Volume Rendering

Gurprit Singh

Philipp Slusalek Karol Myszkowski

Overview

Volumetric Processes:

Absorption

Scattering

Transmittance

Phase Functions

Volumetric Rendering Equation

Volumetric Path Tracing

Woodcock Tracking

Fog

Brassai (Gyula Halasz) 1899-1984

Aerial View

Gurprit Singh

Snow

Gurprit Singh

Fire

Harry Potter/Warner Brothers

Surface or Volume?

Corona Renderer / Chaos Czech a.s. / Chaos Group

Universe

lisamission.org

Defining Participating Media

Media properties are modeled as a probabilistic process

No need to consider individual interactions with particles (won't fit in the memory)

Defining Participating Media

Homoegeneous media:

- Infinite or bounded by a simple surface or simple shape

Krivanek et al. [2014]

Defining Participating Media

Heterogeneous media (spatially varying coefficients):

Disney/Pixar

Defining Participating Media

Heterogeneous media (spatially varying coefficients):

- Procedurally e.g. using a noise function

Disney/Pixar

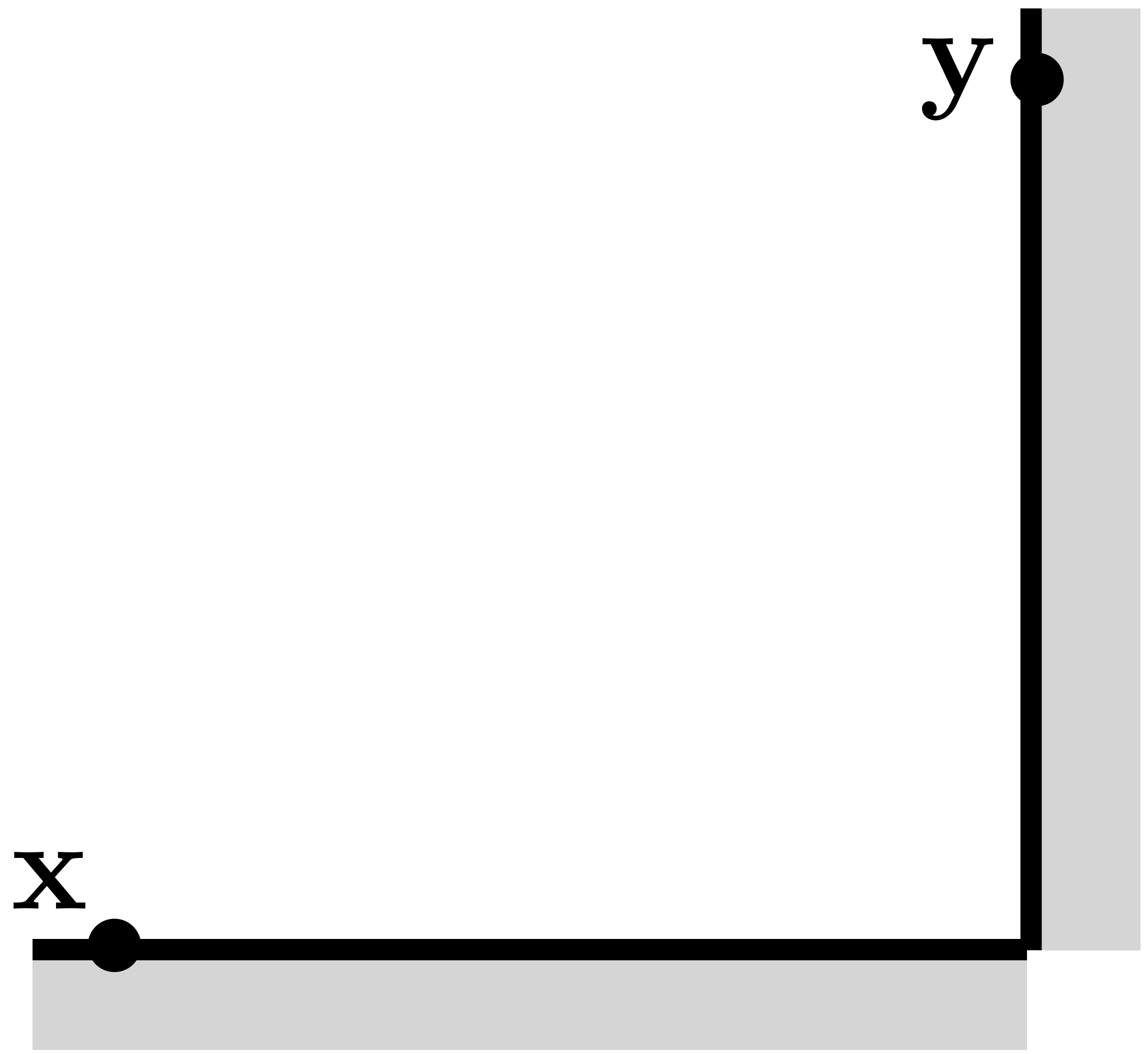
Defining Participating Media

Heterogeneous media (spatially varying coefficients):

- Procedurally e.g. using a noise function
- Simulation + volume discretization, e.g., voxel grid

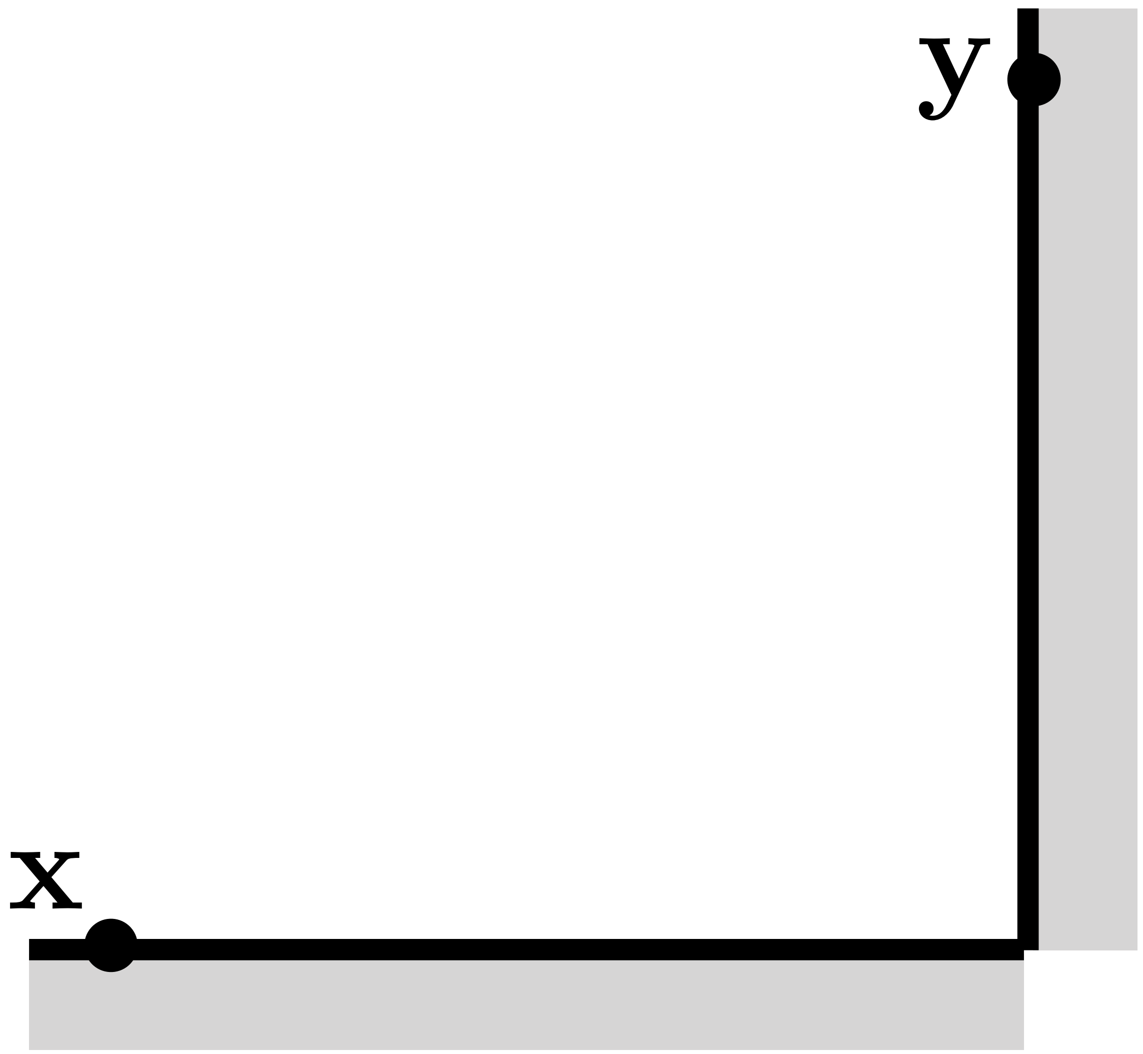
Disney/Pixar

Radiance



Radiance is the main quantity we are interested in for rendering.

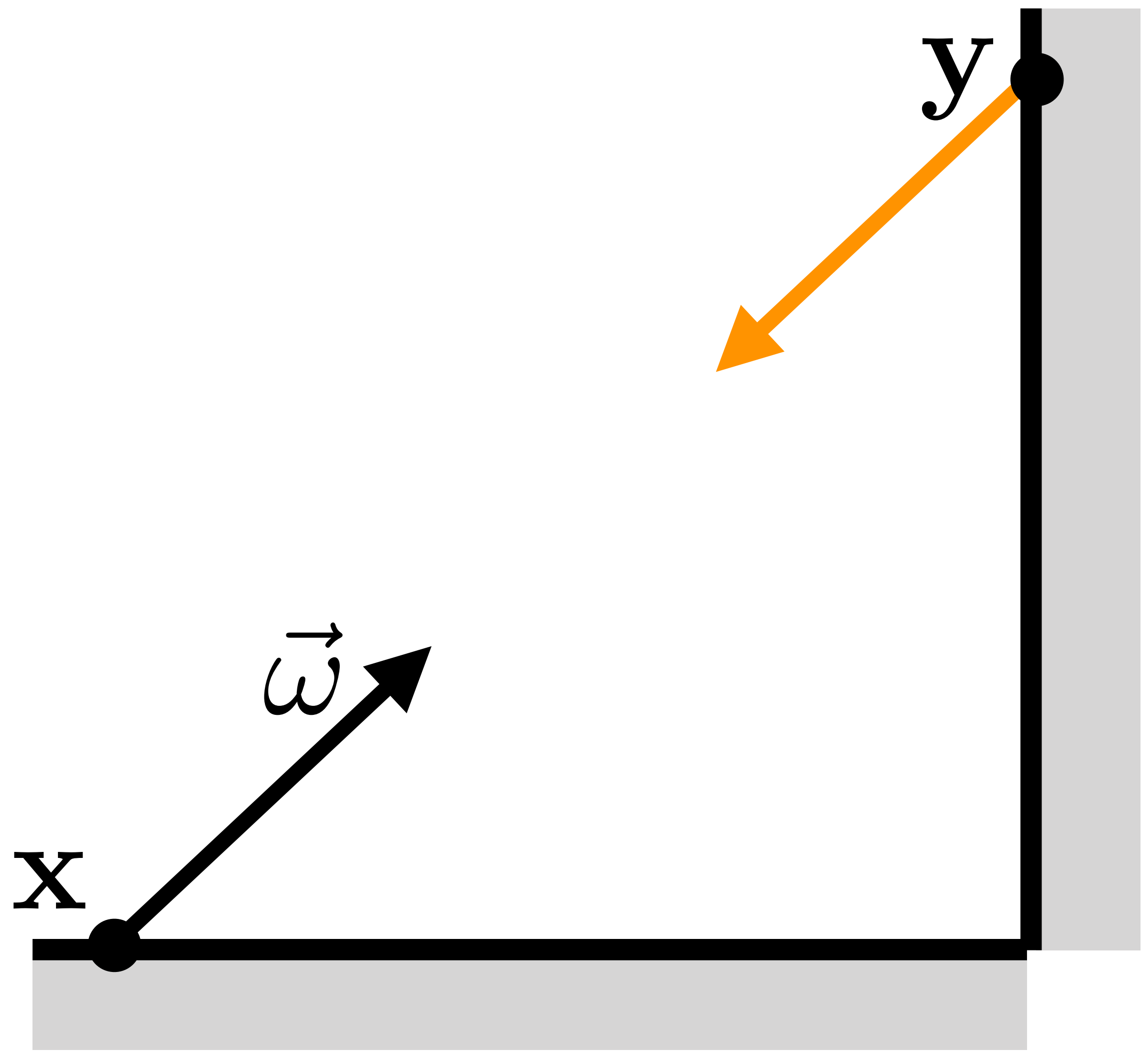
Radiance



Radiance is the main quantity we are interested in for rendering.

In **vaccum**, light transport radiance remains constant along rays between surfaces

Radiance



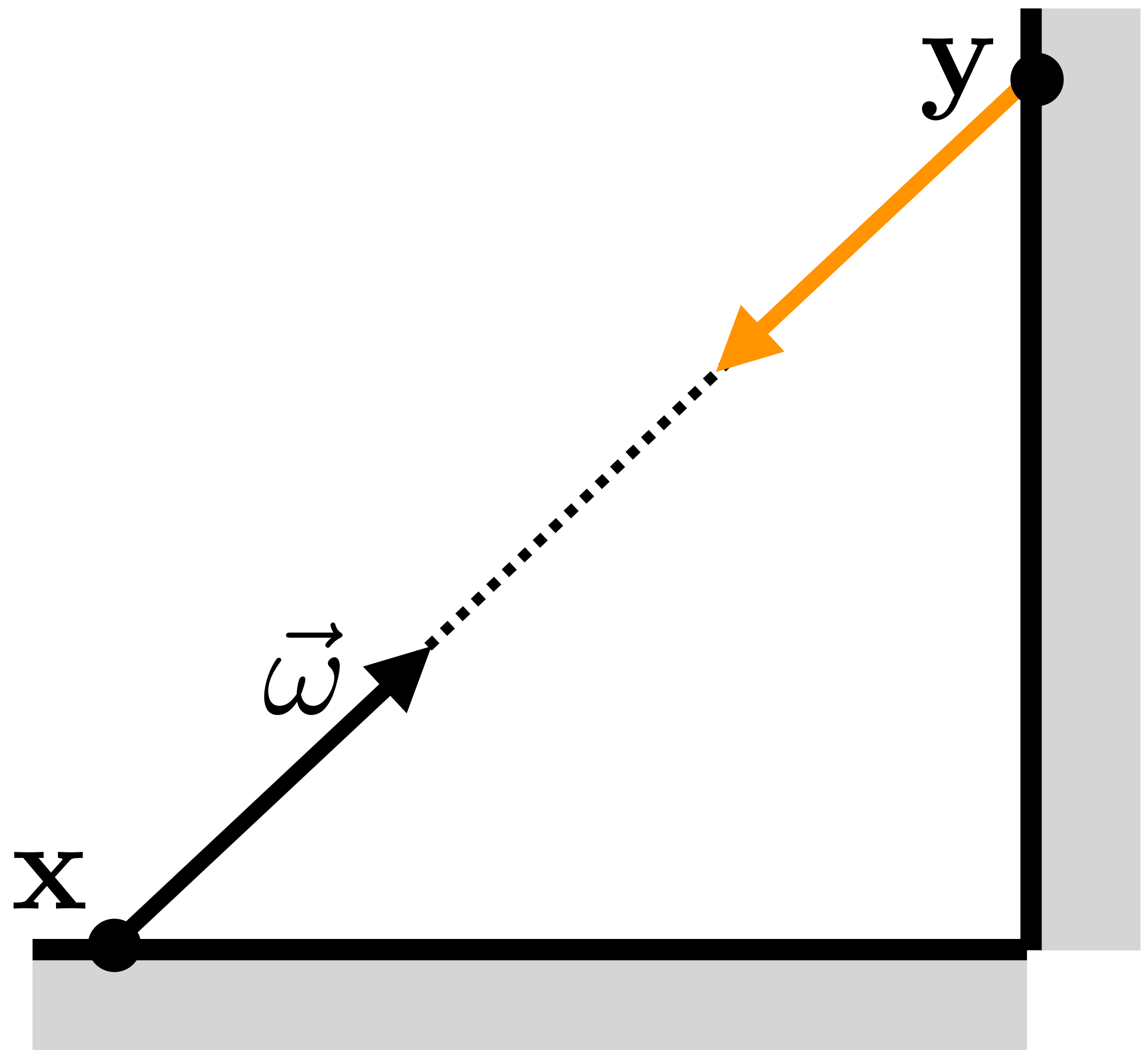
Radiance is the main quantity we are interested in for rendering.

In **vaccum**, light transport radiance remains constant along rays between surfaces

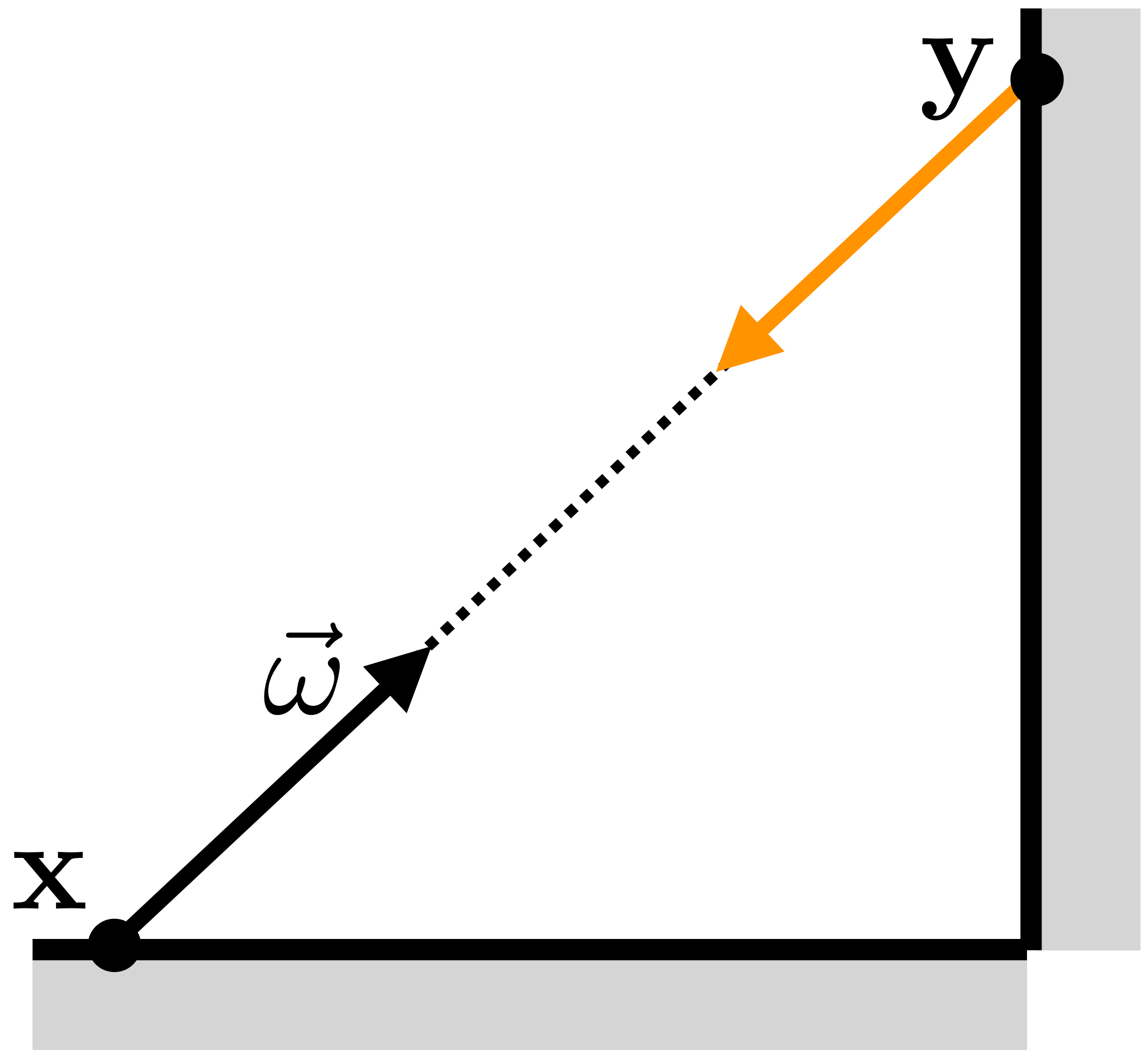
Radiance

Radiance is the main quantity we are interested in for rendering.

In **vaccum**, light transport radiance remains constant along rays between surfaces



Radiance

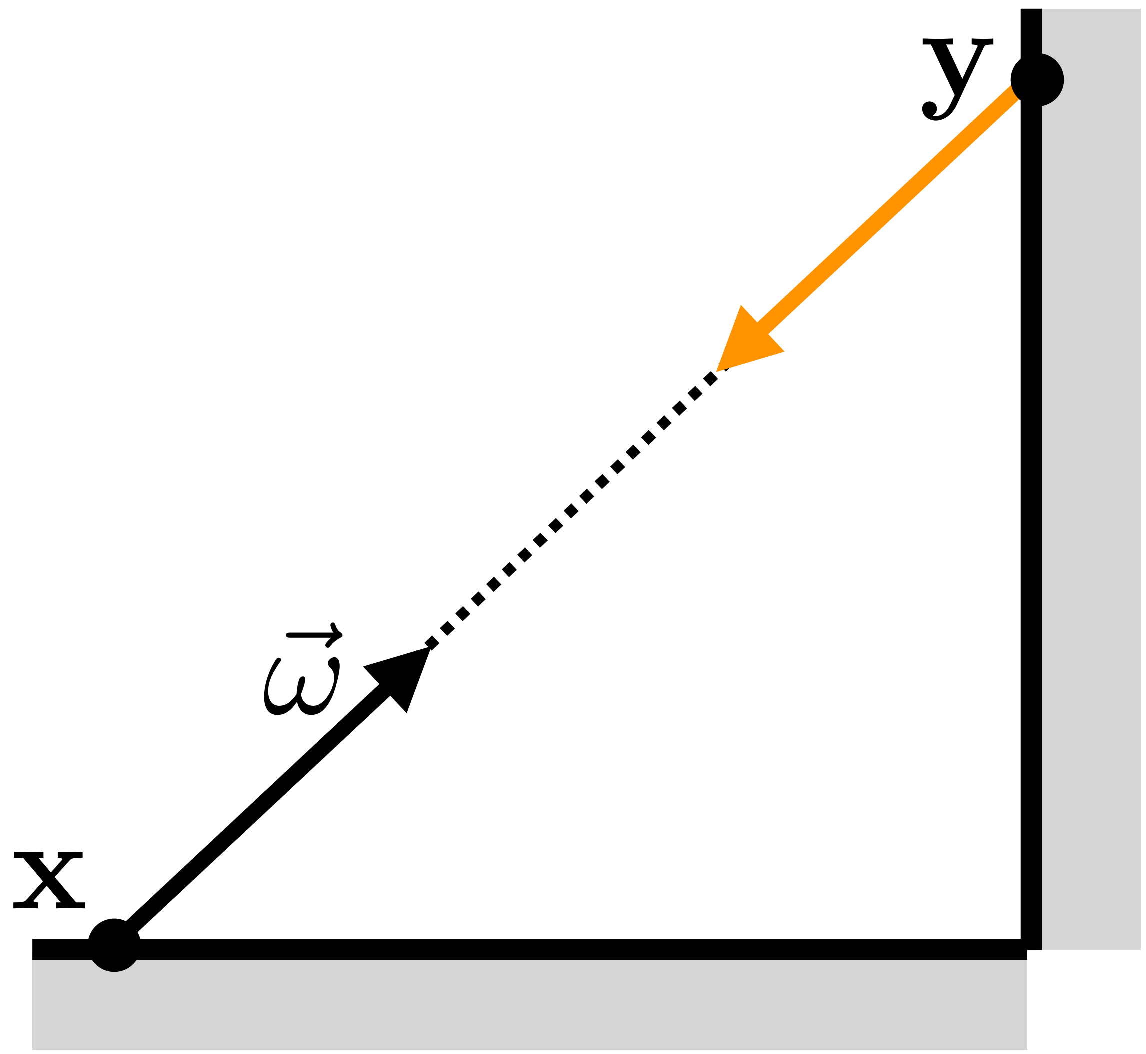


Radiance is the main quantity we are interested in for rendering.

In **vaccum**, light transport radiance remains constant along rays between surfaces

$$L_i(\mathbf{x}, \vec{\omega}) = L_o(\mathbf{y}, -\vec{\omega})$$

Radiance



Radiance is the main quantity we are interested in for rendering.

In **vaccum**, light transport radiance remains constant along rays between surfaces

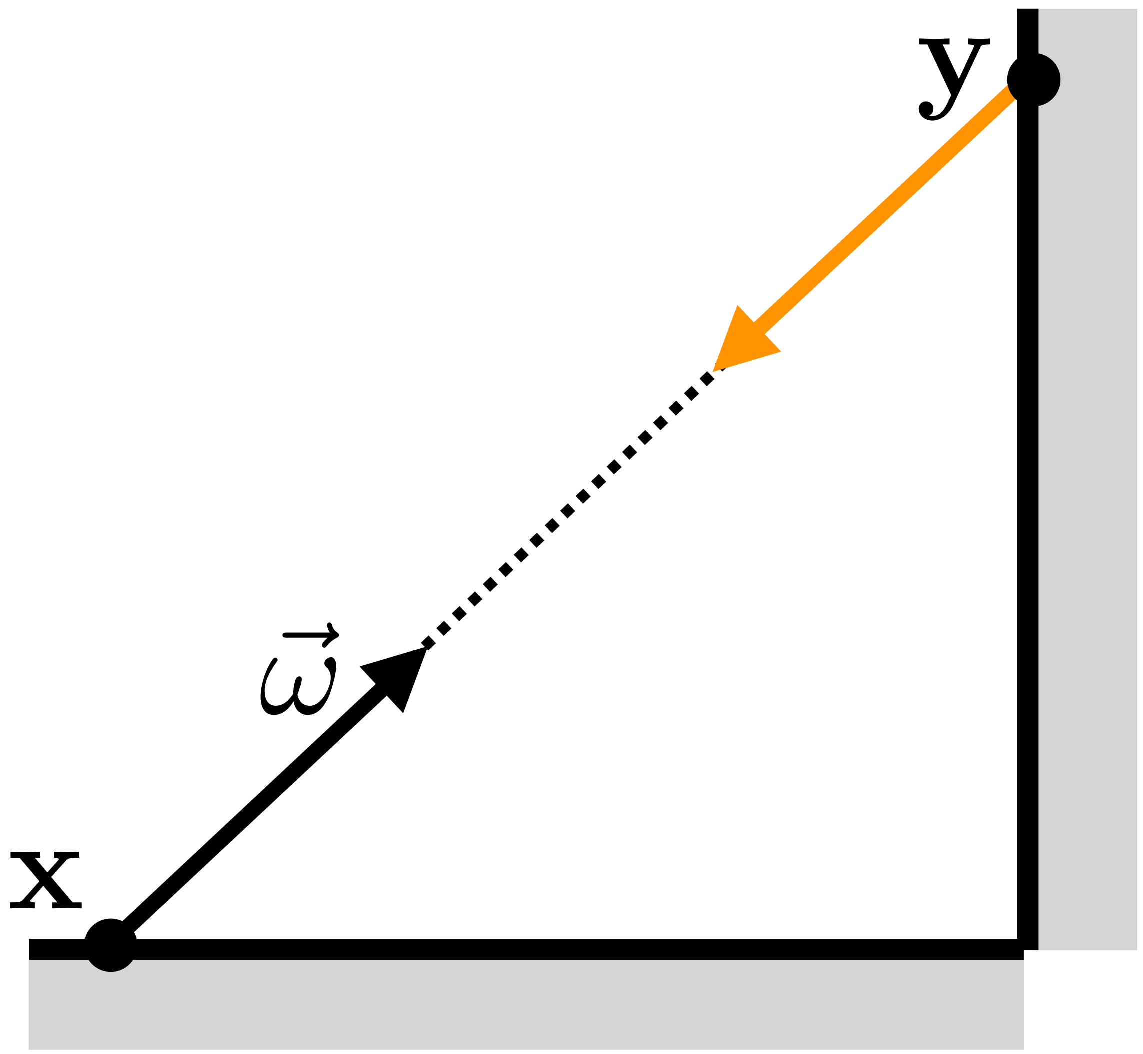
$$L_i(\mathbf{x}, \vec{\omega}) = L_o(\mathbf{y}, -\vec{\omega})$$

$$\mathbf{y} = \mathbf{r}(\mathbf{x}, \vec{\omega})$$

Radiance

Radiance is the main quantity we are interested in for rendering.

In **vaccum**, light transport radiance remains constant along rays between surfaces

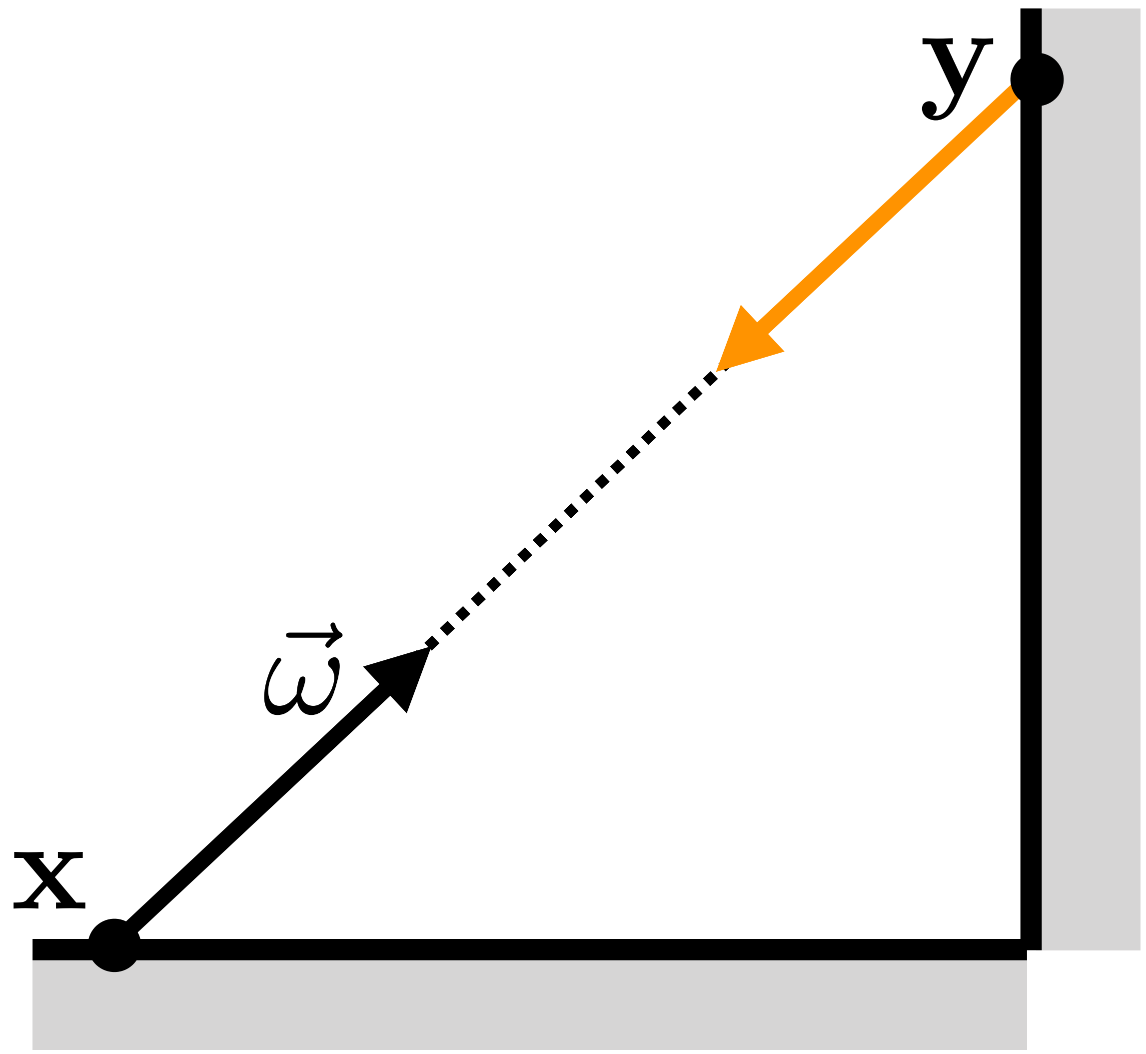


$$L_i(\mathbf{x}, \vec{\omega}) = L_o(\mathbf{y}, -\vec{\omega})$$

$$\mathbf{y} = \mathbf{r}(\mathbf{x}, \vec{\omega})$$

ray tracing function

Radiance



Radiance is the main quantity we are interested in for rendering.

In **vaccum**, light transport radiance remains constant along rays between surfaces

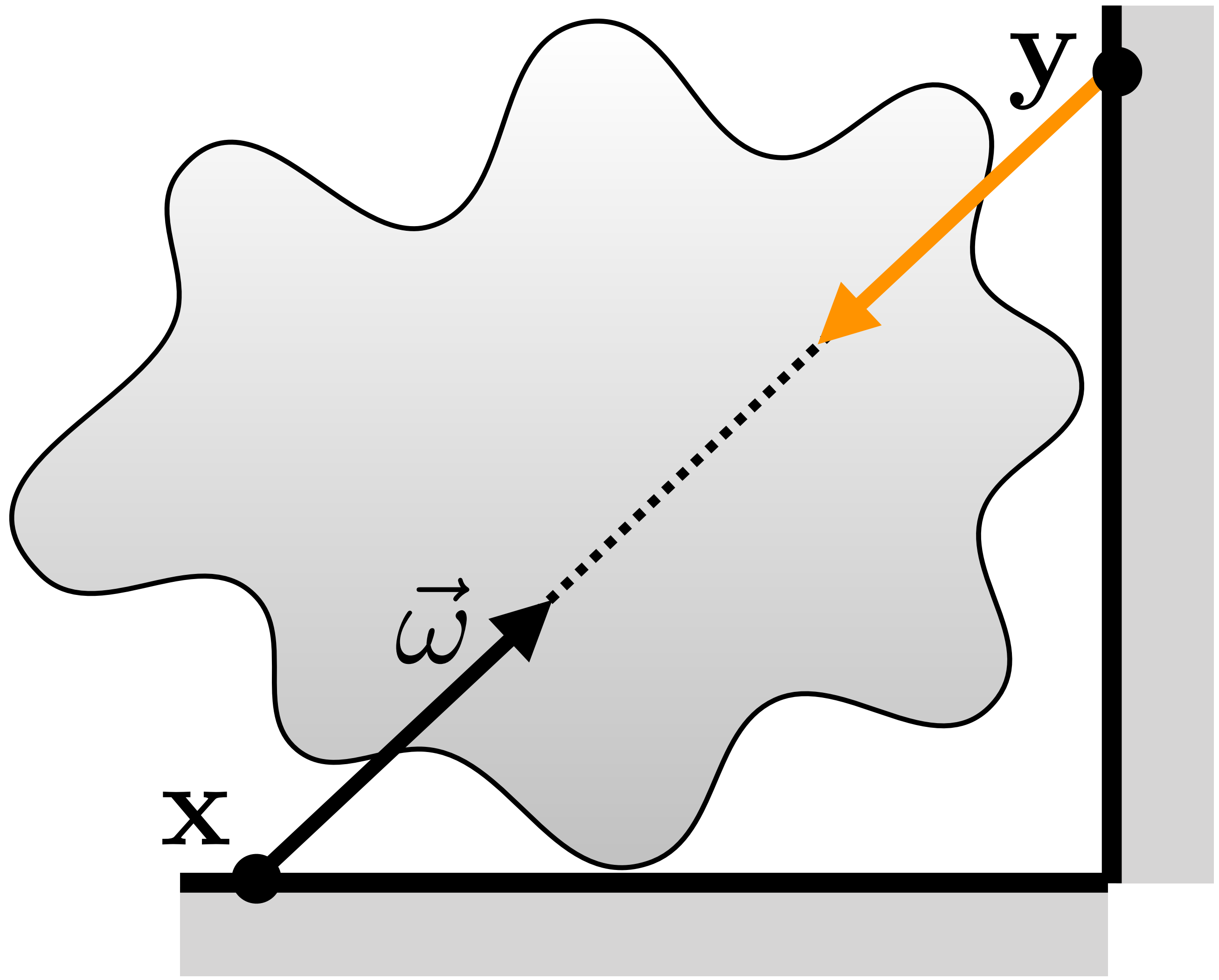
$$L_i(\mathbf{x}, \vec{\omega}) = L_o(\mathbf{y}, -\vec{\omega})$$

$$\mathbf{y} = \mathbf{r}(\mathbf{x}, \vec{\omega})$$

ray tracing function

Radiance

In **participating media**, radiance may change along rays between surfaces



$$L_i(\mathbf{x},\vec{\omega}) \cancel{=} L_o(\mathbf{y},-\vec{\omega})$$

$$\mathbf{y} = \mathbf{r}(\mathbf{x},\vec{\omega})$$

ray tracing function

Volumetric Scattering Processes

Absorption

http://commons.wikimedia.org

Scattering

http://coclouds.com

Emission

http://wikipedia.org

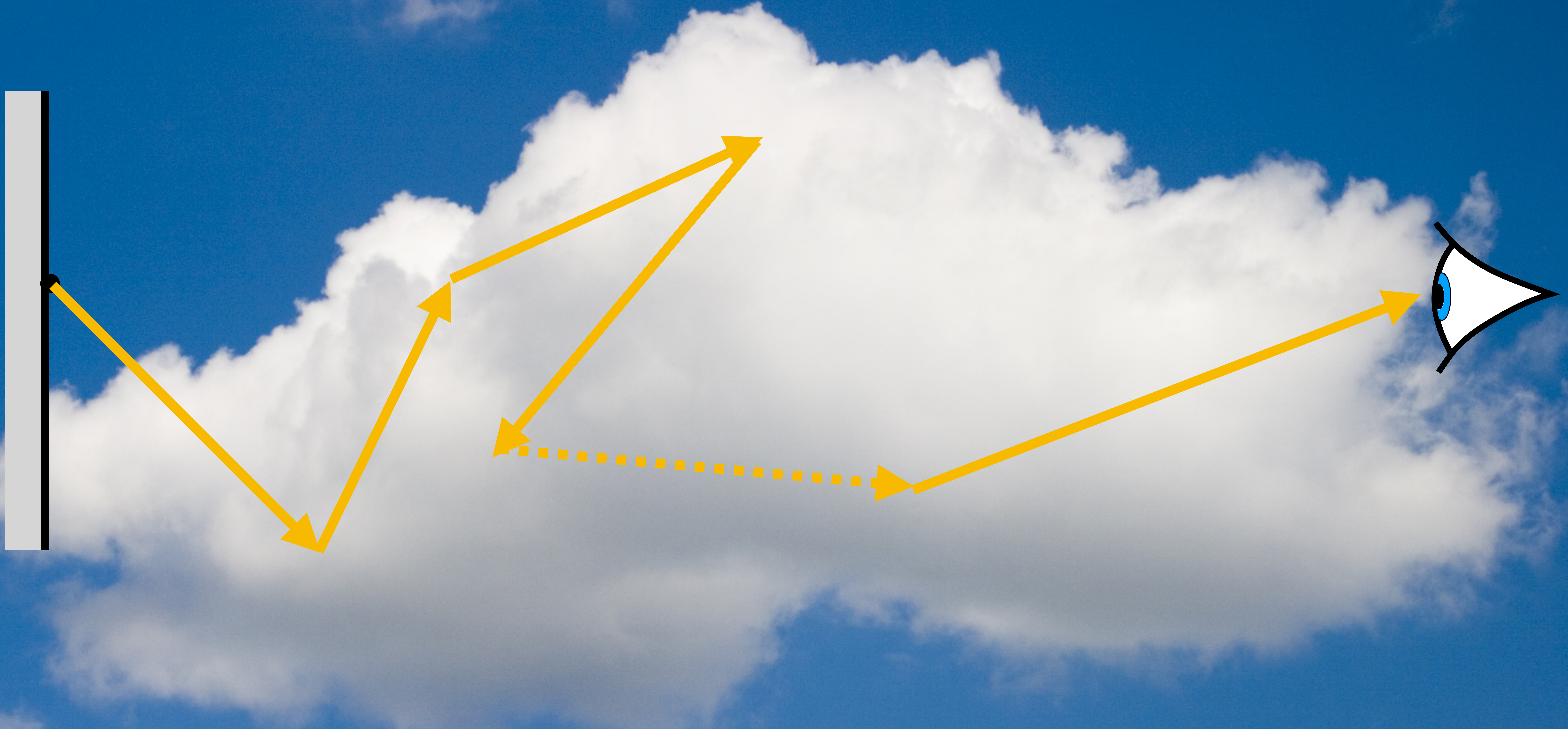
Slide after Jan Novak

Participating Media

Participating Media

Participating Media

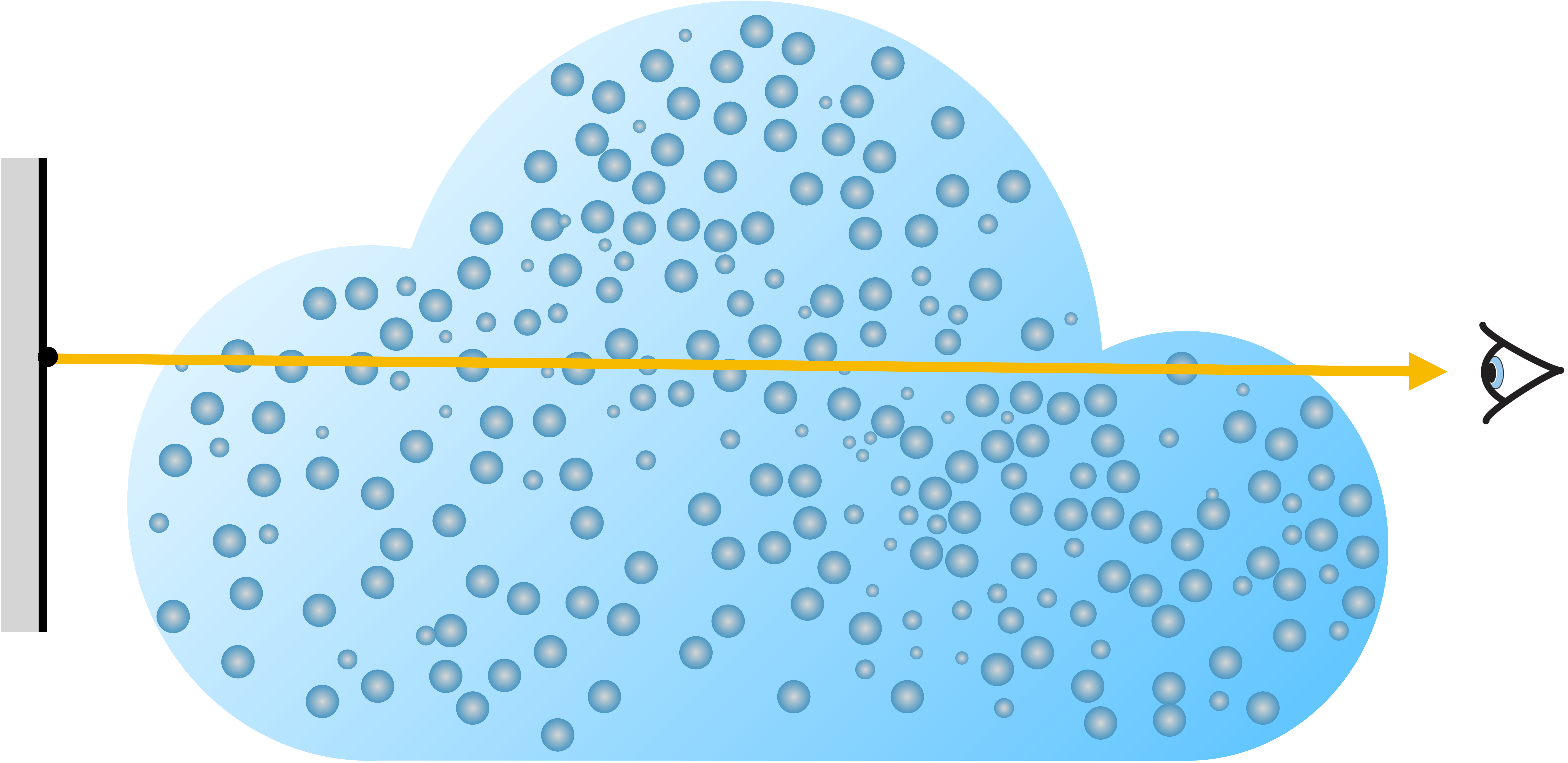
Participating Media



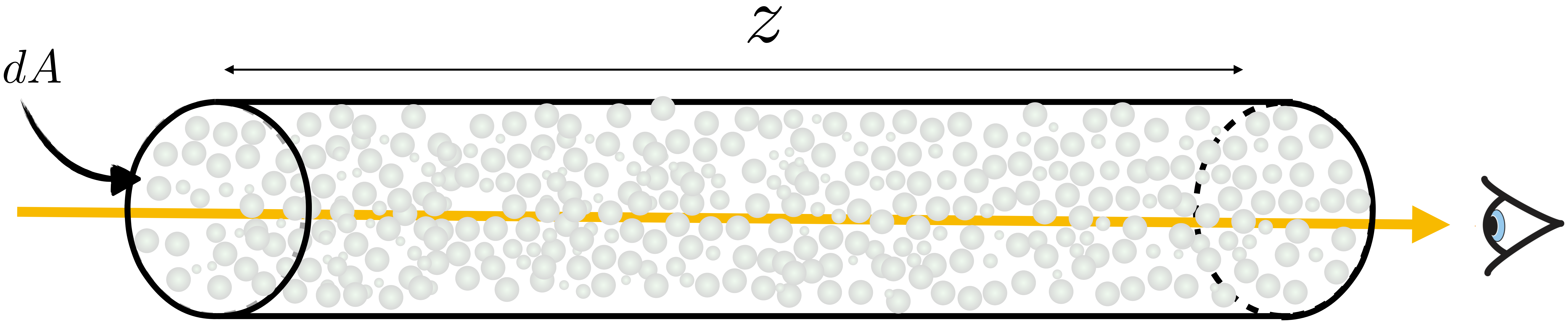
Participating Media

Participating Media

Participating Media

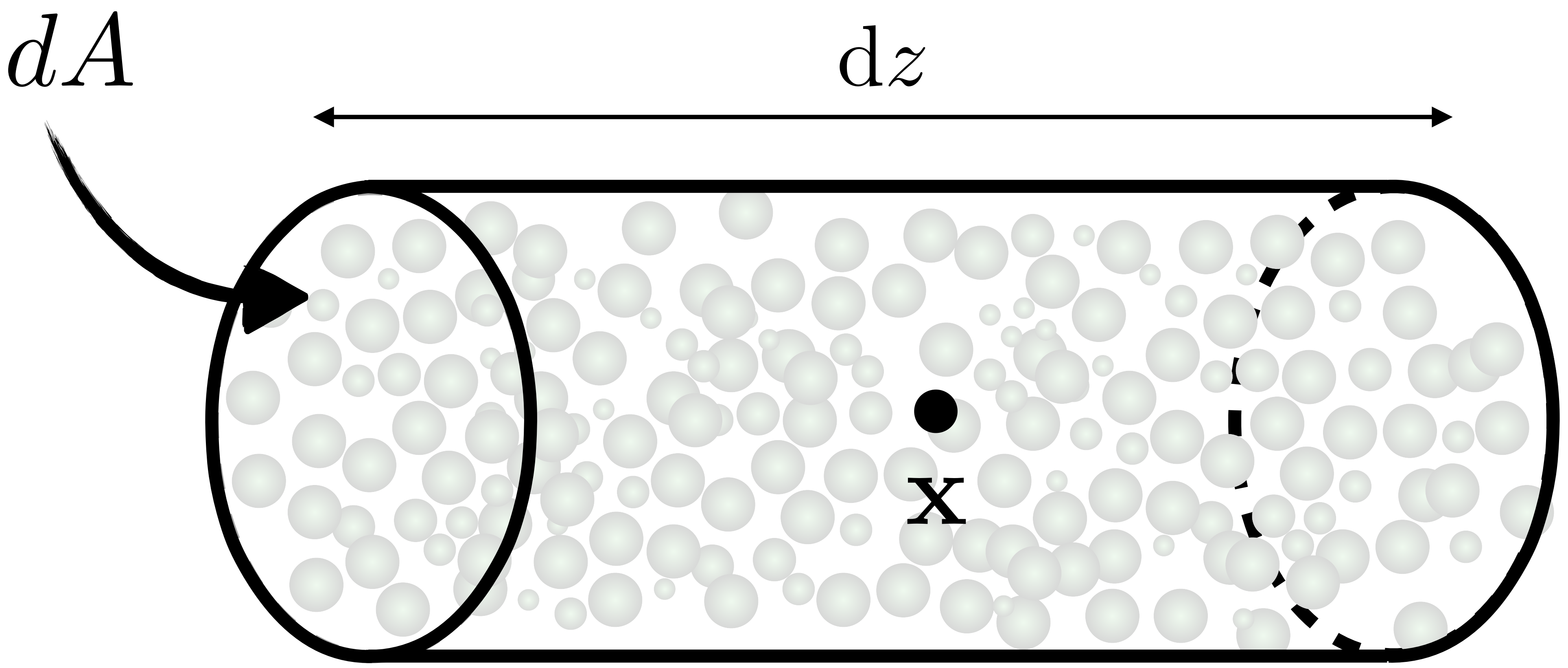


Finite distance Beam

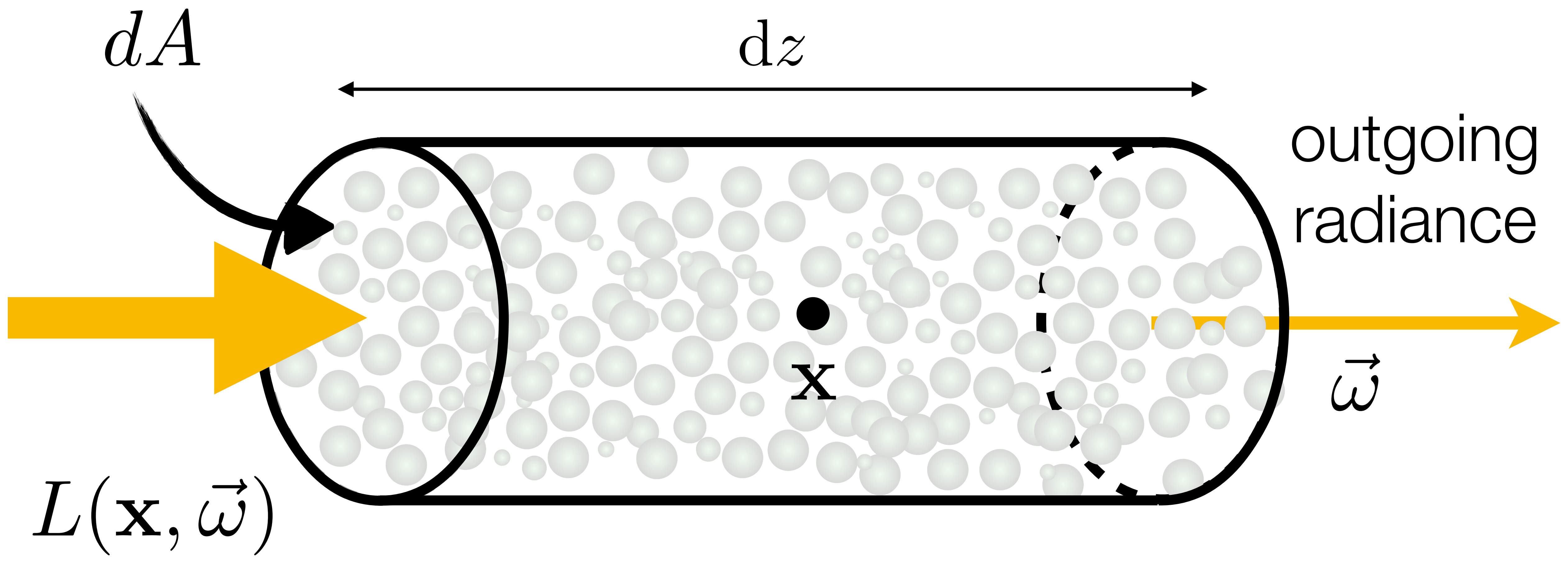


How much light is gained or lost during the travel
through this differential beam due to the interactions with the medium?

