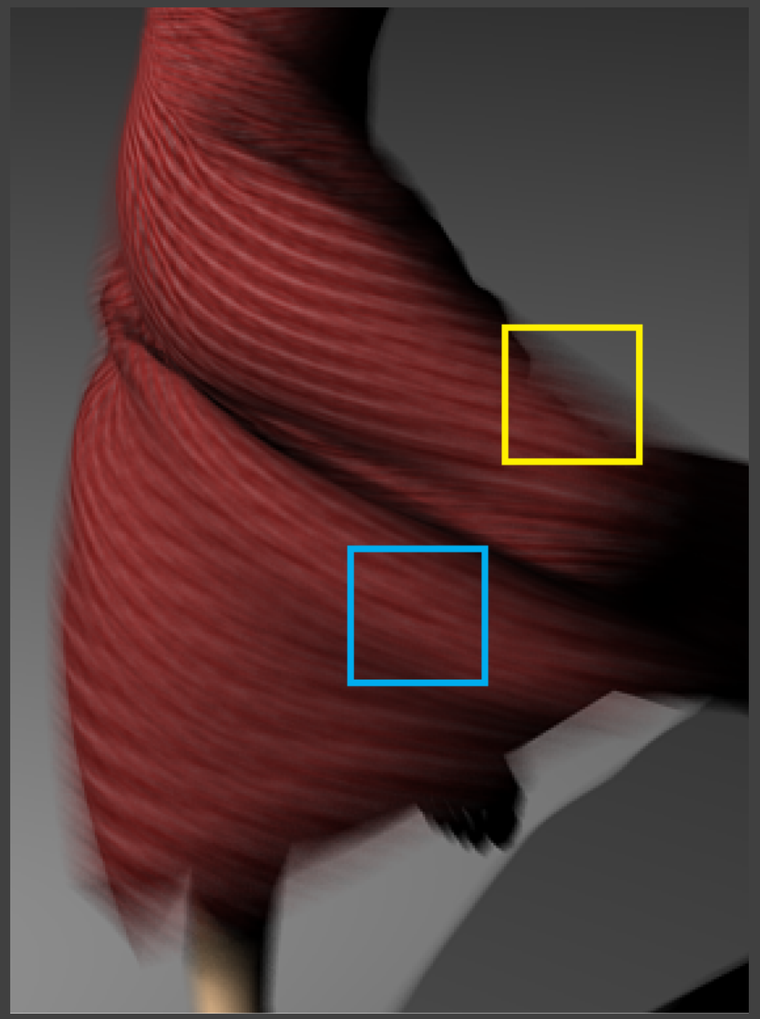
Comparison to Egan et al. [2009]



Egan et al. [2009]
8 samples / pixel

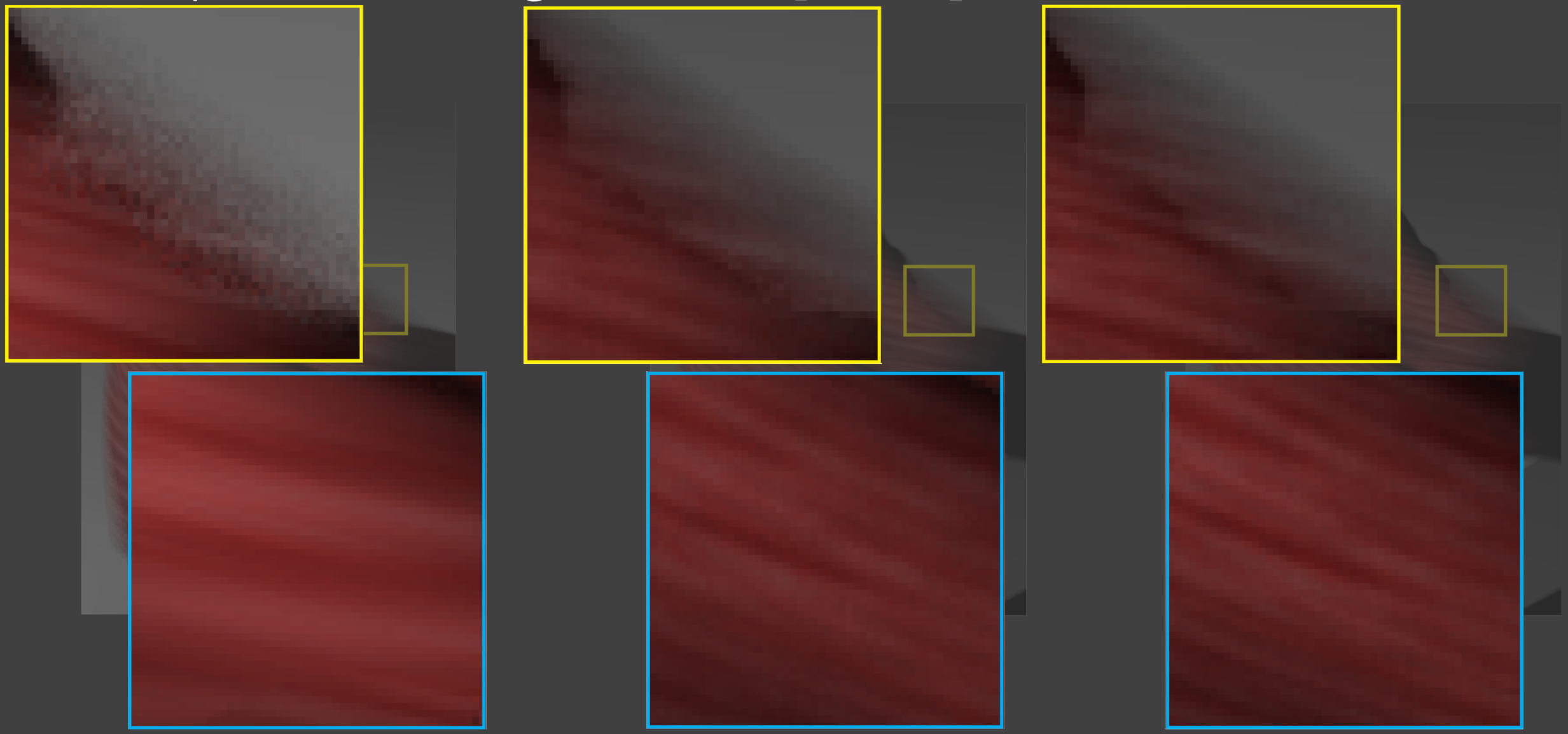


Proposed method
4 samples / pixel



Reference
256 samples / pixel

Comparison to Egan et al. [2009]



Egan et al. [2009]
8 samples / pixel

Our method
4 samples / pixel

Reference
256 samples / pixel

Soft shadows, 4 spp

7D soft shadows with motion and defocus, 4 spp

Acknowledgments

Thanks to everyone below for making the slides available online.

Fredo Durand and colleagues [Frequency analysis of light transport 2005]

Toshiya Hachisuka and colleagues [Multi-dimensional adaptive sampling and reconstruction for ray tracing 2008]

Kevin Egan and colleagues [Frequency Analysis and Sheared Reconstruction for Rendering Motion Blur 2009]

Jakko Lehtinen and colleagues [Temporal Light Field Reconstruction for Rendering Distribution Effects 2011]