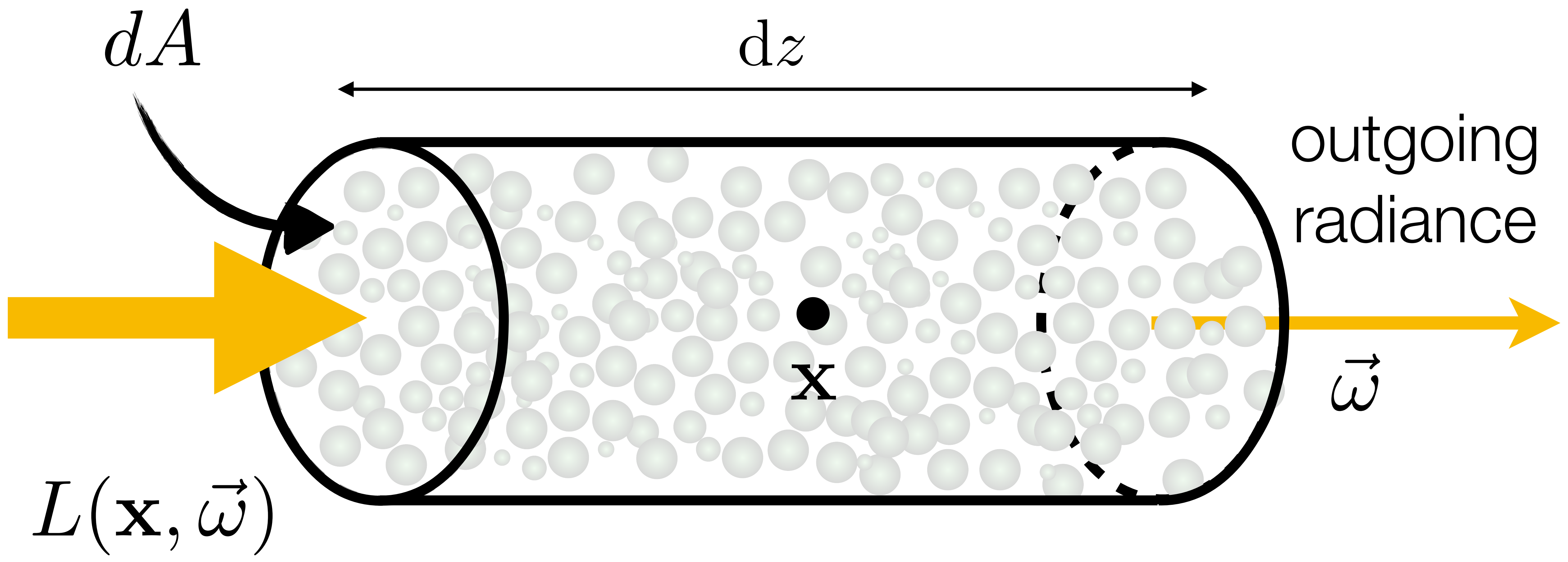
Differential Beam



Absorption

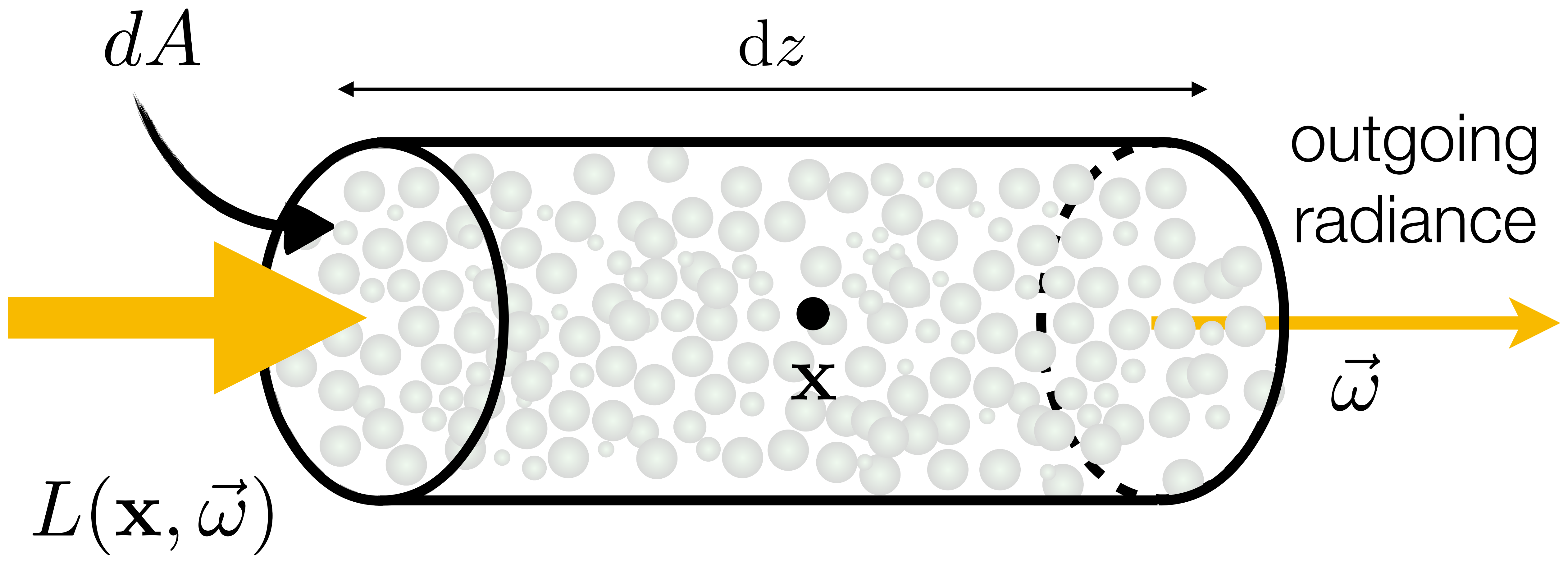

dA
$\mathrm{d}z$
outgoing
radiance
$\mathbf{x}$
$\vec{\omega}$
$L(\mathbf{x}, \vec{\omega})$

Absorption



$$\frac{dL(\mathbf{x},\vec{\omega})}{dz} = -\sigma_a L(\mathbf{x},\vec{\omega})$$

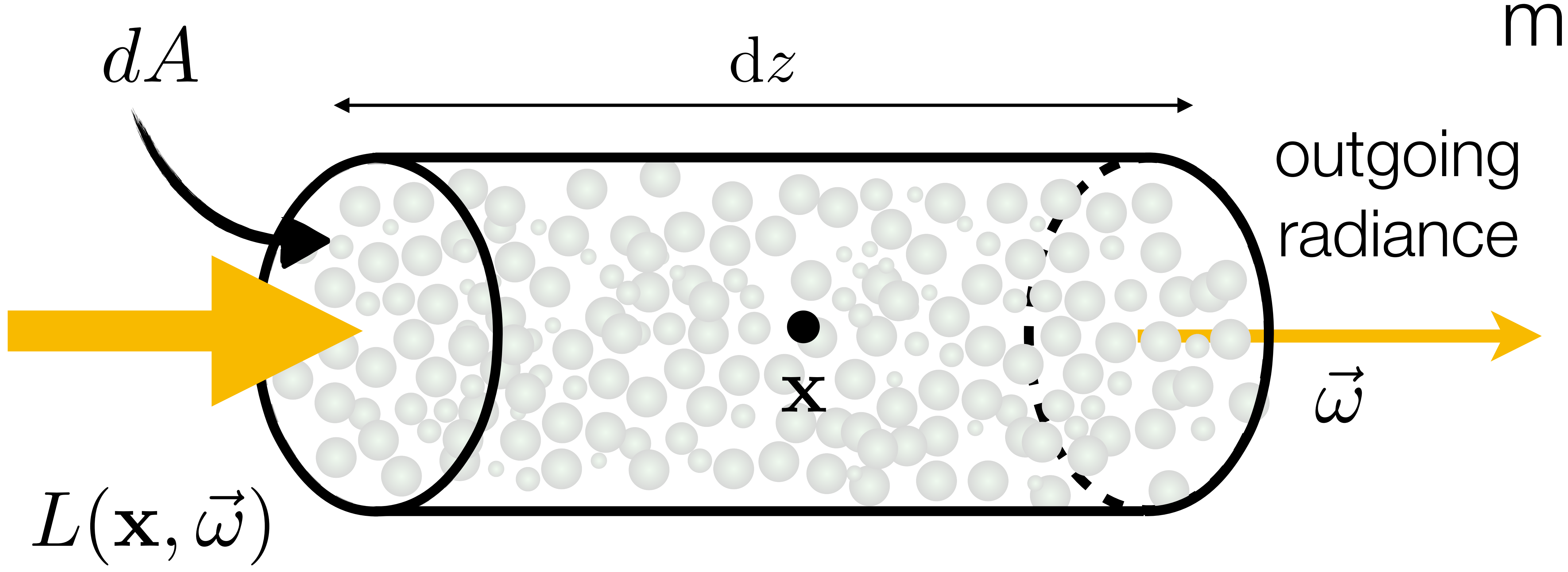
Absorption



$$\frac{dL(\mathbf{x},\vec{\omega})}{dz} = -\sigma_a L(\mathbf{x},\vec{\omega})$$

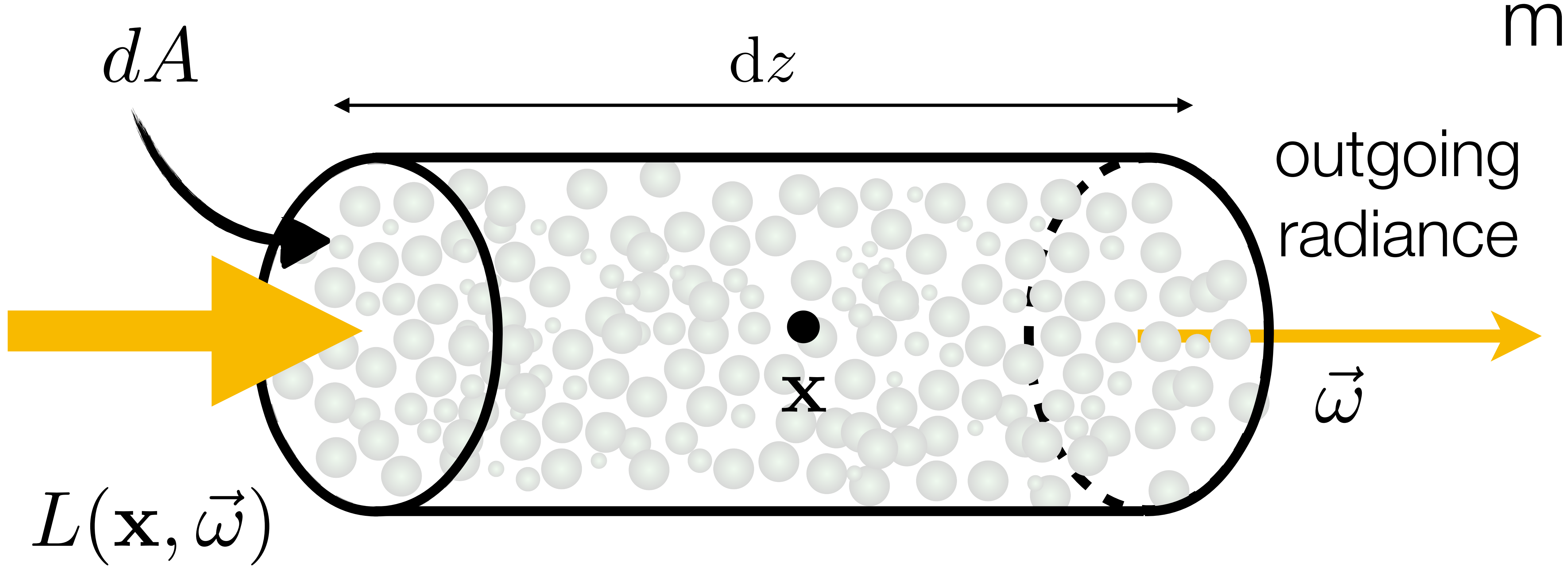
σ_a : absorption coefficient m^{-1}

Absorption



Absorption described by
medium's absorption cross-section σ_a

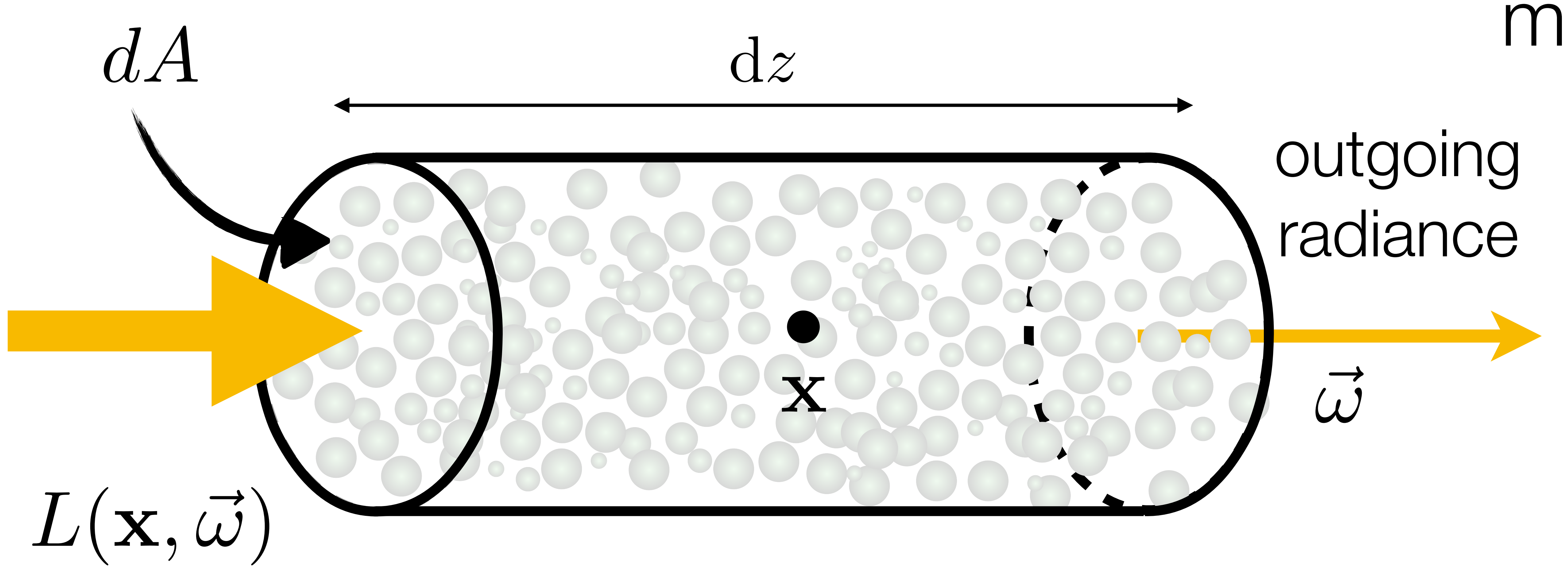
Absorption



Absorption described by
medium's absorption cross-section σ_a

$$\sigma_a \in [0, \infty)$$

Absorption



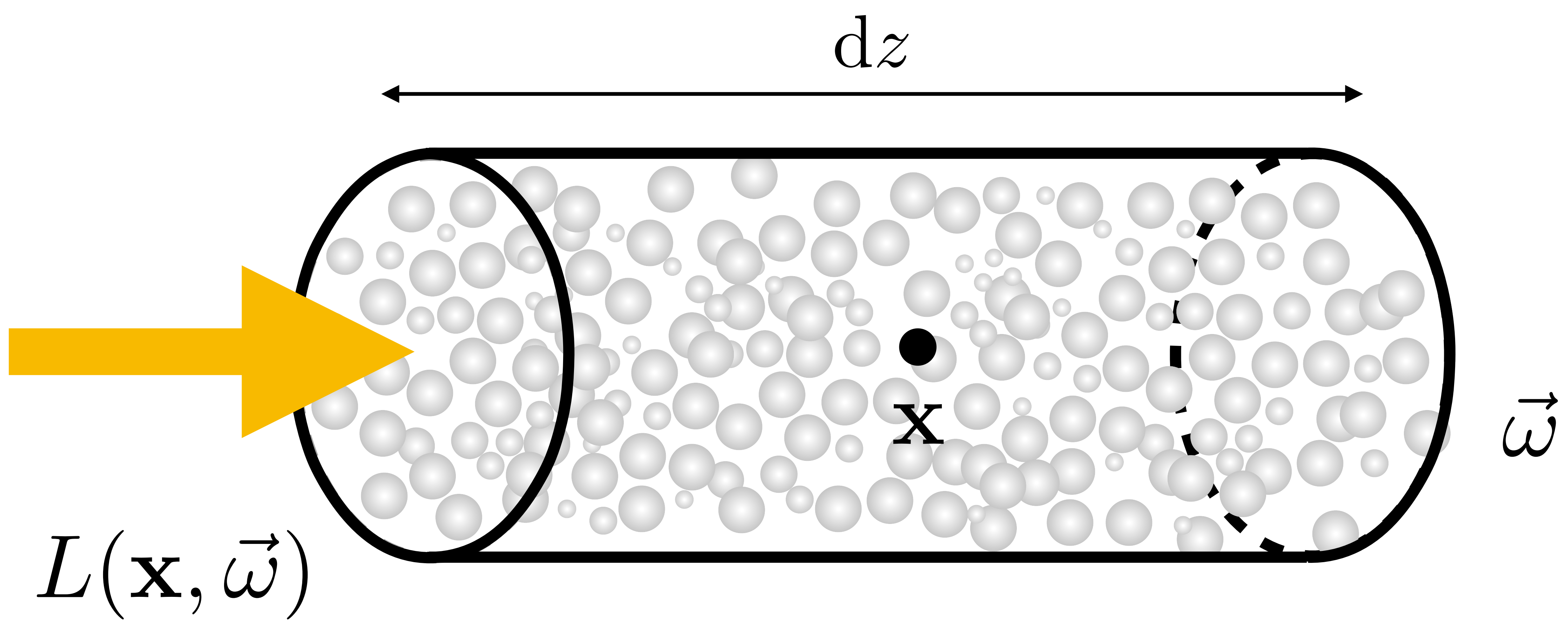
Absorption described by
medium's absorption cross-section σ_a

$$\sigma_a \in [0, \infty)$$

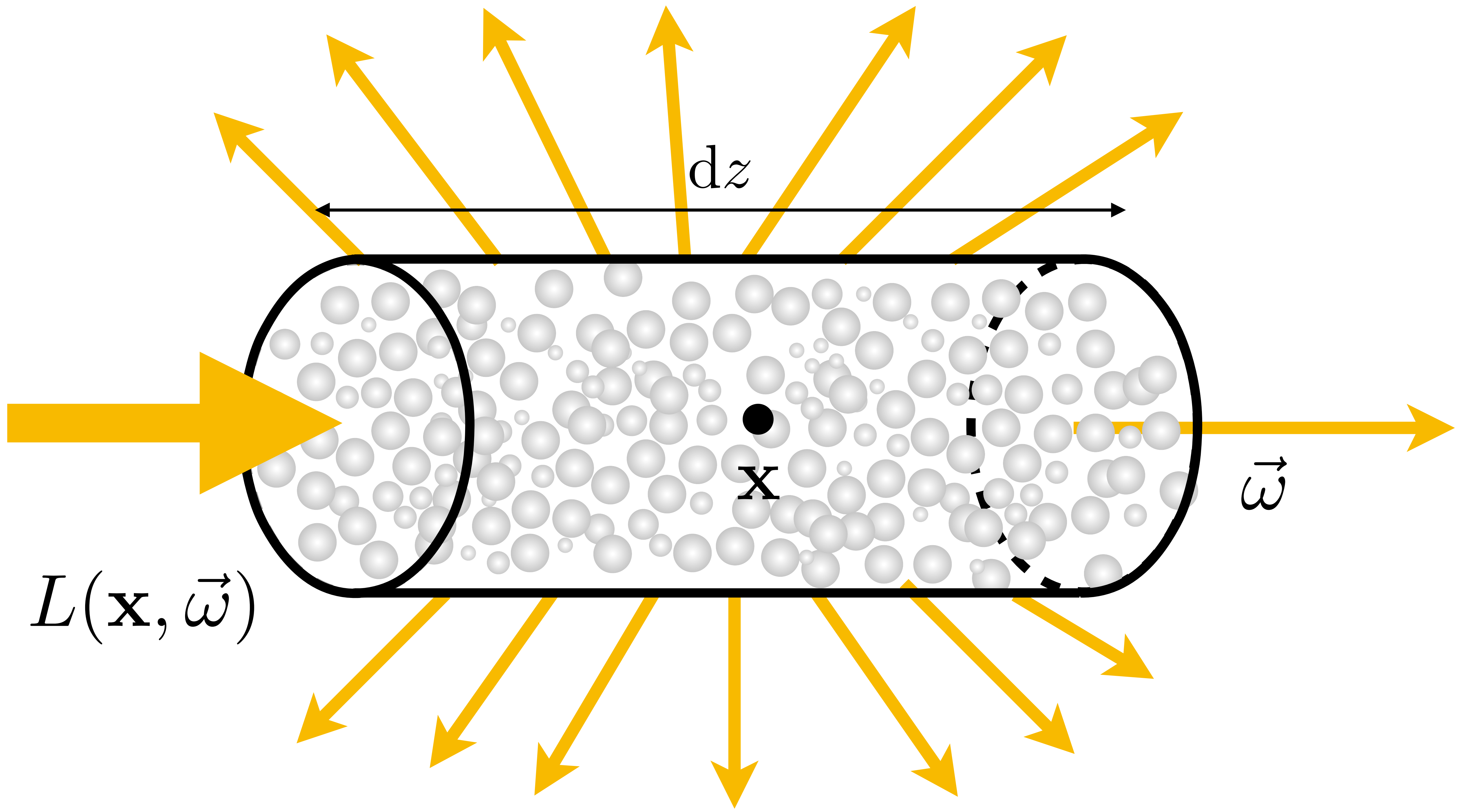
It is the probability density that light is absorbed per unit distance travelled in the medium

It can vary as a position and direction

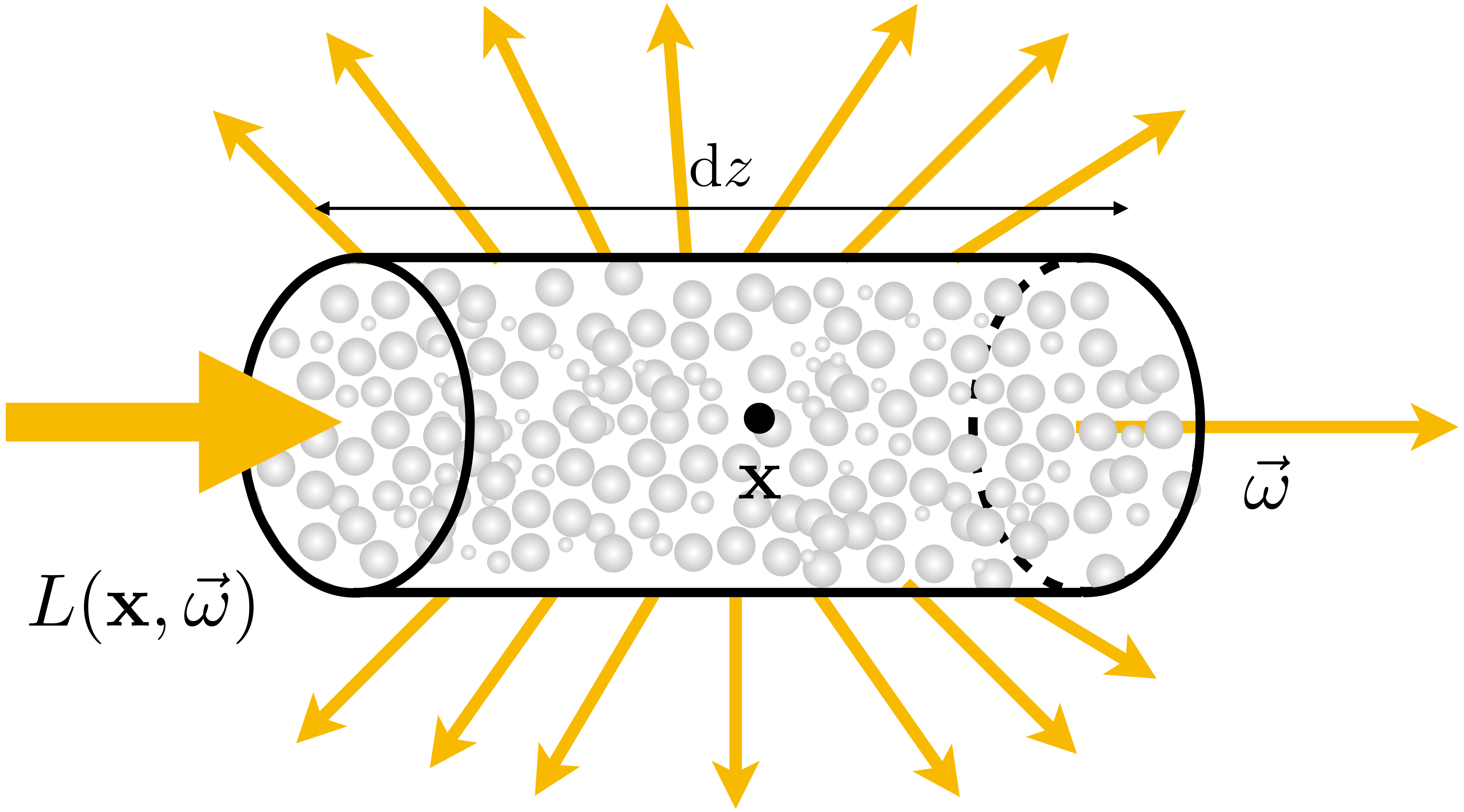
Out-Scattering



Out-Scattering

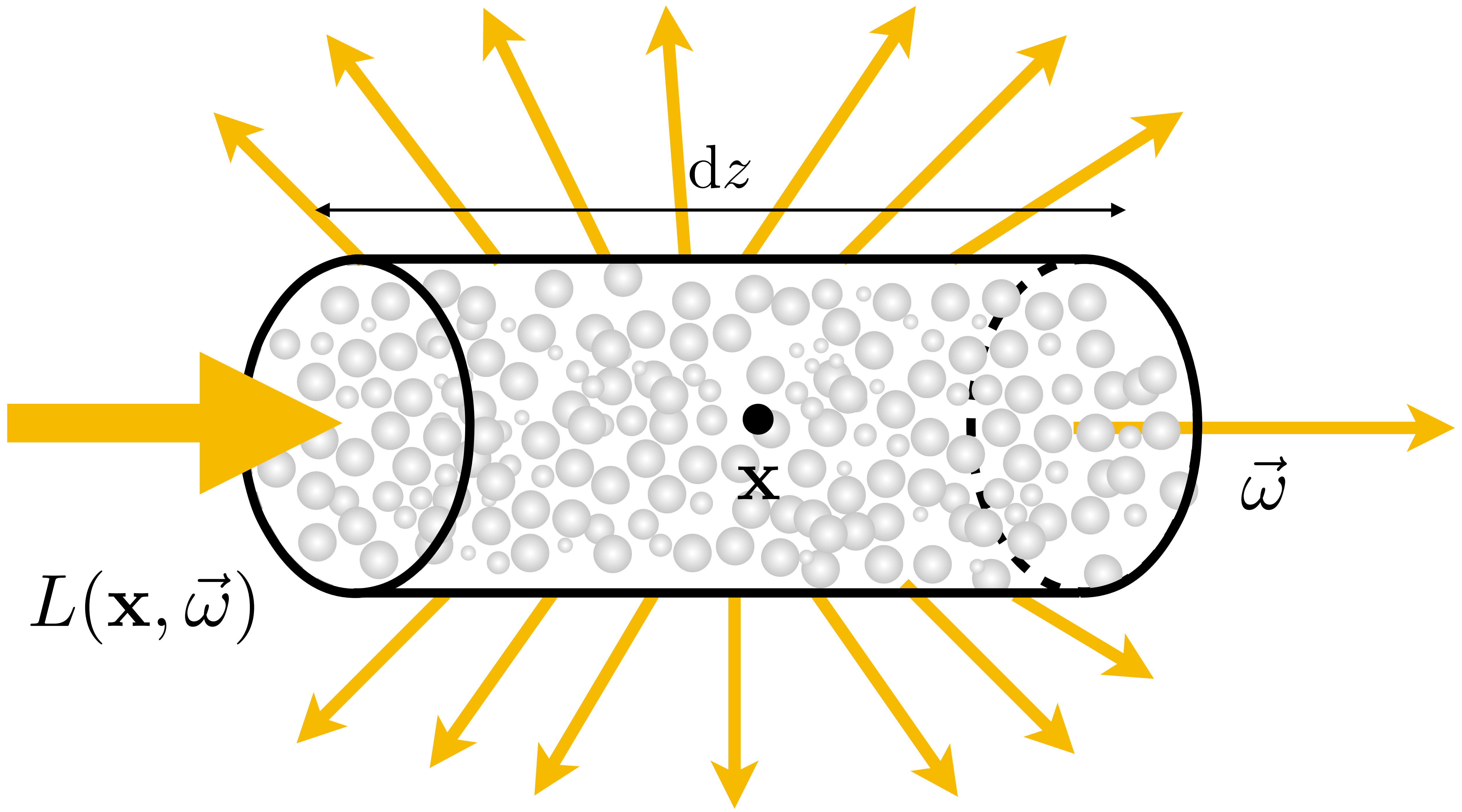


Out-Scattering



$$\frac{dL(\mathbf{x}, \vec{\omega})}{dz} = -\sigma_s L(\mathbf{x}, \vec{\omega})$$

Out-Scattering



$$\frac{dL(\mathbf{x}, \vec{\omega})}{dz} = -\sigma_s L(\mathbf{x}, \vec{\omega})$$

σ_s : scattering coefficient

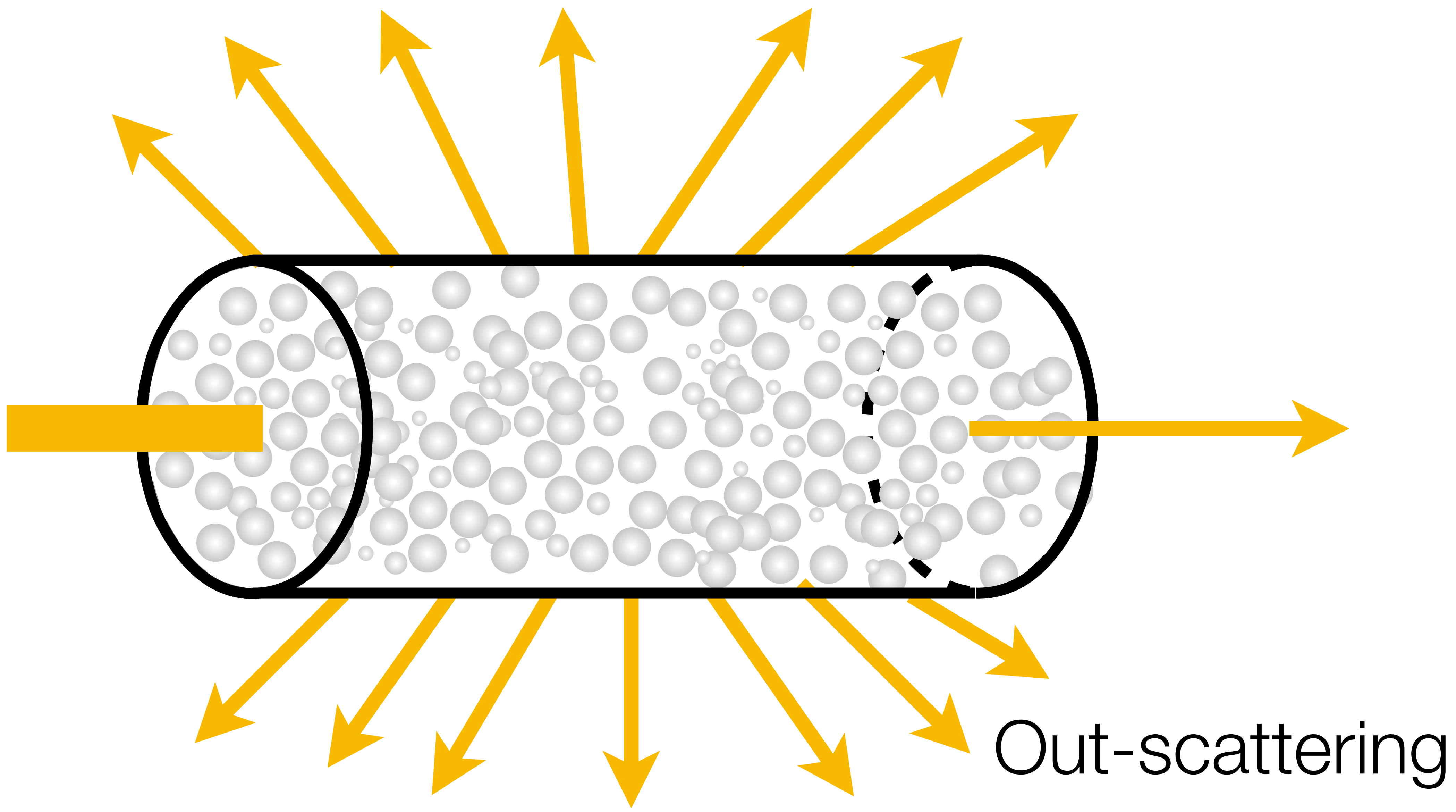
The probability of an out-scattering event occurring per unit distance is given by the scattering coefficient

Attenuation / Extinction

Total reduction in radiance:

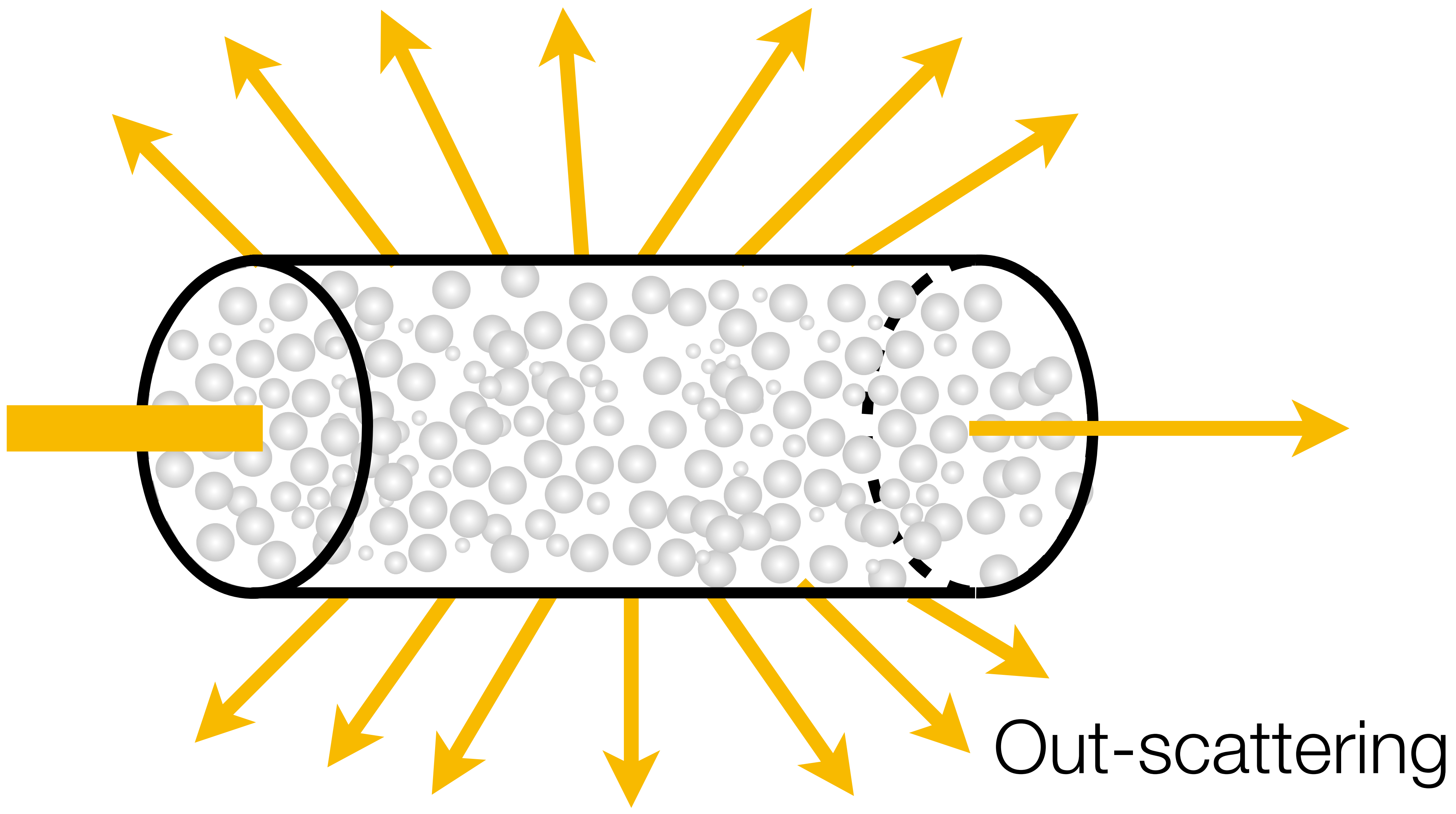
Attenuation / Extinction

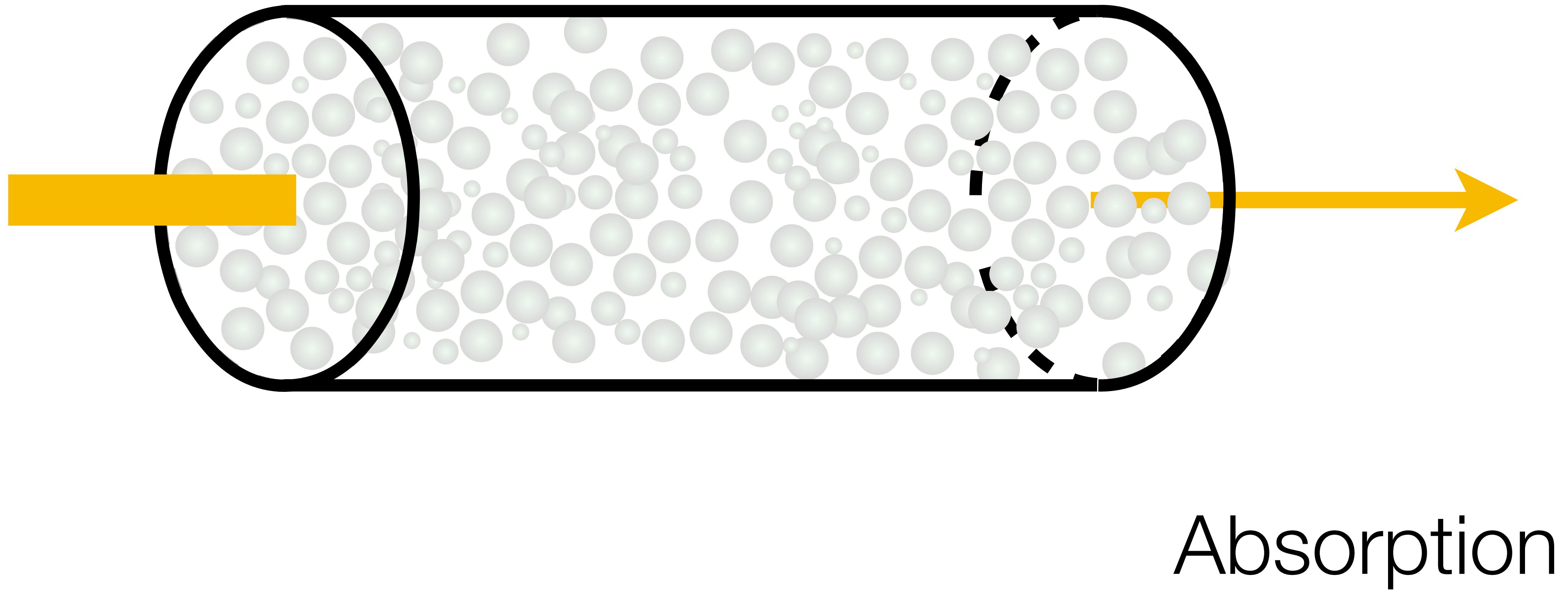
Total reduction in radiance:



Attenuation / Extinction

Total reduction in radiance:



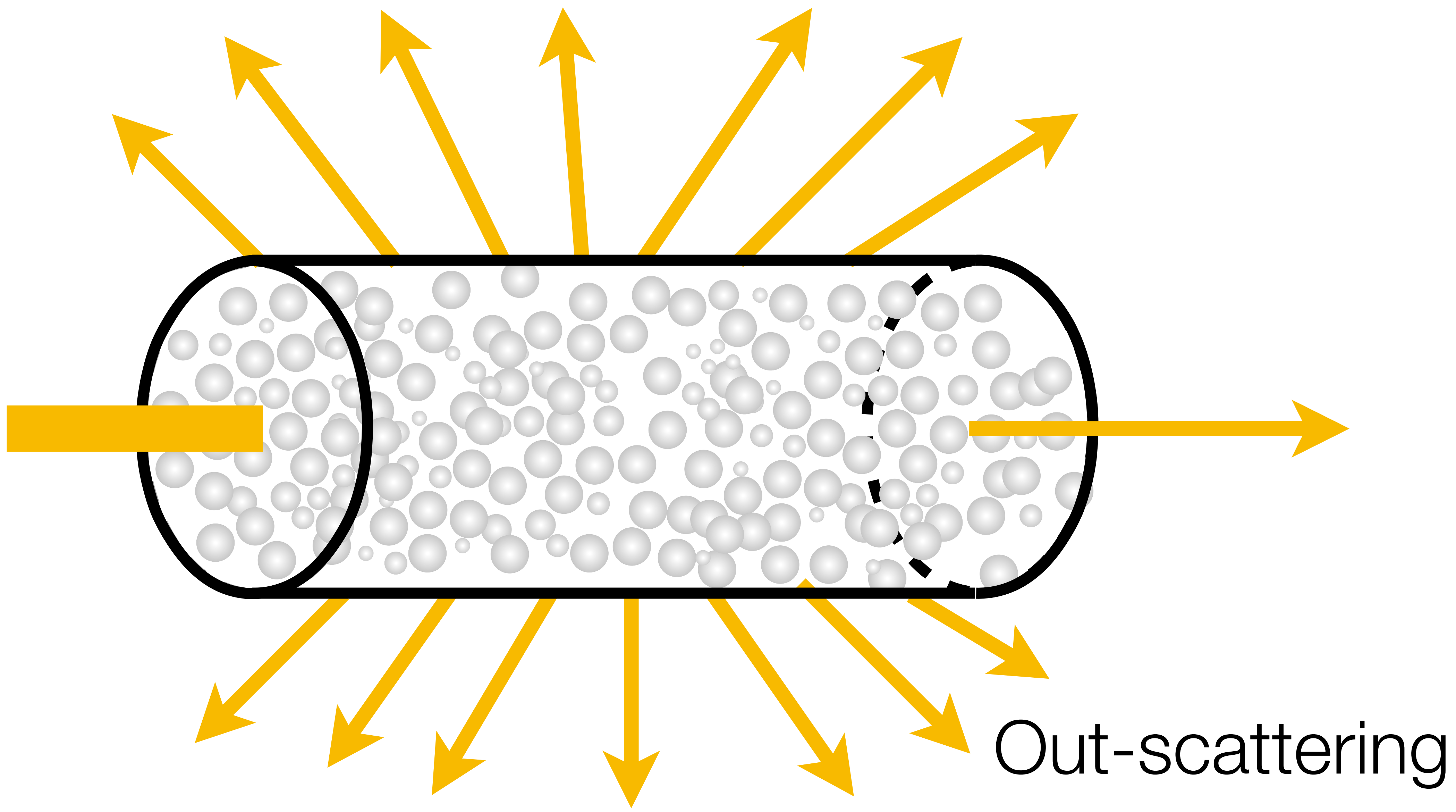


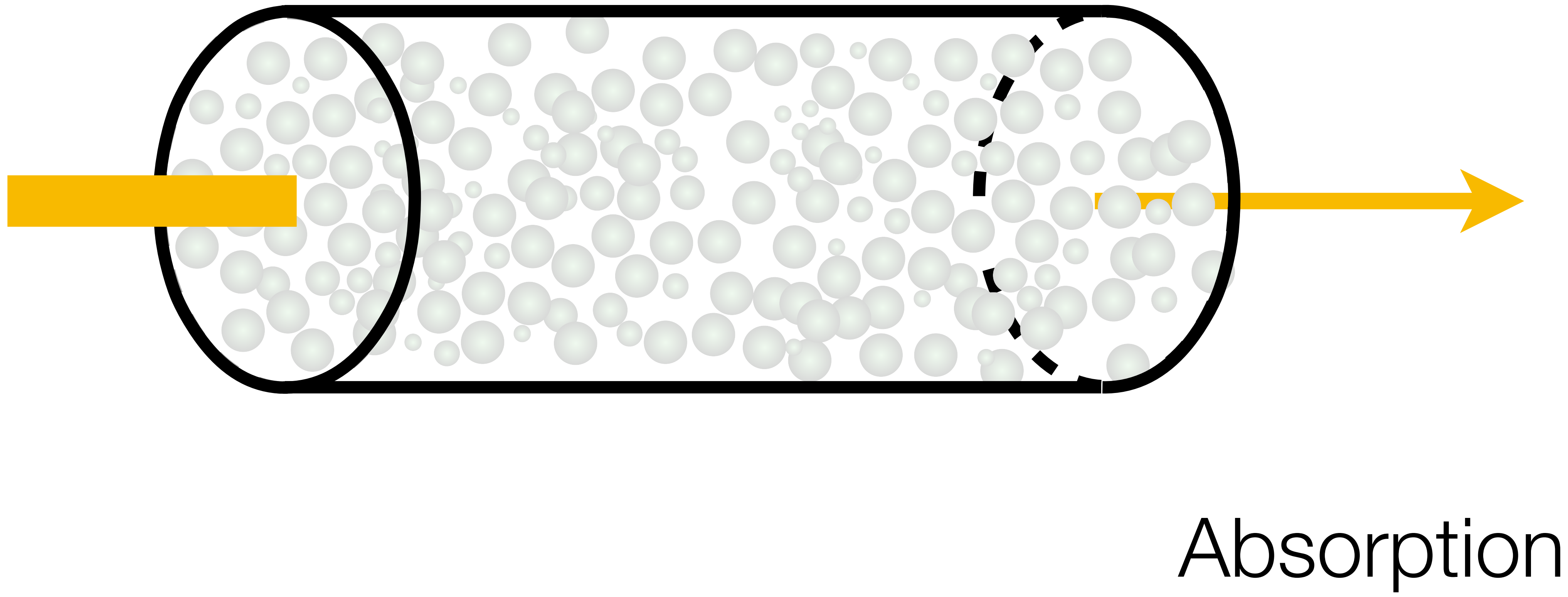
Attenuation / Extinction

Total reduction in radiance:

σ_a : absorption coefficient

σ_s : scattering coefficient





Attenuation / Extinction

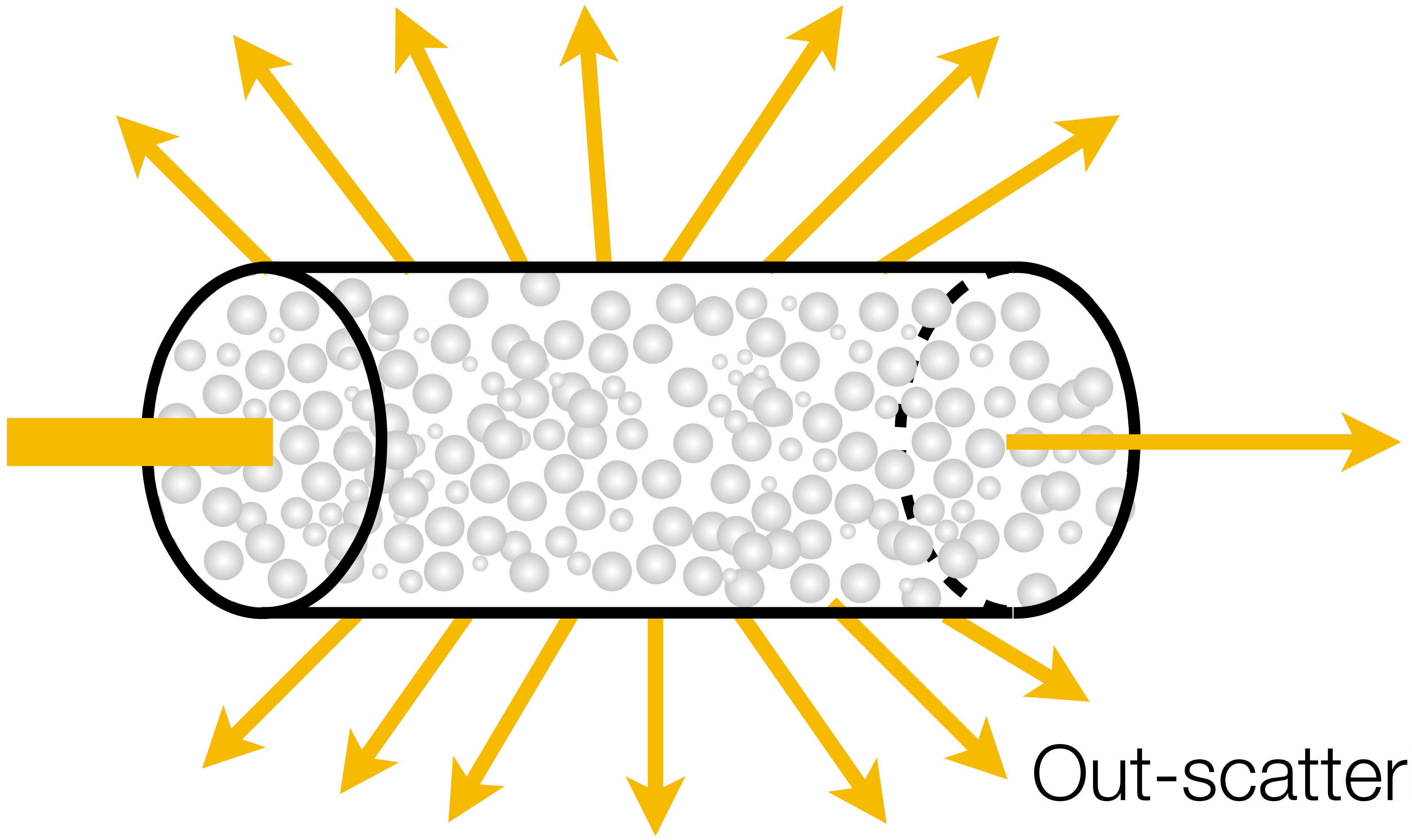
Total reduction in radiance:

$$\sigma_t(\mathbf{x}, \vec{\omega}) = \sigma_a(\mathbf{x}, \vec{\omega}) + \sigma_s(\mathbf{x}, \vec{\omega})$$

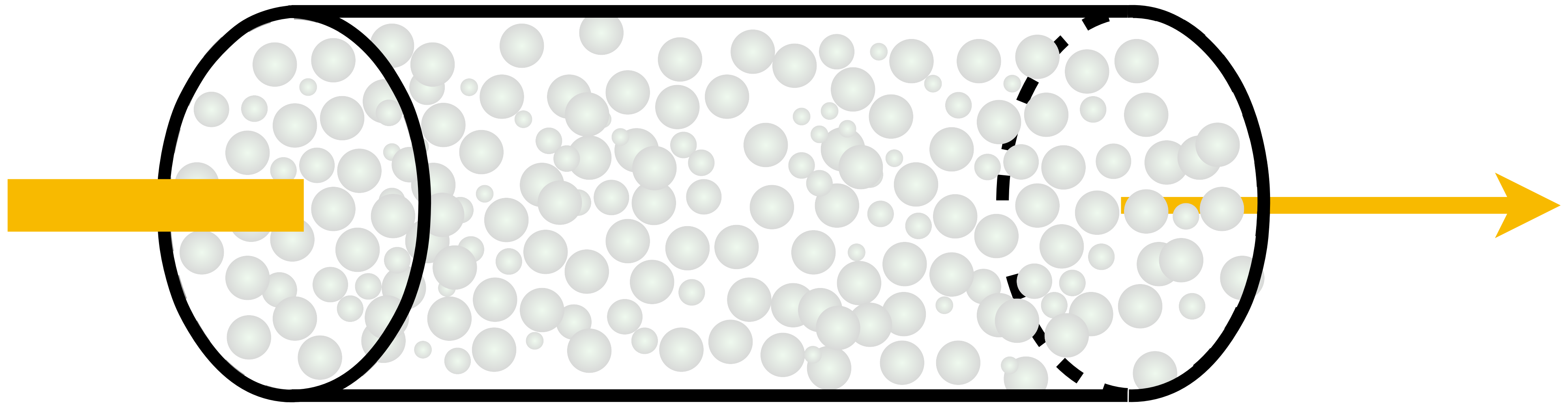
σ_a : absorption coefficient

σ_s : scattering coefficient

σ_t : extinction coefficient



Out-scattering



Absorption

Albedo

$$\alpha(\mathbf{x}) = \frac{\sigma_s(\mathbf{x})}{\sigma_a(\mathbf{x}) + \sigma_s(\mathbf{x})} = \frac{\sigma_s(\mathbf{x})}{\sigma_t(\mathbf{x})}$$

σ_s : scattering coefficient

σ_t : extinction coefficient

Albedo

$$\alpha(\mathbf{x}) = \frac{\sigma_s(\mathbf{x})}{\sigma_a(\mathbf{x}) + \sigma_s(\mathbf{x})} = \frac{\sigma_s(\mathbf{x})}{\sigma_t(\mathbf{x})}$$

σ_s : scattering coefficient

σ_t : extinction coefficient

Albedo

$$\alpha(\mathbf{x}) = \frac{\sigma_s(\mathbf{x})}{\sigma_a(\mathbf{x}) + \sigma_s(\mathbf{x})} = \frac{\sigma_s(\mathbf{x})}{\sigma_t(\mathbf{x})}$$

σ_s : scattering coefficient

σ_t : extinction coefficient

Albedo

$$\alpha(\mathbf{x}) = \frac{\sigma_s(\mathbf{x})}{\sigma_t(\mathbf{x})}$$

The albedo is always between 0 and 1

It describes the probability of scattering (versus absorption) at a scattering event

σ_s : scattering coefficient

σ_t : extinction coefficient

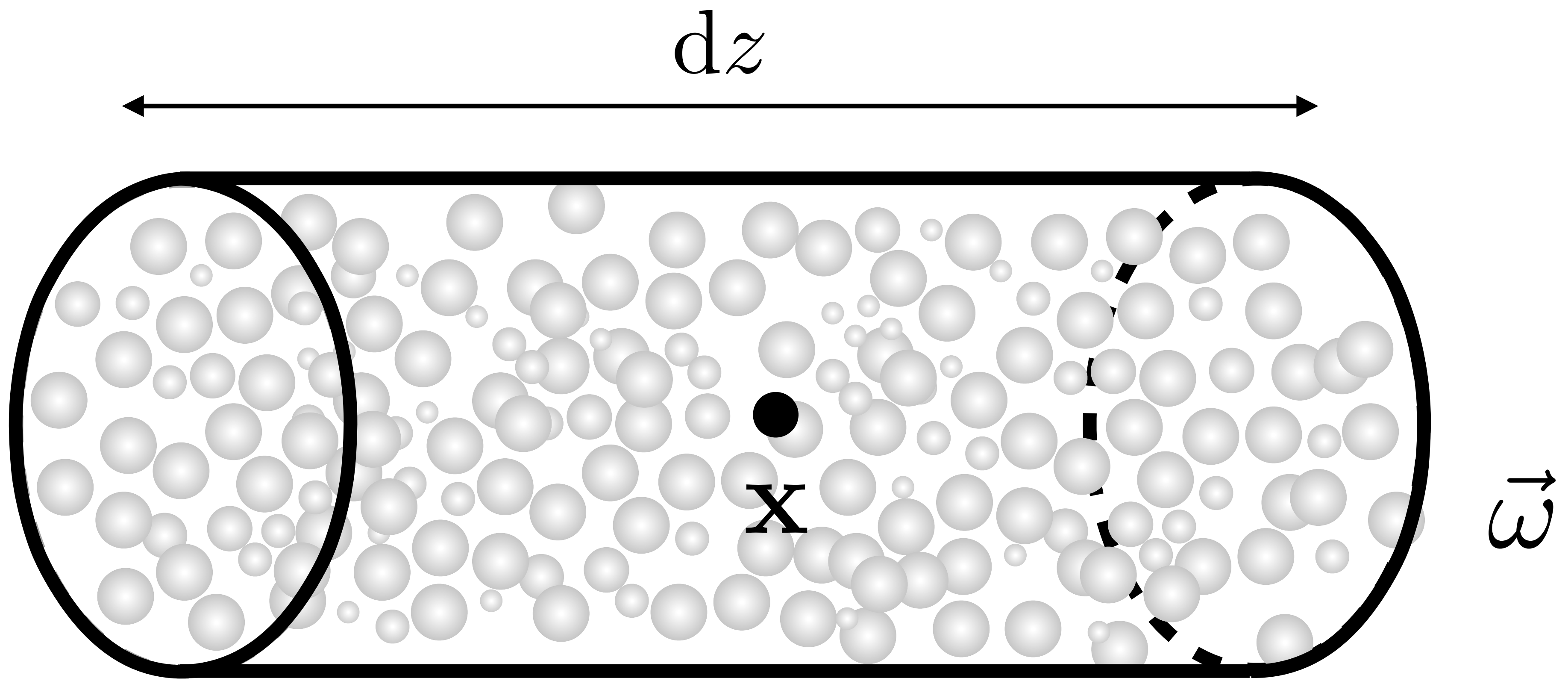
Mean-free path

$$\frac{1}{\sigma_t}$$

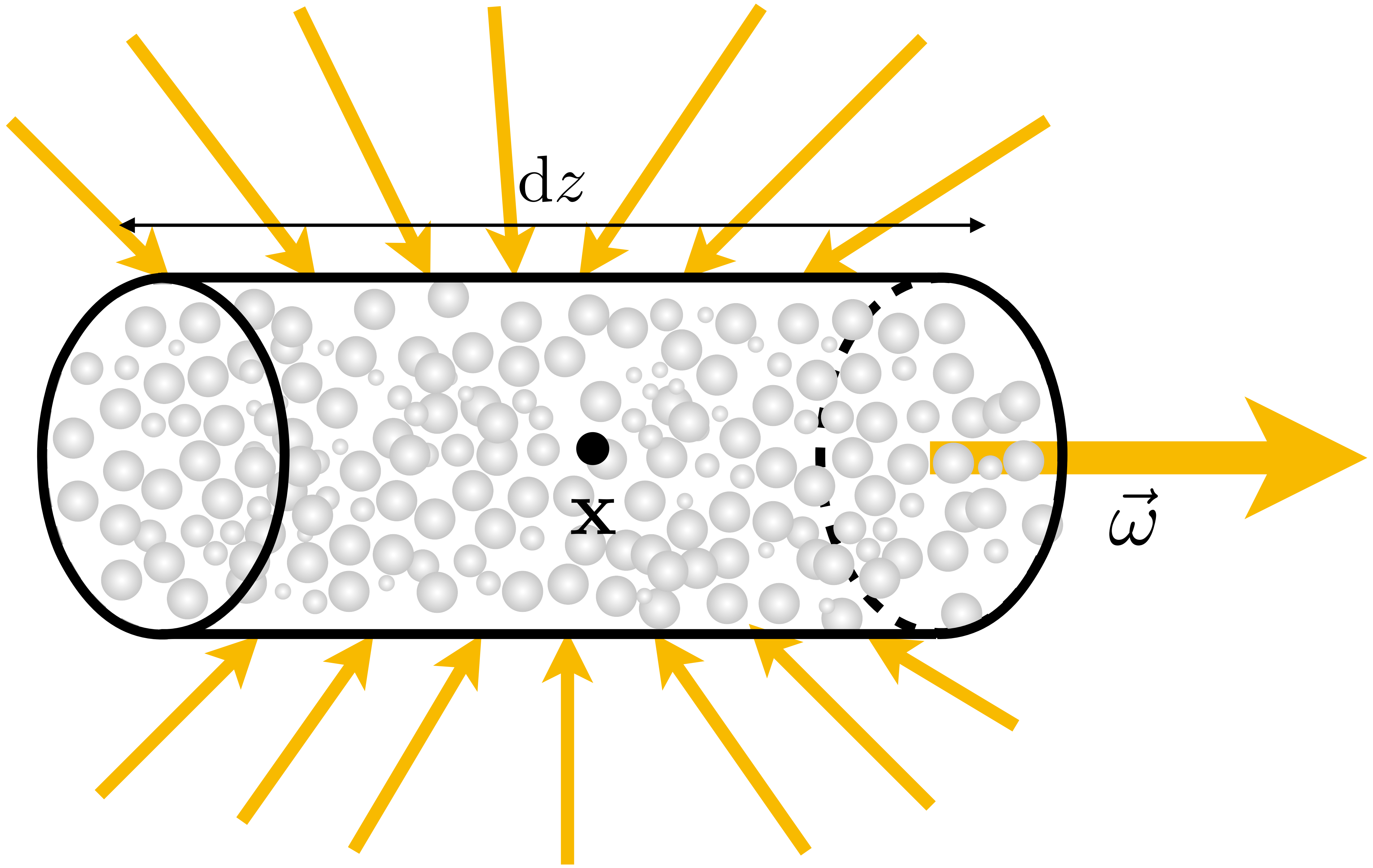
Mean free path gives the average distance travelled by the ray before interacting with a particle

σ_t : extinction coefficient

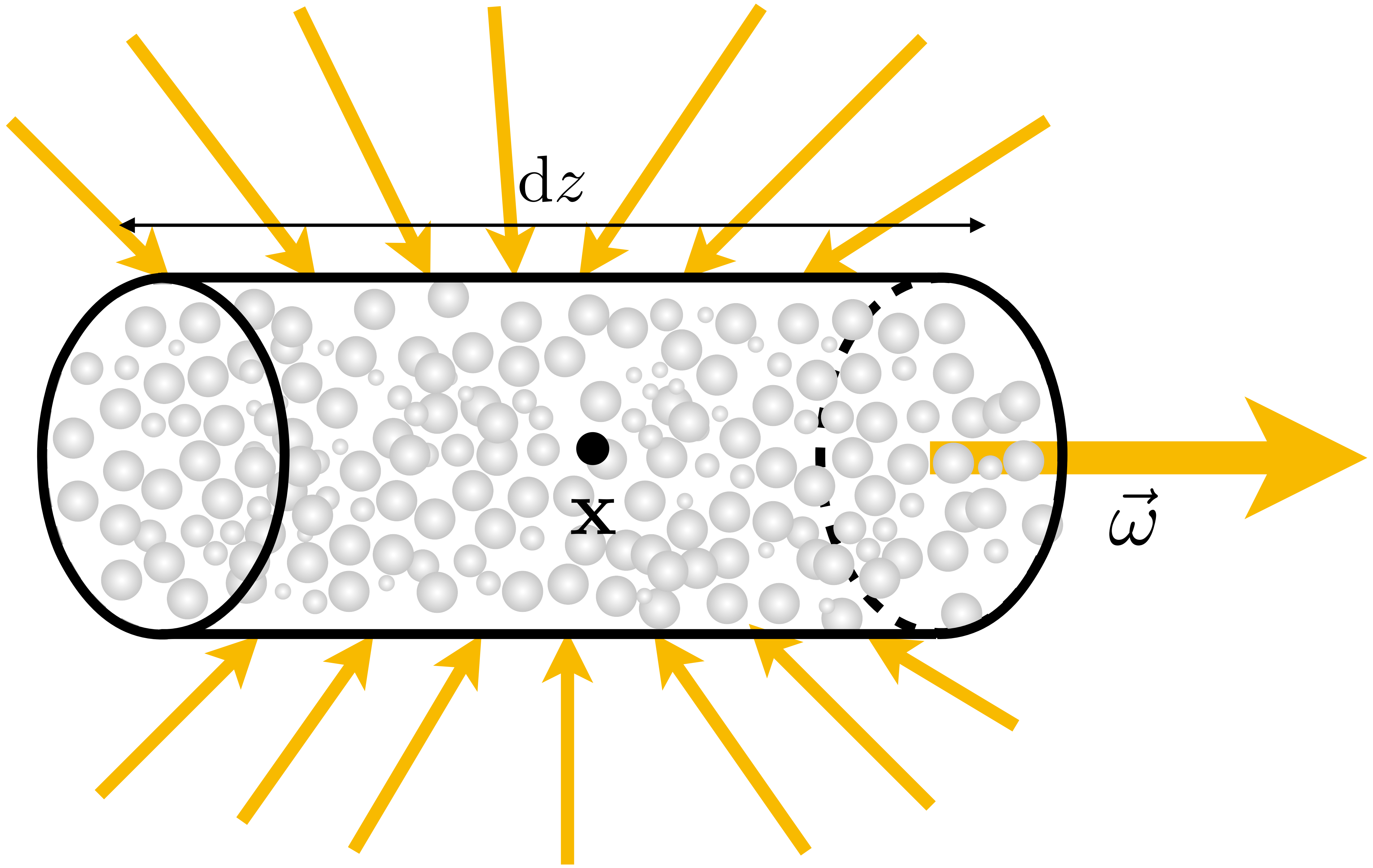
In-Scattering



In-Scattering

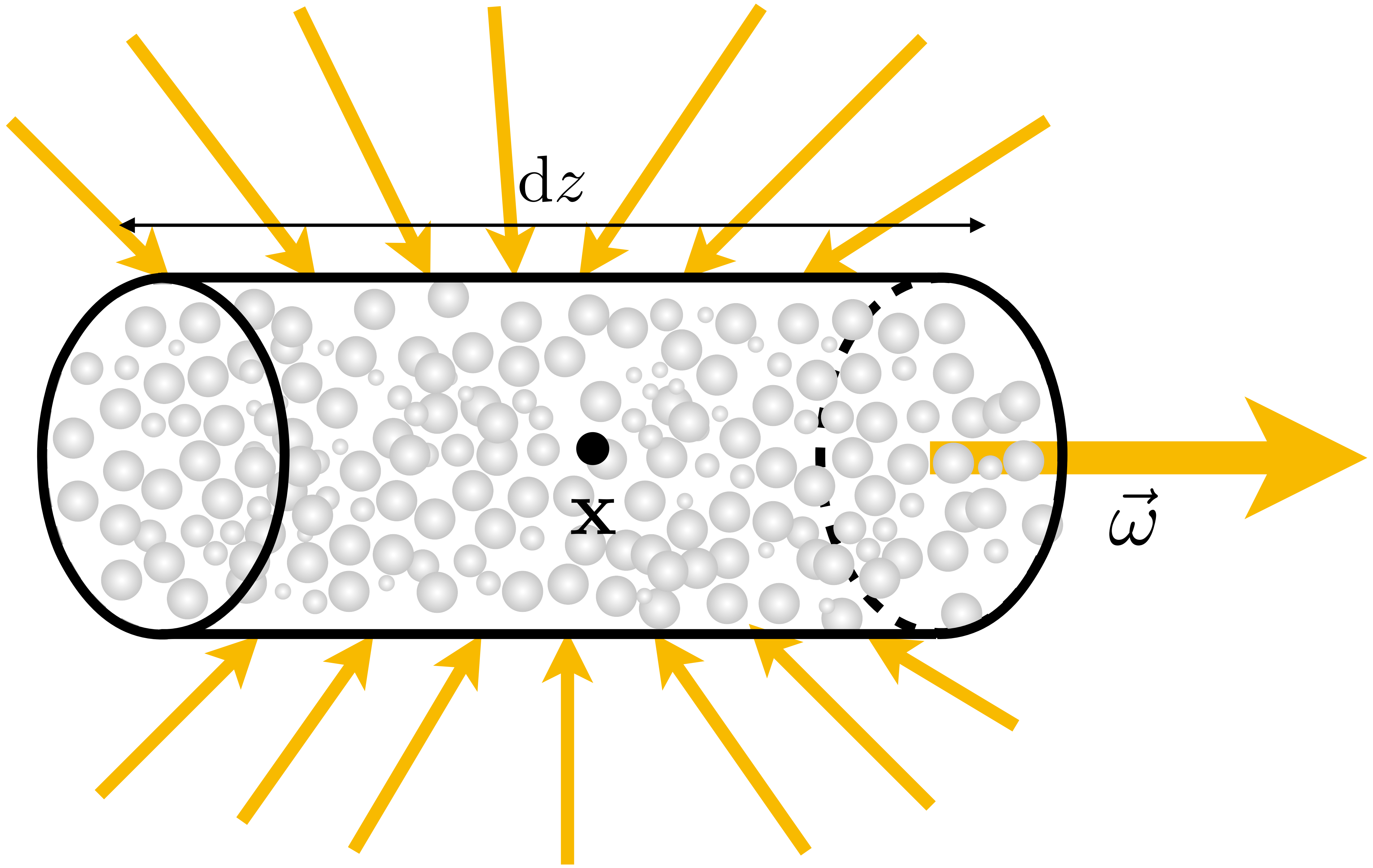


In-Scattering



$$\frac{\mathrm{d}L(\mathbf{x},\vec{\omega})}{\mathrm{d}z} = \sigma_s(\mathbf{x})L_\mathrm{s}(\mathbf{x},\vec{\omega})$$

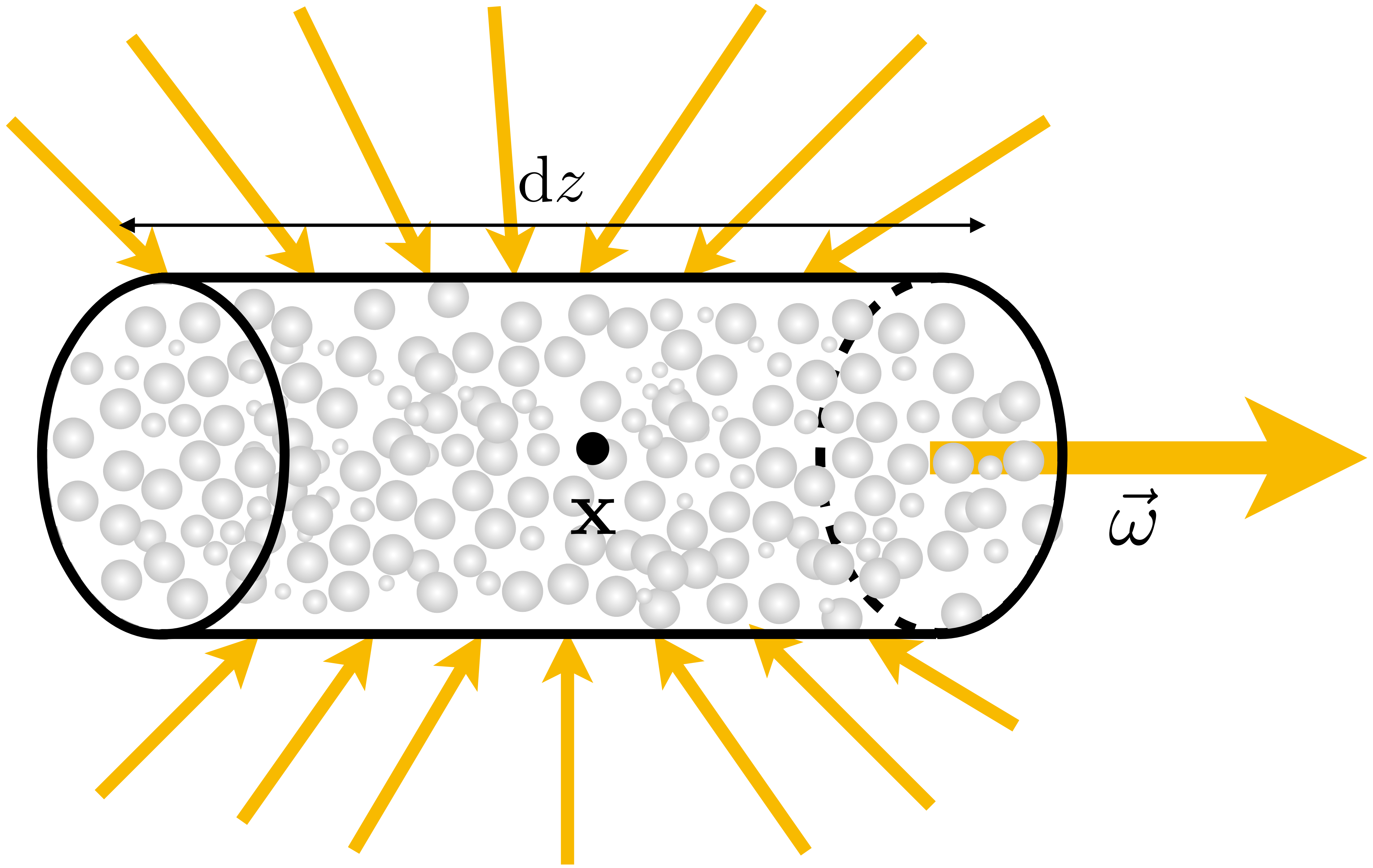
In-Scattering



$$\frac{\mathrm{d}L(\mathbf{x},\vec{\omega})}{\mathrm{d}z} = \sigma_s(\mathbf{x}) L_\mathrm{s}(\mathbf{x},\vec{\omega})$$

$\sigma_s(\mathbf{x})$: scattering coefficient

In-Scattering



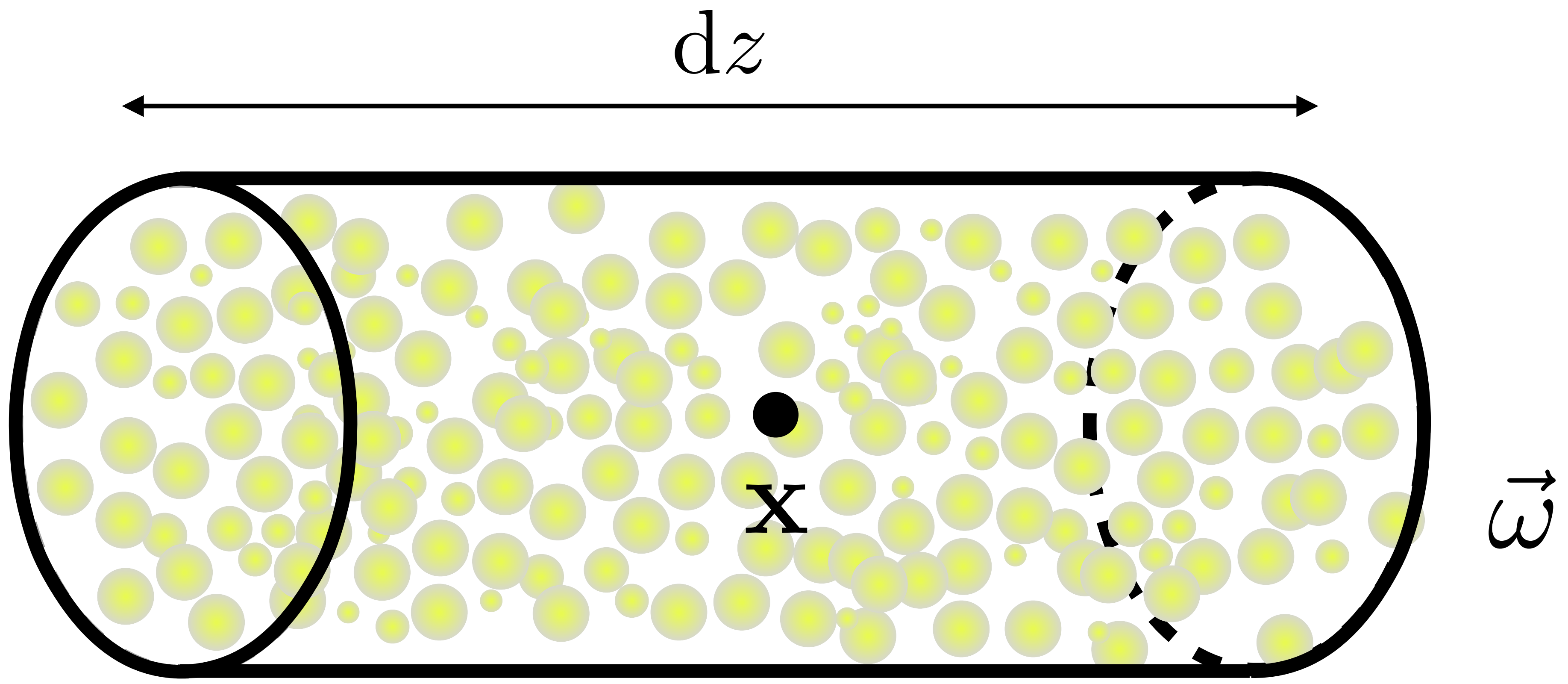
$$\frac{\mathrm{d}L(\mathbf{x},\vec{\omega})}{\mathrm{d}z} = \sigma_s(\mathbf{x})\boxed{L_\mathrm{s}(\mathbf{x},\vec{\omega})}$$

$\sigma_s(\mathbf{x})$: scattering coefficient

In-scattered radiance

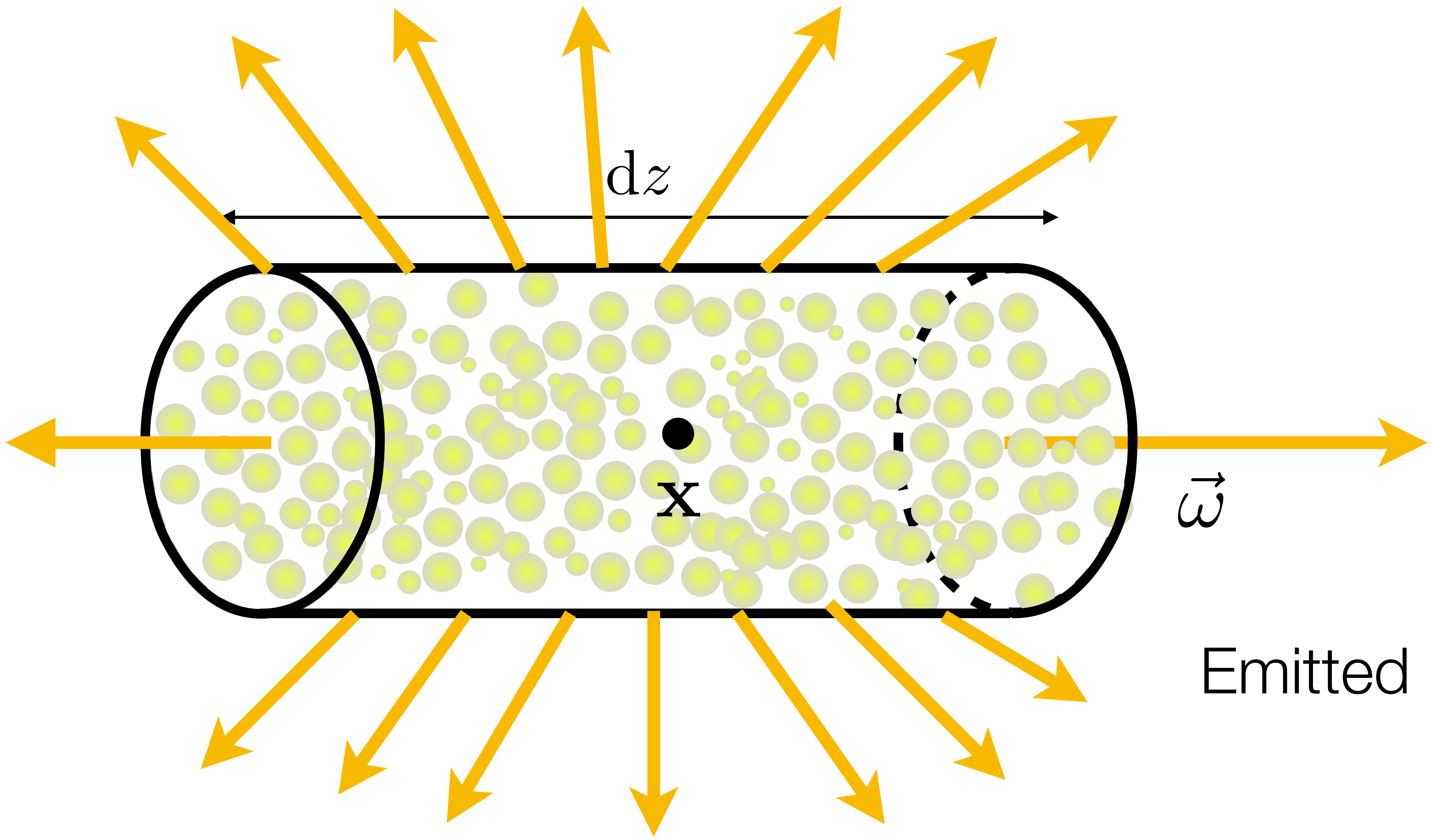
$$\boxed{L_\mathrm{s}(\mathbf{x},\vec{\omega})} = \int_{S^2} f_\mathrm{p}(\vec{\omega},\vec{\omega}')L(\mathbf{x},\vec{\omega}')\mathrm{d}\vec{\omega}'$$

Emission



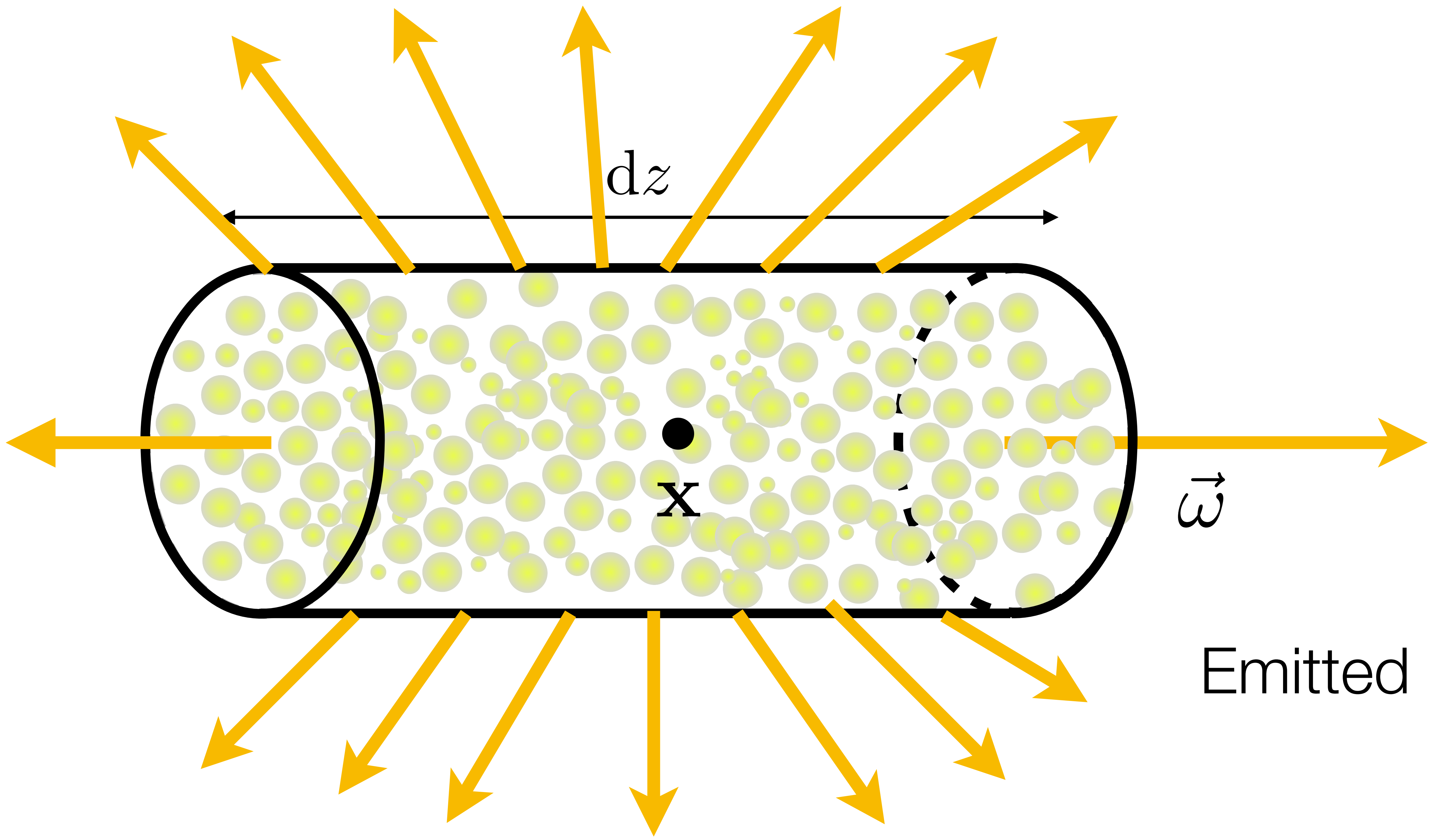
Emitted radiance does not depend on the incoming light L_i

Emission



Emitted radiance does not depend on the incoming light L_i

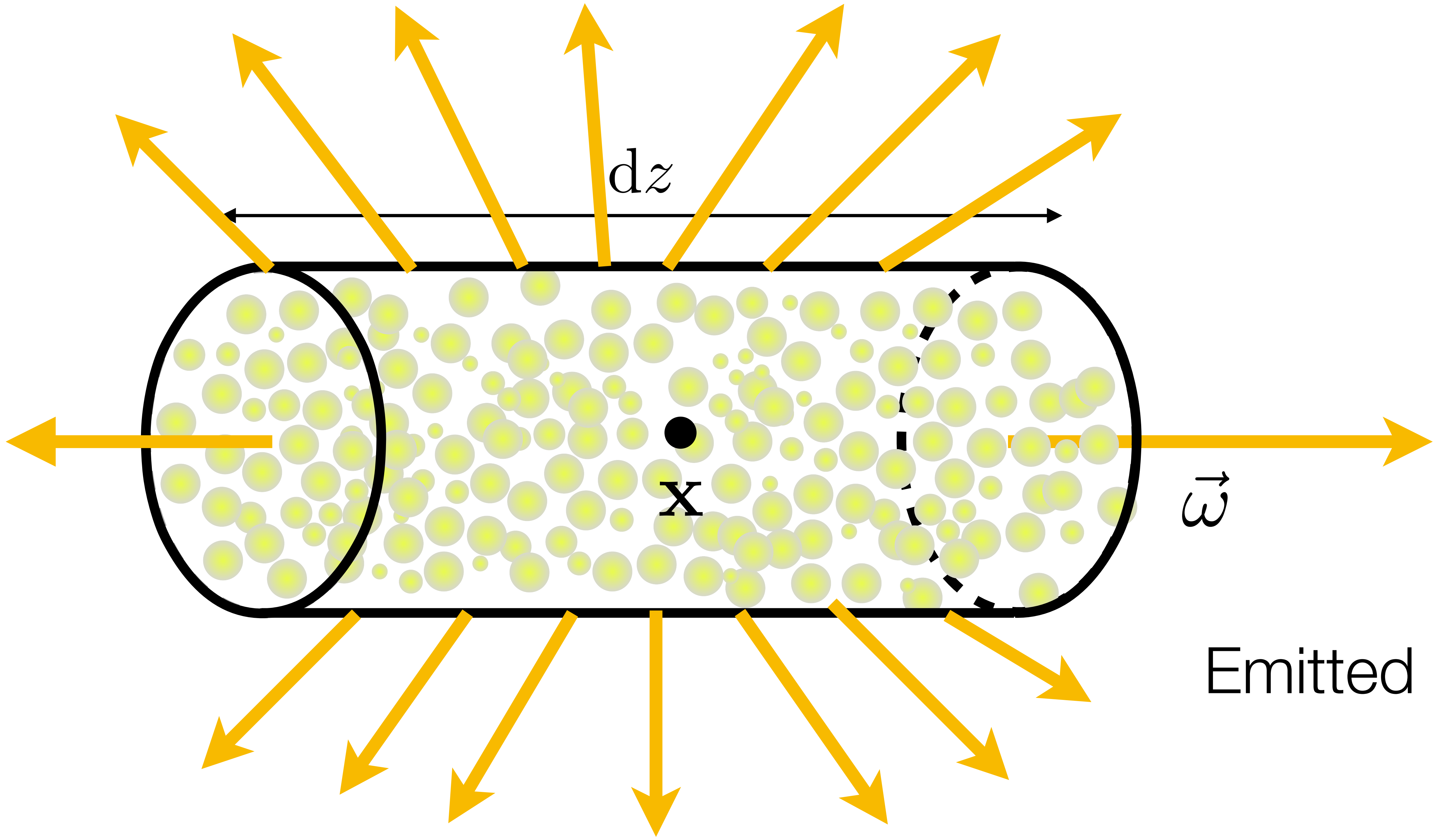
Emission



$$\frac{\mathrm{d}L(\mathbf{x},\vec{\omega})}{\mathrm{d}z} = \sigma_a(\mathbf{x})L_\mathrm{e}(\mathbf{x},\vec{\omega})$$

Emitted radiance does not depend on the incoming light L_i

Emission

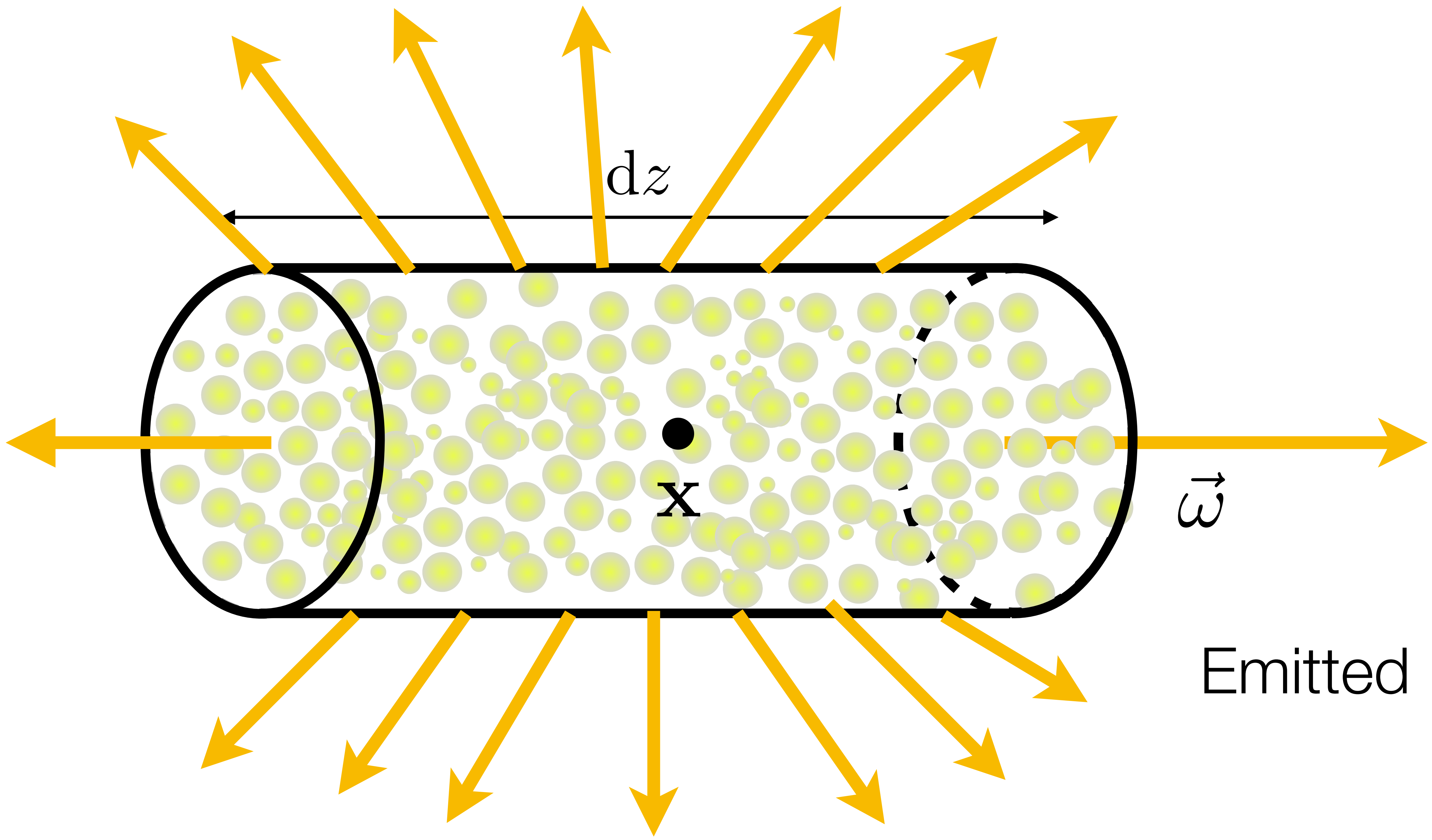


$$\frac{\mathrm{d}L(\mathbf{x},\vec{\omega})}{\mathrm{d}z} = \sigma_a(\mathbf{x})L_{\mathrm{e}}(\mathbf{x},\vec{\omega})$$

$L_e(\mathbf{x},\vec{\omega})$: emitted radiance

Emitted radiance does not depend on the incoming light L_i

Emission



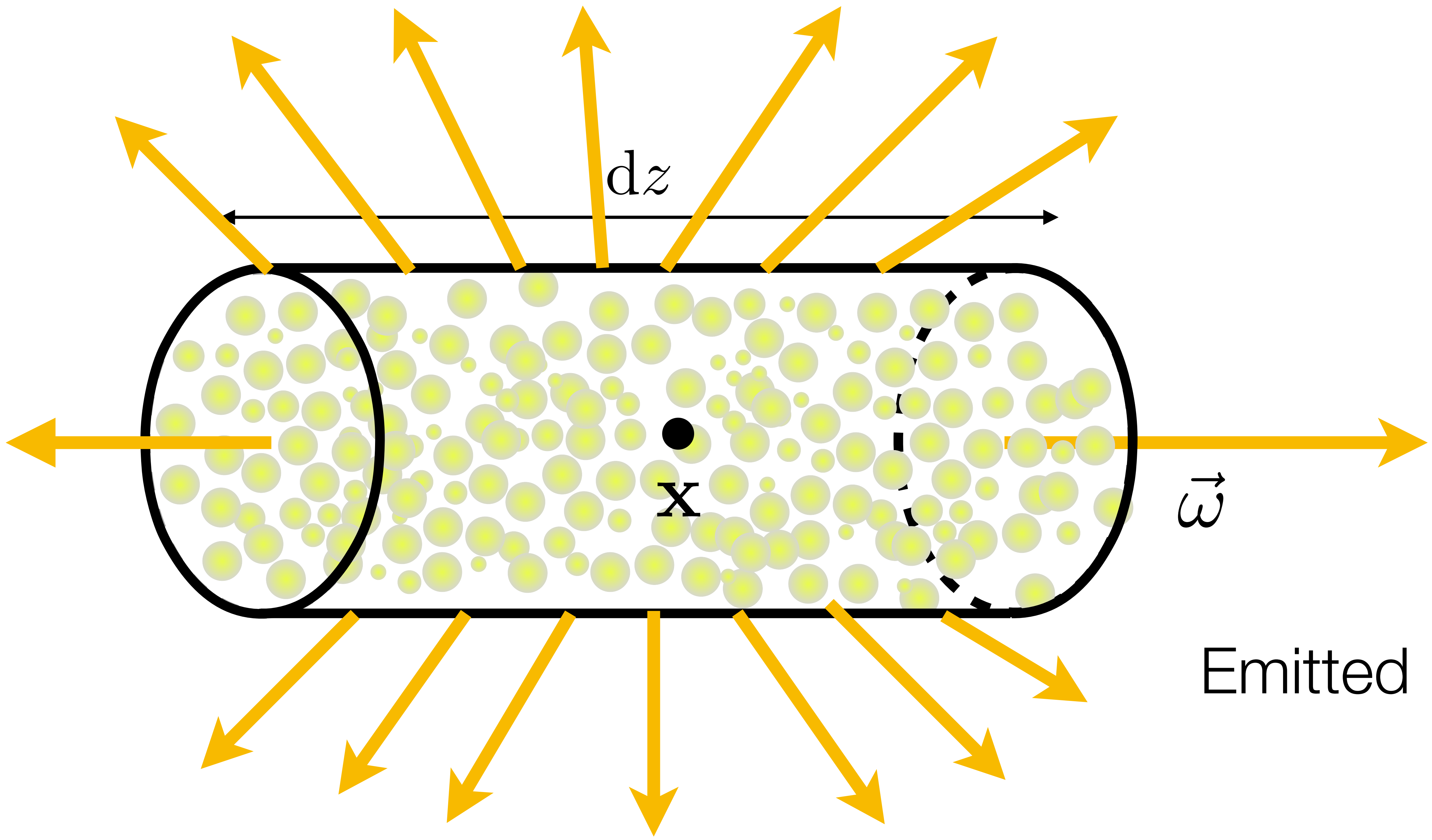
$$\frac{\mathrm{d}L(\mathbf{x},\vec{\omega})}{\mathrm{d}z} = \sigma_a(\mathbf{x})L_\mathrm{e}(\mathbf{x},\vec{\omega})$$

$L_e(\mathbf{x},\vec{\omega})$: emitted radiance

*sometimes modeled without the absorption coefficient term

Emitted radiance does not depend on the incoming light L_i

Emission



$$\frac{\mathrm{d}L(\mathbf{x},\vec{\omega})}{\mathrm{d}z} = \sigma_a(\mathbf{x})L_\mathrm{e}(\mathbf{x},\vec{\omega})$$

$L_e(\mathbf{x},\vec{\omega})$: emitted radiance

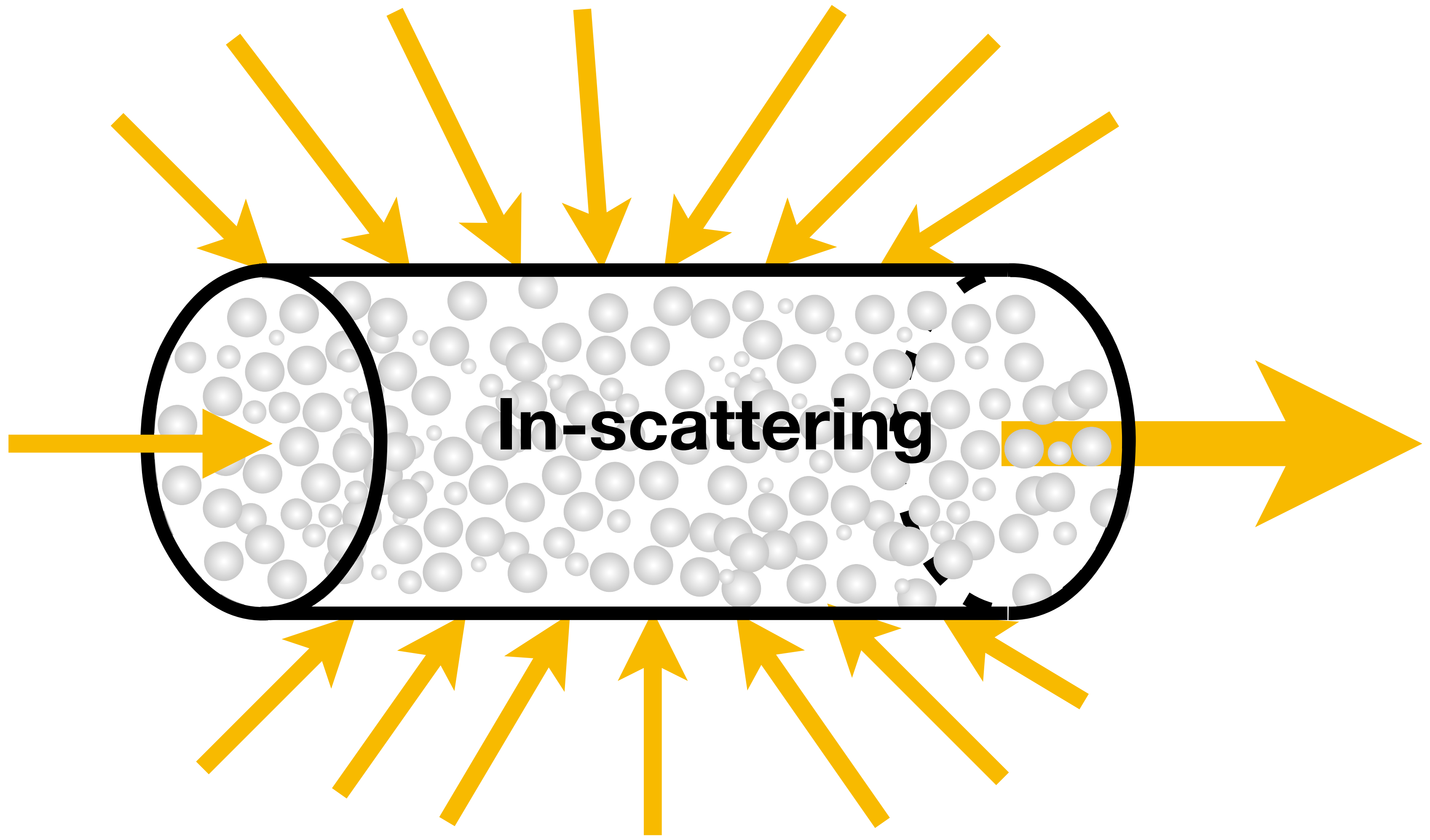
*sometimes modeled without the absorption coefficient term

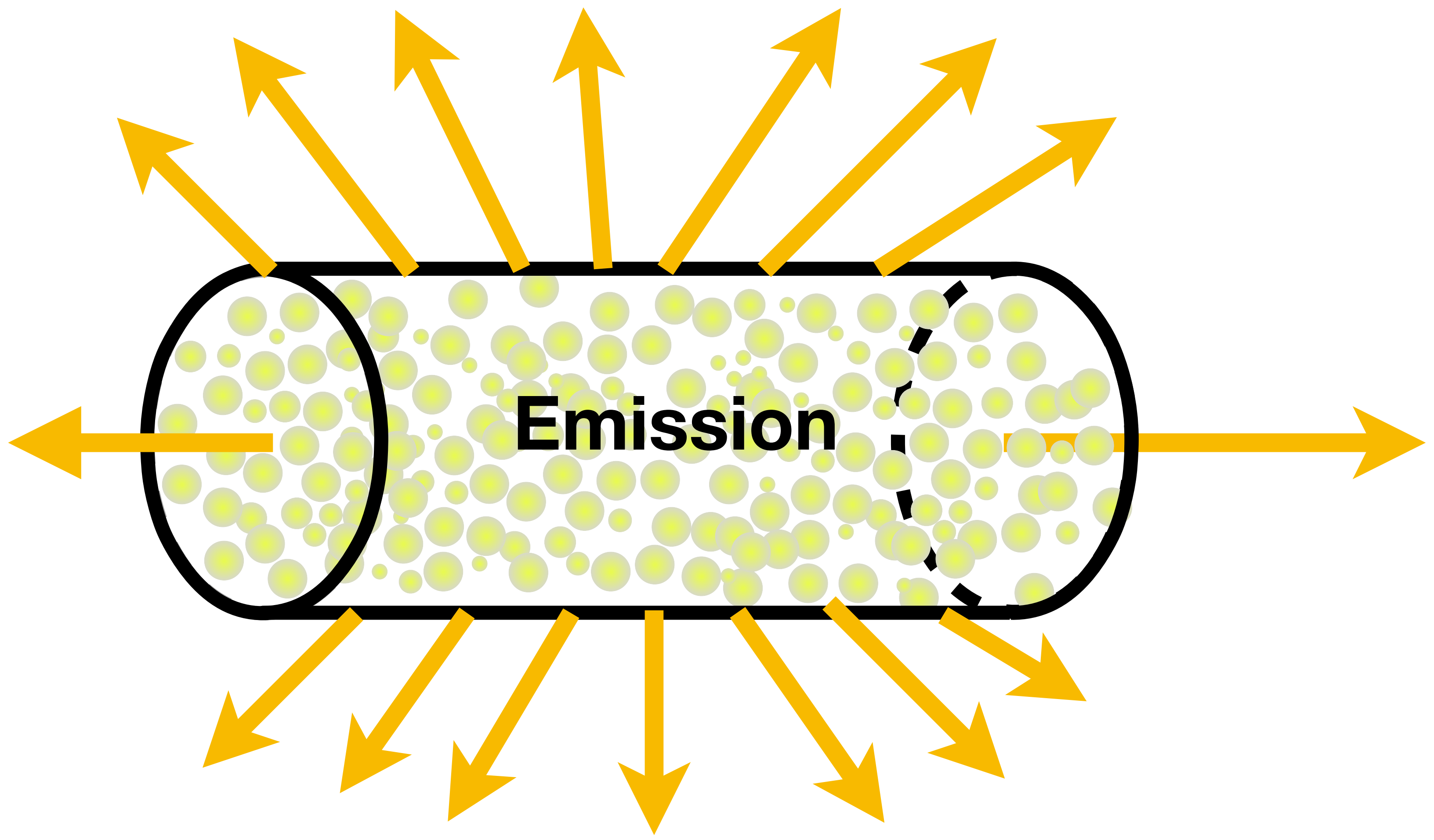
Emitted radiance does not depend on the incoming light L_i

Here we made a choice to represent differential output radiance as a product of emitted radiance and absorption coefficient.

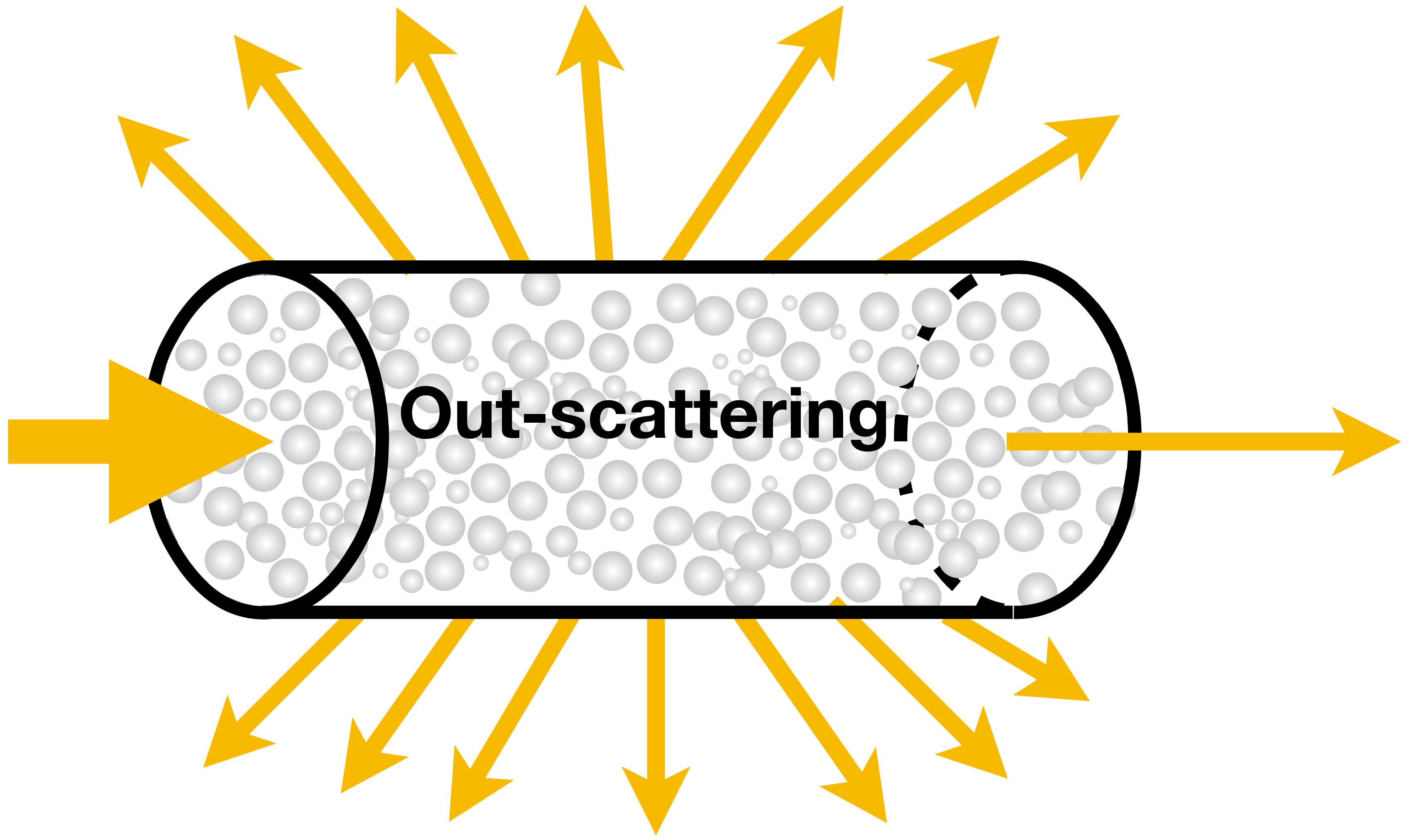
Radiative Transfer Equation

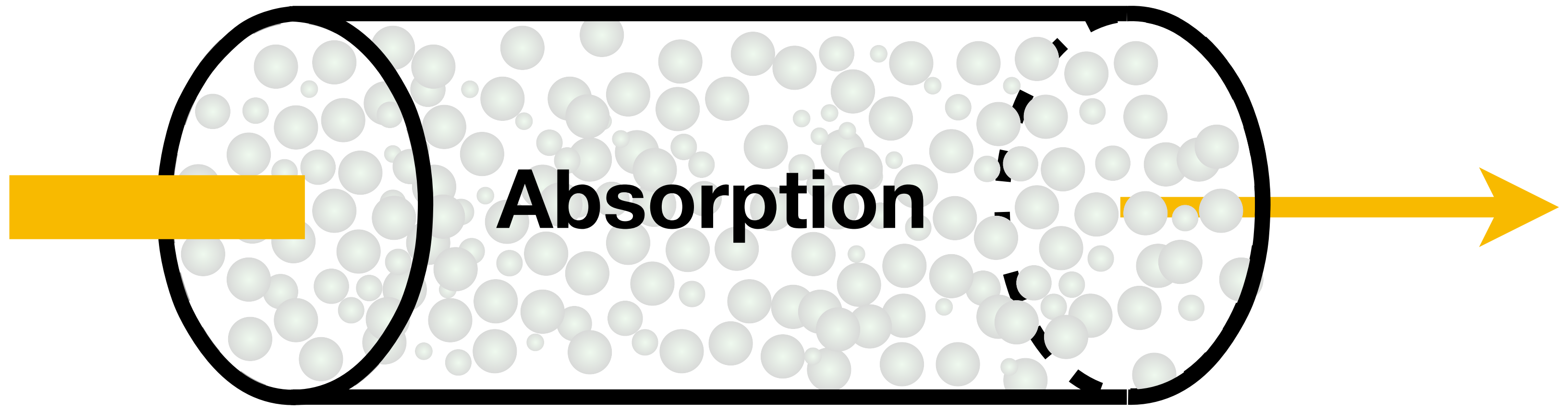
Radiative Transfer Equation (RTE)

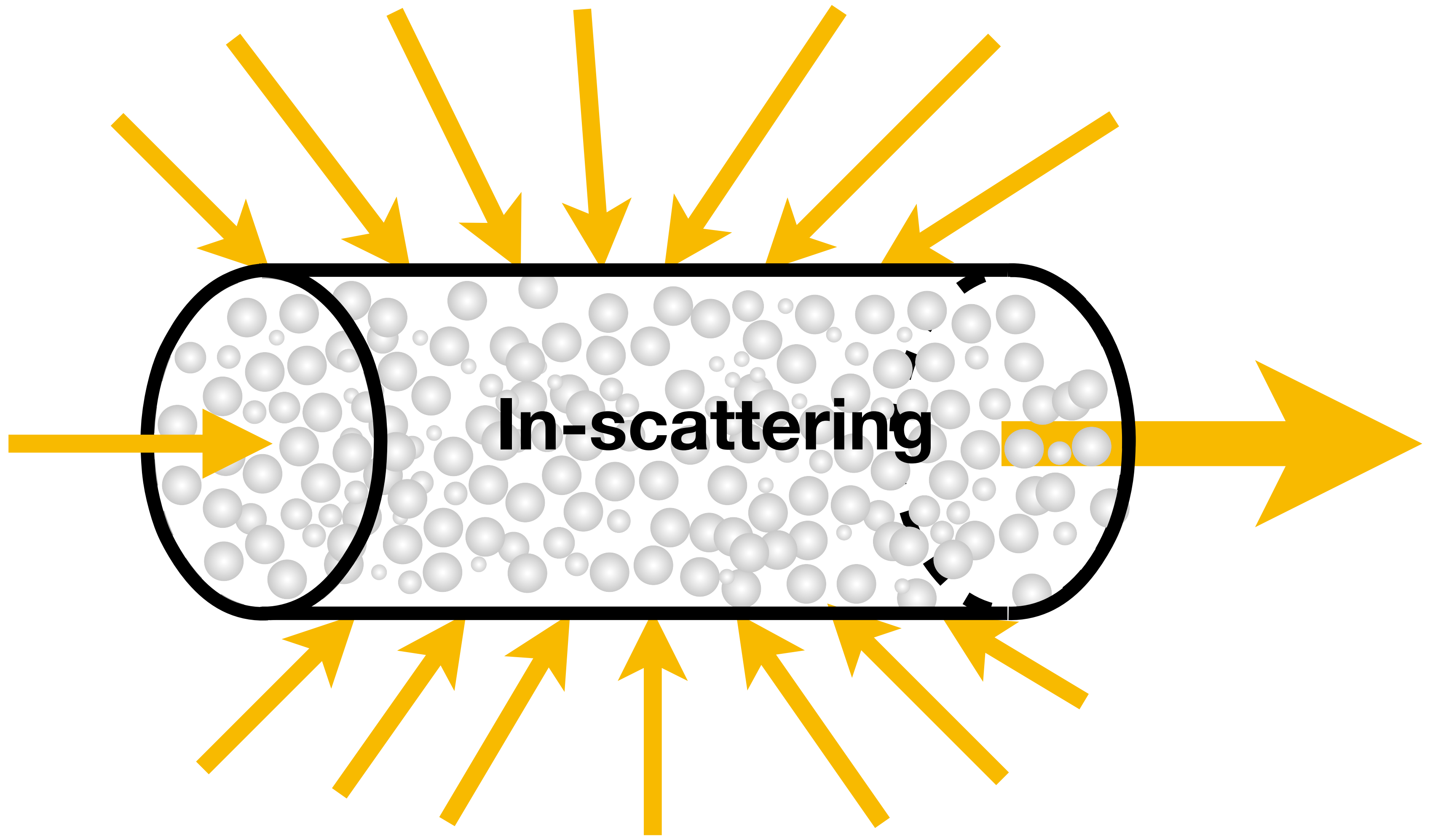


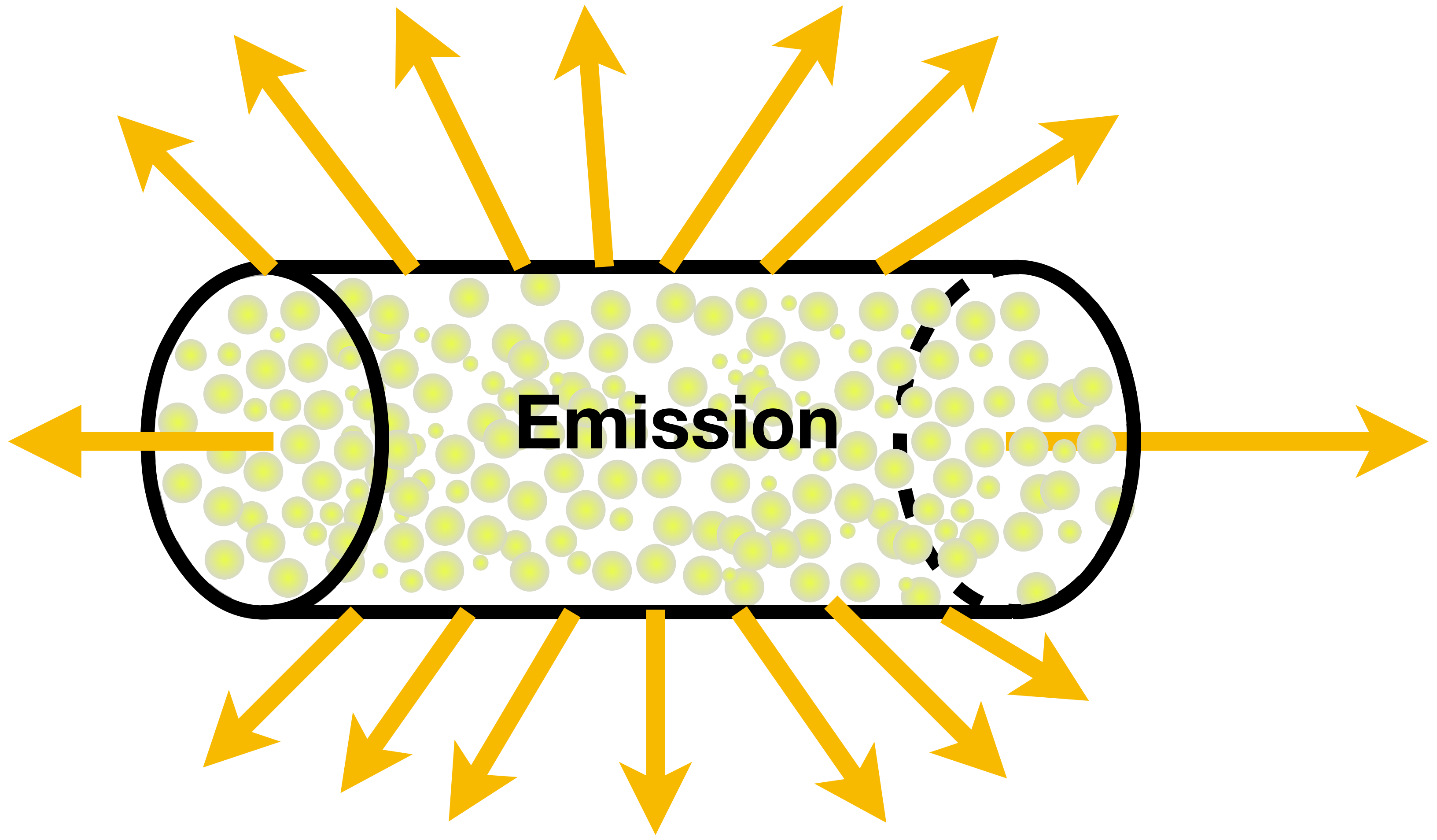


Radiative Transfer Equation (RTE)









Radiative Transfer Equation (RTE)

Out-scattering **Absorption** Losses

$$\frac{\mathrm{d}L(\mathbf{x},\vec{\omega})}{\mathrm{d}z}$$

In-scattering **Emission** Gains

Radiative Transfer Equation (RTE)

$-\sigma_s(\mathbf{x})L(\mathbf{x},\vec{\omega})$

$-\sigma_a(\mathbf{x})L(\mathbf{x},\vec{\omega})$

Out-scattering **Absorption**

Losses

$$\frac{\mathrm{d}L(\mathbf{x},\vec{\omega})}{\mathrm{d}z}$$

$\sigma_s(\mathbf{x})L_s(\mathbf{x},\vec{\omega})$

$\sigma_a(\mathbf{x})L_e(\mathbf{x},\vec{\omega})$

In-scattering **Emission**

Gains

Radiative Transfer Equation (RTE)

Out-scattering **Absorption**

$$\frac{\mathrm{d}L(\mathbf{x},\vec{\omega})}{\mathrm{d}z} \quad -\sigma_s(\mathbf{x})L(\mathbf{x},\vec{\omega}) -\sigma_a(\mathbf{x})L(\mathbf{x},\vec{\omega}) + \sigma_s(\mathbf{x})L_s(\mathbf{x},\vec{\omega}) + \sigma_a(\mathbf{x})L_e(\mathbf{x},\vec{\omega})$$

In-scattering **Emission**

Radiative Transfer Equation (RTE)

Out-scattering **Absorption**

$$\frac{\mathrm{d}L(\mathbf{x},\vec{\omega})}{\mathrm{d}z} \quad -\sigma_s(\mathbf{x})L(\mathbf{x},\vec{\omega}) - \sigma_a(\mathbf{x})L(\mathbf{x},\vec{\omega}) + \sigma_s(\mathbf{x})L_s(\mathbf{x},\vec{\omega}) + \sigma_a(\mathbf{x})L_e(\mathbf{x},\vec{\omega})$$

In-scattering **Emission**

$$\sigma_t(\mathbf{x},\vec{\omega}) = \sigma_a(\mathbf{x},\vec{\omega}) + \sigma_s(\mathbf{x},\vec{\omega})$$

Radiative Transfer Equation (RTE)

Attenuation

$$\frac{\mathrm{d}L(\mathbf{x},\vec{\omega})}{\mathrm{d}z} = \boxed{-\sigma_t(\mathbf{x})L(\mathbf{x},\vec{\omega})} + \boxed{\sigma_s(\mathbf{x})L_s(\mathbf{x},\vec{\omega}) + \sigma_a(\mathbf{x})L_e(\mathbf{x},\vec{\omega})}$$

In-scattering **Emission**

Radiative Transfer Equation (RTE)

Attenuation

$$\frac{\mathrm{d}L(\mathbf{x},\vec{\omega})}{\mathrm{d}z} = -\sigma_t(\mathbf{x})L(\mathbf{x},\vec{\omega}) + \sigma_s(\mathbf{x})L_s(\mathbf{x},\vec{\omega}) + \sigma_a(\mathbf{x})L_e(\mathbf{x},\vec{\omega})$$

In-scattering **Emission**

What about a beam with finite-length z?

Extinction Along a Finite Beam

$$\frac{\mathrm{d}L(\mathbf{x},\vec{\omega})}{\mathrm{d}z} = -\sigma_t(\mathbf{x})L(\mathbf{x},\vec{\omega})$$

Extinction Along a Finite Beam

$$\frac{\mathrm{d}L(\mathbf{x},\vec{\omega})}{\mathrm{d}z} = -\sigma_t(\mathbf{x})L(\mathbf{x},\vec{\omega})$$

$$\frac{\mathrm{d}L(\mathbf{x},\vec{\omega})}{L(\mathbf{x},\vec{\omega})} = -\sigma_t(\mathbf{x})\mathrm{d}\mathbf{z}$$

// Integrate along beam from 0 to $\mathcal{Z}$

Extinction Along a Finite Beam

$$\frac{\mathrm{d}L(\mathbf{x},\vec{\omega})}{\mathrm{d}z} = -\sigma_t(\mathbf{x})L(\mathbf{x},\vec{\omega})$$

$$\frac{\mathrm{d}L(\mathbf{x},\vec{\omega})}{L(\mathbf{x},\vec{\omega})} = -\sigma_t(\mathbf{x})\mathrm{d}\mathbf{z}$$

// Integrate along beam from 0 to $\mathcal{Z}$

$$\log_e L_z - \log_e L_0 = -\sigma_t(\mathbf{x})z$$

Extinction Along a Finite Beam

$$\frac{\mathrm{d}L(\mathbf{x},\vec{\omega})}{\mathrm{d}z} = -\sigma_t(\mathbf{x})L(\mathbf{x},\vec{\omega})$$

$$\frac{\mathrm{d}L(\mathbf{x},\vec{\omega})}{L(\mathbf{x},\vec{\omega})} = -\sigma_t(\mathbf{x})\mathrm{d}\mathbf{z}$$

// Integrate along beam from 0 to $\mathcal{Z}$

$$\log_e L_z - \log_e L_0 = -\sigma_t(\mathbf{x})z$$

$$log_e\left(\frac{L_z}{L_0}\right) = -\sigma_t z$$

// Exponentiate

Extinction Along a Finite Beam

$$\frac{\mathrm{d}L(\mathbf{x},\vec{\omega})}{\mathrm{d}z} = -\sigma_t(\mathbf{x})L(\mathbf{x},\vec{\omega})$$

$$\frac{\mathrm{d}L(\mathbf{x},\vec{\omega})}{L(\mathbf{x},\vec{\omega})} = -\sigma_t(\mathbf{x})\mathrm{d}\mathbf{z}$$

// Integrate along beam from 0 to $\mathcal{Z}$

$$\log_e L_z - \log_e L_0 = -\sigma_t(\mathbf{x})z$$

$$log_e\left(\frac{L_z}{L_0}\right) = -\sigma_t z$$

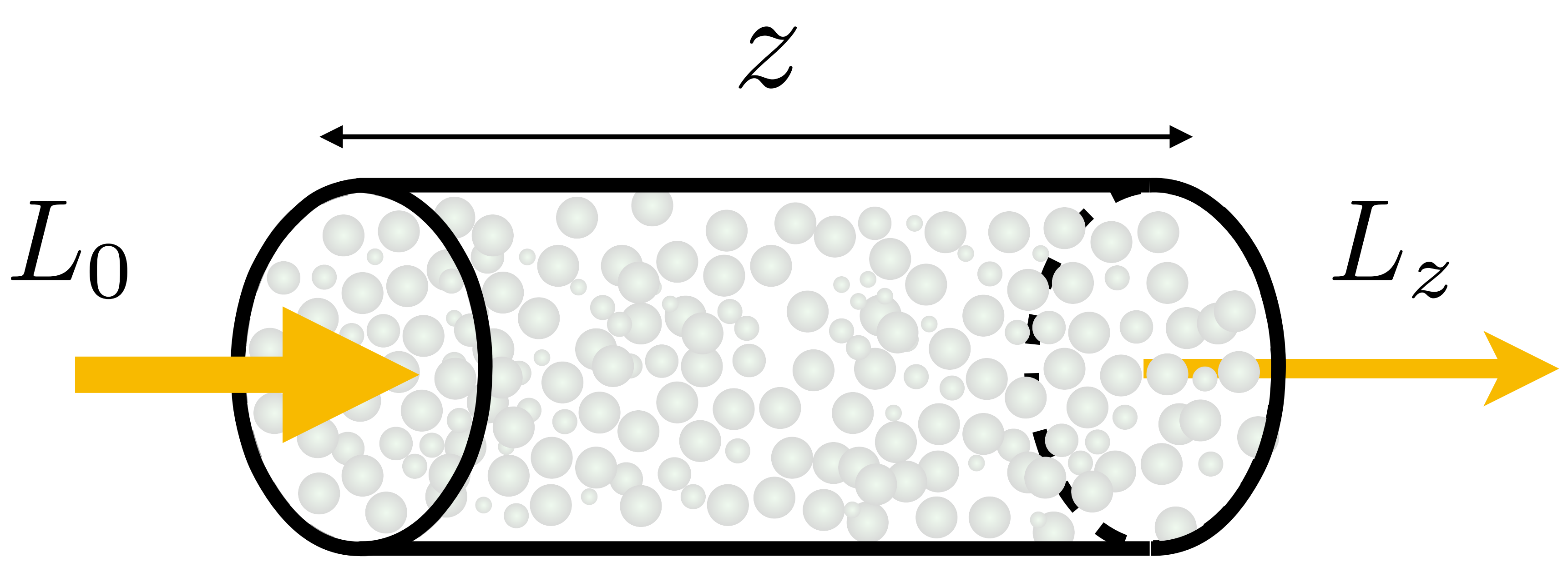
// Exponentiate

$$\frac{L_z}{L_0} = e^{-\sigma_t z}$$

Beer-Lambert Law

The fraction refers to as the *transmittance*

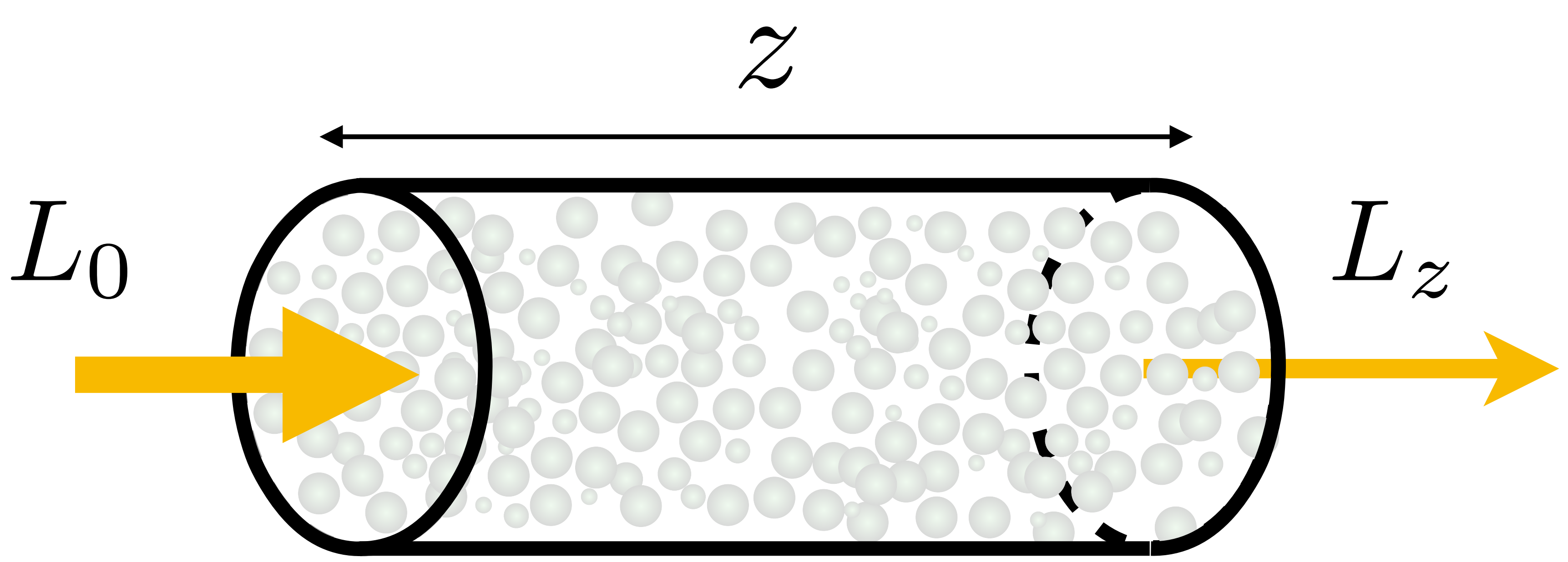
$$\frac{L_z}{L_0} = e^{-\sigma_t z}$$



Think of this as fractional visibility loss between two points

Beer-Lambert Law

The fraction refers to as the *transmittance*



$$\frac{L_z}{L_0} = e^{-\sigma_t z}$$

Radiance at distance 0

Think of this as fractional visibility loss between two points

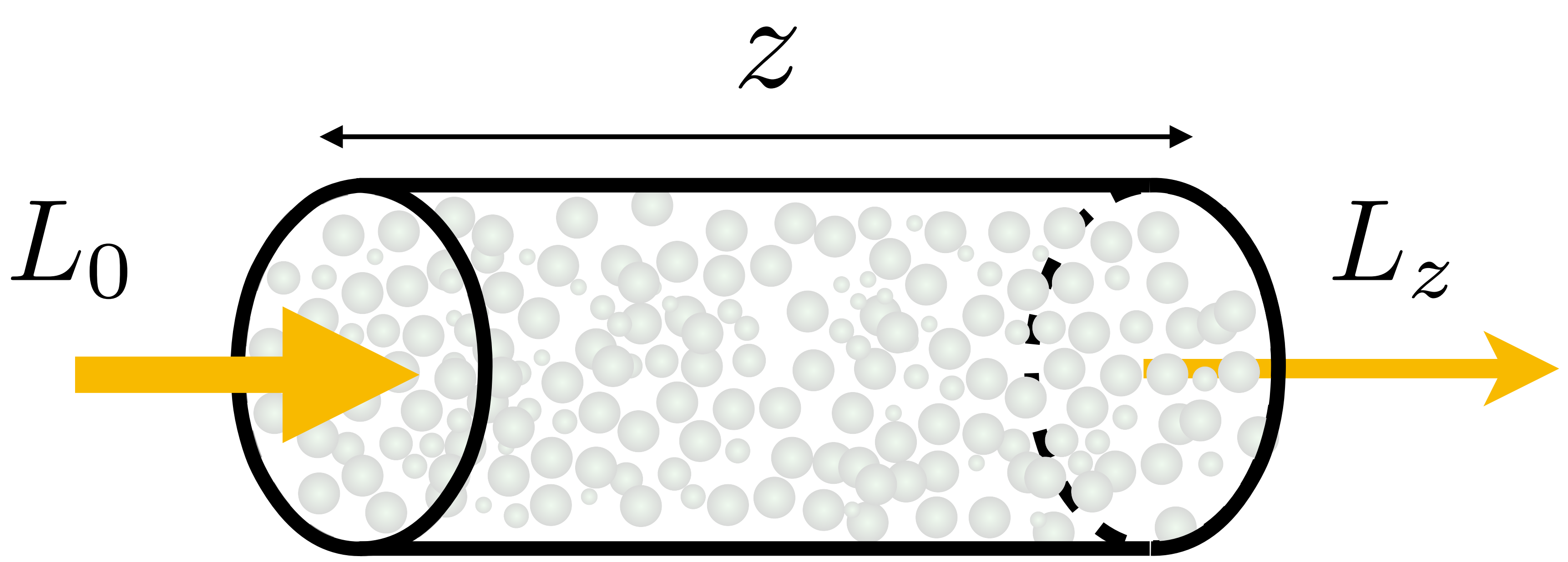
Beer-Lambert Law

The fraction refers to as the *transmittance*

Radiance at distance z

$$\frac{L_z}{L_0} = e^{-\sigma_t z}$$

Radiance at distance 0



Think of this as fractional visibility loss between two points

Beer-Lambert Law

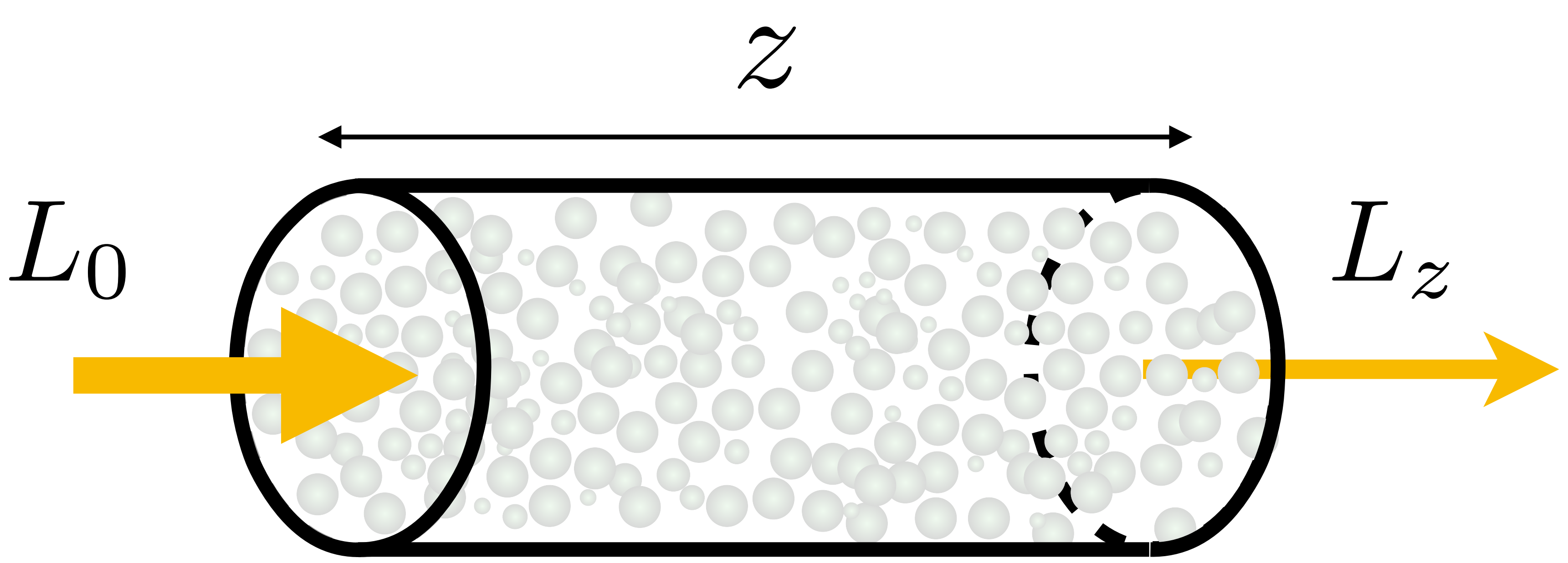
Expresses the remaining radiance after traveling a finite distance through the medium with constant extinction coefficient

The fraction refers to as the *transmittance*

Radiance at distance z

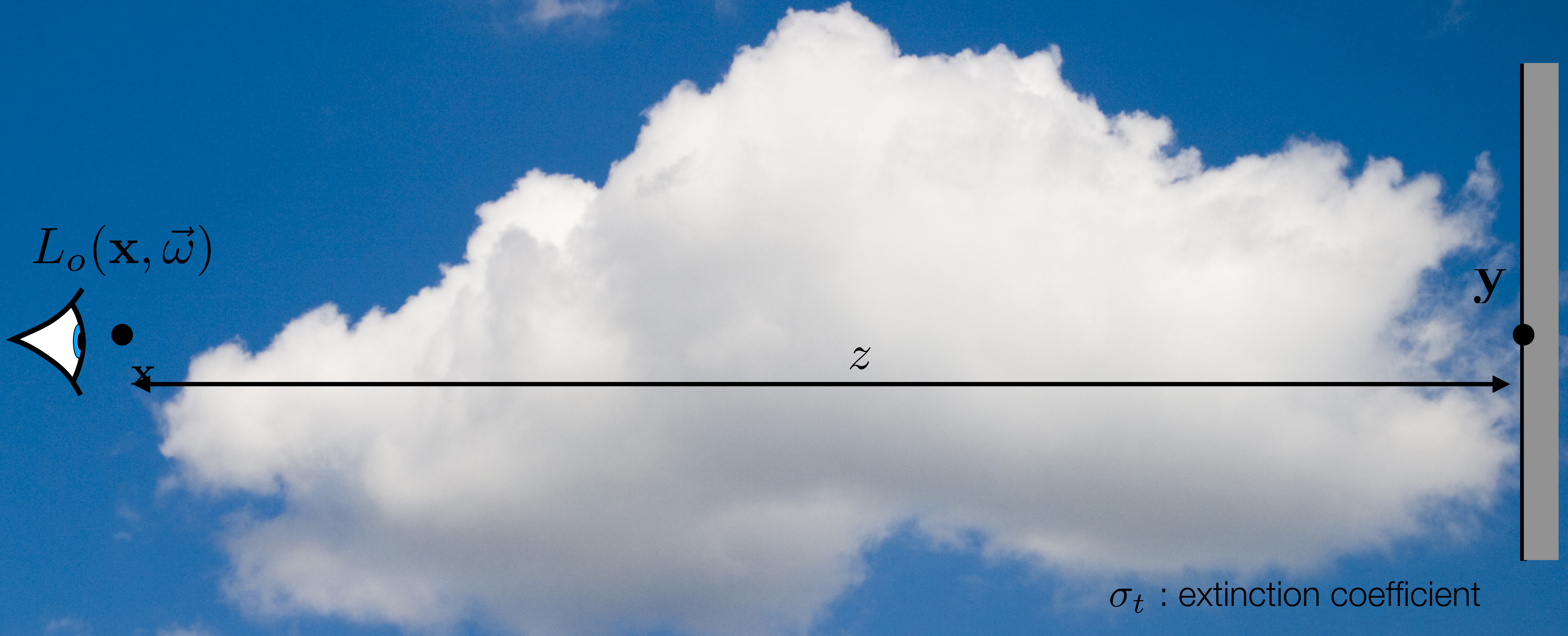
$$\frac{L_z}{L_0} = e^{-\sigma_t z}$$

Radiance at distance 0



Think of this as fractional visibility loss between two points

Beam Transmittance



σ_t : extinction coefficient

Beam Transmittance

σ_t : extinction coefficient

Beam Transmittance

σ_t : extinction coefficient

Beam Transmittance

σ_t : extinction coefficient

Beam Transmittance

Radiance at $\mathbf{y}$

$L_o(\mathbf{x}, \vec{\omega})$

$\mathbf{x}$

$\mathbf{y}$

$T_r(\mathbf{x} \rightarrow \mathbf{x}')L_o(\mathbf{x}, \vec{\omega})$

σ_t : extinction coefficient

Beam Transmittance

$$T_r(\mathbf{x} \to \mathbf{y}) = e^{-\int_0^{||\mathbf{x}-\mathbf{y}||} \sigma_t(t)dt}$$

σ_t : extinction coefficient

Beam Transmittance

$$T_r(\mathbf{x} \to \mathbf{y}) = e^{-\int_0^{||\mathbf{x}-\mathbf{y}||} \sigma_t(t)dt}$$

Radiance at $\mathbf{y}$

$L_o(\mathbf{x}, \vec{\omega})$

$\mathbf{x}$

$\mathbf{y}$

$T_r(\mathbf{x} \to \mathbf{x}')L_o(\mathbf{x}, \vec{\omega})$

σ_t : extinction coefficient

Beam Transmittance

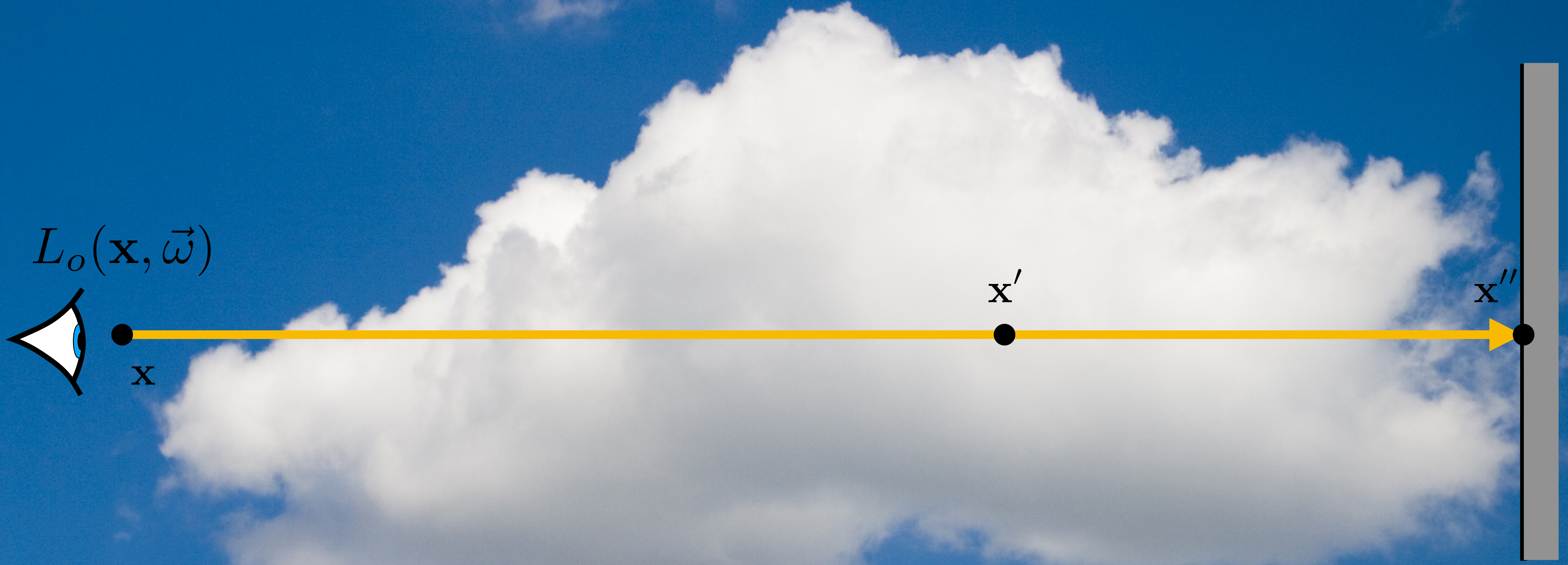
$$T_r(\mathbf{x} \to \mathbf{y}) = e^{-\int_0^{||\mathbf{x}-\mathbf{y}||} \sigma_t(t)dt}$$

σ_t : extinction coefficient

Beam Transmittance: **Multiplicative**

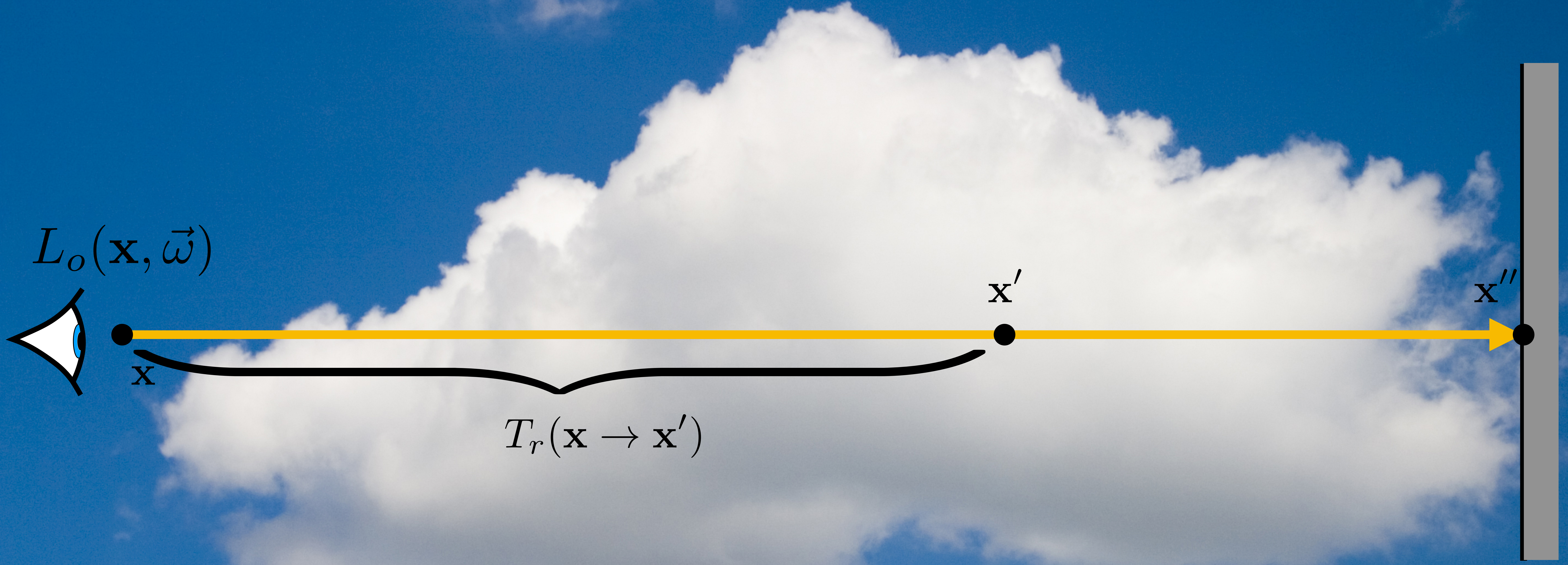
σ_t : extinction coefficient

Beam Transmittance: **Multiplicative**



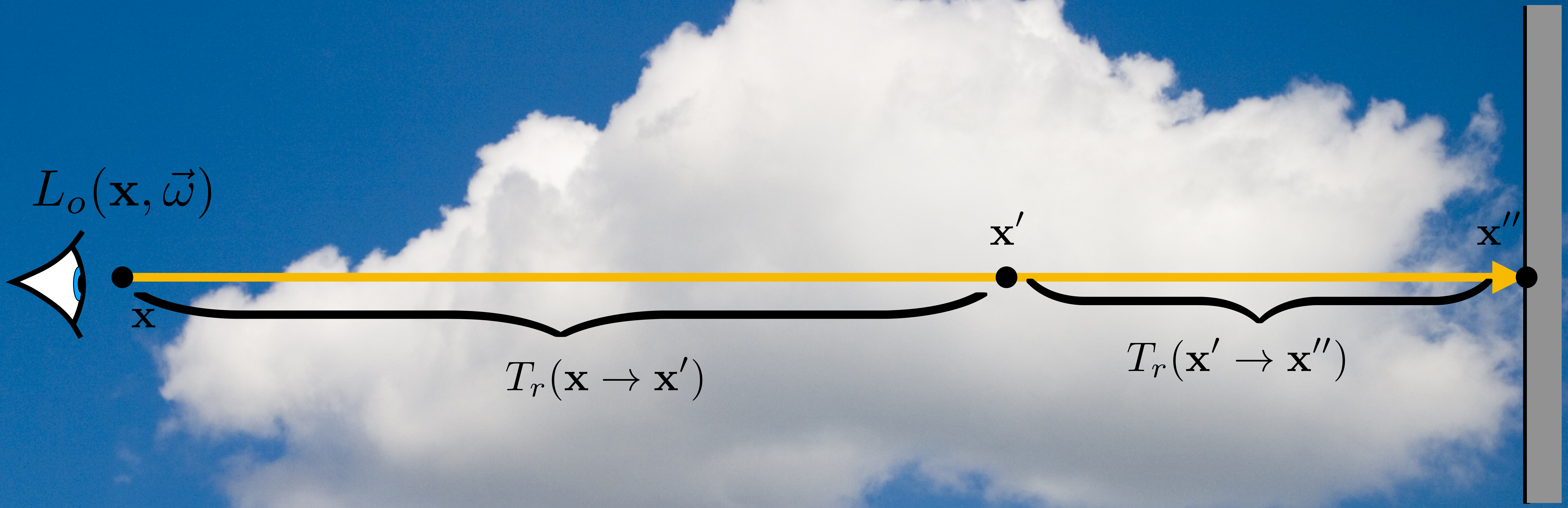
σ_t : extinction coefficient

Beam Transmittance: **Multiplicative**



σ_t : extinction coefficient

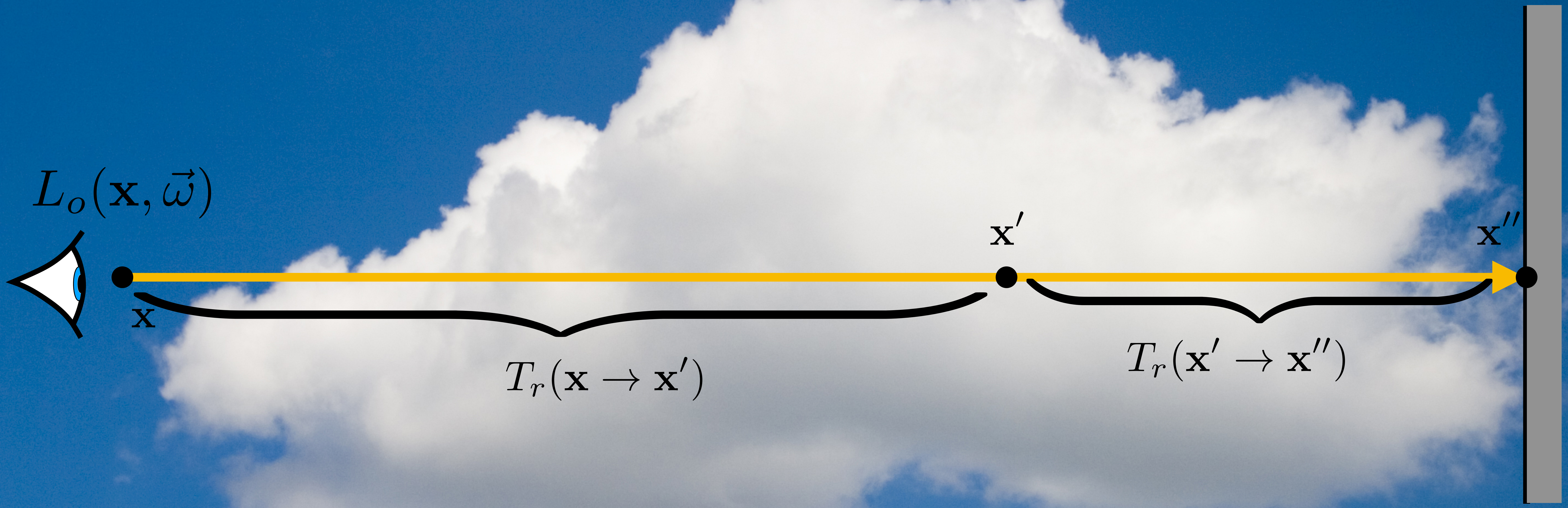
Beam Transmittance: **Multiplicative**



σ_t : extinction coefficient

Beam Transmittance: **Multiplicative**

$$T_r(\mathbf{x} \to \mathbf{x}'') = T_r(\mathbf{x} \to \mathbf{x}')T_r(\mathbf{x}' \to \mathbf{x}'')$$



σ_t : extinction coefficient

Beam Transmittance

In Homogeneous medium σ_t is a constant:

Beam Transmittance

In Homogeneous medium σ_t is a constant:

$$T_r(\mathbf{x} \to \mathbf{y}) = e^{-\sigma_t ||\mathbf{x}-\mathbf{y}||}$$

Beam Transmittance

In Homogeneous medium σ_t is a constant:

$$T_r(\mathbf{x} \to \mathbf{y}) = e^{-\sigma_t ||\mathbf{x}-\mathbf{y}||}$$

In Heterogeneous medium (spatially varying σ_t):

Beam Transmittance

In Homogeneous medium σ_t is a constant:

$$T_r(\mathbf{x} \to \mathbf{y}) = e^{-\sigma_t ||\mathbf{x}-\mathbf{y}||}$$

In Heterogeneous medium (spatially varying σ_t):

$$T_r(\mathbf{x} \to \mathbf{y}) = e^{-\int_0^{||\mathbf{x}-\mathbf{y}||} \sigma_t(t)dt}$$

Beam Transmittance

In Homogeneous medium σ_t is a constant:

$$T_r(\mathbf{x} \to \mathbf{y}) = e^{-\sigma_t ||\mathbf{x}-\mathbf{y}||}$$

In Heterogeneous medium (spatially varying σ_t):

$$T_r(\mathbf{x} \to \mathbf{y}) = e^{-\int_0^{||\mathbf{x}-\mathbf{y}||} \sigma_t(t)dt}$$

Optical thickness

Radiative Transfer Equation (RTE)

Attenuation

$$\frac{\mathrm{d}L(\mathbf{x},\vec{\omega})}{\mathrm{d}z} = -\sigma_t(\mathbf{x})L(\mathbf{x},\vec{\omega}) + \sigma_s(\mathbf{x})L_s(\mathbf{x},\vec{\omega}) + \sigma_a(\mathbf{x})L_e(\mathbf{x},\vec{\omega})$$

In-scattering **Emission**

Radiative Transfer Equation (RTE)

Attenuation

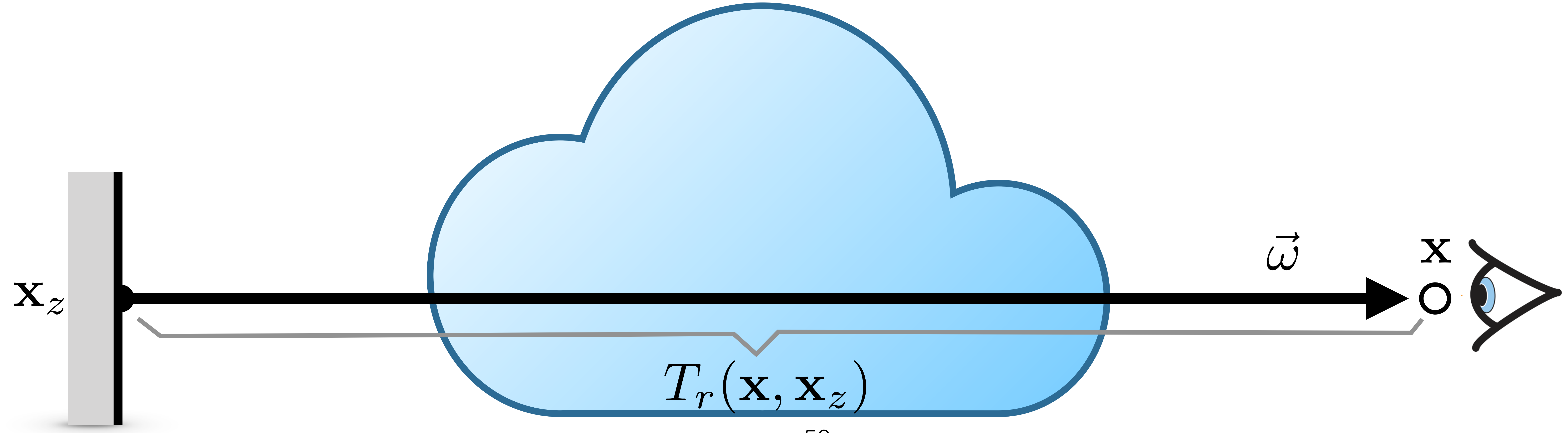
$$\frac{\mathrm{d}L(\mathbf{x},\vec{\omega})}{\mathrm{d}z} = \boxed{-\sigma_t(\mathbf{x})L(\mathbf{x},\vec{\omega})} + \boxed{\sigma_s(\mathbf{x})L_s(\mathbf{x},\vec{\omega}) + \sigma_a(\mathbf{x})L_e(\mathbf{x},\vec{\omega})}$$

In-scattering **Emission**

What about a beam with finite-length z?

Volumetric Rendering Equation

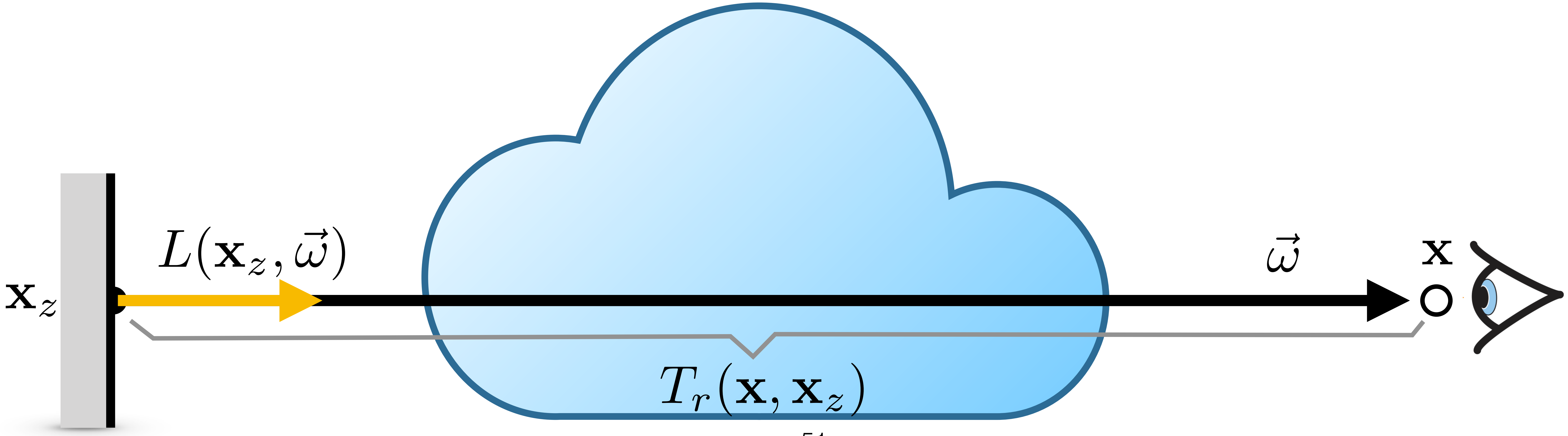
$$L(\mathbf{x},\vec{\omega}) = T_r(\mathbf{x},\mathbf{x}_z)L(\mathbf{x}_z,\vec{\omega})$$



Volumetric Rendering Equation

$$L(\mathbf{x}, \vec{\omega}) = \boxed{T_r(\mathbf{x}, \mathbf{x}_z) L(\mathbf{x}_z, \vec{\omega})}$$

Reduced (background) surface radiance

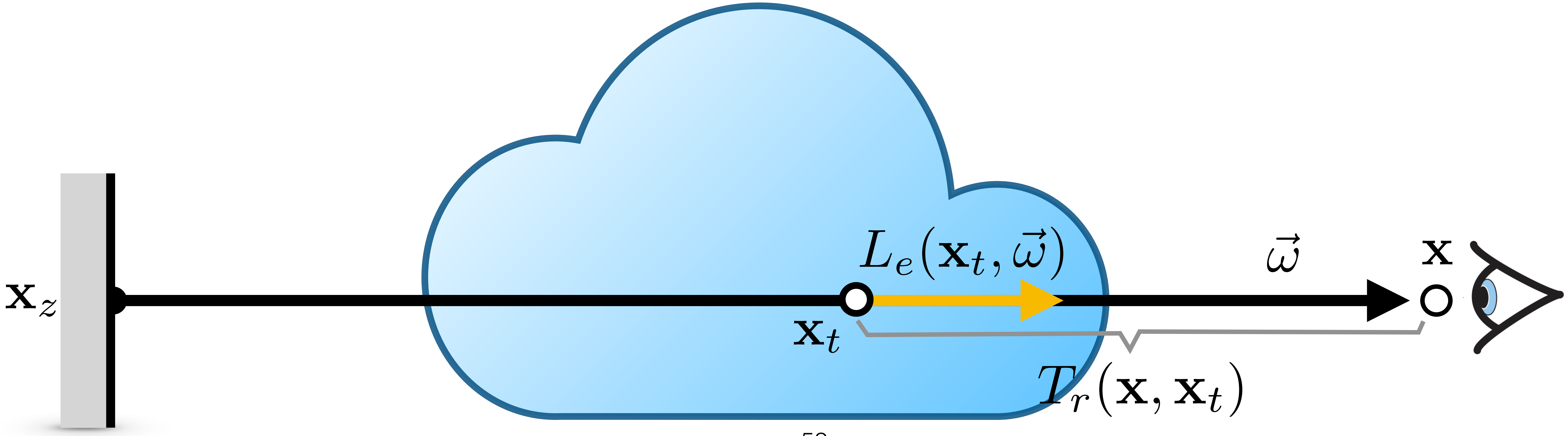


Volumetric Rendering Equation

$$L(\mathbf{x}, \vec{\omega}) = T_r(\mathbf{x}, \mathbf{x}_z) L(\mathbf{x}_z, \vec{\omega})$$

$$+ \int_0^z T_r(\mathbf{x}, \mathbf{x}_t) \sigma_a(\mathbf{x}_t) L_e(\mathbf{x}_t, \vec{\omega}) dt$$

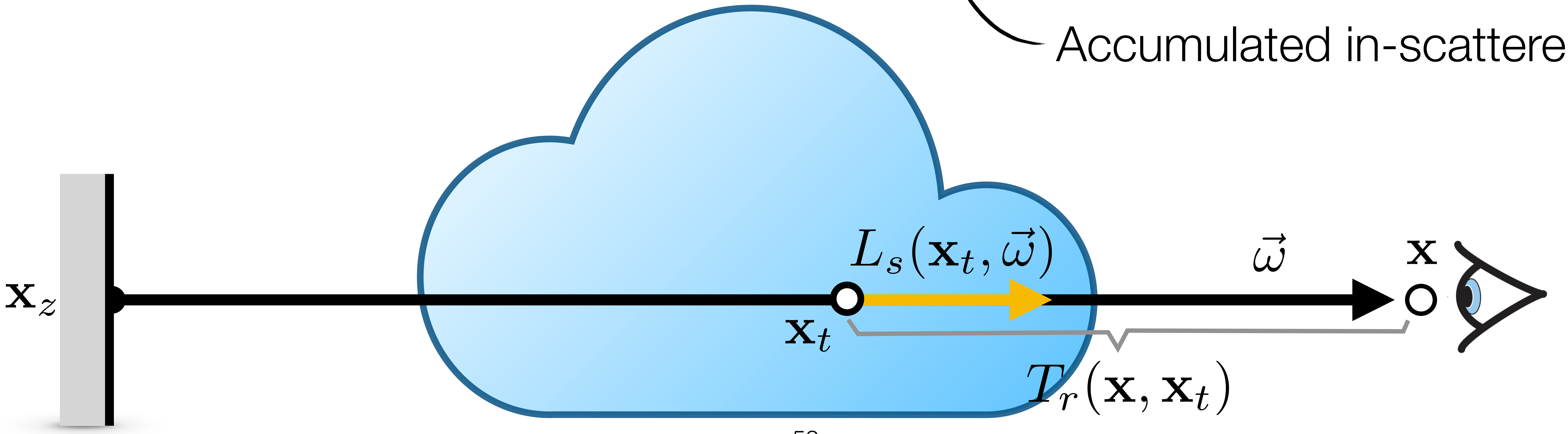
Accumulated emitted radiance



Volumetric Rendering Equation

$$
\begin{aligned}
L(\mathbf{x},\vec{\omega}) &= T_r(\mathbf{x},\mathbf{x}_z)L(\mathbf{x}_z,\vec{\omega}) \\
&\quad + \int_0^z T_r(\mathbf{x},\mathbf{x}_t)\sigma_a(\mathbf{x}_t)L_e(\mathbf{x}_t,\vec{\omega})dt \\
&\quad + \int_0^z T_r(\mathbf{x},\mathbf{x}_t)\sigma_s(\mathbf{x}_t)L_s(\mathbf{x}_t,\vec{\omega})dt
\end{aligned}
$$

Accumulated in-scattered radiance

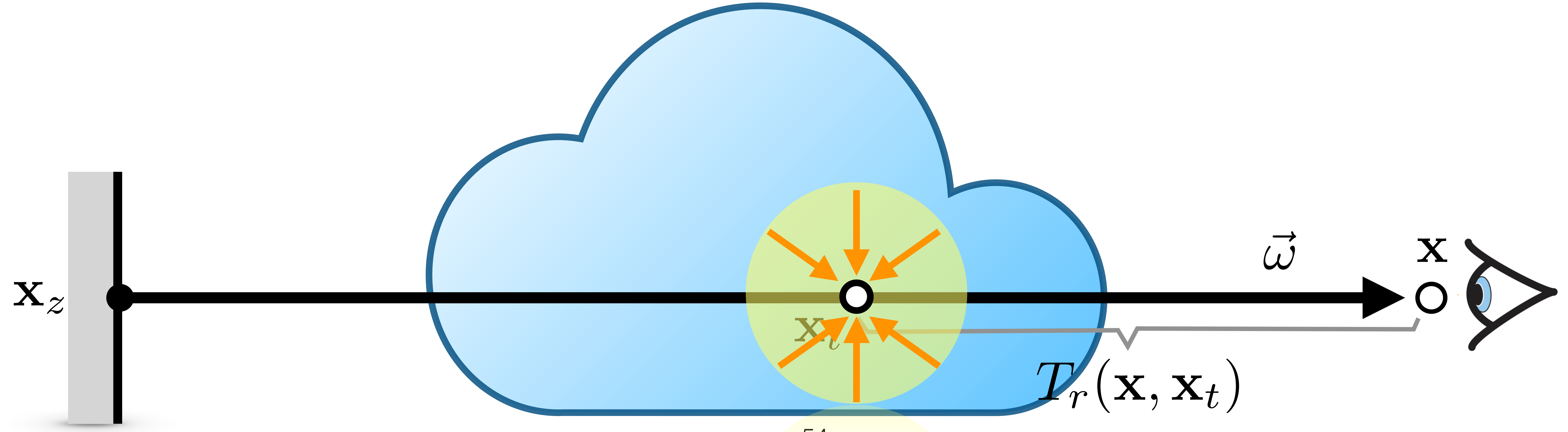


Volumetric Rendering Equation

$$L(\mathbf{x},\vec{\omega}) = T_r(\mathbf{x},\mathbf{x}_z)L(\mathbf{x}_z,\vec{\omega})$$

$$+\int_0^z T_r(\mathbf{x},\mathbf{x}_t)\sigma_a(\mathbf{x}_t)L_e(\mathbf{x}_t,\vec{\omega})dt$$

$$+\int_0^z T_r(\mathbf{x},\mathbf{x}_t)\sigma_s(\mathbf{x}_t)\int_{S^2} f_p(\mathbf{x}_t,\vec{\omega}',\vec{\omega})L_i(\mathbf{x}_t,\vec{\omega}')d\omega' dt$$



Volumetric Rendering Equation

$$
\begin{aligned}
L(\mathbf{x}, \vec{\omega}) &= T_r(\mathbf{x}, \mathbf{x}_z) L(\mathbf{x}_z, \vec{\omega}) \\
&\quad + \int_0^z T_r(\mathbf{x}, \mathbf{x}_t) \sigma_a(\mathbf{x}_t) L_e(\mathbf{x}_t, \vec{\omega}) dt \\
&\quad + \int_0^z T_r(\mathbf{x}, \mathbf{x}_t) \sigma_s(\mathbf{x}_t) \int_{S^2} f_p(\mathbf{x}_t, \vec{\omega}', \vec{\omega}) L_i(\mathbf{x}_t, \vec{\omega}') d\omega' dt
\end{aligned}
$$

Scattering in Media

Phase Functions

It describes the angular distribution of scattered radiation at a point;

Phase Functions

It describes the angular distribution of scattered radiation at a point;

It is the volumetric analog to the BSDF, but it is different from the BSDF.

Phase Functions

It describes the angular distribution of scattered radiation at a point;

It is the volumetric analog to the BSDF, but it is different from the BSDF.

It has a normalization constant:

$$\int_{S^2} f_{\mathrm{p}}(\vec{\omega}, \vec{\omega}')\mathrm{d}\vec{\omega}' = 1 \quad \forall \vec{\omega}$$

Phase Functions

It describes the angular distribution of scattered radiation at a point;

It is the volumetric analog to the BSDF, but it is different from the BSDF.

It has a normalization constant:

$$\int_{S^2} f_{\mathrm{p}}(\vec{\omega}, \vec{\omega}')\mathrm{d}\vec{\omega}' = 1 \quad \forall \vec{\omega}$$

This constraint means that phase functions actually define probability distributions for scattering in a particular direction.

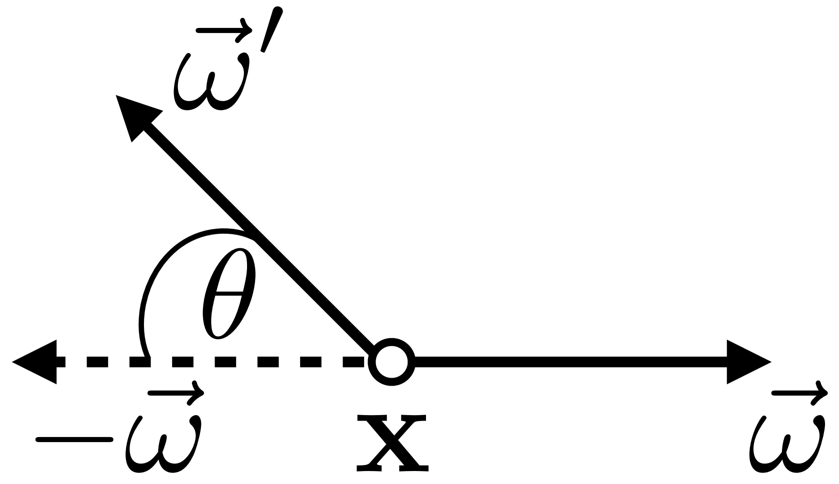
Phase Functions

Isotropic:

$$f_{\mathrm{p}}(\vec{\omega}_o, \vec{\omega}_i) = \frac{1}{4\pi}$$

Uniform scattering, analogous to Lambertian BRDF

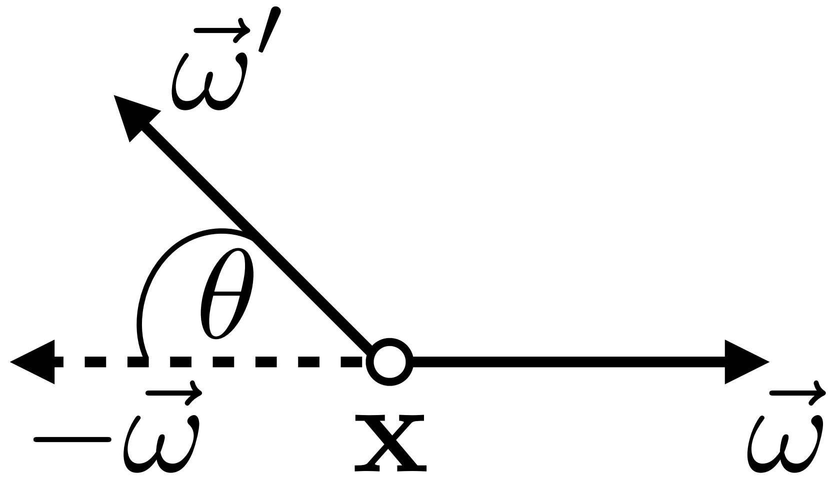
Phase Functions



Phase Functions

Quantifying anisotropy by

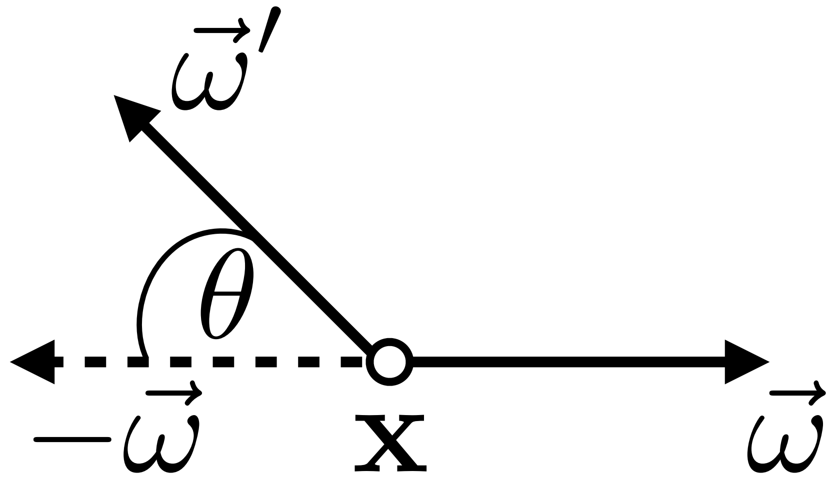
$$g = \int_{S^2} f_{\mathrm{p}}(\mathbf{x}, \vec{\omega}, \vec{\omega}') \cos\theta d\vec{\omega}'$$



Phase Functions

Quantifying anisotropy by

$$g = \int_{S^2} f_{\mathrm{p}}(\mathbf{x}, \vec{\omega}, \vec{\omega}') \cos\theta d\vec{\omega}'$$



where

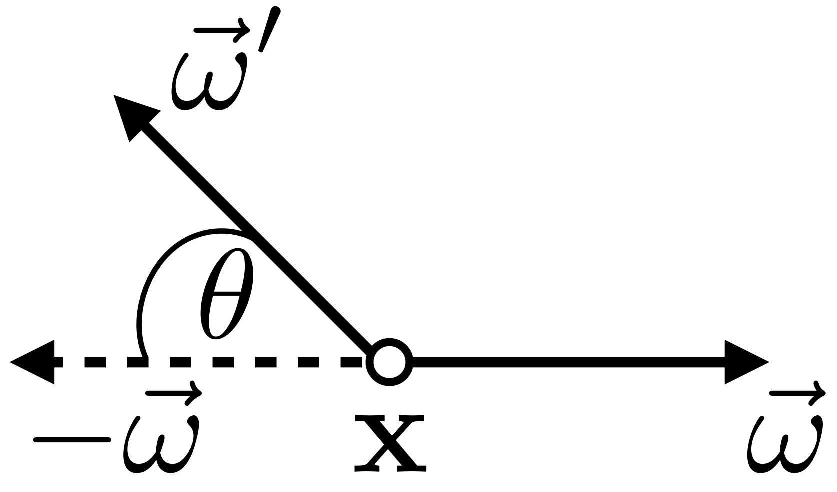
$$\cos\theta = -\vec{\omega} \cdot \vec{\omega}'$$

g is the asymmetry parameter

Phase Functions

Quantifying anisotropy by

$$g = \int_{S^2} f_\mathrm{p}(\mathbf{x}, \vec{\omega}, \vec{\omega}') \cos\theta d\vec{\omega}'$$



where

$$\cos\theta = -\vec{\omega} \cdot \vec{\omega}'$$

$g = 0$: isotropic scattering (on average)

$g > 0$: forward scattering

$g < 0$: backward scattering

g is the asymmetry parameter

Henyey-Greenstein Phase Function

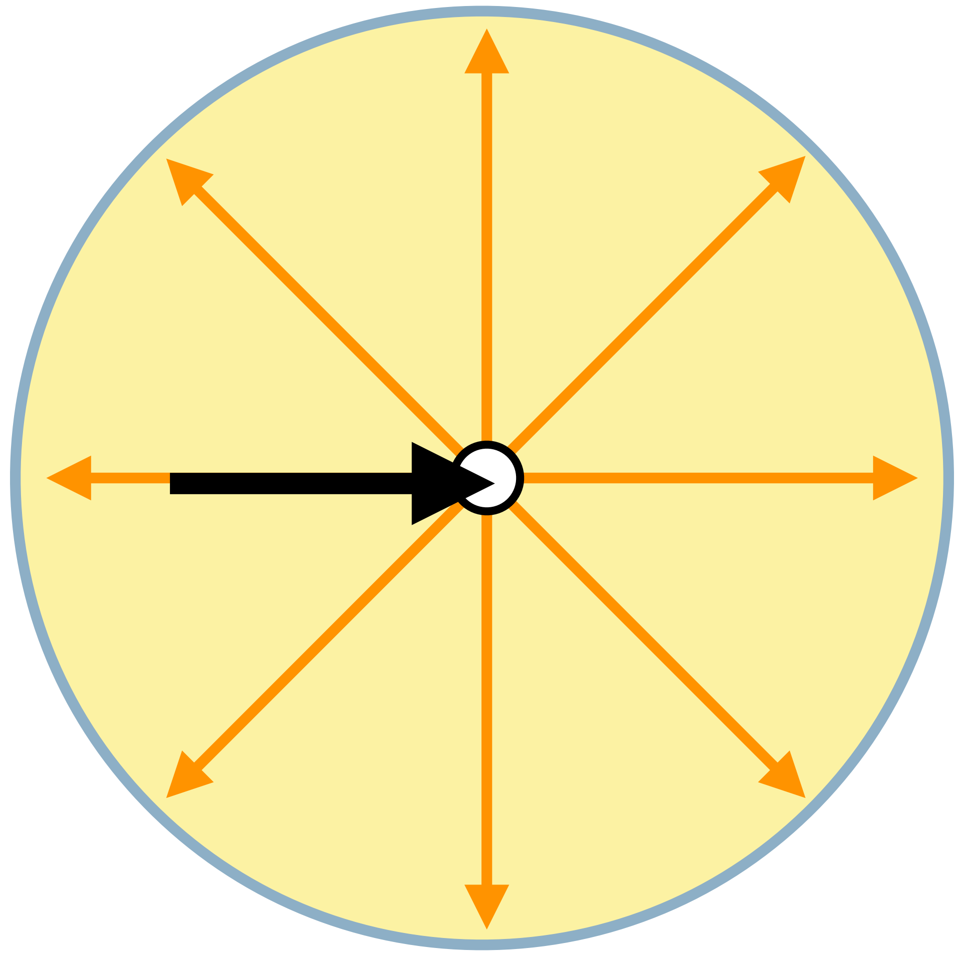
$$f_{\mathrm{p}}(\theta) = \frac{1}{4\pi} \frac{1 - g^2}{(1 + g^2 + 2g(\cos\theta))^{3/2}} \qquad g \in [-1, 1]$$

Henyey-Greenstein Phase Function

$$f_{\mathrm{p}}(\theta) = \frac{1}{4\pi} \frac{1-g^2}{(1+g^2+2g(\cos\theta))^{3/2}}$$

$$g \in [-1, 1]$$

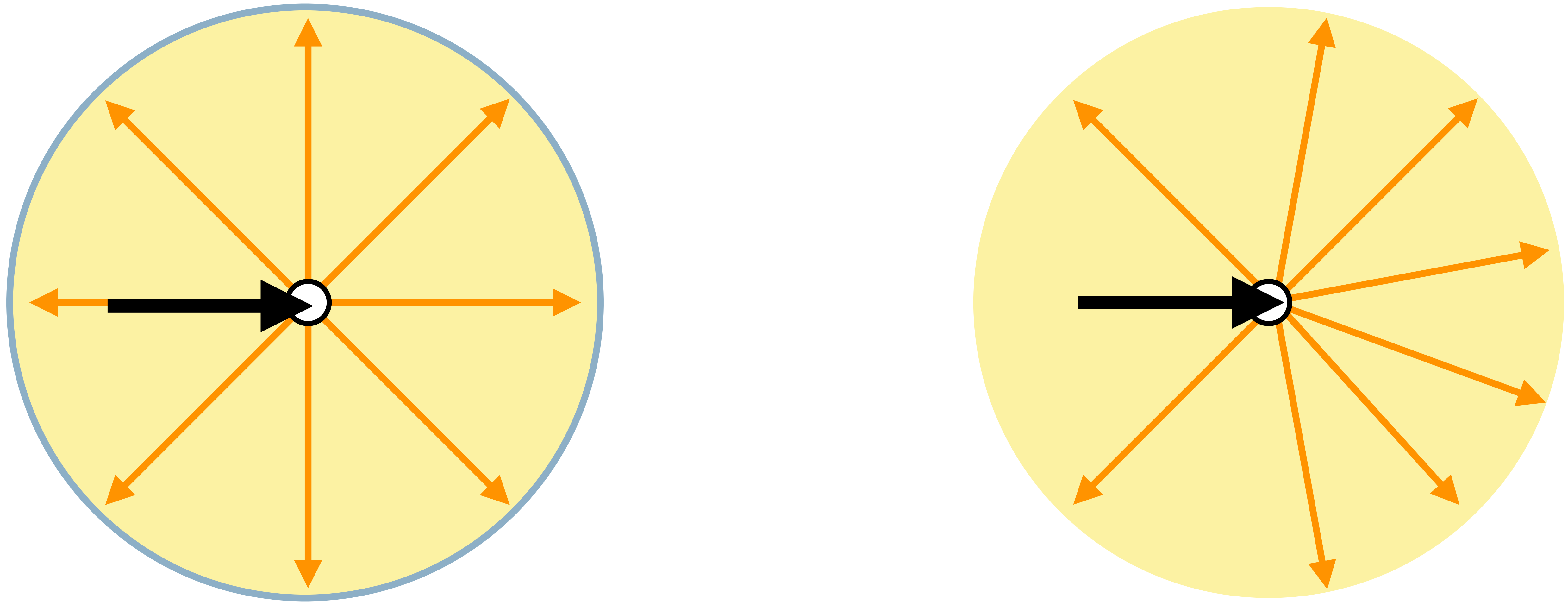
$$g = 0$$



Henyey-Greenstein Phase Function

$$f_{\mathrm{p}}(\theta) = \frac{1}{4\pi} \frac{1 - g^2}{(1 + g^2 + 2g(\cos\theta))^{3/2}} \qquad g \in [-1, 1]$$

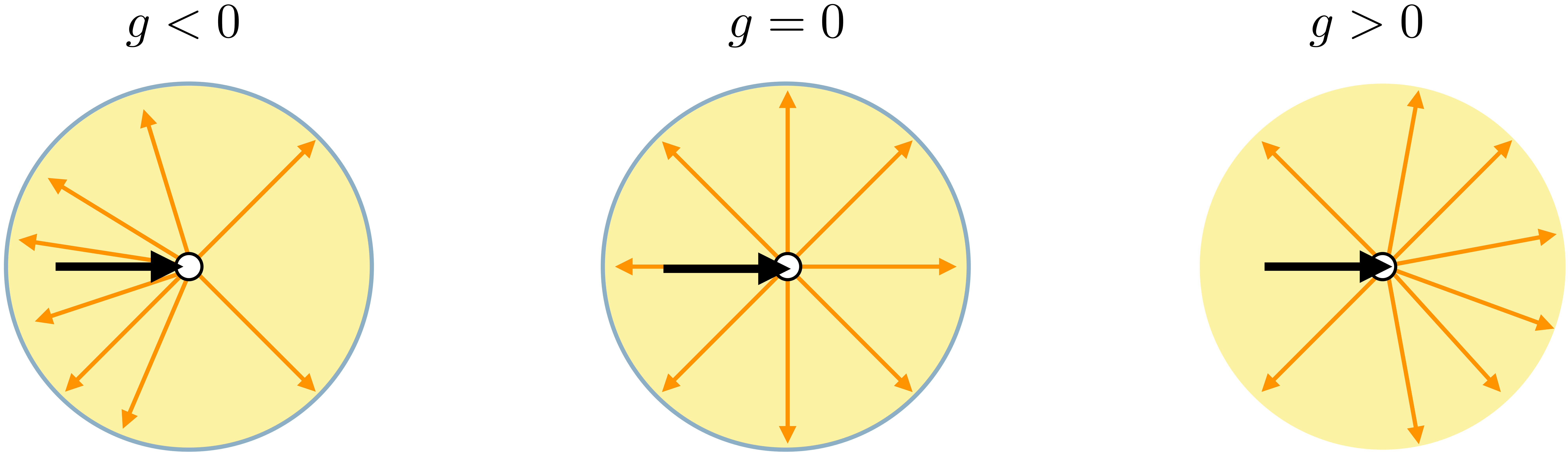
$g = 0$ $\qquad$ $g > 0$



Henyey-Greenstein Phase Function

$$f_{\mathrm{p}}(\theta) = \frac{1}{4\pi} \frac{1 - g^2}{(1 + g^2 + 2g(\cos\theta))^{3/2}}$$

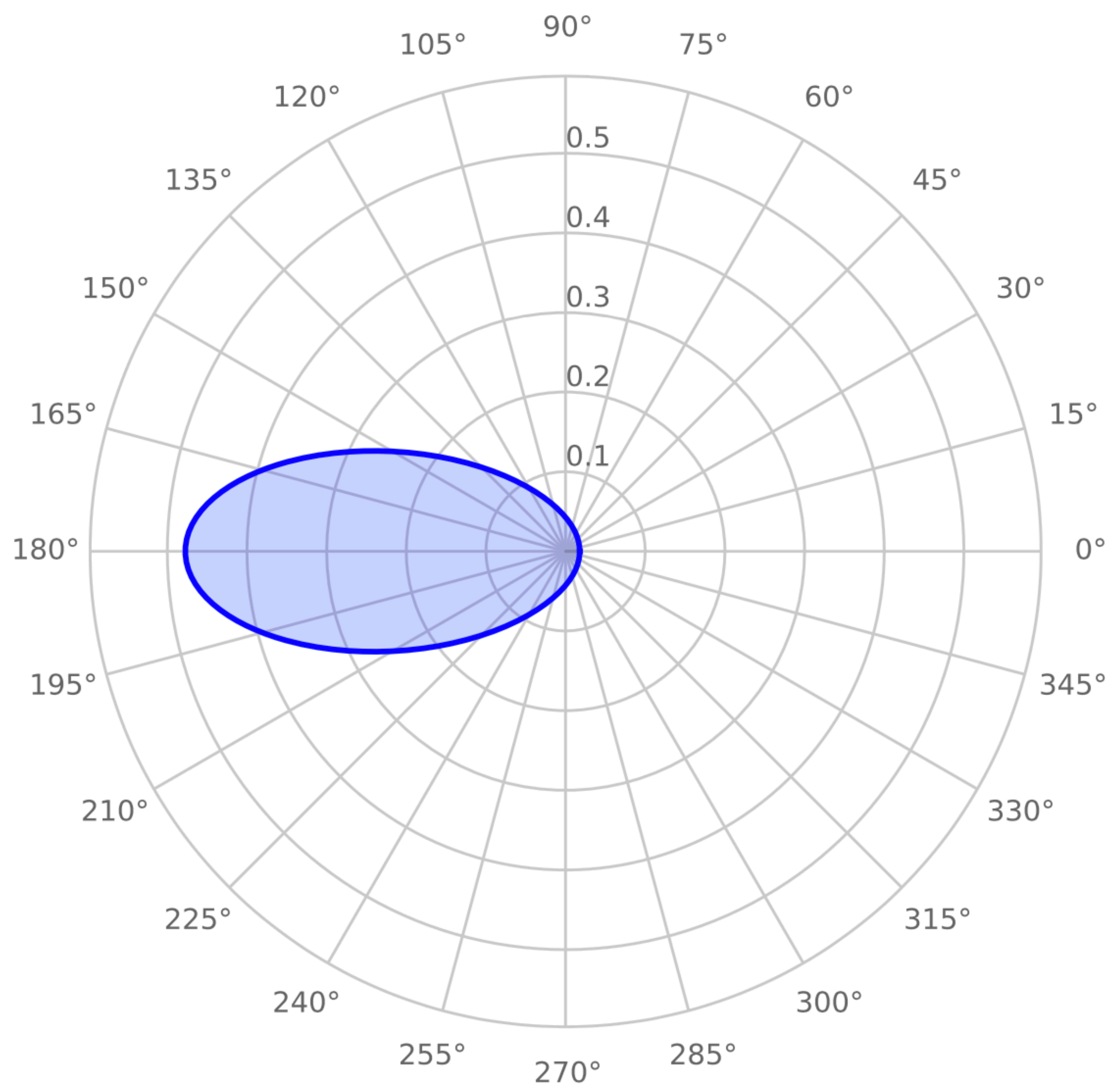
$$g \in [-1, 1]$$



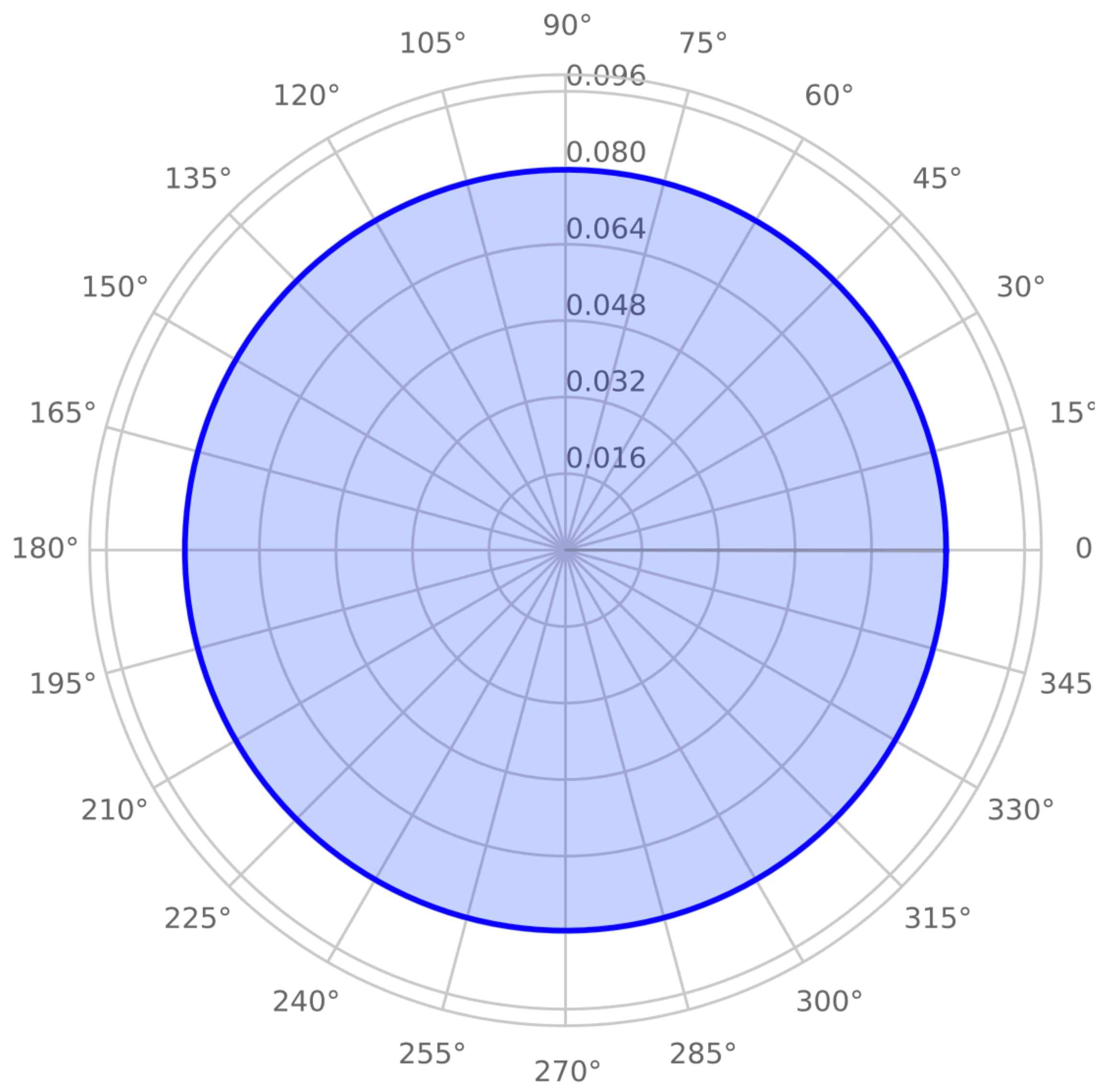
Henyey-Greenstein Phase Function

$$f_{\mathrm{p}}(\theta) = \frac{1}{4\pi}\frac{1-g^2}{(1+g^2+2g(\cos\theta))^{3/2}}$$

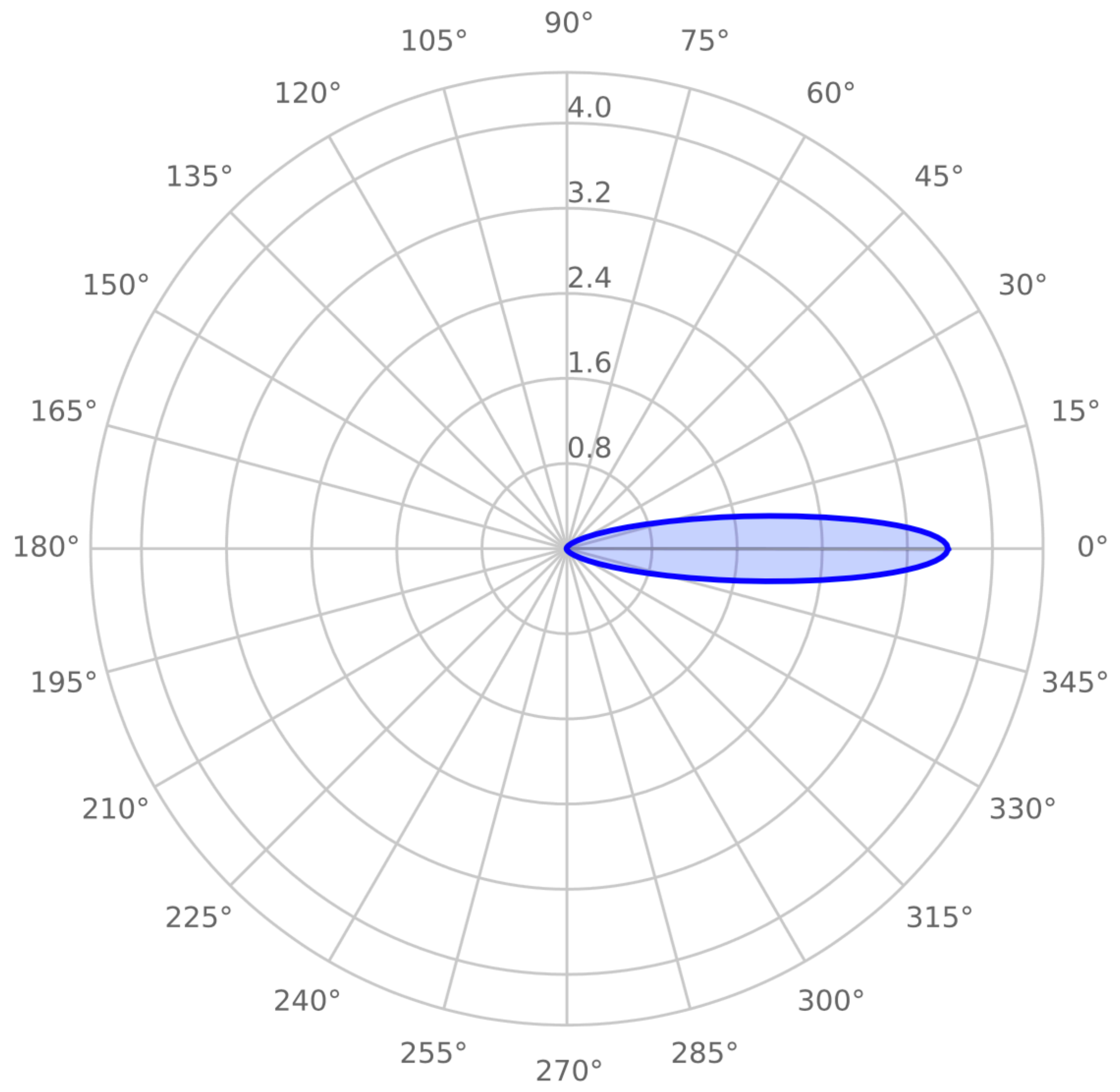
$g = -0.5$



$g = 0$



$g = 0.8$



Henyey-Greenstein Phase Function

$g = -0.7$

$g = 0.7$

Strong forward scattering

Strong backward scattering

PBRTv3 [2016]

Schlick's Phase Function

Empirical Phase Function

Faster approximation to HG

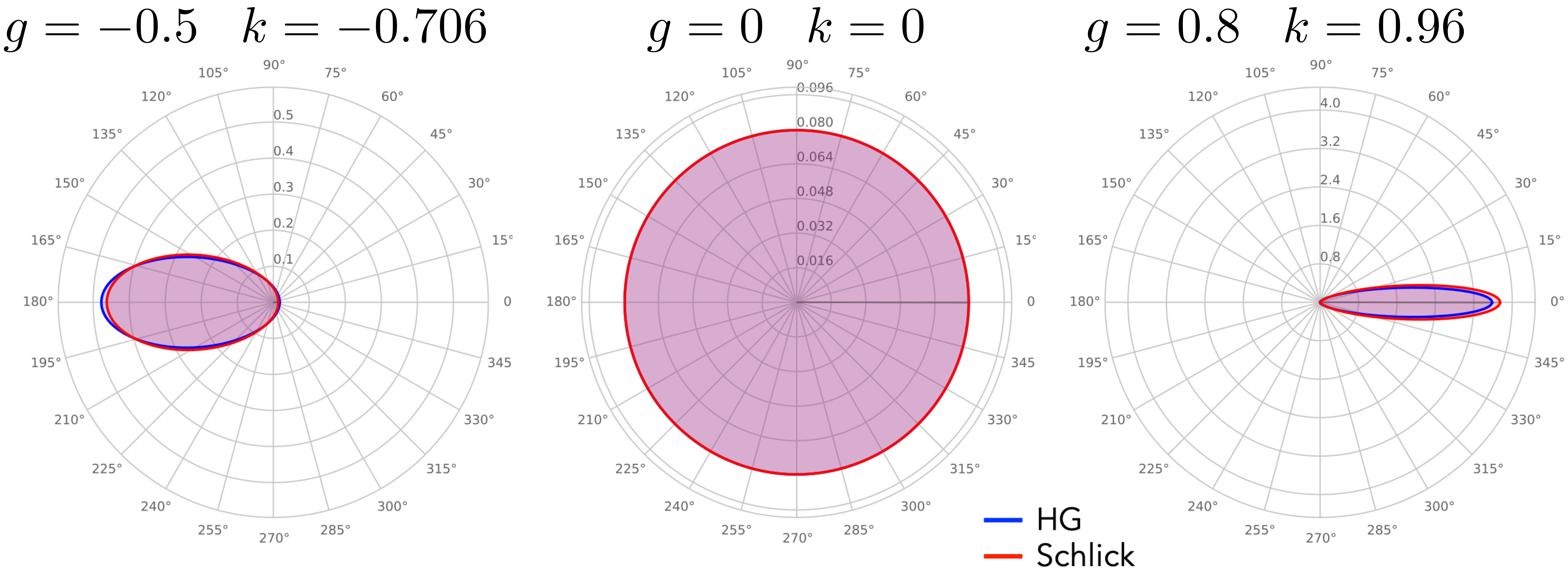
$$f_{\mathrm{p}}(\theta) = \frac{1}{4\pi} \frac{1 - k^2}{(1 - k\cos\theta)^2}$$

$$k = 1.55g - 0.55g^3$$

Schlick's Phase Function

Empirical Phase Function

Faster approximation to HG


$g = -0.5 \quad k = -0.706$

$g = 0 \quad k = 0$

$g = 0.8 \quad k = 0.96$

Rainbows

Lorenz-Mie Scattering

For large-size particles (scatterers), we cannot ignore the wave nature of light

Solution to Maxwell's equations for scattering from many spherical dielectric particles

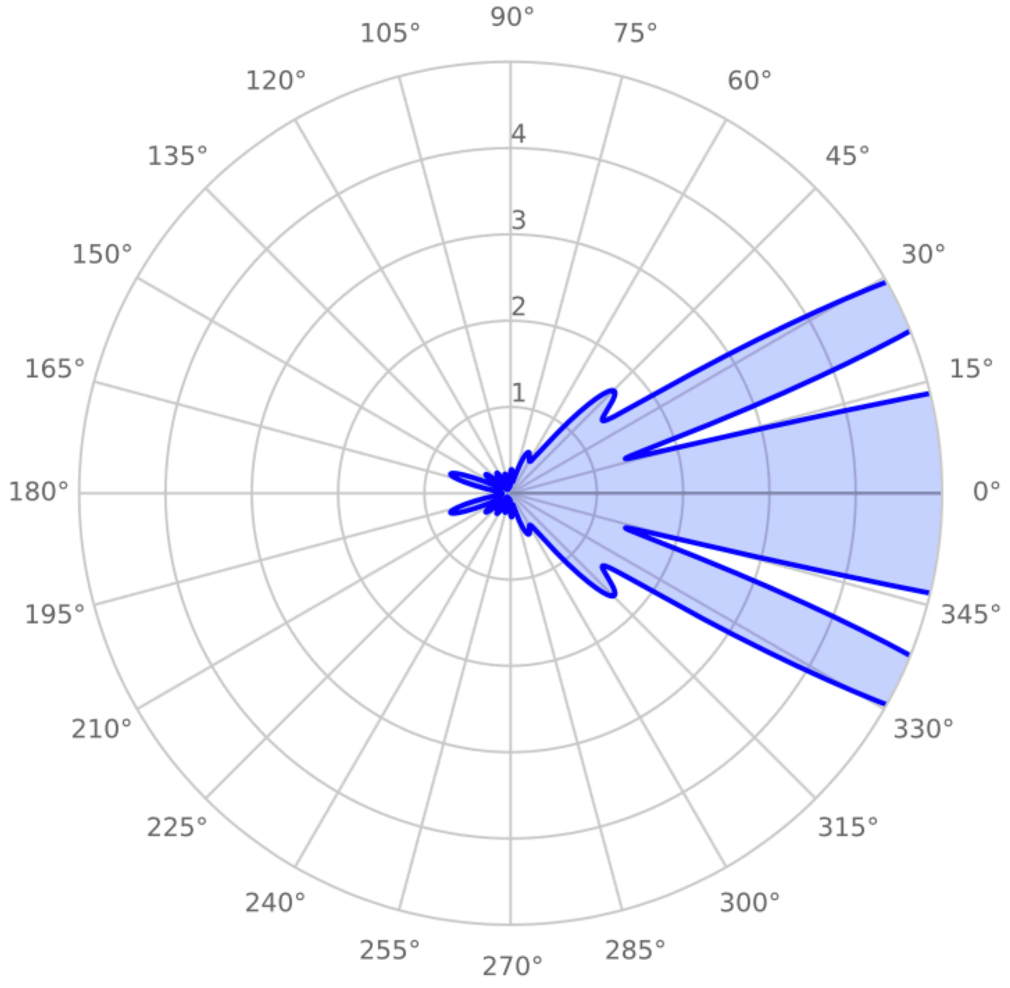
Explains many phenomena

Complicated: solution is an infinite analytic series

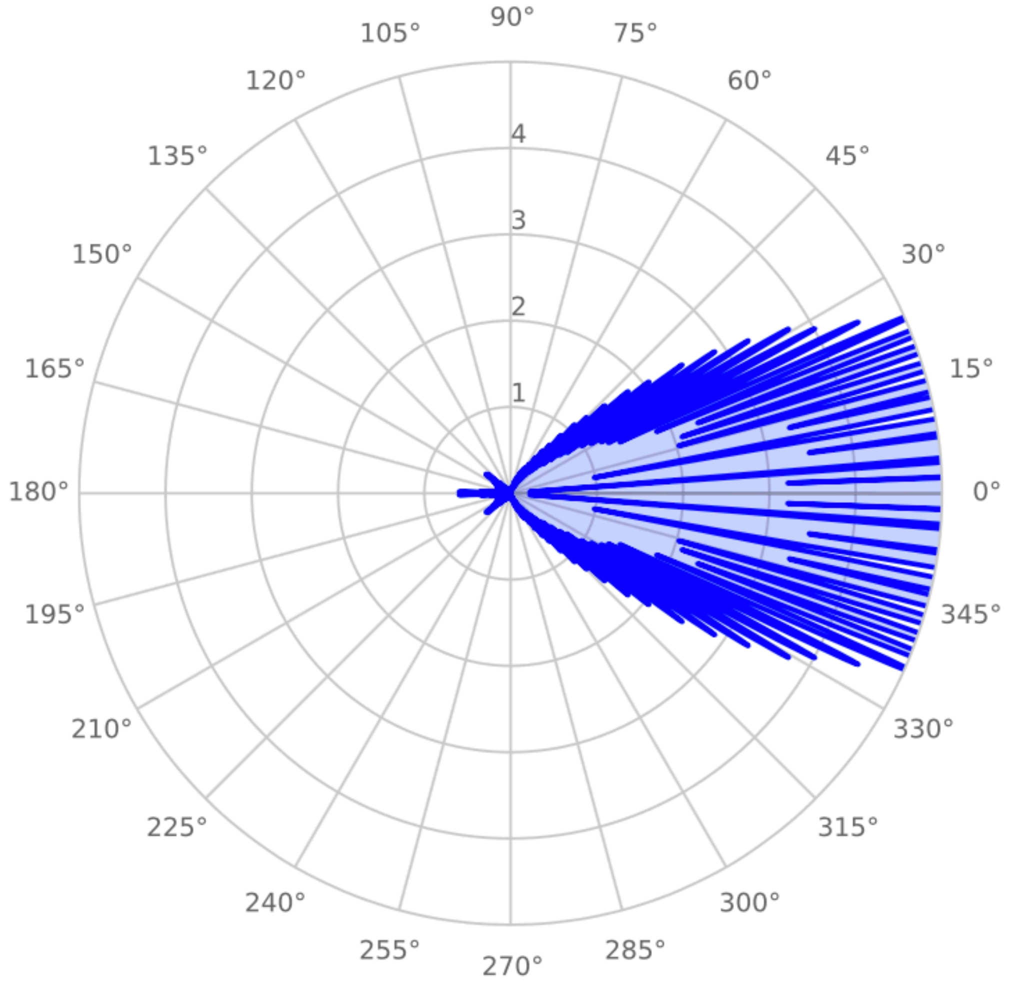
Lorenz-Mie Scattering

Linear plot

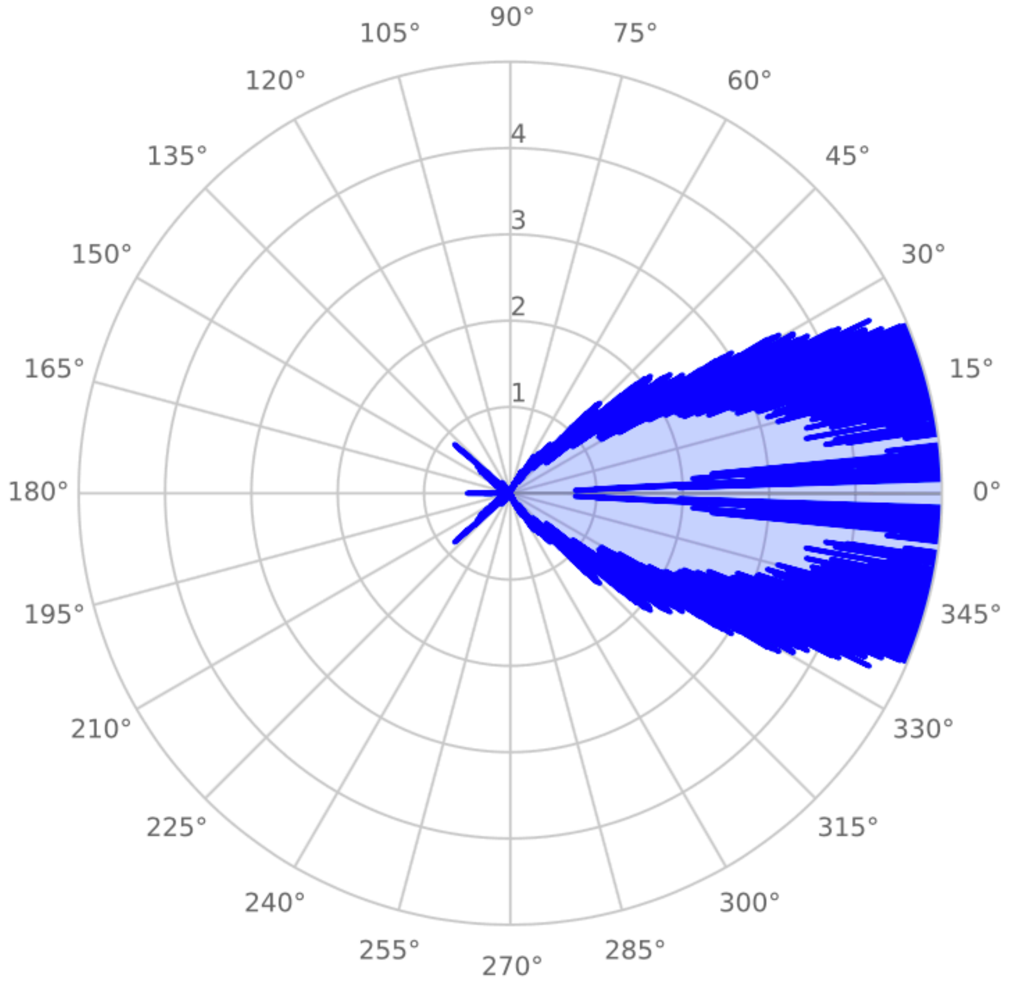
Sphere diameter $=1\mu m$



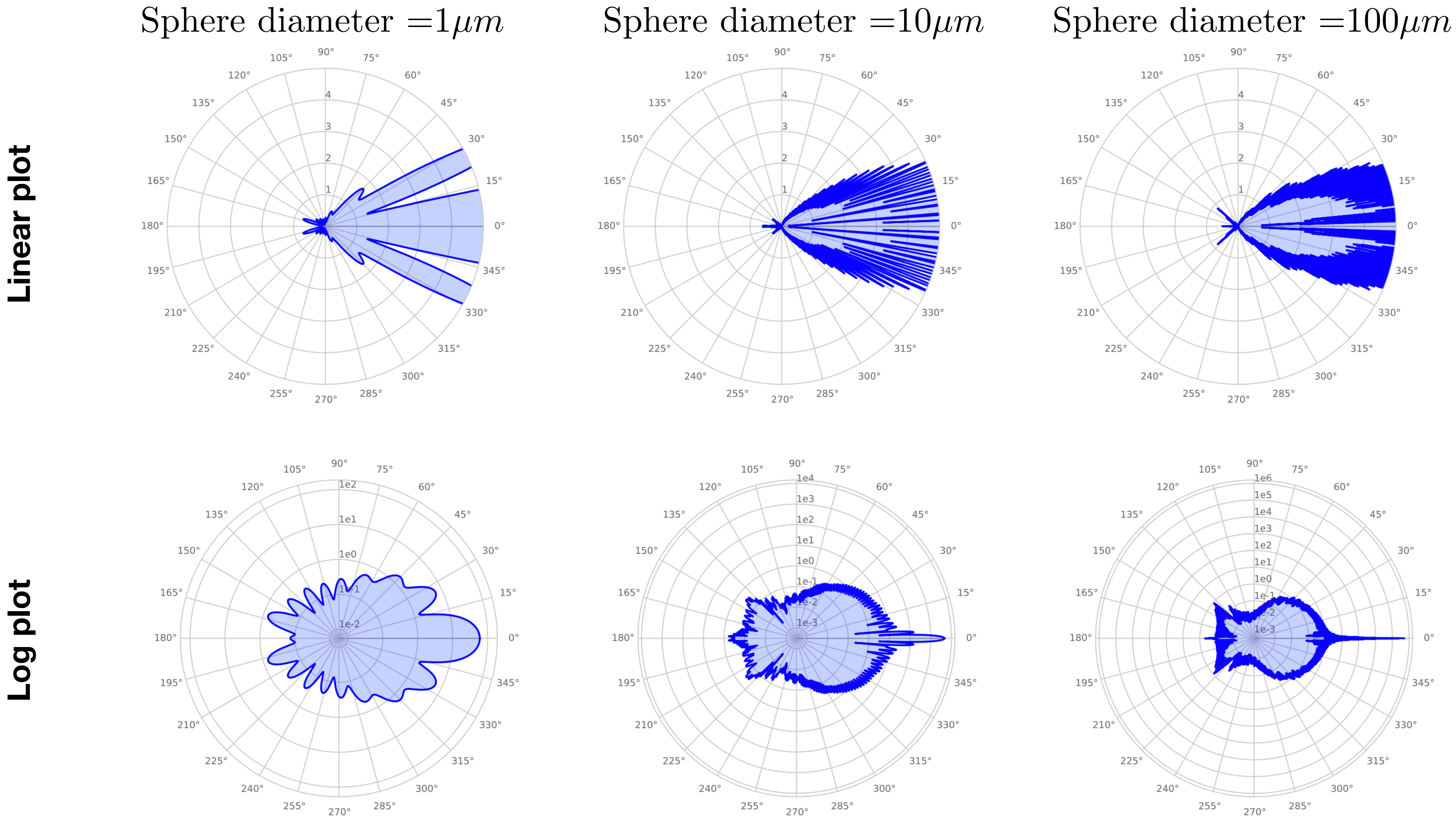
Sphere diameter $=10\mu m$



Sphere diameter $=100\mu m$



Lorenz-Mie Scattering



Sphere diameter $=1\mu m$

Sphere diameter $=10\mu m$

Sphere diameter $=100\mu m$

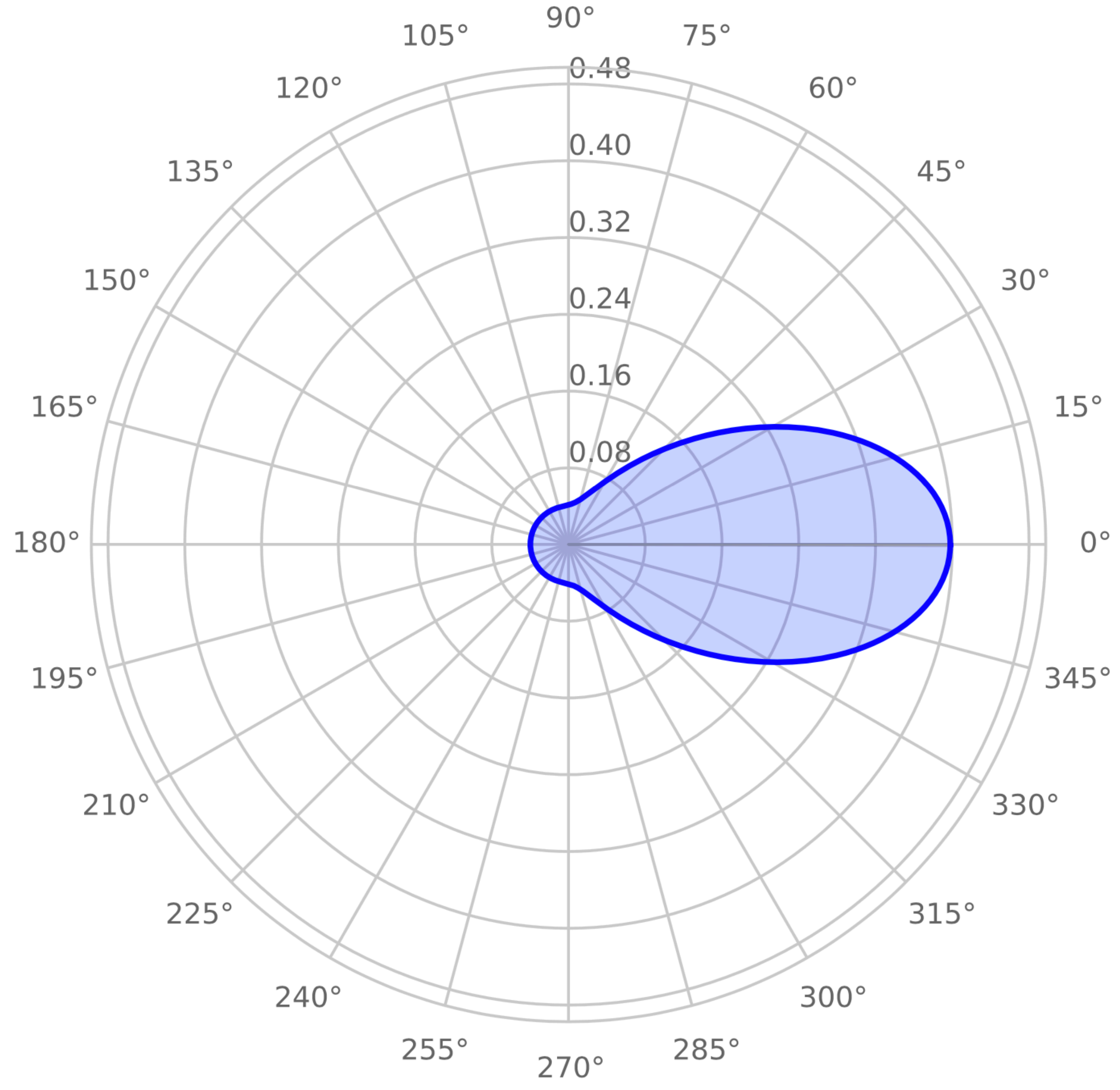
Linear plot

Log plot

Lorenz-Mie Approximations

Hazy atmosphere

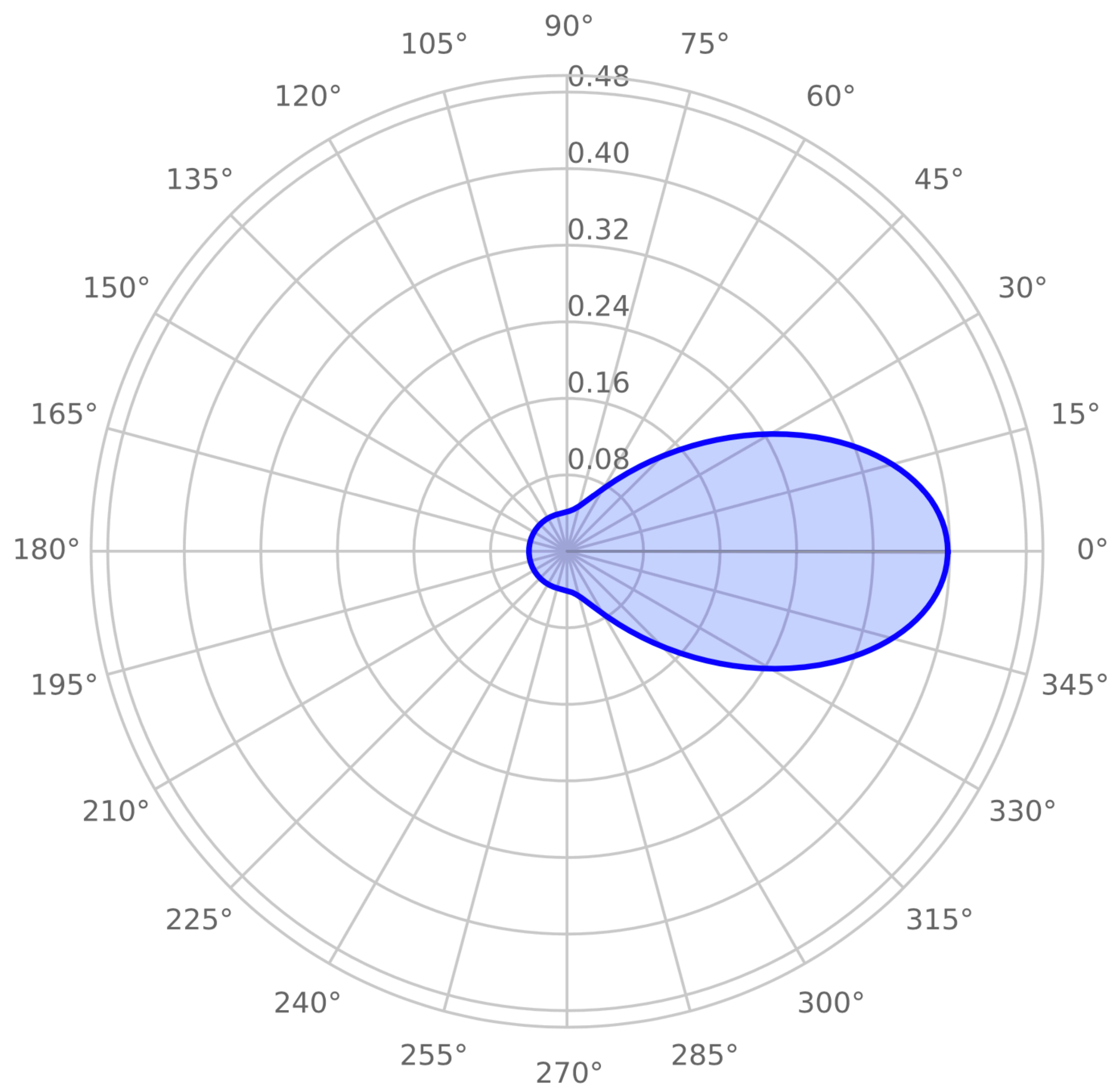
$$f_p^{\text{hazy}}(\theta) = \frac{1}{4\pi}\left(5 + \left(\frac{1+\cos\theta}{2}\right)^8\right)$$



Lorenz-Mie Approximations

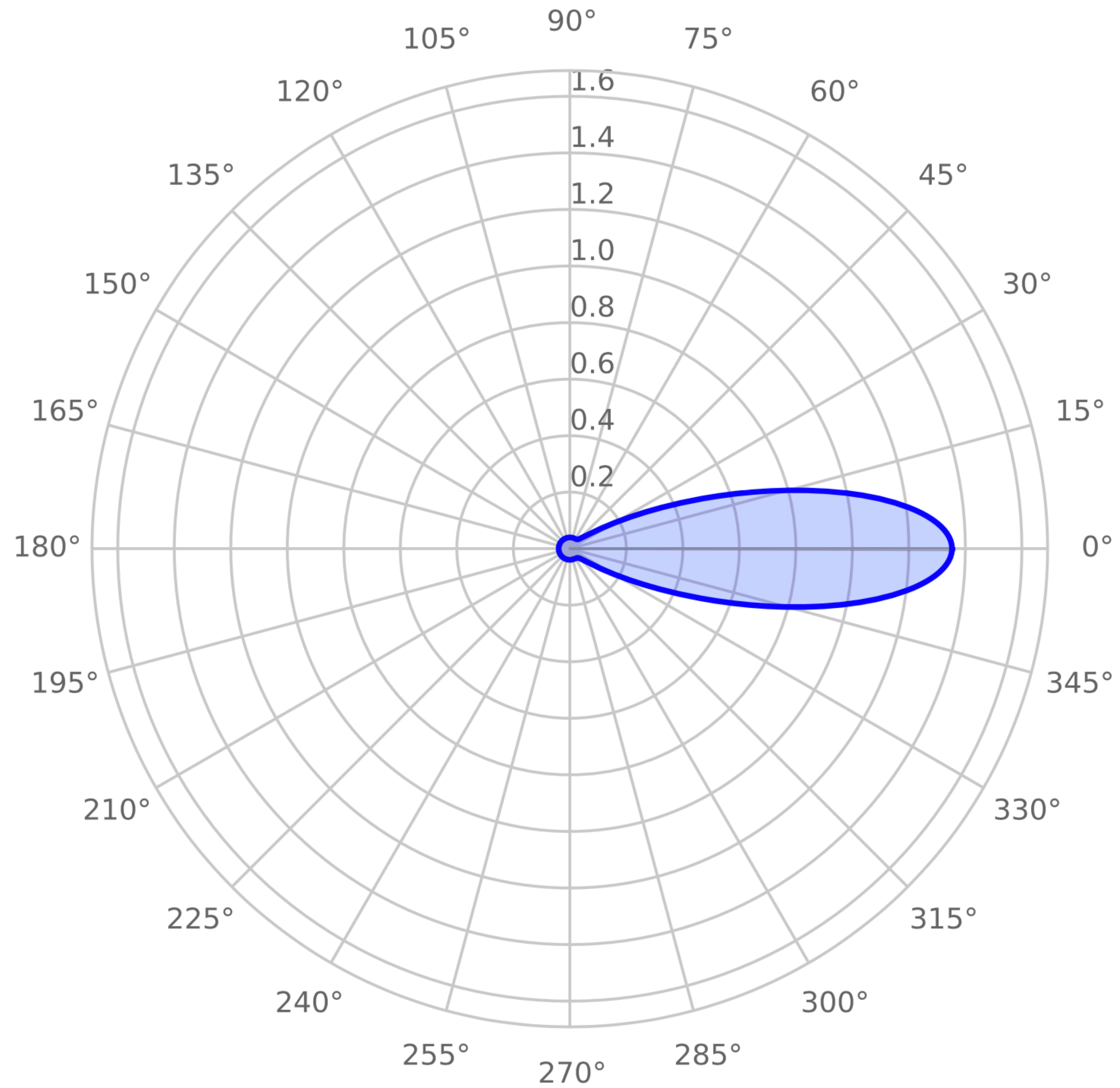
Hazy atmosphere

$$f_p^{\text{hazy}}(\theta) = \frac{1}{4\pi}\left(5 + \left(\frac{1+\cos\theta}{2}\right)^8\right)$$

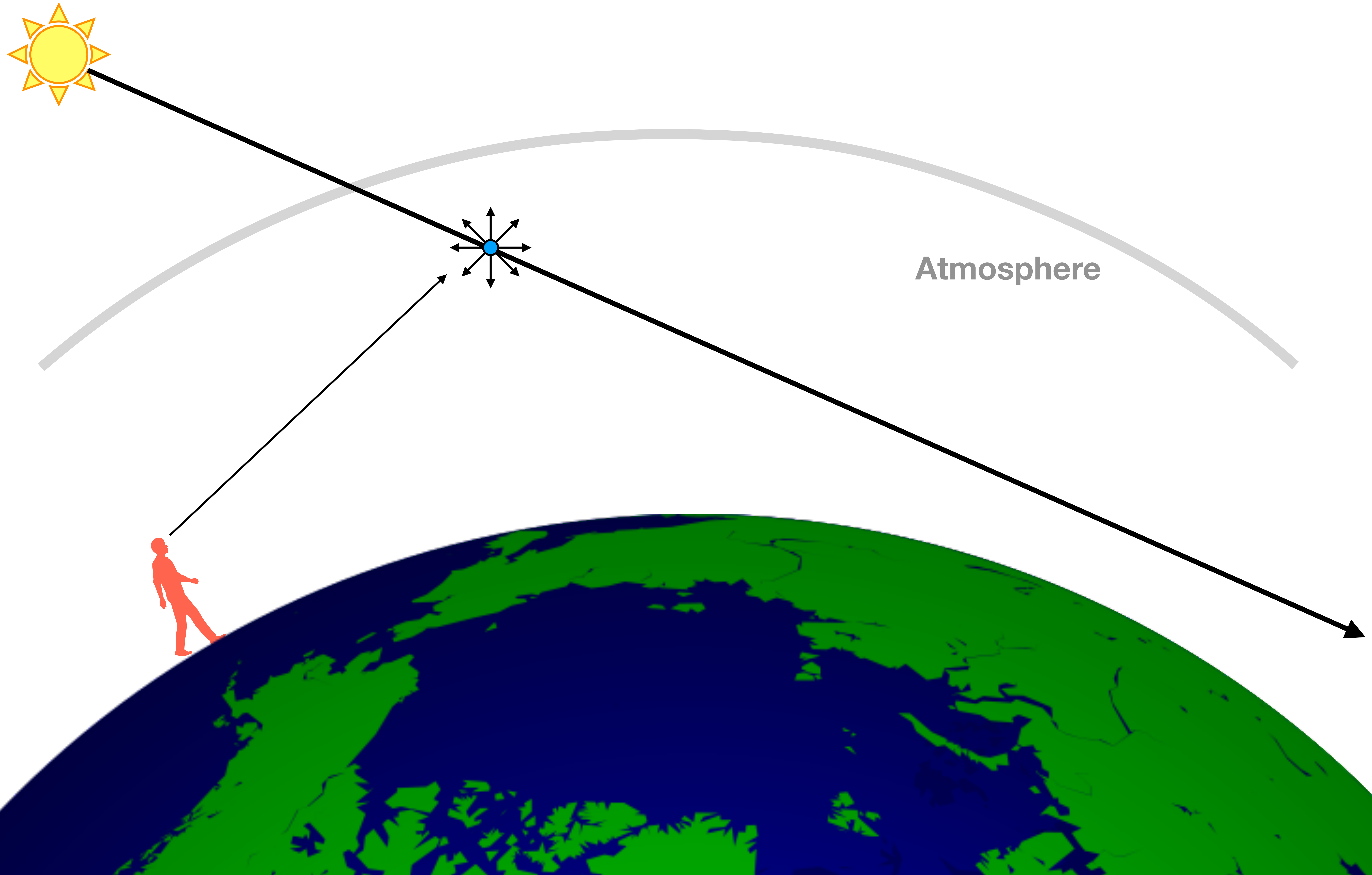


Murky atmosphere

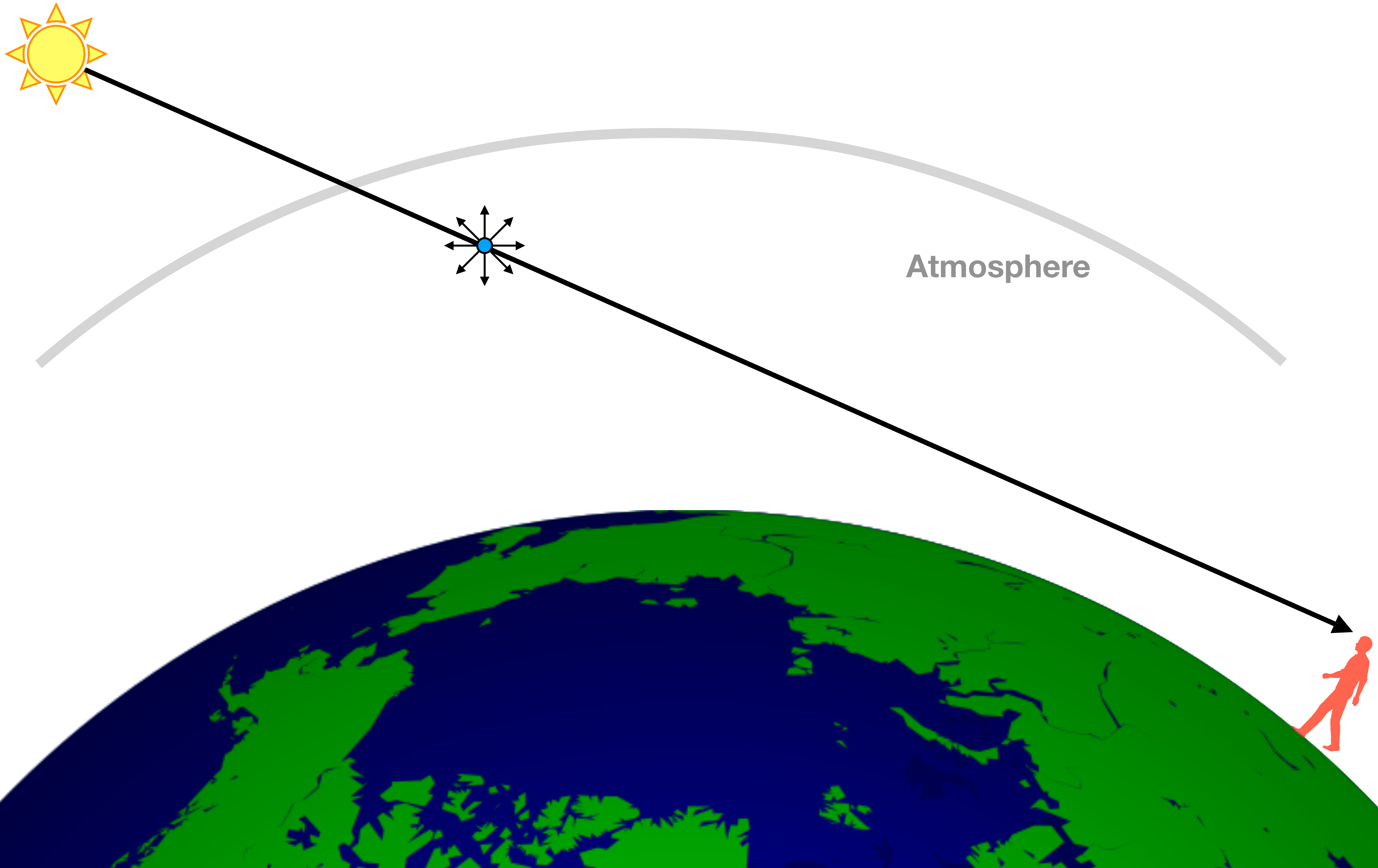
$$f_p^{\text{murky}}(\theta) = \frac{1}{4\pi}\left(17 + \left(\frac{1+\cos\theta}{2}\right)^{32}\right)$$



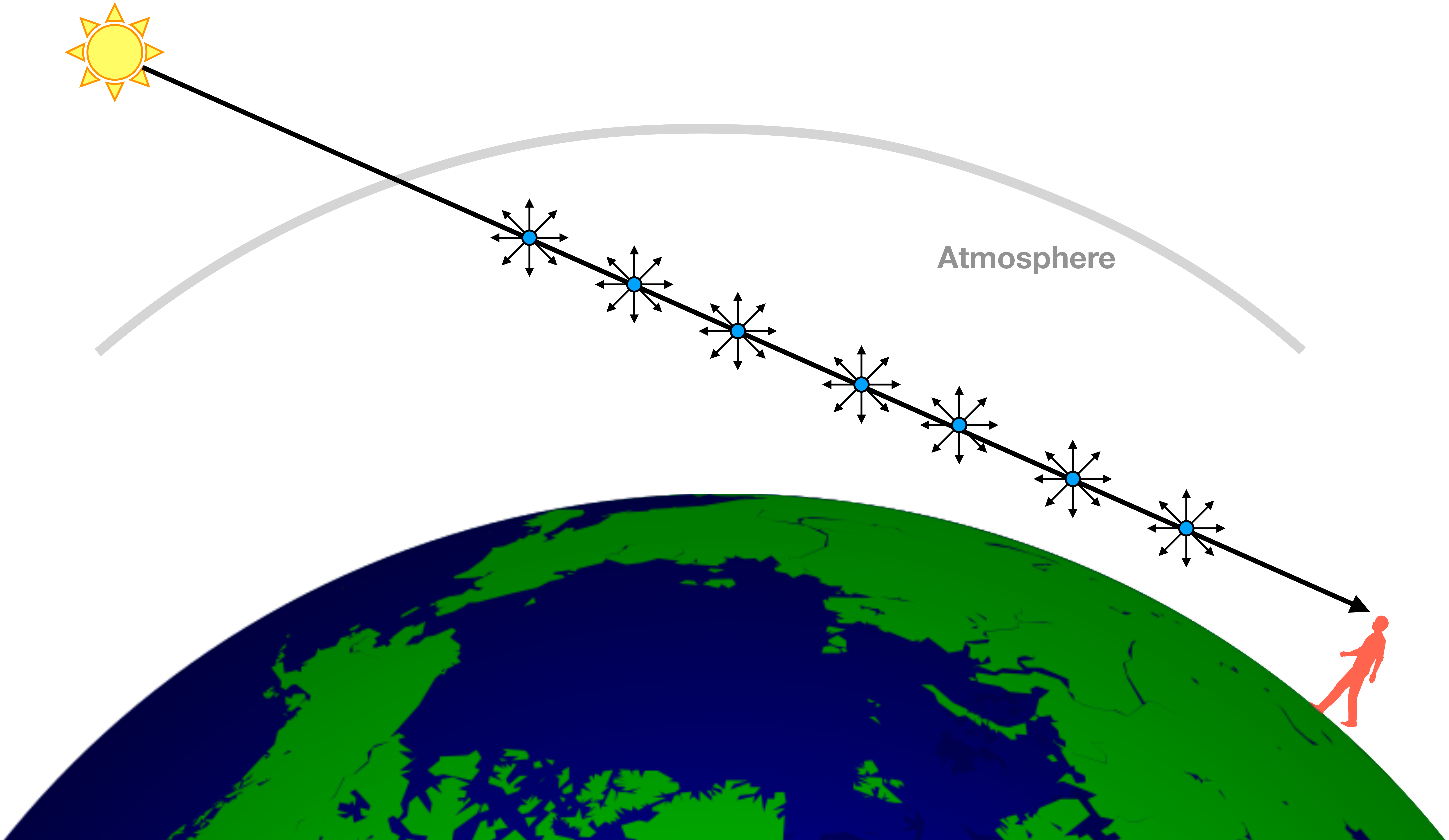
Why is the Sky Blue?



Why is the Sunset Red?



Why is the Sunset Red?



Rayleigh Scattering

forbes.com

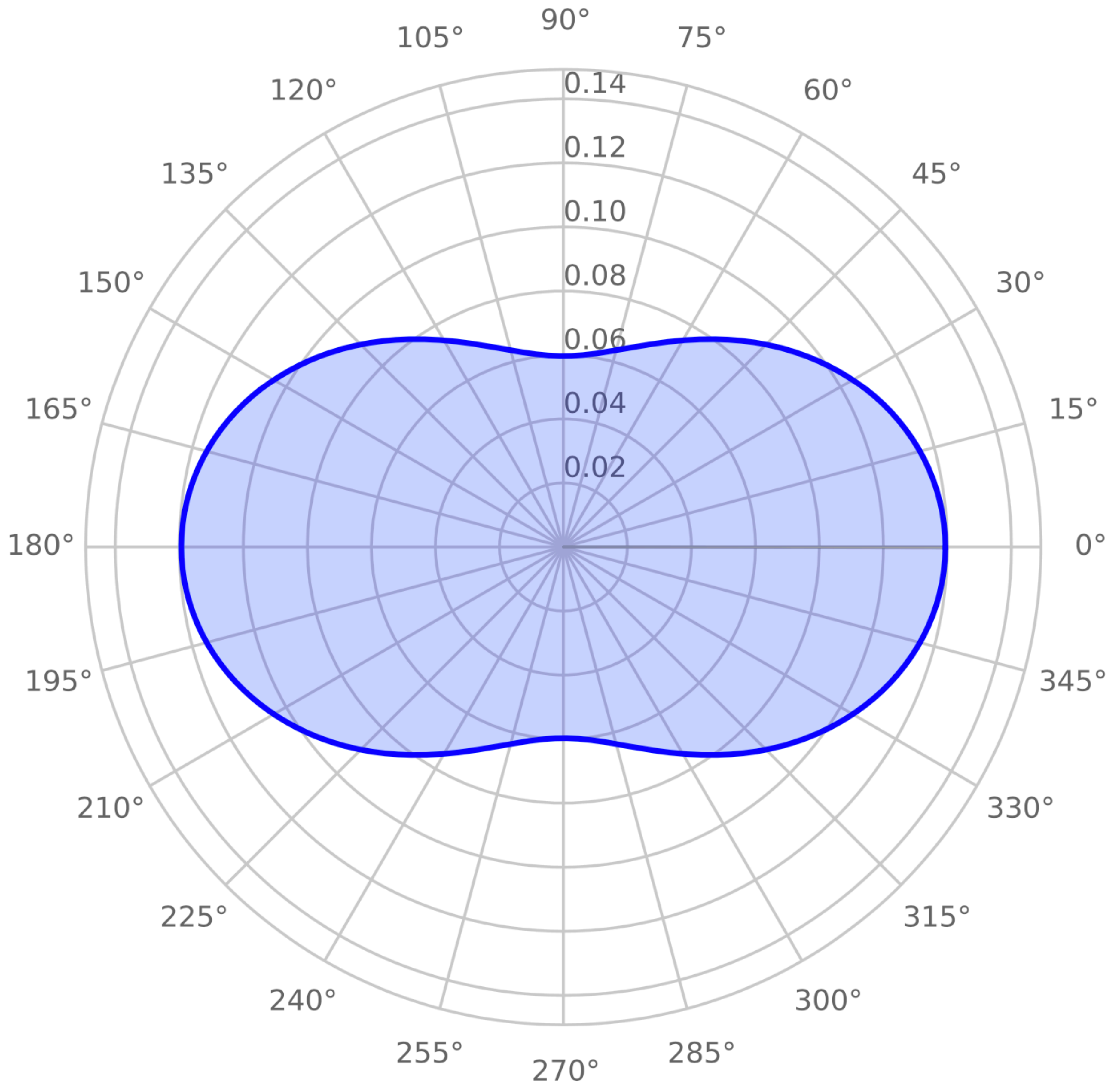
Rayleigh Scattering

Approximation of Lorenz-Mie for tiny particles (scatterers) that are typically smaller than 1/10th the wavelength of visible light

Used for atmospheric scattering, gasses, transparent solids

Highly wavelength dependent

Rayleigh Phase Function



$$f_p^{\mathrm{Rayleigh}}(\theta) = \frac{3}{16\pi}\left(1 + \cos^2\theta\right)$$

Scattering at right angles is half as likely as scattering forward or backward

Rayleigh Scattering

$$\beta_s^{\text{Rayleigh}}(\lambda, d, \eta, \rho) = \rho \frac{2\pi^5 d^6}{3\lambda^4} \left(\frac{\eta^2 - 1}{\eta^2 + 2} \right)^2$$

Rayleigh Scattering

Wavelength

$$\beta_s^{\text{Rayleigh}}(\lambda, d, \eta, \rho) = \rho \frac{2\pi^5 d^6}{3\lambda^4} \left(\frac{\eta^2 - 1}{\eta^2 + 2} \right)^2$$

Rayleigh Scattering

Wavelength

$$\beta_s^{\text{Rayleigh}}(\lambda, d, \eta, \rho) = \rho \frac{2\pi^5 d^6}{3\lambda^4} \left(\frac{\eta^2 - 1}{\eta^2 + 2} \right)^2$$

Diameter of particles

Rayleigh Scattering

Wavelength

Index of refraction

$$\beta_s^{\text{Rayleigh}}(\lambda, d, \eta, \rho) = \rho \frac{2\pi^5 d^6}{3\lambda^4} \left(\frac{\eta^2 - 1}{\eta^2 + 2} \right)^2$$

Diameter of particles

Rayleigh Scattering

Wavelength

Index of refraction

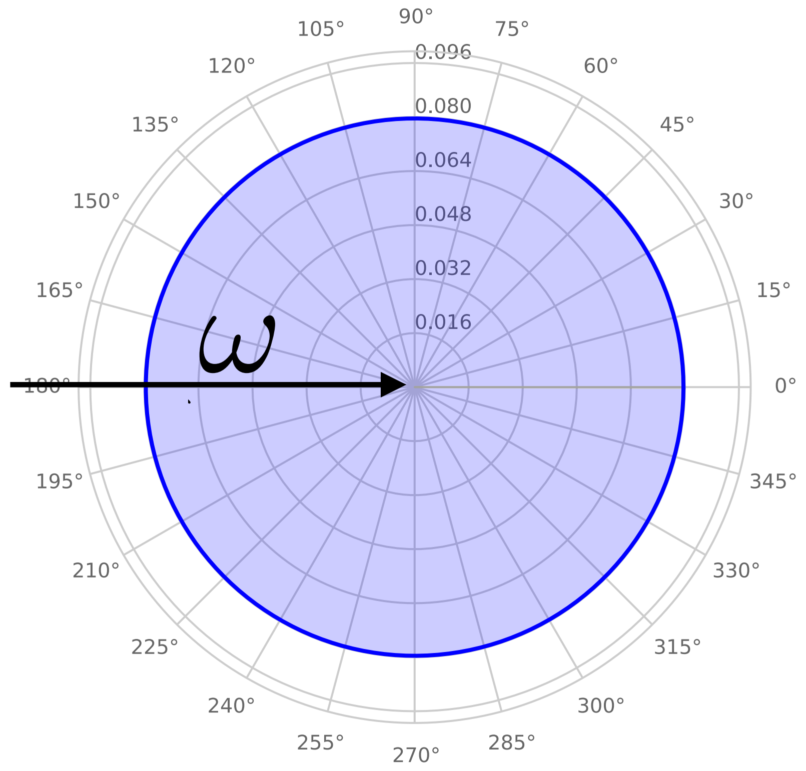
$$\beta_s^{\text{Rayleigh}}(\lambda, d, \eta, \rho) = \rho \frac{2\pi^5 d^6}{3\lambda^4} \left(\frac{\eta^2 - 1}{\eta^2 + 2} \right)^2$$

Density of particles

Diameter of particles

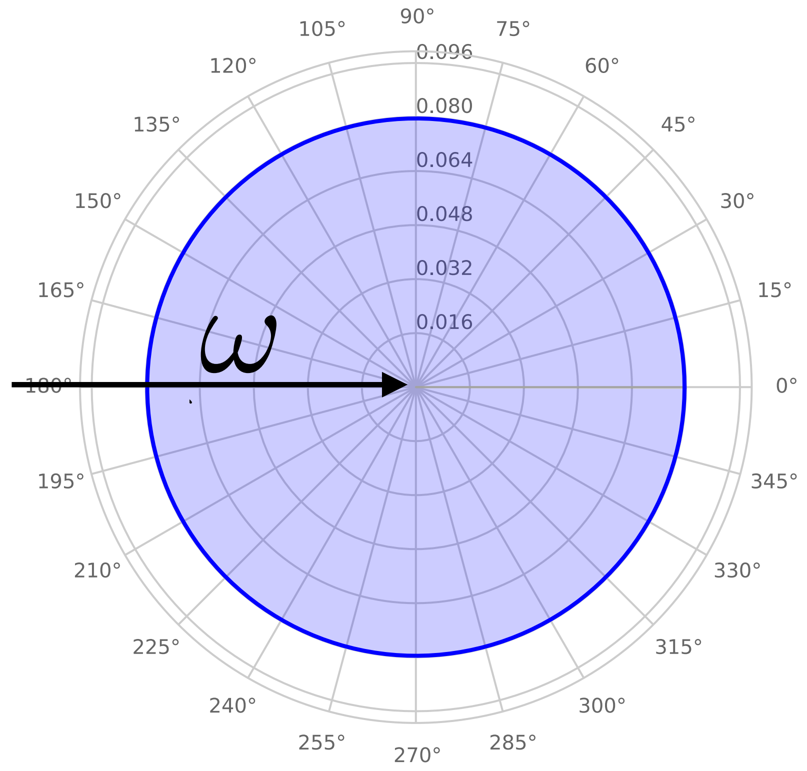
Recap: Phase Functions

Recap: Phase Functions

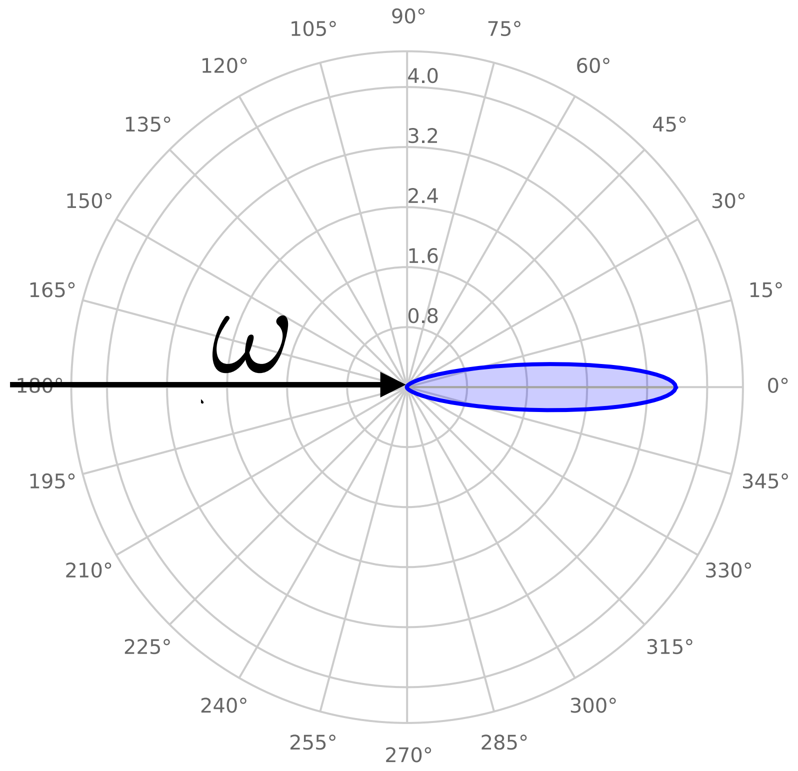


Isotropic

Recap: Phase Functions

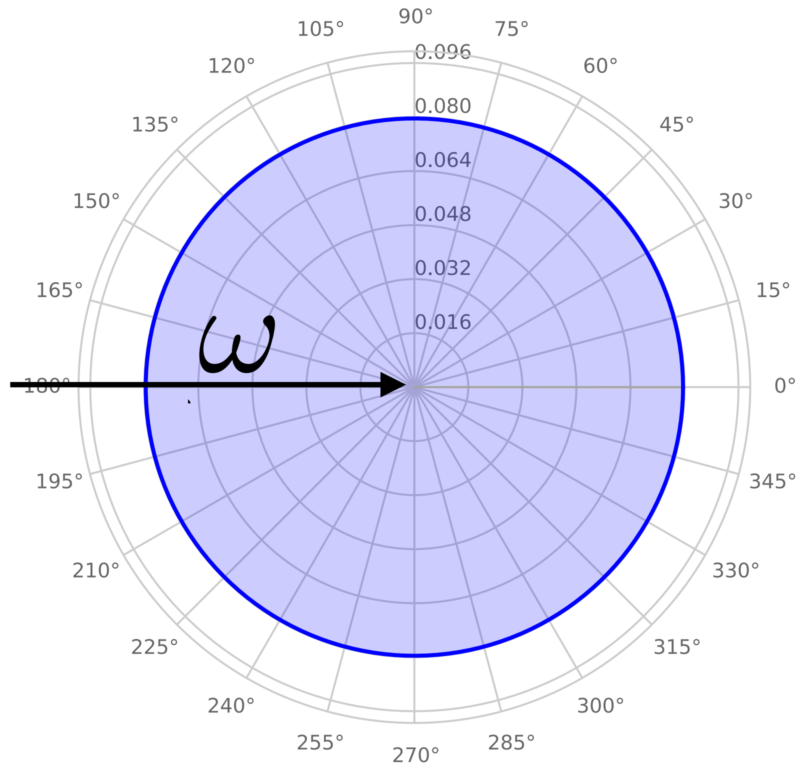


Isotropic

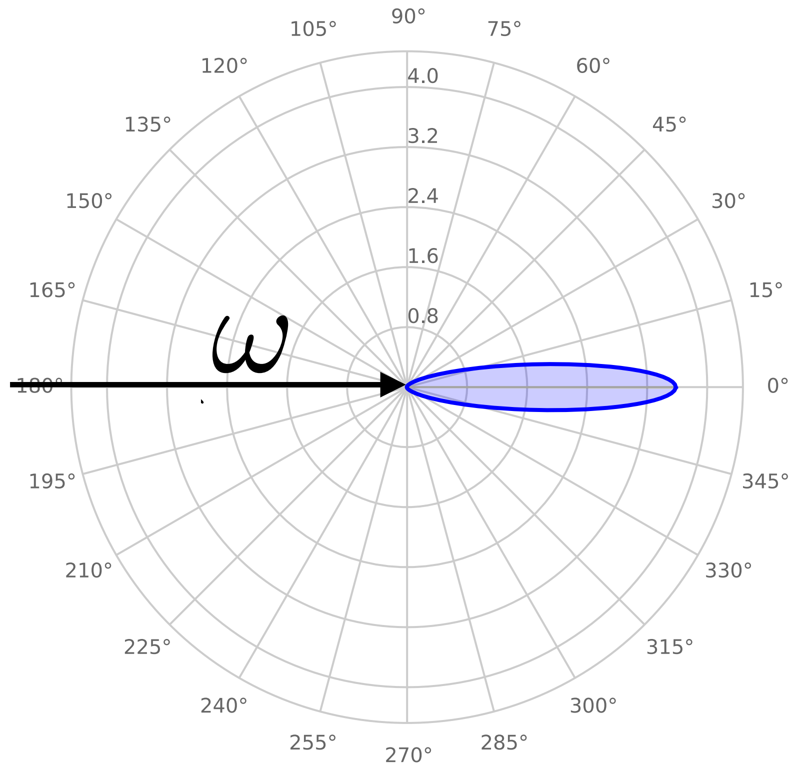


Henyey-Greenstein

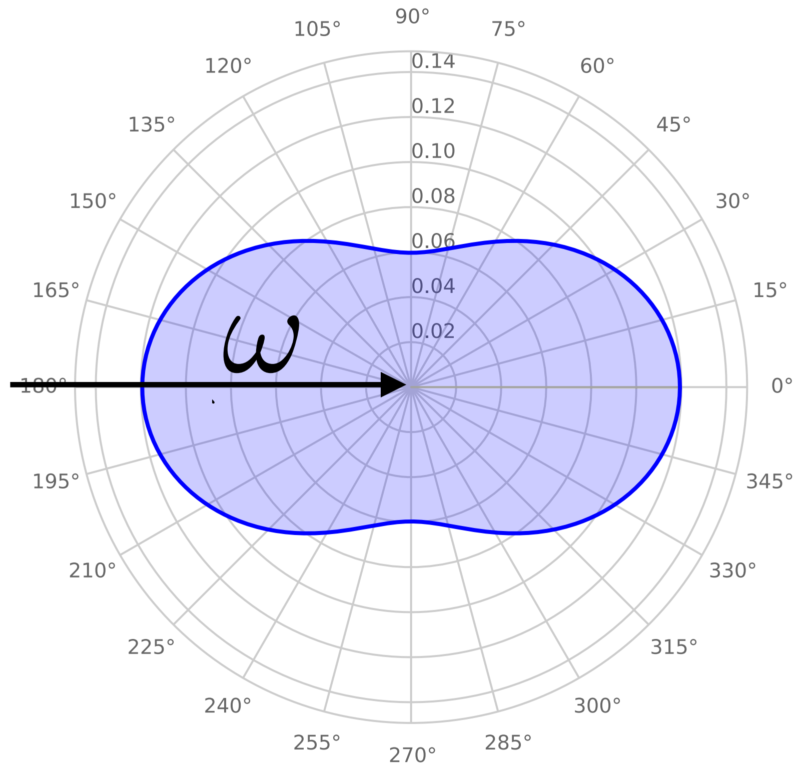
Recap: Phase Functions



Isotropic

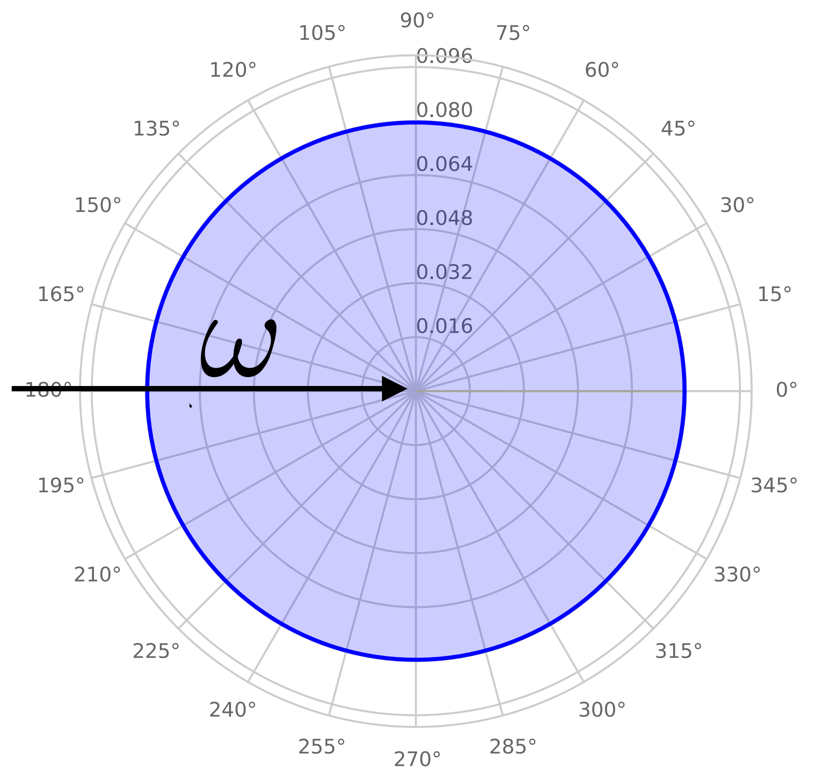


Henyey-Greenstein

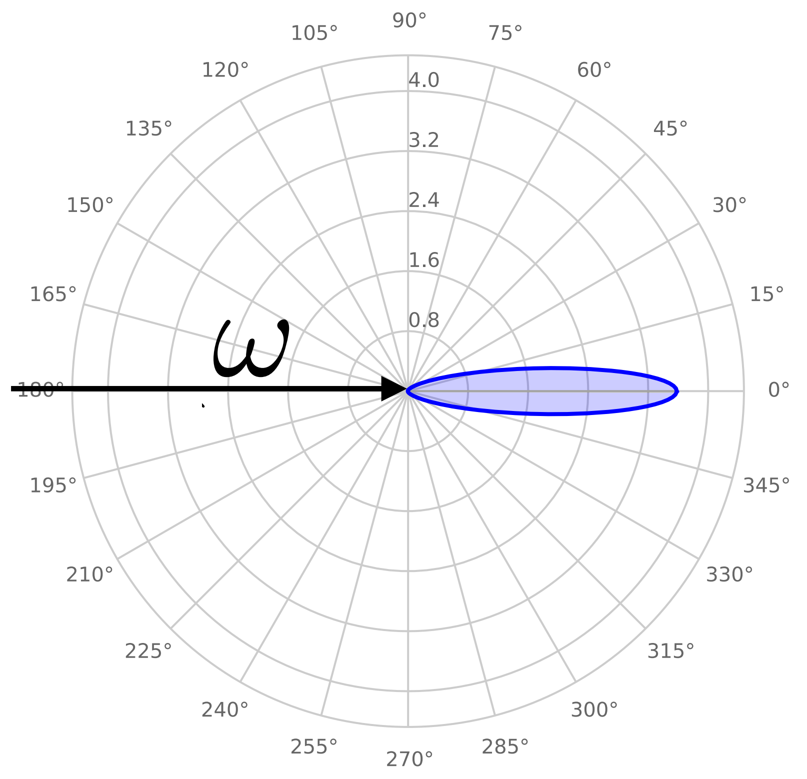


Rayleigh

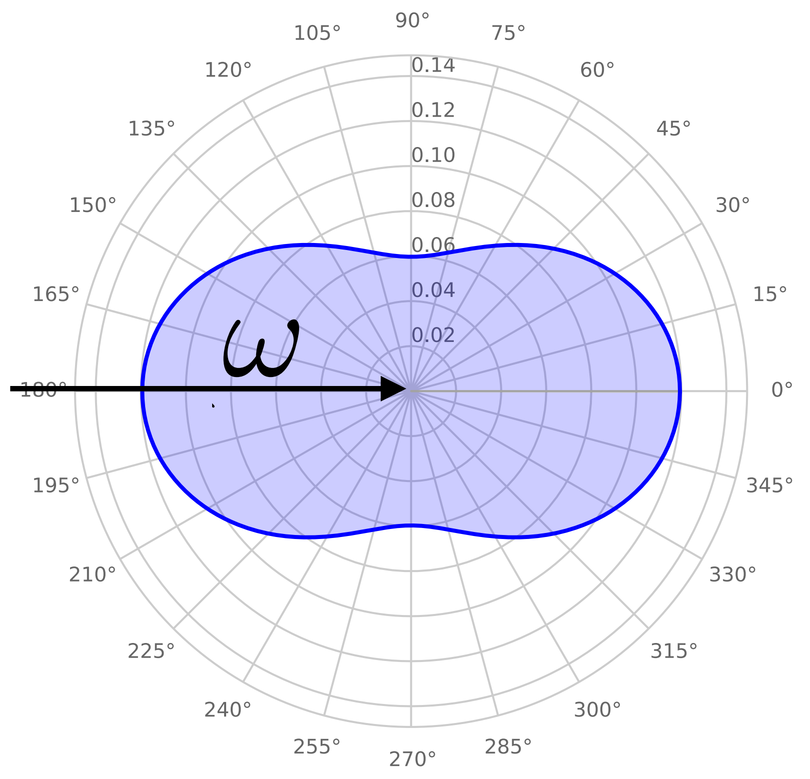
Recap: Phase Functions



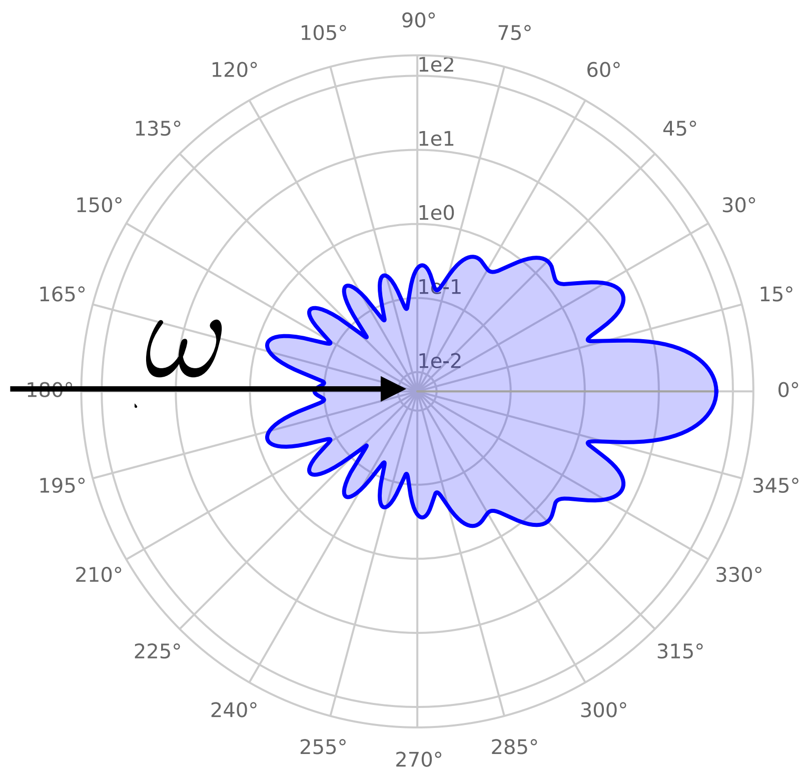
Isotropic



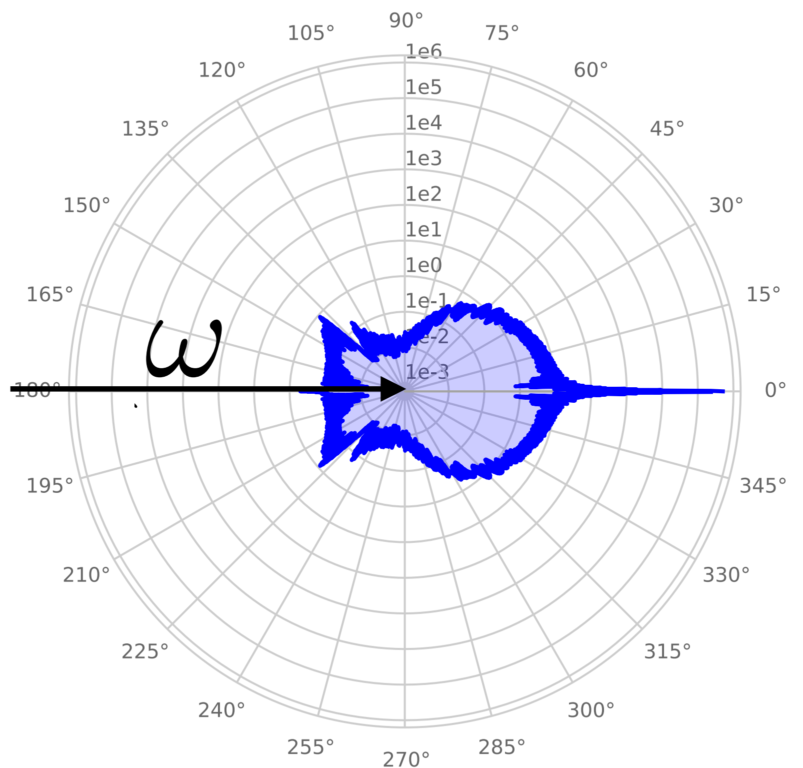
Henyey-Greenstein



Rayleigh



Lorenz-Mie small particles



Lorenz-Mie large particles

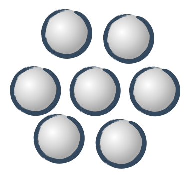
Anisotropy: Phase Function vs. Medium

Isotropic Medium

Slide after Jan Novak

Anisotropy: Phase Function vs. Medium

Isotropic Medium

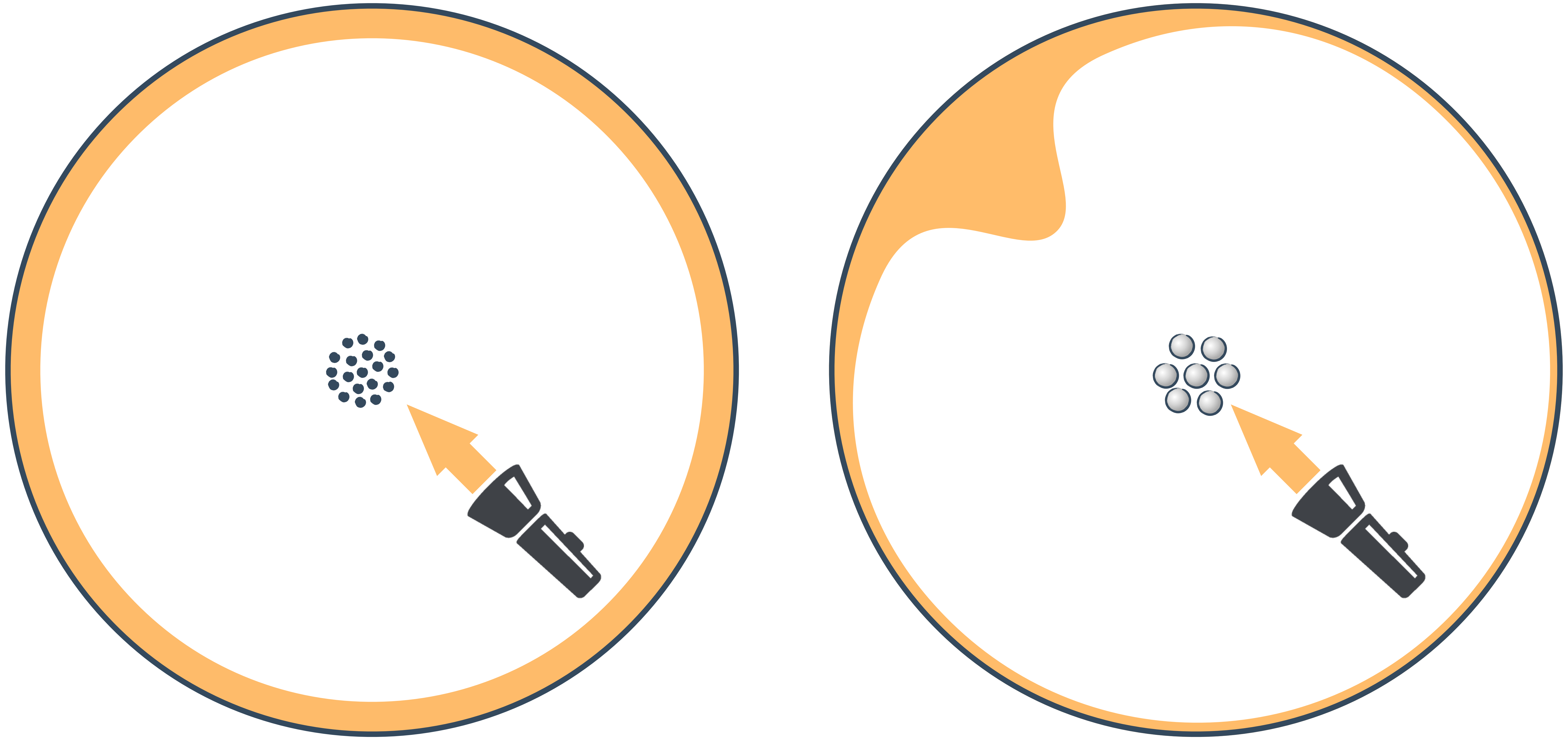
Isotropic phase function	Anisotropic phase function
	

Slide after Jan Novak

Anisotropy: Phase Function vs. Medium

Isotropic Medium

Isotropic phase function	Anisotropic phase function

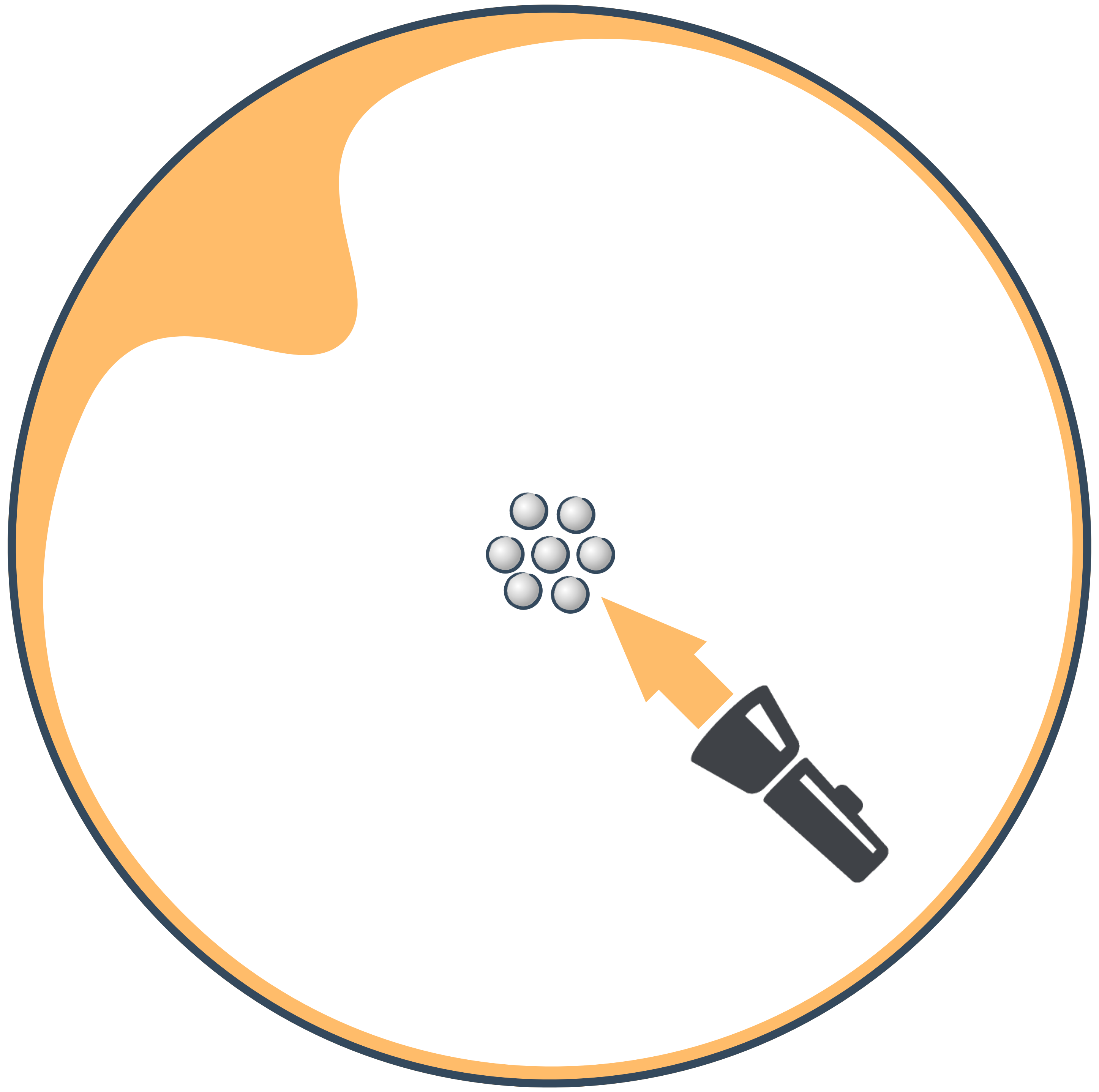


Slide after Jan Novak

Anisotropy: Phase Function vs. Medium

Isotropic Medium

Isotropic phase function | Anisotropic phase function



Slide after Jan Novak

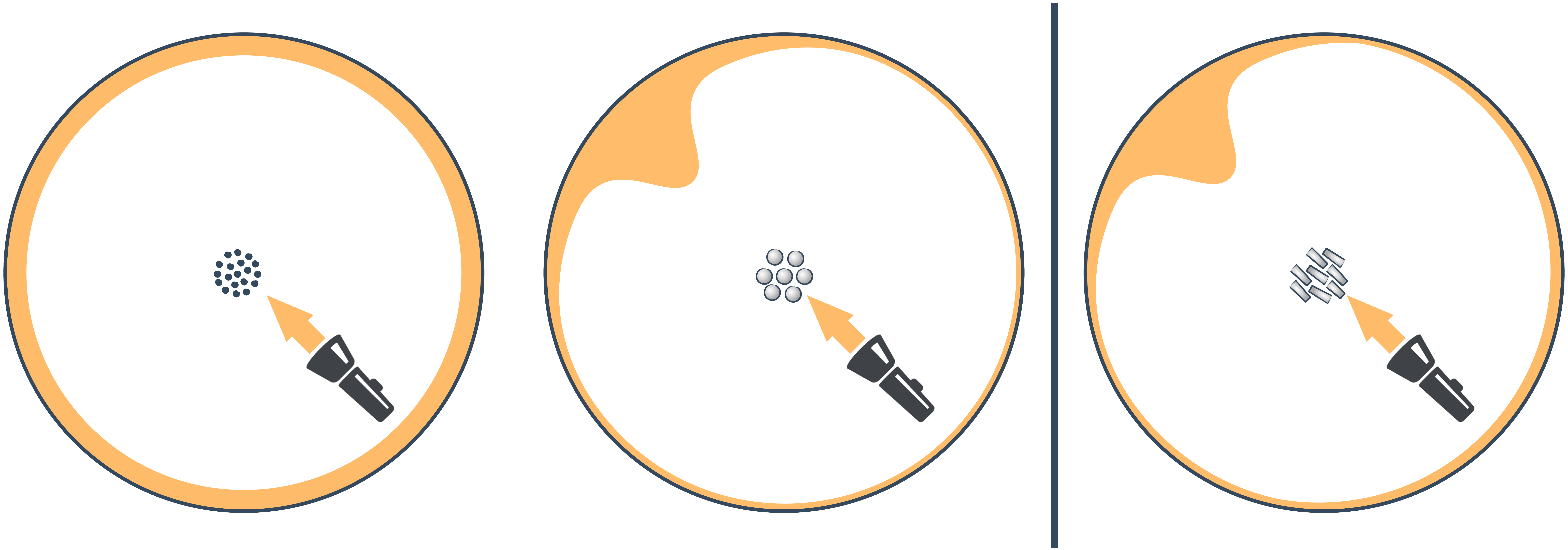
Anisotropy: Phase Function vs. Medium

Isotropic Medium

Isotropic phase function

Anisotropic phase function

Anisotropic Medium



Slide after Jan Novak

Recap: Media Properties

Given:

Absorption coefficient	$\sigma_a(\mathbf{x})$	$[m^{-1}]$
Scattering coefficient	$\sigma_s(\mathbf{x})$	$[m^{-1}]$
Phase function	$f_p(\mathbf{x}, \vec{\omega}, \vec{\omega}')$	$[sr^{-1}]$

Recap: Media Properties

Given:

Absorption coefficient	$\sigma_a(\mathbf{x})$	$[m^{-1}]$
Scattering coefficient	$\sigma_s(\mathbf{x})$	$[m^{-1}]$
Phase function	$f_p(\mathbf{x}, \vec{\omega}, \vec{\omega}')$	$[sr^{-1}]$

Derived:

Extinction coefficient	$\sigma_t(\mathbf{x}) = \sigma_a(\mathbf{x}) + \sigma_s(\mathbf{x})$	$[m^{-1}]$
Albedo	$\alpha(\mathbf{x}) = \sigma_s(\mathbf{x})/\sigma_t(\mathbf{x})$	$[\text{None}]$
Mean-free path	$1/\sigma_t(\mathbf{x})$	$[m]$
Transmittance	$T_r(\mathbf{x}, \mathbf{y}) = e^{-\int_0^{\|\|\mathbf{x}-\mathbf{y}\|\|} \sigma_t(t)dt}$	$[\text{None}]$

For Homogeneous Isotropic Medium

Given:

Absorption coefficient	σ_a	$[m^{-1}]$
Scattering coefficient	σ_s	$[m^{-1}]$
Phase function	$\frac{1}{4\pi}$	$[sr^{-1}]$

Derived:

Extinction coefficient	$\sigma_t = \sigma_a + \sigma_s$	$[m^{-1}]$
Albedo	$\alpha = \sigma_s / \sigma_t$	$[\text{None}]$
Mean-free path	$1/\sigma_t$	$[m]$
Transmittance	$T_r(\mathbf{x}, \mathbf{y}) = e^{-\sigma_t \|\|\mathbf{x} - \mathbf{y}\|\|}$	$[\text{None}]$

Solving the Volumetric Rendering Equation

Complexity

Homogeneous vs. Heterogeneous

Scattering

- none
- fake
- single scattering
- multiple scattering

Volumetric Rendering Equation

$$
\begin{aligned}
L(\mathbf{x},\vec{\omega}) &= T_r(\mathbf{x},\mathbf{x}_z)L(\mathbf{x}_z,\vec{\omega}) \\
&\quad + \int_0^z T_r(\mathbf{x},\mathbf{x}_t)\sigma_a(\mathbf{x}_t)L_e(\mathbf{x}_t,\vec{\omega})dt \\
&\quad + \int_0^z T_r(\mathbf{x},\mathbf{x}_t)\sigma_s(\mathbf{x}_t)\int_{S^2} f_p(\mathbf{x}_t,\vec{\omega}',\vec{\omega})L_i(\mathbf{x}_t,\vec{\omega}')d\omega' dt
\end{aligned}
$$

Volumetric Rendering Equation

$$
\begin{aligned}
L(\mathbf{x}, \vec{\omega}) = & \; T_r(\mathbf{x}, \mathbf{x}_z) L(\mathbf{x}_z, \vec{\omega}) \\
& + \int_0^z T_r(\mathbf{x}, \mathbf{x}_t) \sigma_a(\mathbf{x}_t) L_e(\mathbf{x}_t, \vec{\omega}) dt \\
& + \int_0^z T_r(\mathbf{x}, \mathbf{x}_t) \sigma_s(\mathbf{x}_t) \int_{S^2} f_p(\mathbf{x}_t, \vec{\omega}', \vec{\omega}) L_i(\mathbf{x}_t, \vec{\omega}') d\omega' dt
\end{aligned}
$$

Attenuated background radiance

Volumetric Rendering Equation

$$L(\mathbf{x},\vec{\omega}) = T_r(\mathbf{x},\mathbf{x}_z)L(\mathbf{x}_z,\vec{\omega})$$

Attenuated background radiance

$$+\int_0^z T_r(\mathbf{x},\mathbf{x}_t)\sigma_a(\mathbf{x}_t)L_e(\mathbf{x}_t,\vec{\omega})dt$$

Accumulated emitted radiance

$$+\int_0^z T_r(\mathbf{x},\mathbf{x}_t)\sigma_s(\mathbf{x}_t)\int_{S^2} f_p(\mathbf{x}_t,\vec{\omega}',\vec{\omega})L_i(\mathbf{x}_t,\vec{\omega}')d\omega' dt$$

Volumetric Rendering Equation

$$L(\mathbf{x},\vec{\omega}) = T_r(\mathbf{x},\mathbf{x}_z)L(\mathbf{x}_z,\vec{\omega})$$

Attenuated background radiance

$$+\int_0^z T_r(\mathbf{x},\mathbf{x}_t)\sigma_a(\mathbf{x}_t)L_e(\mathbf{x}_t,\vec{\omega})dt$$

Accumulated emitted radiance

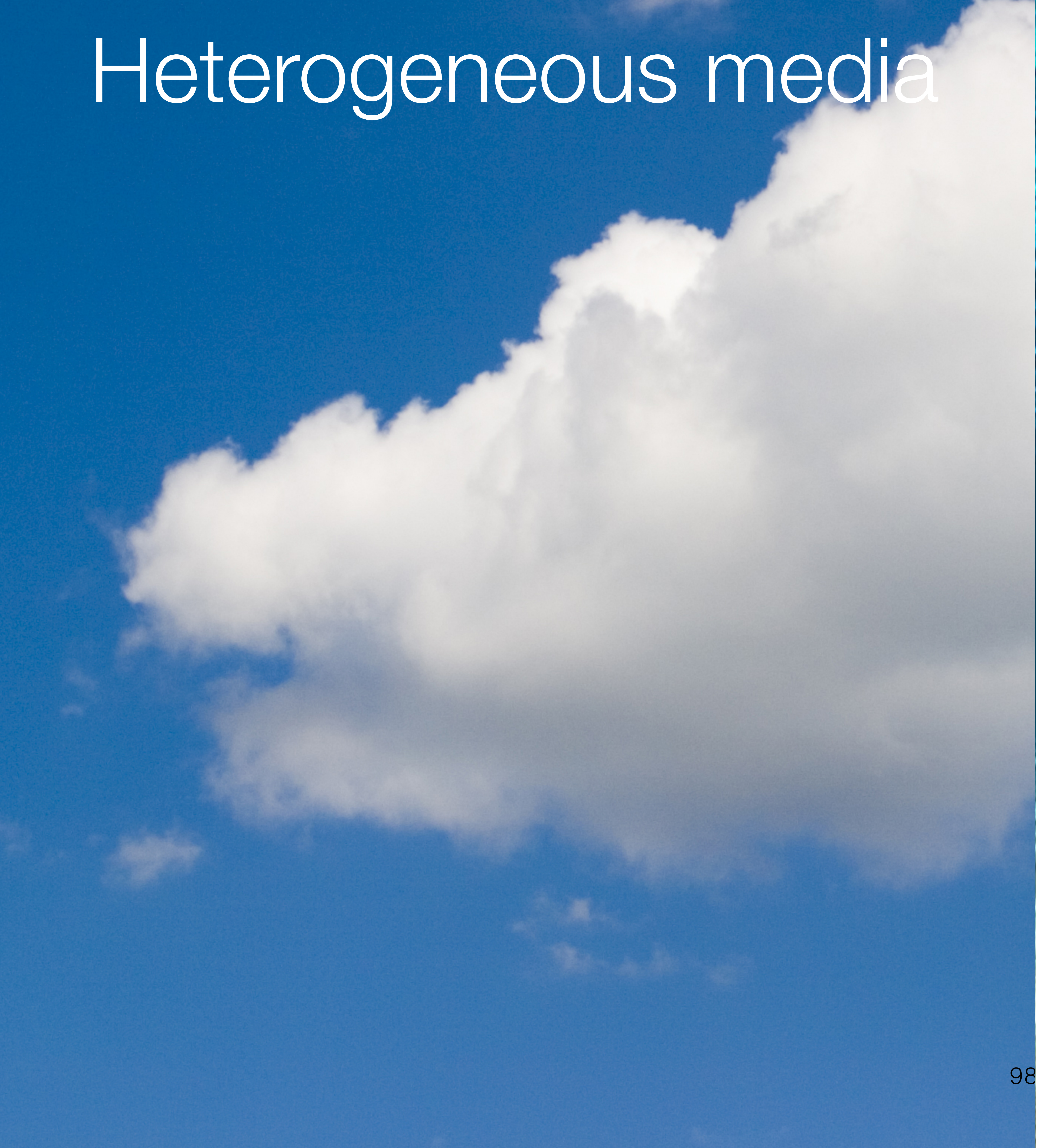
$$+\int_0^z T_r(\mathbf{x},\mathbf{x}_t)\sigma_s(\mathbf{x}_t)\int_{S^2} f_p(\mathbf{x}_t,\vec{\omega}',\vec{\omega})L_i(\mathbf{x}_t,\vec{\omega}')d\omega' dt$$

Accumulated in-scattered radiance

Heterogeneous/Homogeneous media

Homogeneous media

Heterogeneous media



Homogeneous media

Participating Media: Heterogeneous

Participating Media: Heterogeneous

Participating Media: Heterogeneous

$$L(\mathbf{x},\vec{\omega}) = \int_0^s T_r(\mathbf{x} \leftrightarrow \mathbf{x}_t)\sigma_s(\mathbf{x}_t)L_i(\mathbf{x}_t,\vec{\omega})dt + T_r(\mathbf{x} \leftrightarrow \mathbf{x}_s)L(\mathbf{x}_s,\vec{\omega})$$

Participating Media: Heterogeneous

$$L(\mathbf{x}, \vec{\omega}) = \int_0^s T_r(\mathbf{x} \leftrightarrow \mathbf{x}_t)\sigma_s(\mathbf{x}_t)L_i(\mathbf{x}_t, \vec{\omega})dt + T_r(\mathbf{x} \leftrightarrow \mathbf{x}_s)L(\mathbf{x}_s, \vec{\omega})$$

Participating Media: Homogeneous

$$L(\mathbf{x},\vec{\omega}) = \int_0^s T_r(\mathbf{x} \leftrightarrow \mathbf{x}_t)\sigma_s(\mathbf{x}_t)L_i(\mathbf{x}_t,\vec{\omega})dt + T_r(\mathbf{x} \leftrightarrow \mathbf{x}_s)L(\mathbf{x}_s,\vec{\omega})$$

Participating Media: Homogeneous

$$L(\mathbf{x},\vec{\omega}) = \int_0^s T_r(\mathbf{x} \leftrightarrow \mathbf{x}_t)\sigma_s(\mathbf{x}_t)L_i(\mathbf{x}_t,\vec{\omega})dt + T_r(\mathbf{x} \leftrightarrow \mathbf{x}_s)L(\mathbf{x}_s,\vec{\omega})$$

$$L(\mathbf{x},\vec{\omega}) = \sigma_s \int_0^s T_r(\mathbf{x} \leftrightarrow \mathbf{x}_t)L_i(\mathbf{x}_t,\vec{\omega})dt + T_r(\mathbf{x} \leftrightarrow \mathbf{x}_s)L(\mathbf{x}_s,\vec{\omega})$$

Participating Media: Homogeneous

$$L(\mathbf{x}, \vec{\omega}) = \int_0^s T_r(\mathbf{x} \leftrightarrow \mathbf{x}_t)\sigma_s(\mathbf{x}_t)L_i(\mathbf{x}_t, \vec{\omega})dt + T_r(\mathbf{x} \leftrightarrow \mathbf{x}_s)L(\mathbf{x}_s, \vec{\omega})$$

$$L(\mathbf{x}, \vec{\omega}) = \sigma_s \int_0^s T_r(\mathbf{x} \leftrightarrow \mathbf{x}_t)L_i(\mathbf{x}_t, \vec{\omega})dt + T_r(\mathbf{x} \leftrightarrow \mathbf{x}_s)L(\mathbf{x}_s, \vec{\omega})$$

Participating Media: Homogeneous

$$L(\mathbf{x},\vec{\omega}) = \int_0^s T_r(\mathbf{x} \leftrightarrow \mathbf{x}_t)\sigma_s(\mathbf{x}_t)L_i(\mathbf{x}_t,\vec{\omega})dt + T_r(\mathbf{x} \leftrightarrow \mathbf{x}_s)L(\mathbf{x}_s,\vec{\omega})$$

$$L(\mathbf{x},\vec{\omega}) = \sigma_s \int_0^s T_r(\mathbf{x} \leftrightarrow \mathbf{x}_t)L_i(\mathbf{x}_t,\vec{\omega})dt + T_r(\mathbf{x} \leftrightarrow \mathbf{x}_s)L(\mathbf{x}_s,\vec{\omega})$$

$$L(\mathbf{x},\vec{\omega}) = \sigma_s \int_0^s e^{-t\sigma_t}L_i(\mathbf{x}_t,\vec{\omega})dt + e^{-s\sigma_t}L(\mathbf{x}_s,\vec{\omega})$$

Participating Media: Homogeneous

$$L(\mathbf{x},\vec{\omega}) = \int_0^s T_r(\mathbf{x} \leftrightarrow \mathbf{x}_t)\sigma_s(\mathbf{x}_t)L_i(\mathbf{x}_t,\vec{\omega})dt + T_r(\mathbf{x} \leftrightarrow \mathbf{x}_s)L(\mathbf{x}_s,\vec{\omega})$$

$$L(\mathbf{x},\vec{\omega}) = \sigma_s \int_0^s T_r(\mathbf{x} \leftrightarrow \mathbf{x}_t)L_i(\mathbf{x}_t,\vec{\omega})dt + T_r(\mathbf{x} \leftrightarrow \mathbf{x}_s)L(\mathbf{x}_s,\vec{\omega})$$

$$L(\mathbf{x},\vec{\omega}) = \sigma_s \int_0^s e^{-t\sigma_t}L_i(\mathbf{x}_t,\vec{\omega})dt + e^{-s\sigma_t}L(\mathbf{x}_s,\vec{\omega})$$

Participating Media: Homogeneous

$$L(\mathbf{x},\vec{\omega}) = \sigma_s \int_0^s e^{-t\sigma_t} L_i(\mathbf{x}_t,\vec{\omega})dt + e^{-s\sigma_t} L(\mathbf{x}_s,\vec{\omega})$$

Homogeneous Ambient Media

$$L(\mathbf{x},\vec{\omega}) = \sigma_s \int_0^s e^{-t\sigma_t} L_i(\mathbf{x}_t,\vec{\omega})dt + e^{-s\sigma_t} L(\mathbf{x}_s,\vec{\omega})$$

Homogeneous Ambient Media

Assume in-scattered radiance is an ambient constant

$$L(\mathbf{x},\vec{\omega}) = \sigma_s \int_0^s e^{-t\sigma_t} \boxed{L_i(\mathbf{x}_t,\vec{\omega})dt} + e^{-s\sigma_t} L(\mathbf{x}_s,\vec{\omega})$$

Homogeneous Ambient Media

Assume in-scattered radiance is an ambient constant

$$L(\mathbf{x}, \vec{\omega}) = \sigma_s \int_0^s e^{-t\sigma_t} L_i(\mathbf{x}_t, \vec{\omega}) dt + e^{-s\sigma_t} L(\mathbf{x}_s, \vec{\omega})$$

$$L(\mathbf{x}, \vec{\omega}) = \sigma_s \boxed{L_i} \int_0^s e^{-t\sigma_t} dt + e^{-s\sigma_t} L(\mathbf{x}_s, \vec{\omega})$$

Homogeneous Ambient Media

Assume in-scattered radiance is an ambient constant

$$L(\mathbf{x},\vec{\omega}) = \sigma_s \int_0^s e^{-t\sigma_t} L_i(\mathbf{x}_t,\vec{\omega})dt + e^{-s\sigma_t} L(\mathbf{x}_s,\vec{\omega})$$

$$L(\mathbf{x},\vec{\omega}) = \sigma_s L_i \boxed{\int_0^s e^{-t\sigma_t} dt} + e^{-s\sigma_t} L(\mathbf{x}_s,\vec{\omega})$$

Homogeneous Ambient Media

Assume in-scattered radiance is an ambient constant

$$L(\mathbf{x},\vec{\omega}) = \sigma_s \int_0^s e^{-t\sigma_t} L_i(\mathbf{x}_t,\vec{\omega})dt + e^{-s\sigma_t} L(\mathbf{x}_s,\vec{\omega})$$

$$L(\mathbf{x},\vec{\omega}) = \sigma_s L_i \int_0^s e^{-t\sigma_t} dt + e^{-s\sigma_t} L(\mathbf{x}_s,\vec{\omega})$$

$$L(\mathbf{x},\vec{\omega}) = \sigma_s L_i \boxed{\frac{1-e^{-s\sigma_t}}{\sigma_t}} + e^{-s\sigma_t} L(\mathbf{x}_s,\vec{\omega})$$

Homogeneous Ambient Media

Assume in-scattered radiance is an ambient constant

$$L(\mathbf{x},\vec{\omega}) = \sigma_s \int_0^s e^{-t\sigma_t} L_i(\mathbf{x}_t,\vec{\omega})dt + e^{-s\sigma_t} L(\mathbf{x}_s,\vec{\omega})$$

$$L(\mathbf{x},\vec{\omega}) = \sigma_s L_i \int_0^s e^{-t\sigma_t} dt + e^{-s\sigma_t} L(\mathbf{x}_s,\vec{\omega})$$

$$L(\mathbf{x},\vec{\omega}) = \sigma_s L_i \frac{1 - e^{-s\sigma_t}}{\sigma_t} + e^{-s\sigma_t} L(\mathbf{x}_s,\vec{\omega})$$

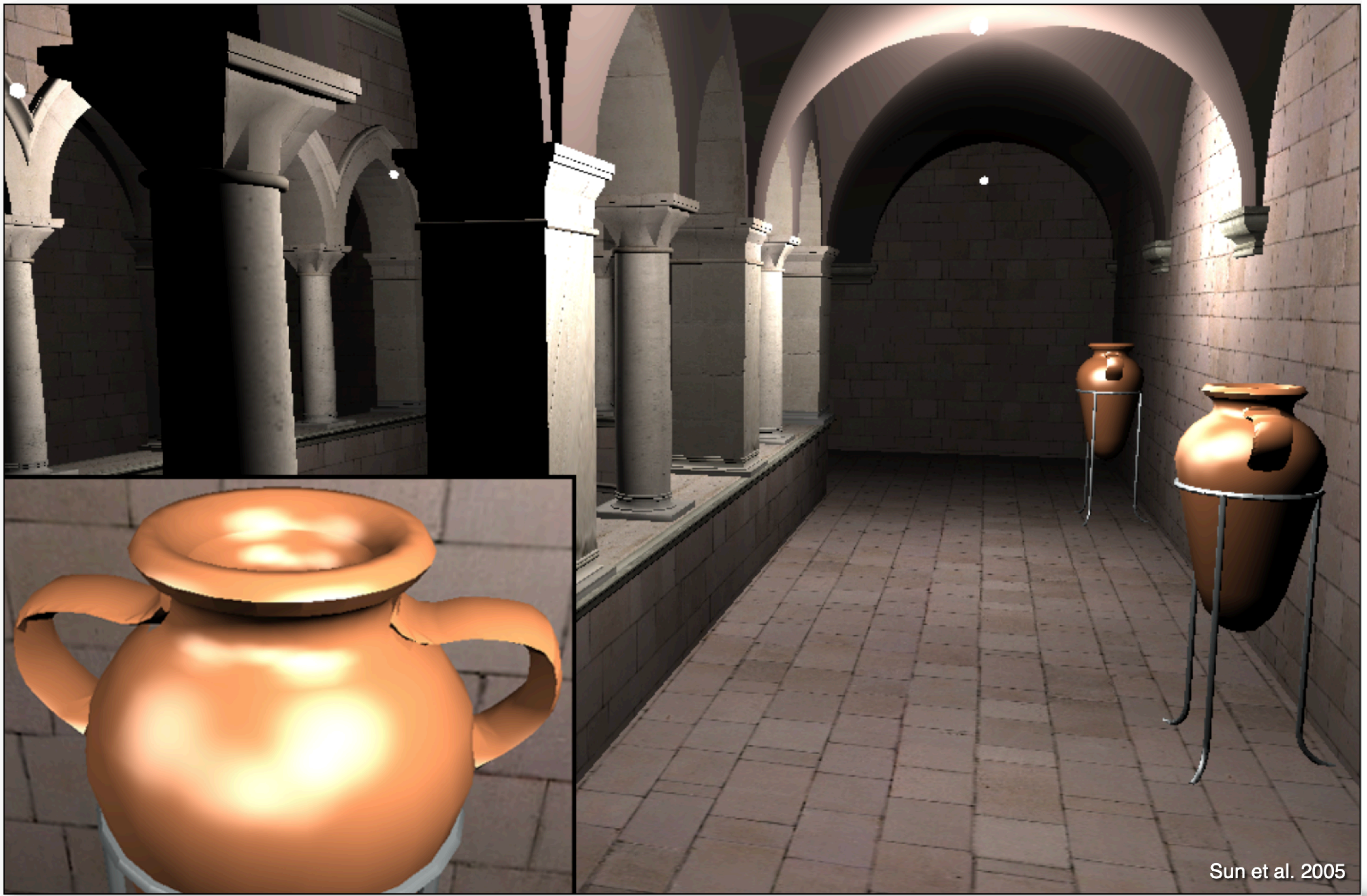
$$L(\mathbf{x},\vec{\omega}) = \mathrm{lerp}\left(\frac{\sigma_s}{\sigma_t} L_i, L(\mathbf{x}_s,\vec{\omega}), e^{-s\sigma_t}\right)$$

Homogeneous Ambient Media

Fog

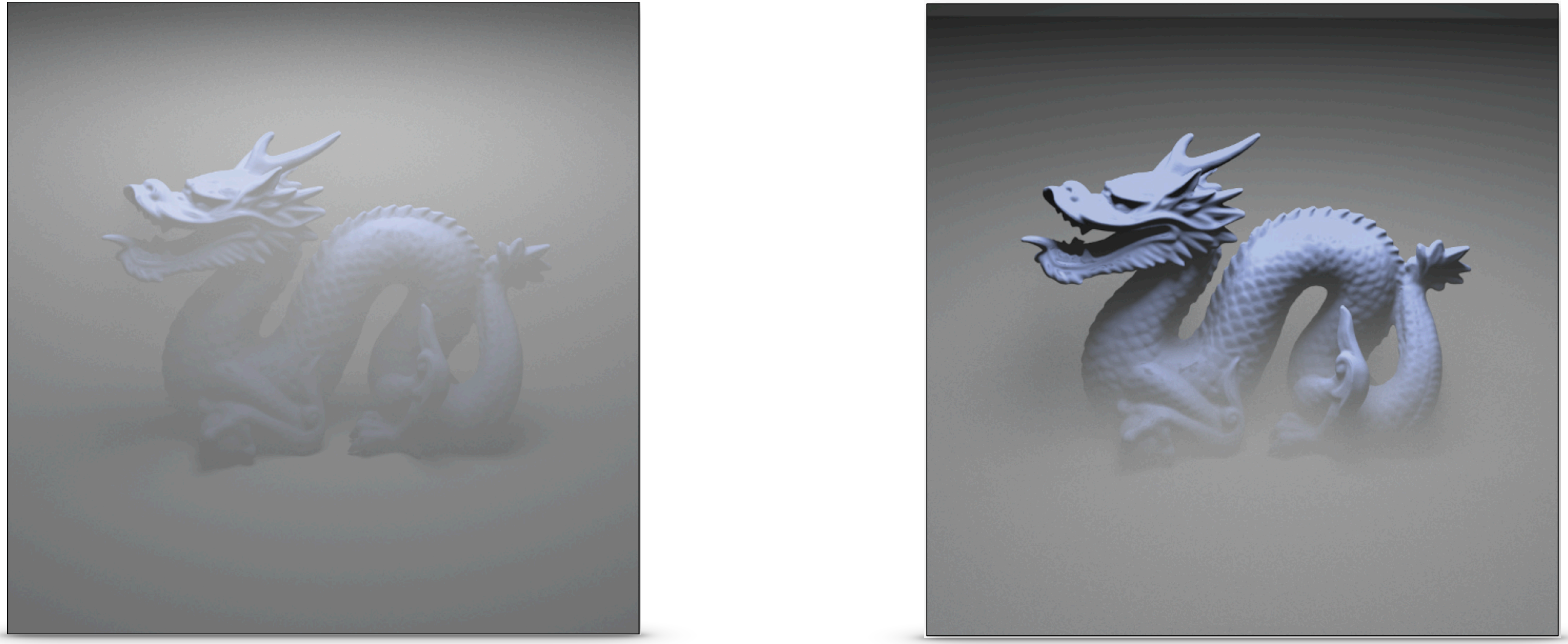
Sun et al. 2005

Clear Day



Sun et al. 2005

Fog



Volumetric Rendering Equation

$$L(\mathbf{x},\vec{\omega}) = T_r(\mathbf{x},\mathbf{x}_z)L(\mathbf{x}_z,\vec{\omega}) + \int_0^z T_r(\mathbf{x},\mathbf{x}_t)\sigma_a(\mathbf{x}_t)L_e(\mathbf{x}_t,\vec{\omega})dt + \int_0^z T_r(\mathbf{x},\mathbf{x}_t)\sigma_s(\mathbf{x}_t)\int_{S^2} f_p(\mathbf{x}_t,\vec{\omega}',\vec{\omega})L_i(\mathbf{x}_t,\vec{\omega}')d\omega' dt$$

Accumulated in-scattered radiance

In-scattered Radiance

$$L(\mathbf{x},\omega) = \int_0^z T_r(\mathbf{x},\mathbf{x}_t)\sigma_s(\mathbf{x}_t)\int_{S^2} f_p(\mathbf{x}_t,\vec{\omega}',\vec{\omega})L_i(\mathbf{x}_t,\vec{\omega}')d\omega' dt$$

In-scattered Radiance

$$L(\mathbf{x}, \omega) = \int_0^z T_r(\mathbf{x}, \mathbf{x}_t)\sigma_s(\mathbf{x}_t) \int_{S^2} f_p(\mathbf{x}_t, \vec{\omega}', \vec{\omega}) L_i(\mathbf{x}_t, \vec{\omega}') d\omega' dt$$

In-scattered Radiance

$$L(\mathbf{x}, \omega) = \int_0^z T_r(\mathbf{x}, \mathbf{x}_t)\sigma_s(\mathbf{x}_t) \int_{S^2} f_p(\mathbf{x}_t, \vec{\omega}', \vec{\omega}) L_i(\mathbf{x}_t, \vec{\omega}') d\omega' dt$$

$$L_s(\mathbf{x}, \omega) = \int_{S^2} f_p(\mathbf{x}_t, \vec{\omega}', \vec{\omega}) L_i(\mathbf{x}_t, \vec{\omega}') d\omega' dt$$

In-scattered Radiance

$$L(\mathbf{x},\omega) = \int_0^z T_r(\mathbf{x},\mathbf{x}_t)\sigma_s(\mathbf{x}_t)\int_{S^2} f_p(\mathbf{x}_t,\vec{\omega}',\vec{\omega})L_i(\mathbf{x}_t,\vec{\omega}')d\omega' dt$$

$$L_s(\mathbf{x},\omega) = \int_{S^2} f_p(\mathbf{x}_t,\vec{\omega}',\vec{\omega})L_i(\mathbf{x}_t,\vec{\omega}')d\omega' dt$$

Single scattering L_i arrives directly from a light source (direct illumination)

In-scattered Radiance

$$L(\mathbf{x}, \omega) = \int_0^z T_r(\mathbf{x}, \mathbf{x}_t)\sigma_s(\mathbf{x}_t) \int_{S^2} f_p(\mathbf{x}_t, \vec{\omega}', \vec{\omega}) L_i(\mathbf{x}_t, \vec{\omega}') d\omega' dt$$

$$L_s(\mathbf{x}, \omega) = \int_{S^2} f_p(\mathbf{x}_t, \vec{\omega}', \vec{\omega}) L_i(\mathbf{x}_t, \vec{\omega}') d\omega' dt$$

Single scattering

L_i arrives directly from a light source (direct illumination)

$$L_i(\mathbf{x}, \vec{\omega}) = T_r(\mathbf{x}, r(\mathbf{x}, \vec{\omega})) L_e(r(\mathbf{x}, \vec{\omega}), -\vec{\omega})$$

In-scattered Radiance

$$L(\mathbf{x},\omega) = \int_0^z T_r(\mathbf{x},\mathbf{x}_t)\sigma_s(\mathbf{x}_t)\int_{S^2} f_p(\mathbf{x}_t,\vec{\omega}',\vec{\omega})L_i(\mathbf{x}_t,\vec{\omega}')d\omega' dt$$

$$L_s(\mathbf{x},\omega) = \int_{S^2} f_p(\mathbf{x}_t,\vec{\omega}',\vec{\omega})L_i(\mathbf{x}_t,\vec{\omega}')d\omega' dt$$

Single scattering

L_i arrives directly from a light source (direct illumination)

$$L_i(\mathbf{x},\vec{\omega}) = T_r(\mathbf{x},r(\mathbf{x},\vec{\omega}))L_e(r(\mathbf{x},\vec{\omega}),-\vec{\omega})$$

Multiple scattering

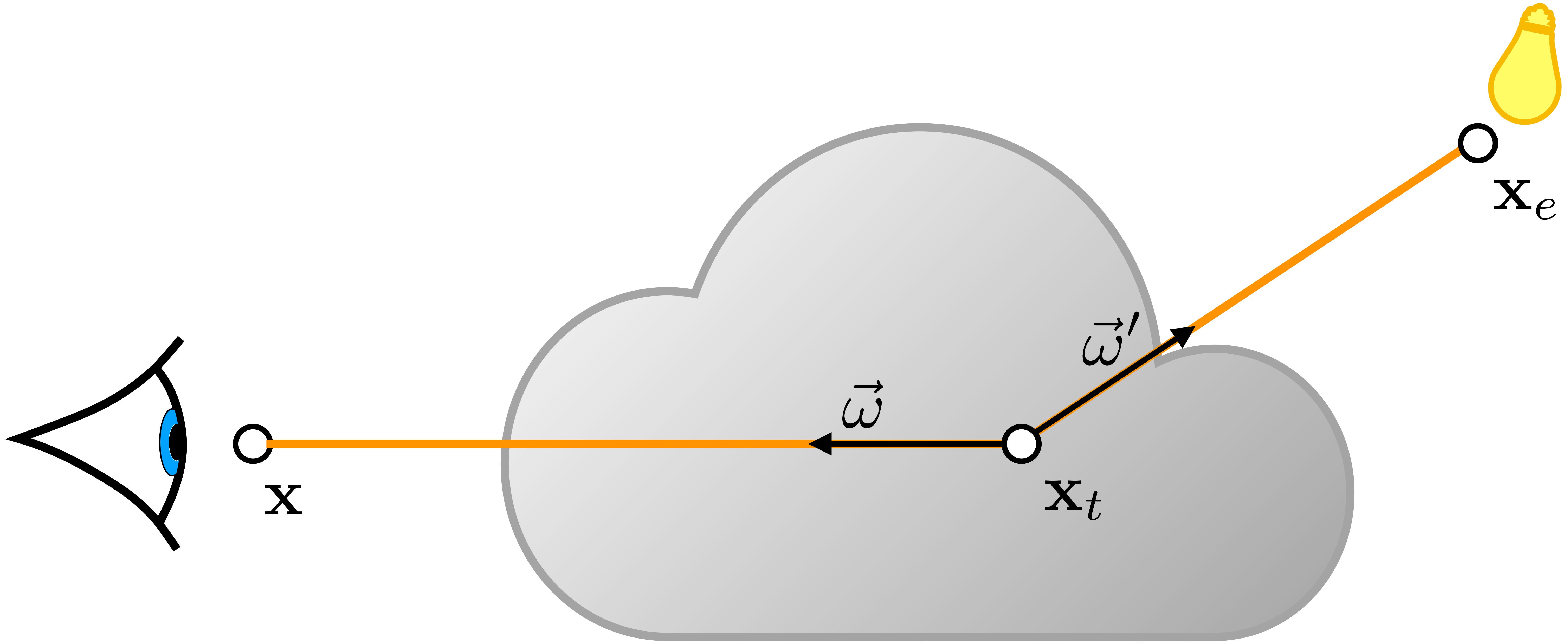
arrives through multiple bounces (indirect illumination)

Single Scattering

$$L(\mathbf{x},\vec{\omega}) = \int_0^z T_r(\mathbf{x},\mathbf{x}_t)\sigma_s(\mathbf{x}_t)\int_{S^2} f_p(\mathbf{x},\vec{\omega},\vec{\omega}')T_r(\mathbf{x}_t,\mathbf{x}_e)L_e(\mathbf{x}_e,-\vec{\omega})d\vec{\omega}'dt$$

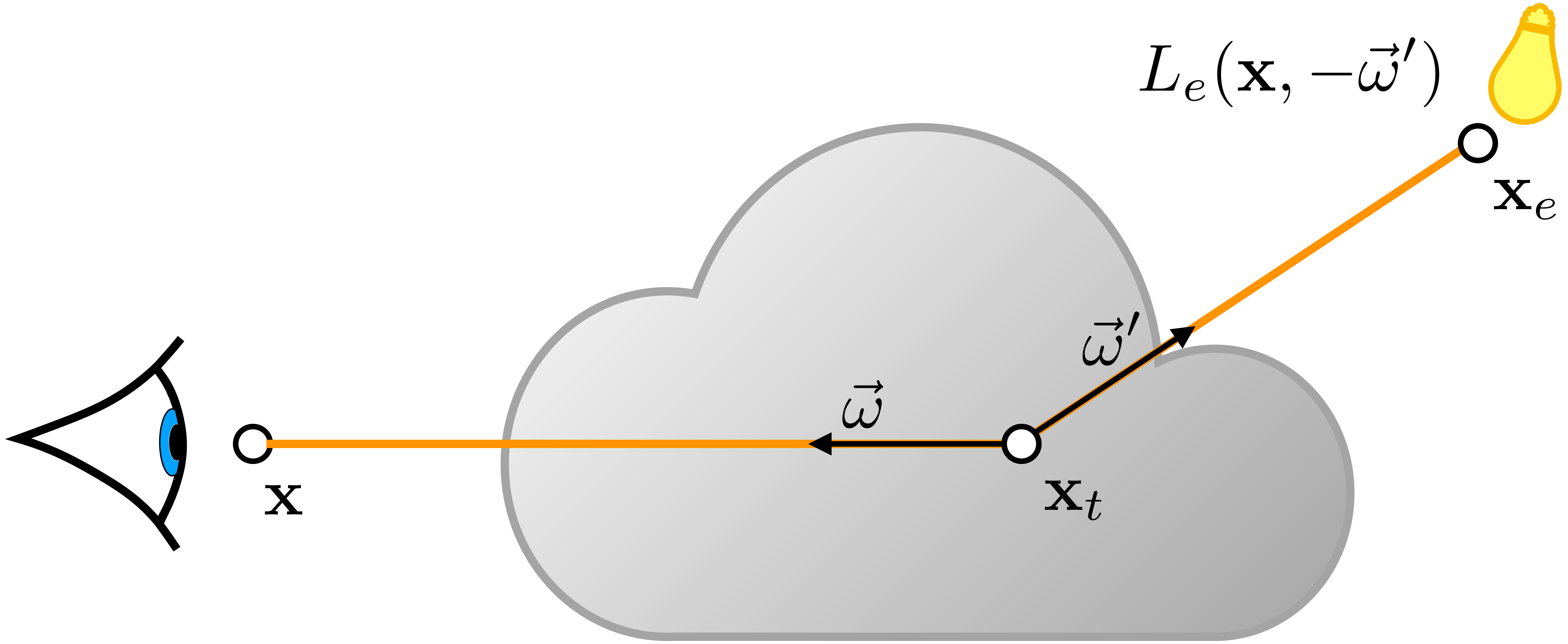
Single Scattering

$$L(\mathbf{x},\vec{\omega}) = \int_0^z T_r(\mathbf{x},\mathbf{x}_t)\sigma_s(\mathbf{x}_t)\int_{S^2} f_p(\mathbf{x},\vec{\omega},\vec{\omega}')T_r(\mathbf{x}_t,\mathbf{x}_e)L_e(\mathbf{x}_e,-\vec{\omega})d\vec{\omega}'dt$$



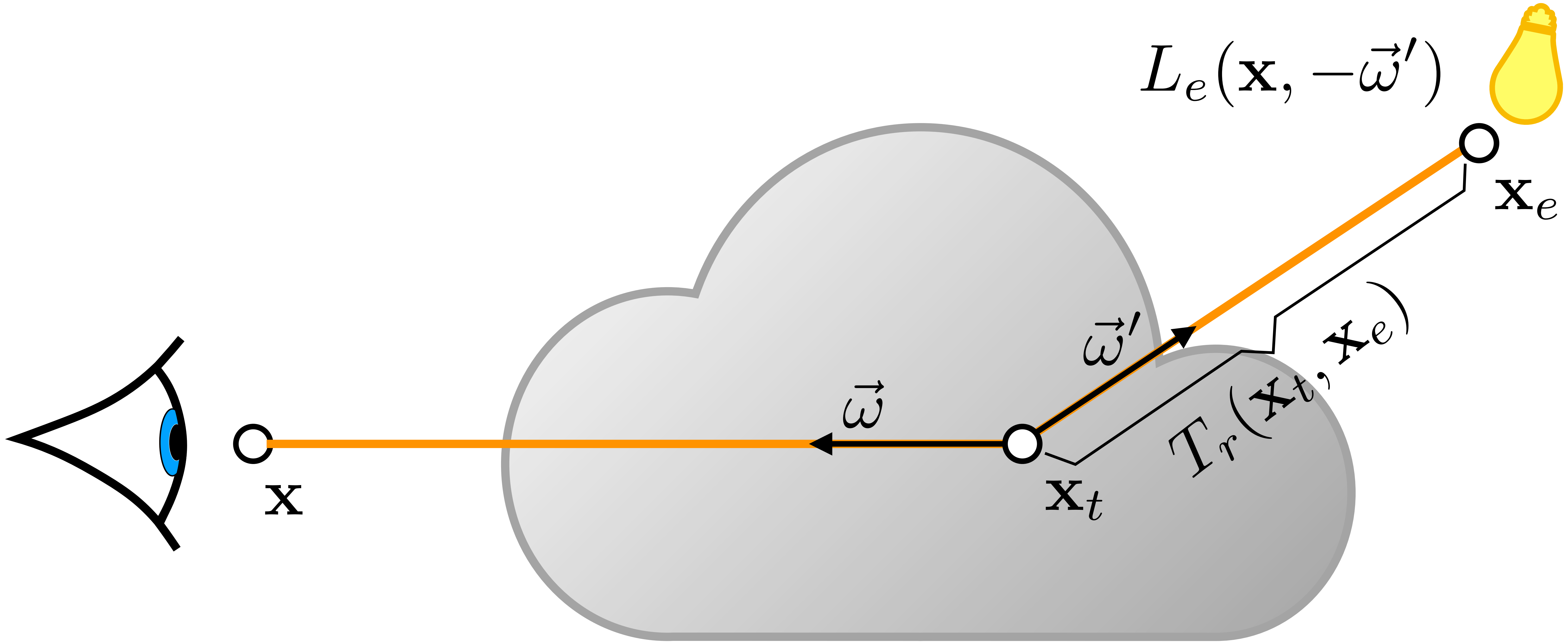
Single Scattering

$$L(\mathbf{x},\vec{\omega}) = \int_0^z T_r(\mathbf{x},\mathbf{x}_t)\sigma_s(\mathbf{x}_t)\int_{S^2} f_p(\mathbf{x},\vec{\omega},\vec{\omega}')T_r(\mathbf{x}_t,\mathbf{x}_e)L_e(\mathbf{x}_e,-\vec{\omega})d\vec{\omega}'dt$$



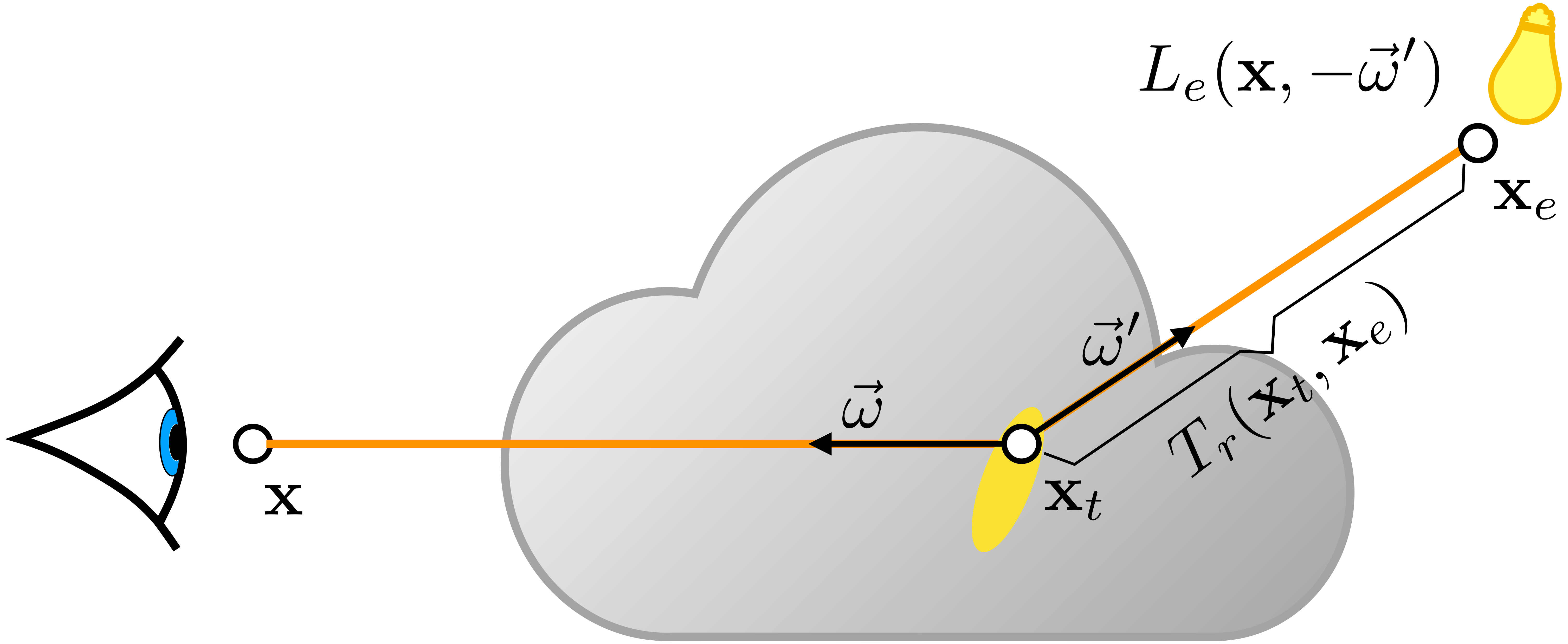
Single Scattering

$$L(\mathbf{x},\vec{\omega}) = \int_0^z T_r(\mathbf{x},\mathbf{x}_t)\sigma_s(\mathbf{x}_t)\int_{S^2} f_p(\mathbf{x},\vec{\omega},\vec{\omega}')T_r(\mathbf{x}_t,\mathbf{x}_e)L_e(\mathbf{x}_e,-\vec{\omega})d\vec{\omega}'dt$$



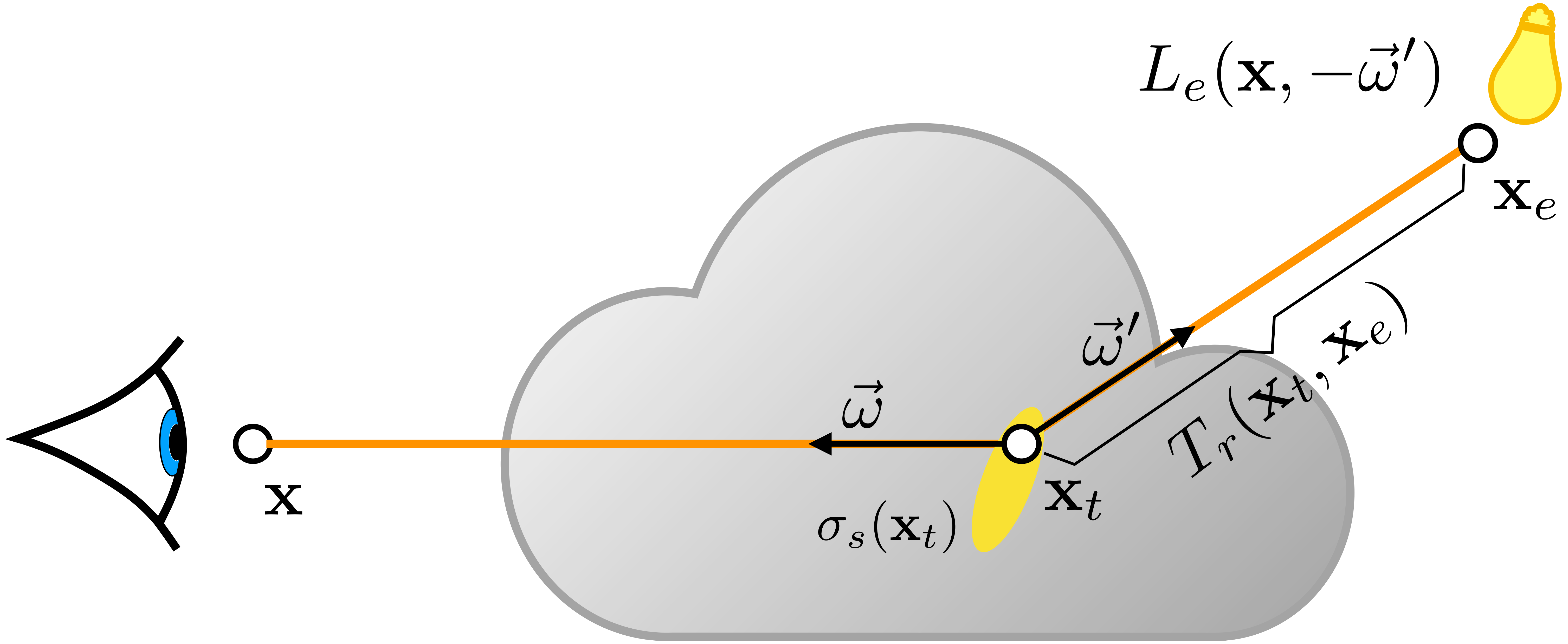
Single Scattering

$$L(\mathbf{x},\vec{\omega}) = \int_0^z T_r(\mathbf{x},\mathbf{x}_t)\sigma_s(\mathbf{x}_t) \int_{S^2} f_p(\mathbf{x},\vec{\omega},\vec{\omega}')T_r(\mathbf{x}_t,\mathbf{x}_e)L_e(\mathbf{x}_e,-\vec{\omega})d\vec{\omega}'dt$$



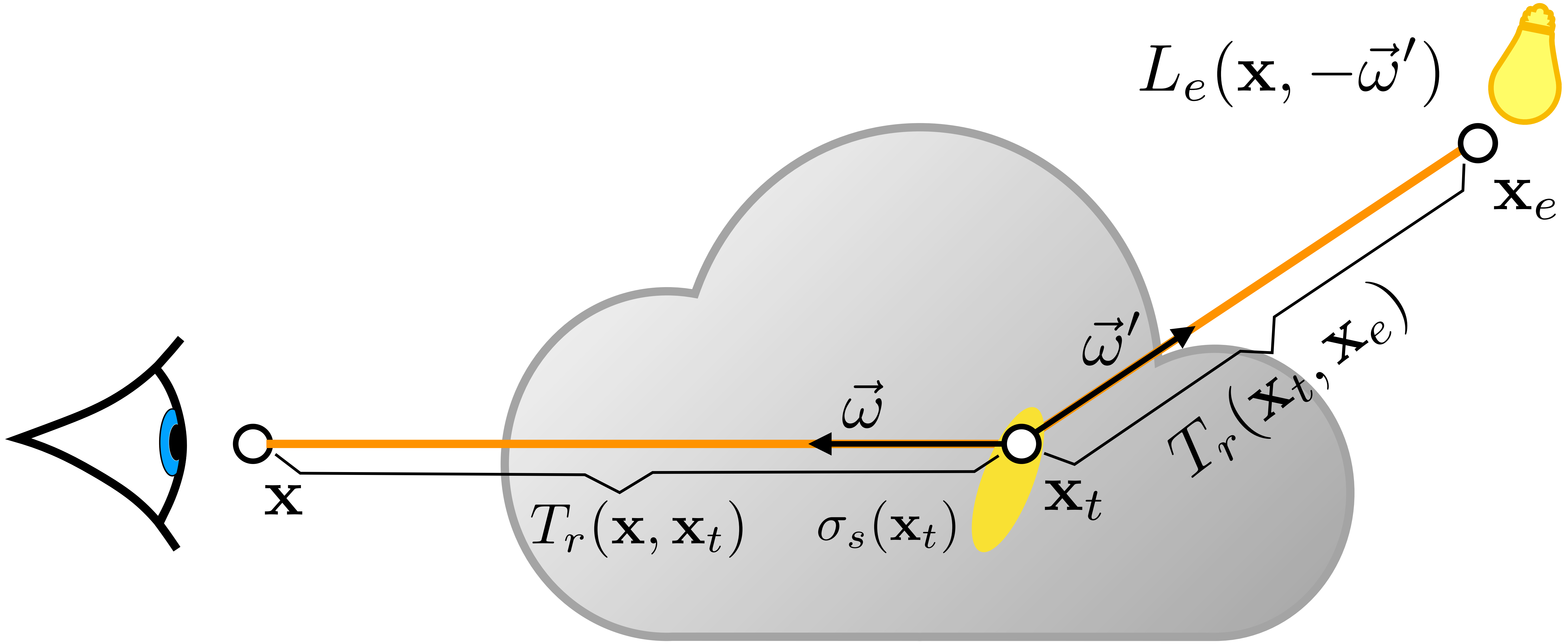
Single Scattering

$$L(\mathbf{x},\vec{\omega}) = \int_0^z T_r(\mathbf{x},\mathbf{x}_t)\sigma_s(\mathbf{x}_t)\int_{S^2} f_p(\mathbf{x},\vec{\omega},\vec{\omega}')T_r(\mathbf{x}_t,\mathbf{x}_e)L_e(\mathbf{x}_e,-\vec{\omega})d\vec{\omega}'dt$$



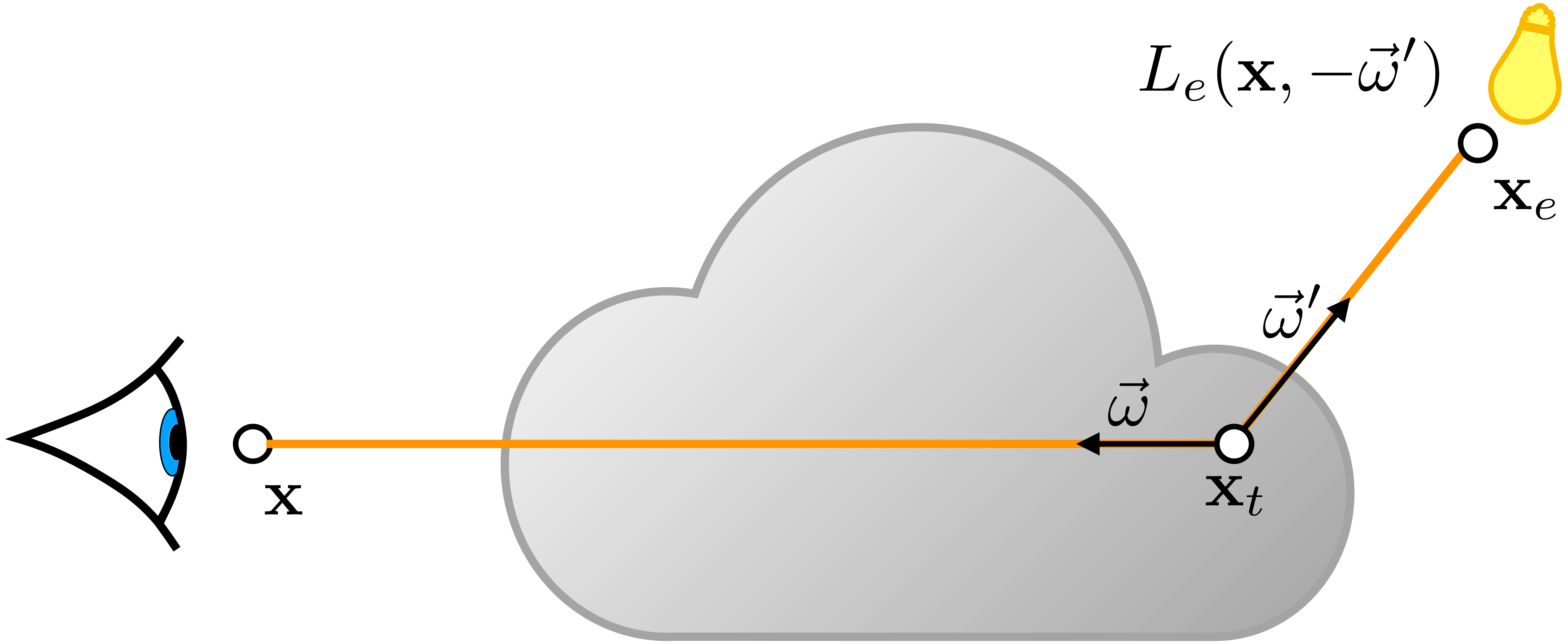
Single Scattering

$$L(\mathbf{x},\vec{\omega}) = \int_0^z T_r(\mathbf{x},\mathbf{x}_t)\sigma_s(\mathbf{x}_t)\int_{S^2} f_p(\mathbf{x},\vec{\omega},\vec{\omega}')T_r(\mathbf{x}_t,\mathbf{x}_e)L_e(\mathbf{x}_e,-\vec{\omega})d\vec{\omega}'dt$$



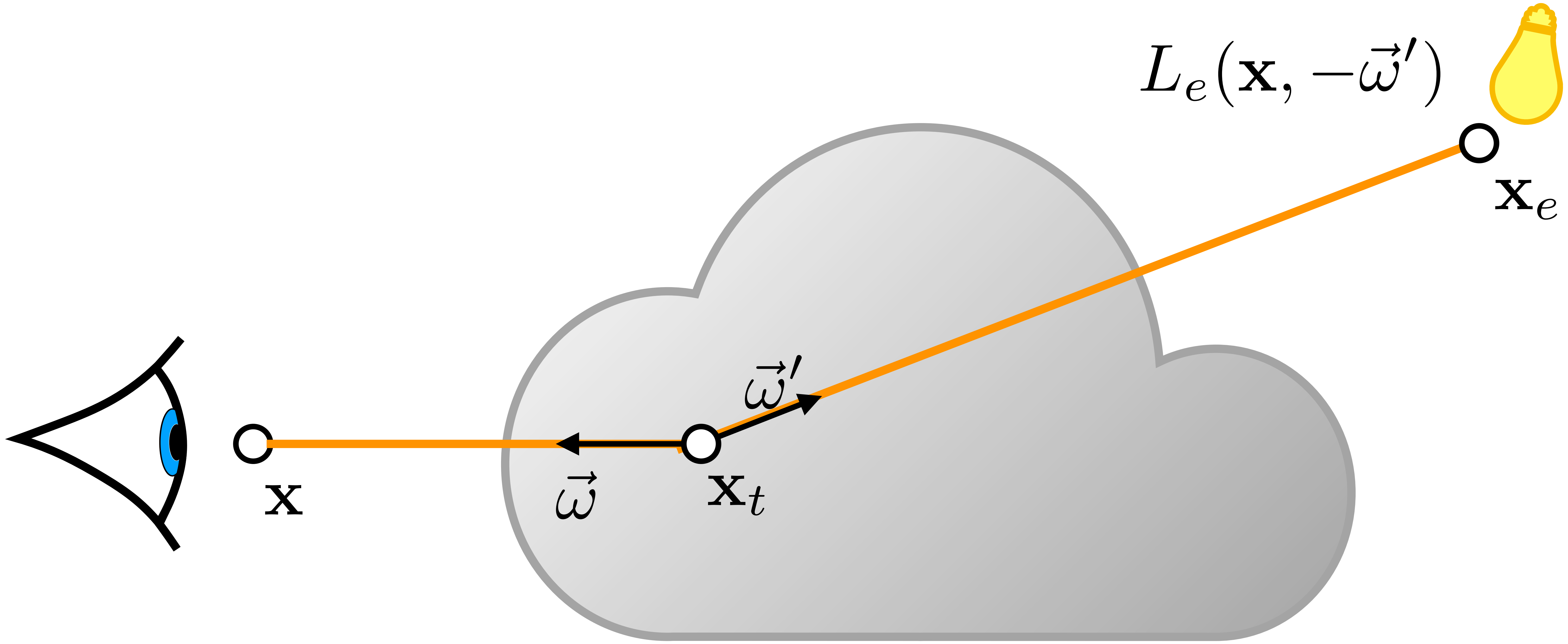
Single Scattering

$$L(\mathbf{x},\vec{\omega}) = \int_0^z T_r(\mathbf{x},\mathbf{x}_t)\sigma_s(\mathbf{x}_t)\int_{S^2} f_p(\mathbf{x},\vec{\omega},\vec{\omega}')T_r(\mathbf{x}_t,\mathbf{x}_e)L_e(\mathbf{x}_e,-\vec{\omega})d\vec{\omega}'dt$$



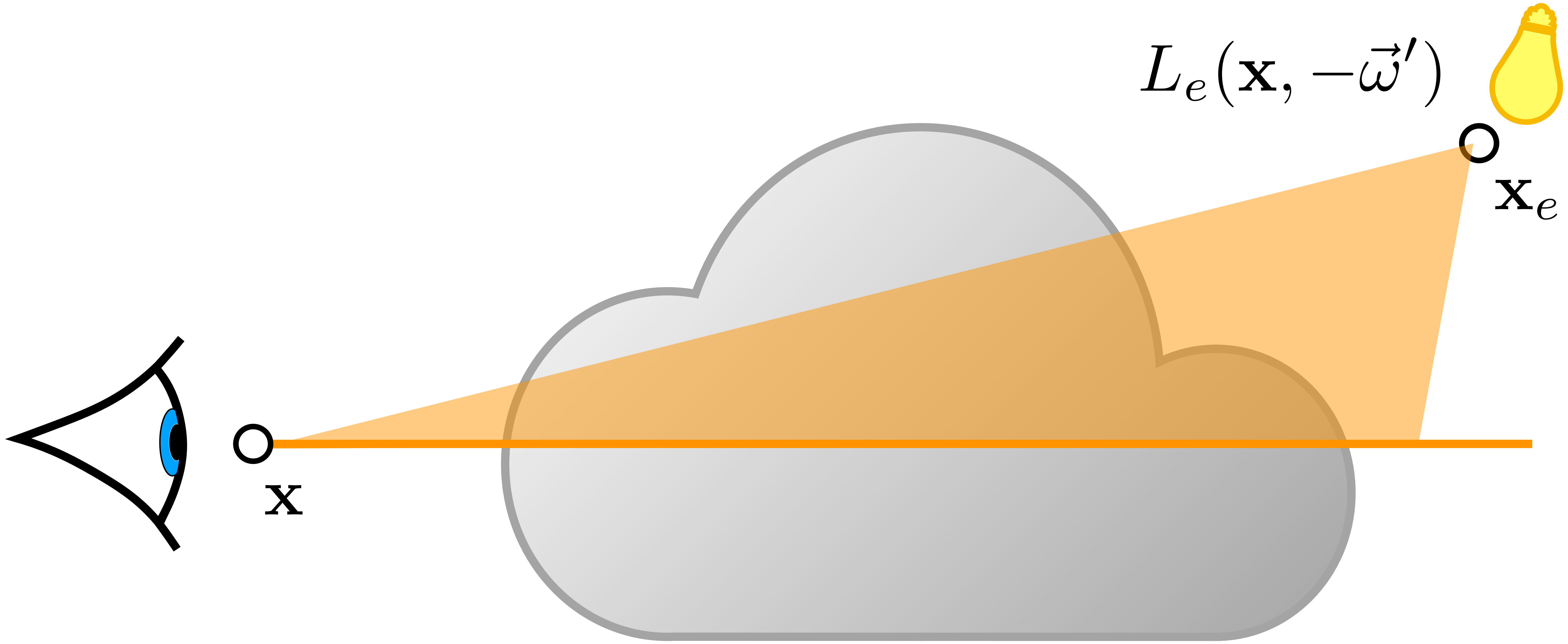
Single Scattering

$$L(\mathbf{x},\vec{\omega}) = \int_0^z T_r(\mathbf{x},\mathbf{x}_t)\sigma_s(\mathbf{x}_t)\int_{S^2} f_p(\mathbf{x},\vec{\omega},\vec{\omega}')T_r(\mathbf{x}_t,\mathbf{x}_e)L_e(\mathbf{x}_e,-\vec{\omega})d\vec{\omega}'dt$$



Single Scattering

$$L(\mathbf{x}, \vec{\omega}) = \int_0^z T_r(\mathbf{x}, \mathbf{x}_t)\sigma_s(\mathbf{x}_t) \int_{S^2} f_p(\mathbf{x}, \vec{\omega}, \vec{\omega}')T_r(\mathbf{x}_t, \mathbf{x}_e)L_e(\mathbf{x}_e, -\vec{\omega})d\vec{\omega}'dt$$



Single Scattering

$$L(\mathbf{x},\vec{\omega}) = \int_0^z T_r(\mathbf{x},\mathbf{x}_t)\sigma_s(\mathbf{x}_t)\int_{S^2} f_p(\mathbf{x},\vec{\omega},\vec{\omega}')T_r(\mathbf{x}_t,\mathbf{x}_e)L_e(\mathbf{x}_e,-\vec{\omega})d\vec{\omega}'dt$$

Single Scattering

$$L(\mathbf{x},\vec{\omega}) = \int_0^z T_r(\mathbf{x},\mathbf{x}_t)\sigma_s(\mathbf{x}_t)\int_{S^2} f_p(\mathbf{x},\vec{\omega},\vec{\omega}')T_r(\mathbf{x}_t,\mathbf{x}_e)L_e(\mathbf{x}_e,-\vec{\omega})d\vec{\omega}'dt$$

Semi-analytic solutions

Sun et al. [2005]

Pegoraro et al. [2009, 2010]

Single Scattering

$$L(\mathbf{x},\vec{\omega}) = \int_0^z T_r(\mathbf{x},\mathbf{x}_t)\sigma_s(\mathbf{x}_t)\int_{S^2} f_p(\mathbf{x},\vec{\omega},\vec{\omega}')T_r(\mathbf{x}_t,\mathbf{x}_e)L_e(\mathbf{x}_e,-\vec{\omega})d\vec{\omega}'dt$$

Semi-analytic solutions

- Sun et al. [2005]
- Pegoraro et al. [2009, 2010]

Numerical solutions

- Ray marching
- Equiangular sampling

Analytic Single Scattering

$$L(\mathbf{x}, \vec{\omega}) = \int_0^z T_r(\mathbf{x}, \mathbf{x}_t)\sigma_s(\mathbf{x}_t) \int_{S^2} f_p(\mathbf{x}, \vec{\omega}, \vec{\omega}')T_r(\mathbf{x}_t, \mathbf{x}_e)L_e(\mathbf{x}_e, -\vec{\omega})d\vec{\omega}'dt$$

Assumptions:

Analytic Single Scattering

$$L(\mathbf{x},\vec{\omega}) = \int_0^z T_r(\mathbf{x},\mathbf{x}_t)\sigma_s(\mathbf{x}_t)\int_{S^2} f_p(\mathbf{x},\vec{\omega},\vec{\omega}')T_r(\mathbf{x}_t,\mathbf{x}_e)L_e(\mathbf{x}_e,-\vec{\omega})d\vec{\omega}'dt$$

Assumptions:

Homogeneous

Analytic Single Scattering

$$L(\mathbf{x},\vec{\omega}) = \int_0^z T_r(\mathbf{x},\mathbf{x}_t)\sigma_s(\mathbf{x}_t)\int_{S^2} f_p(\mathbf{x},\vec{\omega},\vec{\omega}')T_r(\mathbf{x}_t,\mathbf{x}_e)L_e(\mathbf{x}_e,-\vec{\omega})d\vec{\omega}'dt$$

Assumptions:

Homogeneous

Point or spot light

Analytic Single Scattering

$$L(\mathbf{x}, \vec{\omega}) = \int_0^z T_r(\mathbf{x}, \mathbf{x}_t)\sigma_s(\mathbf{x}_t) \int_{S^2} f_p(\mathbf{x}, \vec{\omega}, \vec{\omega}')T_r(\mathbf{x}_t, \mathbf{x}_e)L_e(\mathbf{x}_e, -\vec{\omega})d\vec{\omega}' dt$$

Assumptions:

Homogeneous

Point or spot light

Relatively simple phase function

Analytic Single Scattering

$$L(\mathbf{x},\vec{\omega}) = \int_0^z T_r(\mathbf{x},\mathbf{x}_t)\sigma_s(\mathbf{x}_t)\int_{S^2} f_p(\mathbf{x},\vec{\omega},\vec{\omega}')T_r(\mathbf{x}_t,\mathbf{x}_e)L_e(\mathbf{x}_e,-\vec{\omega})d\vec{\omega}'dt$$

Assumptions:

Homogeneous

Point or spot light

Relatively simple phase function

No occlusion

Analytic Single Scattering

$$L(\mathbf{x},\vec{\omega}) = \int_0^z T_r(\mathbf{x},\mathbf{x}_t)\sigma_s(\mathbf{x}_t)\int_{S^2} f_p(\mathbf{x},\vec{\omega},\vec{\omega}')T_r(\mathbf{x}_t,\mathbf{x}_e)L_e(\mathbf{x}_e,-\vec{\omega})d\vec{\omega}'dt$$

Assumptions:

$$\left.\begin{array}{l}\text{Homogeneous}\\ \text{Point or spot light}\\ \text{Relatively simple phase function}\\ \text{No occlusion}\end{array}\right\}$$

Analytic Single Scattering

$$L(\mathbf{x},\vec{\omega}) = \int_0^z T_r(\mathbf{x},\mathbf{x}_t)\sigma_s(\mathbf{x}_t)\int_{S^2} f_p(\mathbf{x},\vec{\omega},\vec{\omega}')T_r(\mathbf{x}_t,\mathbf{x}_e)L_e(\mathbf{x}_e,-\vec{\omega})d\vec{\omega}'dt$$

Assumptions:

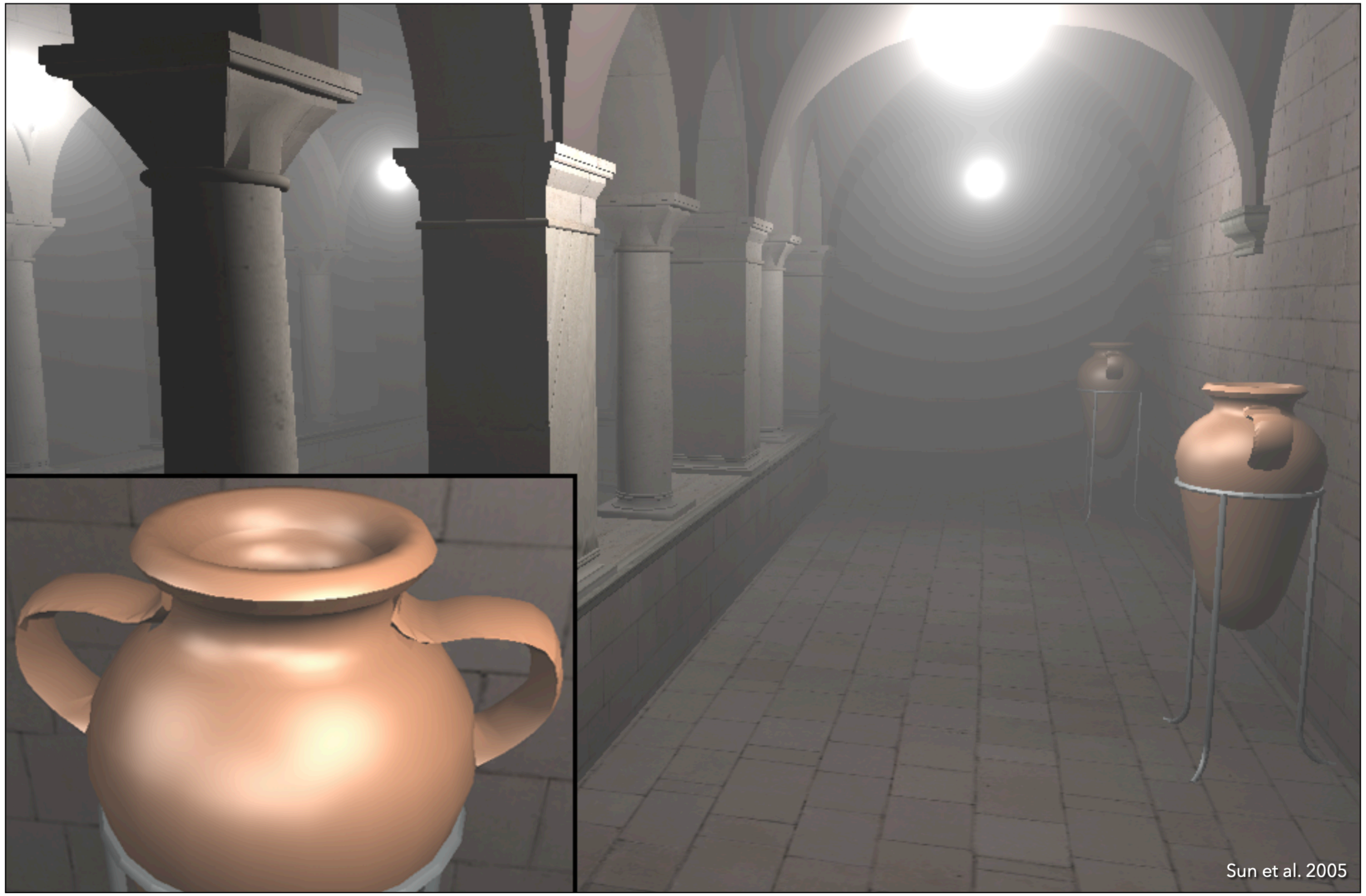
- Homogeneous
- Point or spot light
- Relatively simple phase function
- No occlusion

$$L_{(}\mathbf{x},\vec{\omega}) = \frac{\Phi}{4\pi}\frac{1}{4\pi}\int_0^z e^{-\sigma_t||\mathbf{x},\mathbf{x}_t||}\frac{e^{-\sigma_t||\mathbf{x}_t,\mathbf{x}_p||}}{e^{-\sigma_t||\mathbf{x}_t,\mathbf{x}_p||^2}}dt$$

OpenGL Fog

Sun et al. 2005

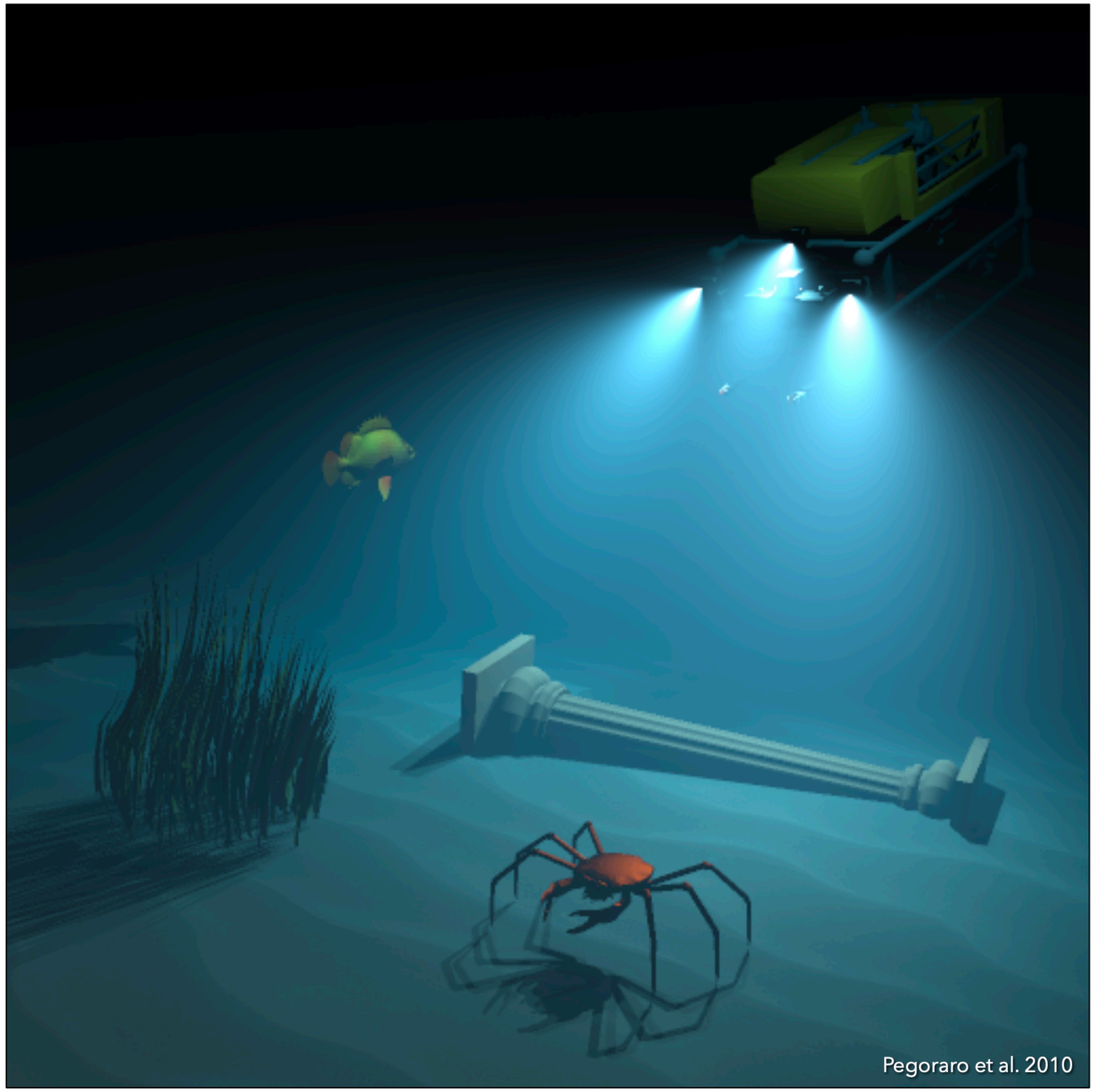
Analytic Single Scattering



Sun et al. 2005

Analytic Single Scattering

Pegoraro et al. 2009



Pegoraro et al. 2010

Analytic Single Scattering

$$L_m(x_a, x_b, \vec{\omega}) = \frac{\kappa_s}{h} e^{\kappa_t (x_a - x_h)} 2 \sum_{n=0}^{N-1} c(n) \sum_{k=0}^{2n} d(n,k) \int_{v_a}^{v_b} \frac{e^{-Hv}}{(v^2+1)^{n+1}} v^k \mathrm{d}v$$

$$\begin{aligned}\int \frac{e^{av}}{(v^2+1)^m} v^n \mathrm{d}v = \frac{1}{2^{m-1}} \sum_{l=0}^{m-1} \frac{1}{2^l} \binom{m-1+l}{m-1} \Bigg(\sum_{k=0}^{\min\{m-1-l,n\}} \binom{n}{k} \Bigg(\frac{a^{m-1-l-k}}{(m-1-l-k)!} E(a, v, m-n-l+k) \\ - e^{av} \sum_{j=1}^{m-1-l-k} \frac{(j-1)!}{(m-1-l-k)!} \frac{a^{m-1-l-k-j}}{(v^2+1)^j} \sum_{\substack{i=(m-n-l+k-j) \bmod 2 \\ i+=2}}^{\leq j} (-1)^{\frac{m-n-l+k-j+i}{2}} \binom{j}{i} v^i \Bigg) \\ + \frac{e^{av}}{a} \sum_{k=0}^{\leq n-m+l} \binom{n}{k} \sum_{j=0}^{n-m+l-k} \frac{(n-m+l-k)!}{j!} \frac{1}{(-a)^{n-m+l-k-j}} \sum_{\substack{i=(-m+l+k-j) \bmod 2 \\ i+=2}}^{\leq j} (-1)^{\frac{-m+l+k-j+i}{2}} \binom{j}{i} v^i \Bigg)\end{aligned}$$

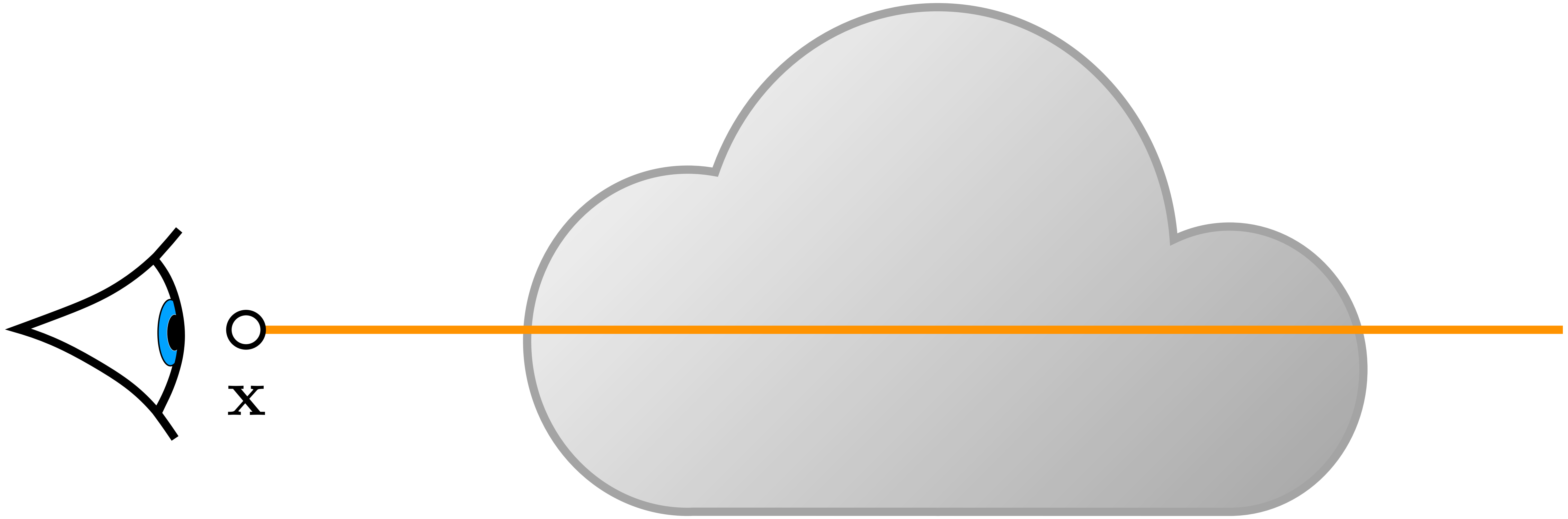
No shadows, implementation nightmare, computationally intensive,...

Let's try brute force!

Ray Marching

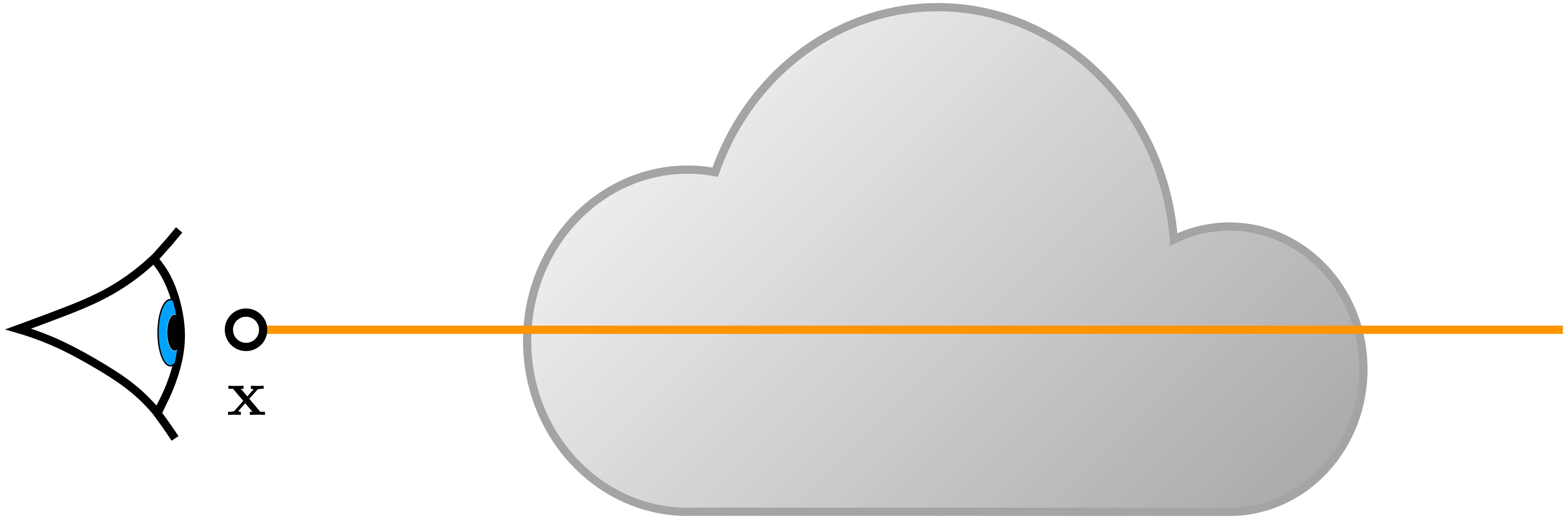
$$L(\mathbf{x}, \vec{\omega}) = \int_0^z T_r(\mathbf{x}, \mathbf{x}_t)\sigma_s(\mathbf{x}_t)L_s(\mathbf{x}_t, \vec{\omega})dt$$

Approximate with Riemann summation



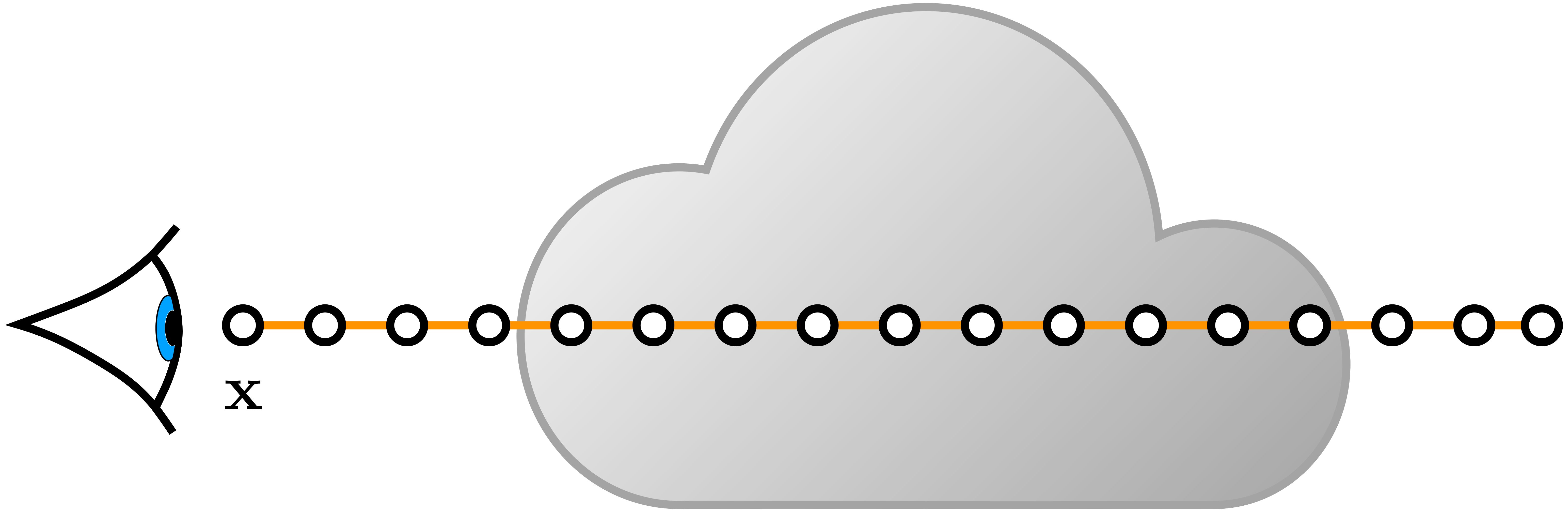
Ray Marching

$$L_(\mathbf{x},\vec{\omega}) \approx \sum_{k=0}^{N} T_r(\mathbf{x},\mathbf{x}_{t,k})\sigma_s(\mathbf{x}_{t,k})L_s(\mathbf{x}_{t,k},\vec{\omega})\Delta t$$



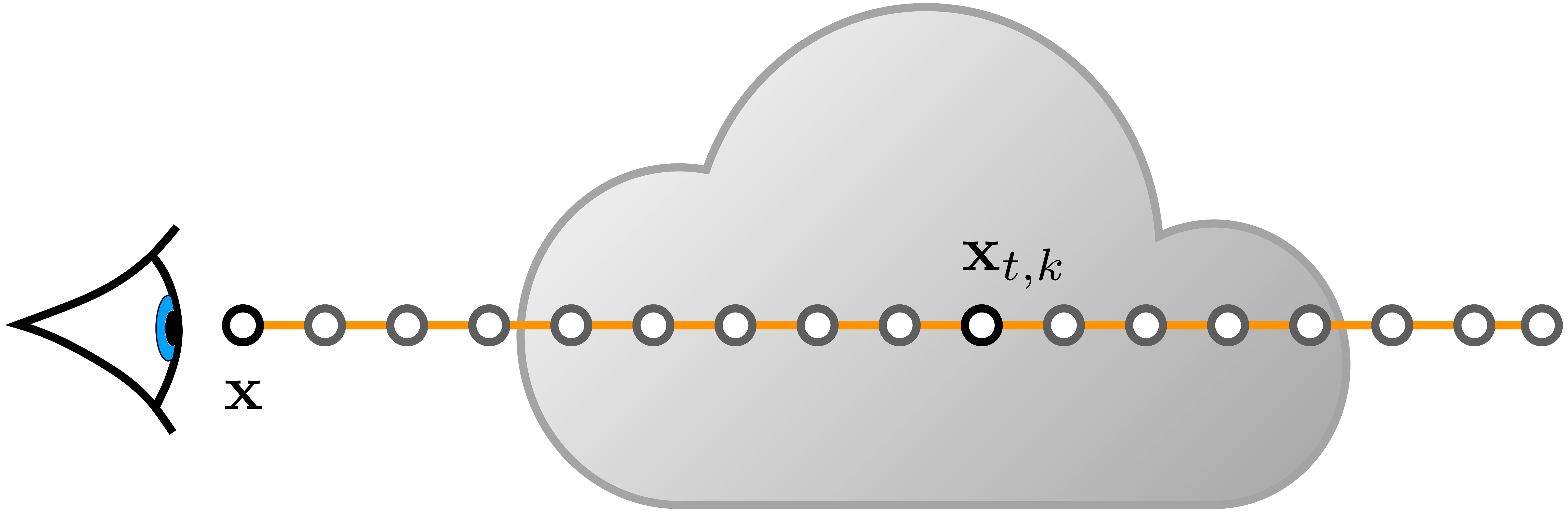
Ray Marching

$$L_(\mathbf{x}, \vec{\omega}) \approx \sum_{k=0}^{N} T_r(\mathbf{x}, \mathbf{x}_{t,k})\sigma_s(\mathbf{x}_{t,k})L_s(\mathbf{x}_{t,k}, \vec{\omega})\Delta t$$



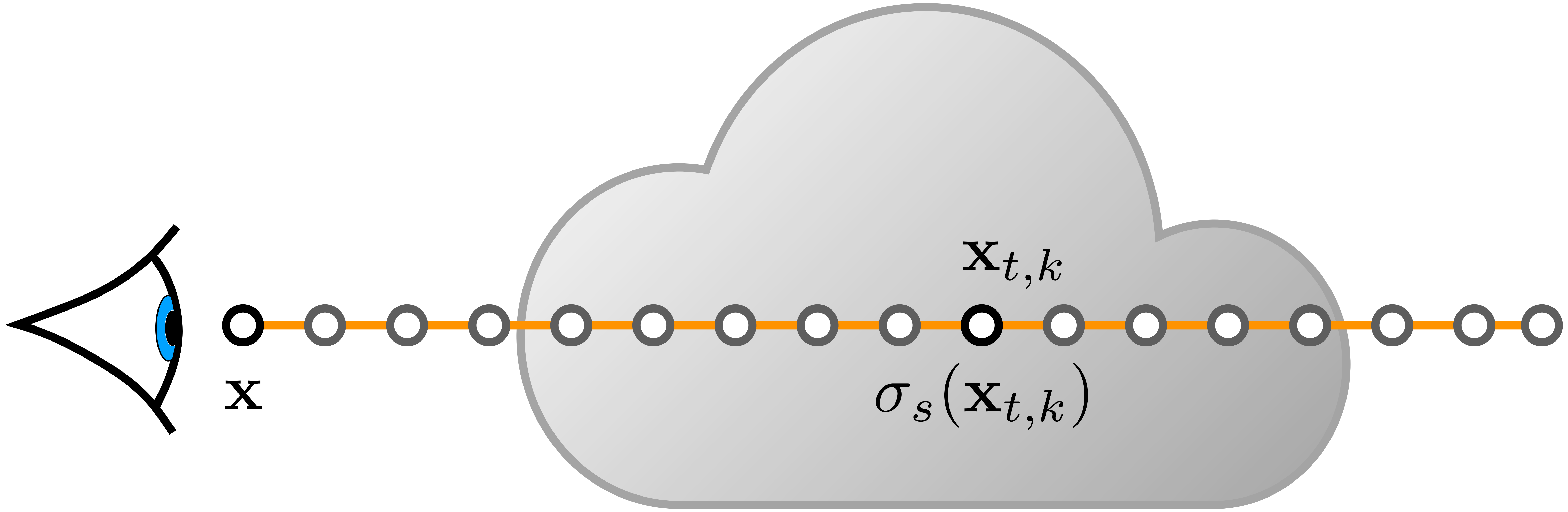
Ray Marching

$$L_(\mathbf{x},\vec{\omega}) \approx \sum_{k=0}^{N} T_r(\mathbf{x},\mathbf{x}_{t,k})\sigma_s(\mathbf{x}_{t,k})L_s(\mathbf{x}_{t,k},\vec{\omega})\Delta t$$



Ray Marching

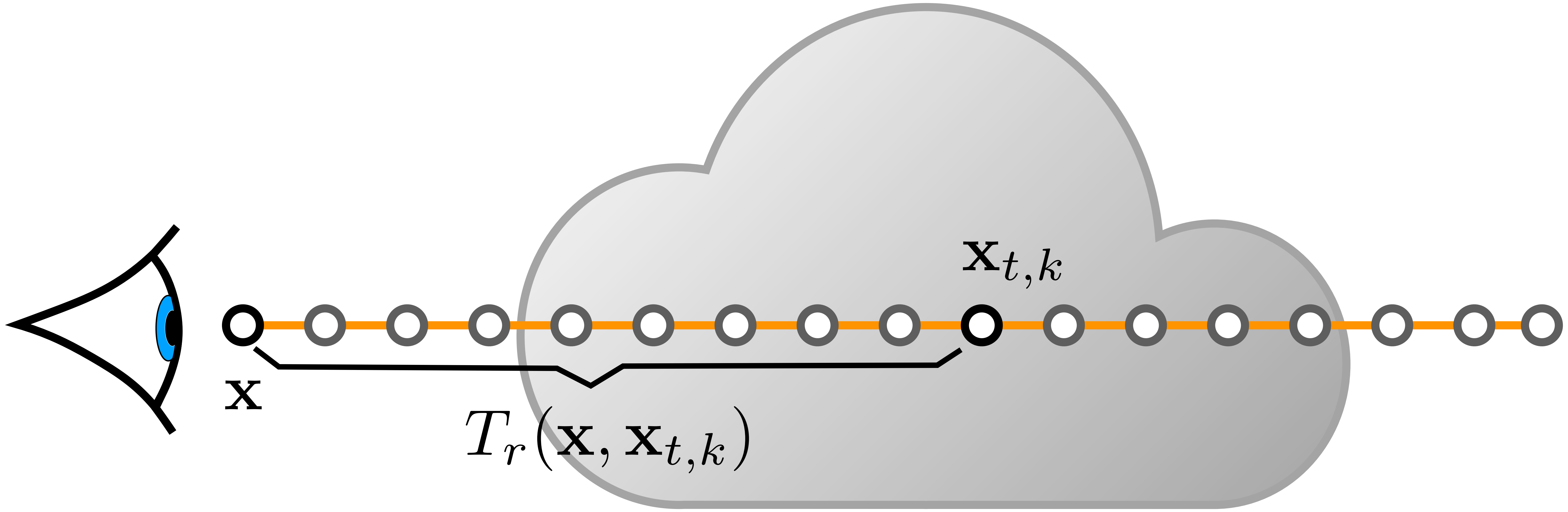
$$L_{(}\mathbf{x}, \vec{\omega}) \approx \sum_{k=0}^{N} T_r(\mathbf{x}, \mathbf{x}_{t,k})\sigma_s(\mathbf{x}_{t,k})L_s(\mathbf{x}_{t,k}, \vec{\omega})\Delta t$$



Ray Marching

$$L_{(}\mathbf{x}, \vec{\omega}) \approx \sum_{k=0}^{N} T_r(\mathbf{x}, \mathbf{x}_{t,k}) \sigma_s(\mathbf{x}_{t,k}) L_s(\mathbf{x}_{t,k}, \vec{\omega}) \Delta t$$

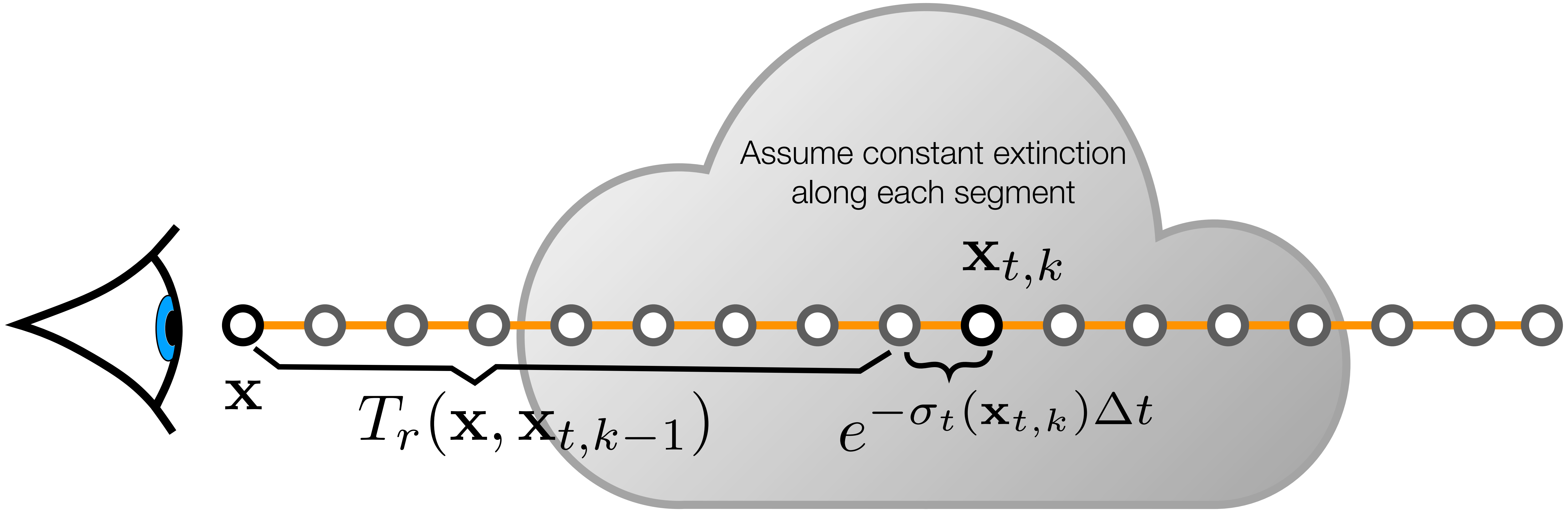
Homogeneous volume: $T_r(\mathbf{x}, \mathbf{x}_{t,k}) = e^{-\sigma_t ||\mathbf{x}, \mathbf{x}_{t,k}||}$



Ray Marching

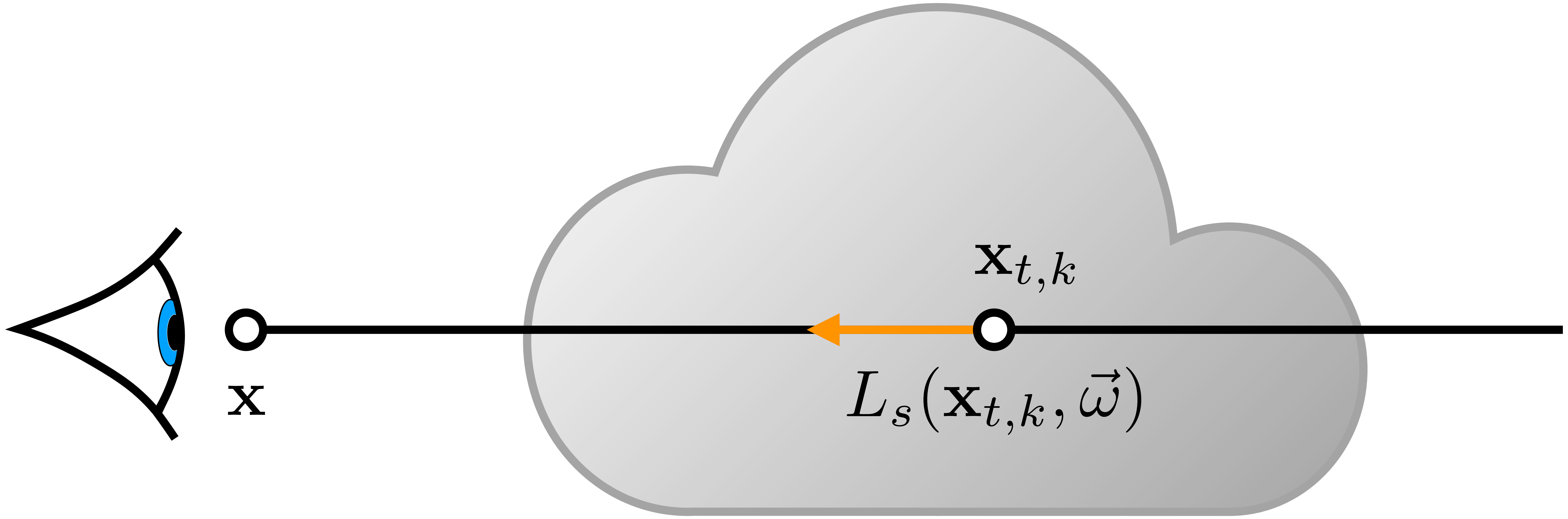
$$L_(\mathbf{x}, \vec{\omega}) \approx \sum_{k=0}^{N} T_r(\mathbf{x}, \mathbf{x}_{t,k})\sigma_s(\mathbf{x}_{t,k})L_s(\mathbf{x}_{t,k}, \vec{\omega})\Delta t$$

Heterogeneous volume: $T_r(\mathbf{x}, \mathbf{x}_{t,k}) = T_r(\mathbf{x}, \mathbf{x}_{t,k-1})e^{-\sigma_t(\mathbf{x}_{t,k})\Delta t}$


Assume constant extinction along each segment

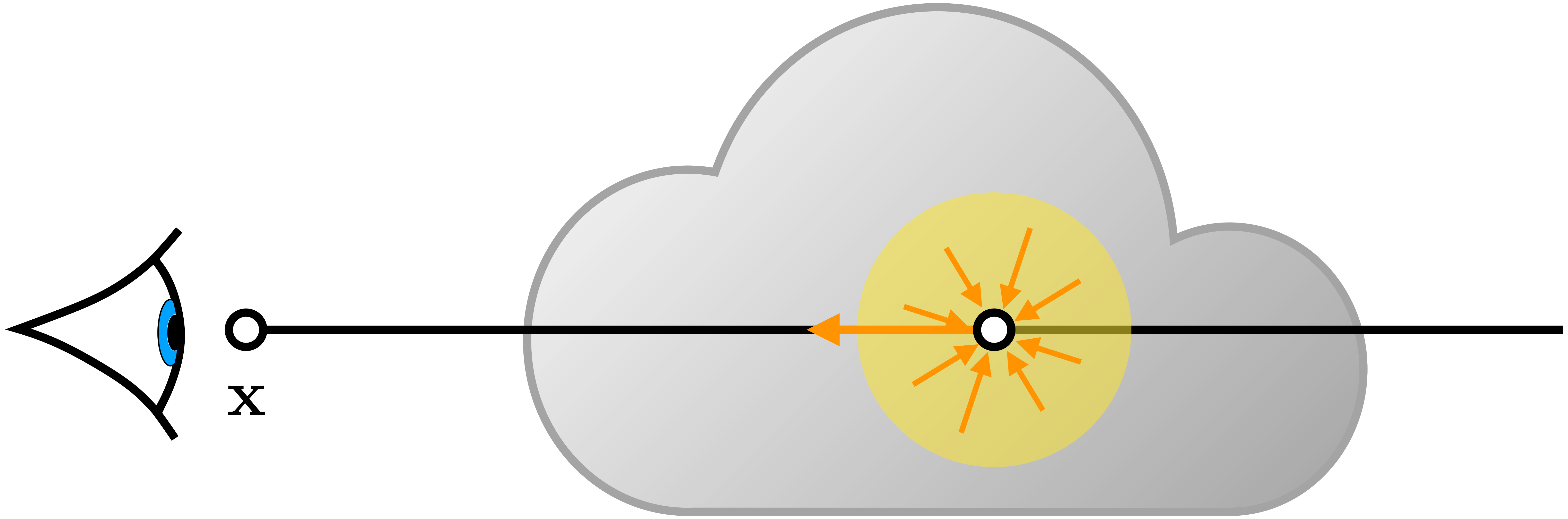
Ray Marching

$$L_(\mathbf{x}, \vec{\omega}) \approx \sum_{k=0}^{N} T_r(\mathbf{x}, \mathbf{x}_{t,k})\sigma_s(\mathbf{x}_{t,k})\boxed{L_s(\mathbf{x}_{t,k}, \vec{\omega})}\Delta t$$



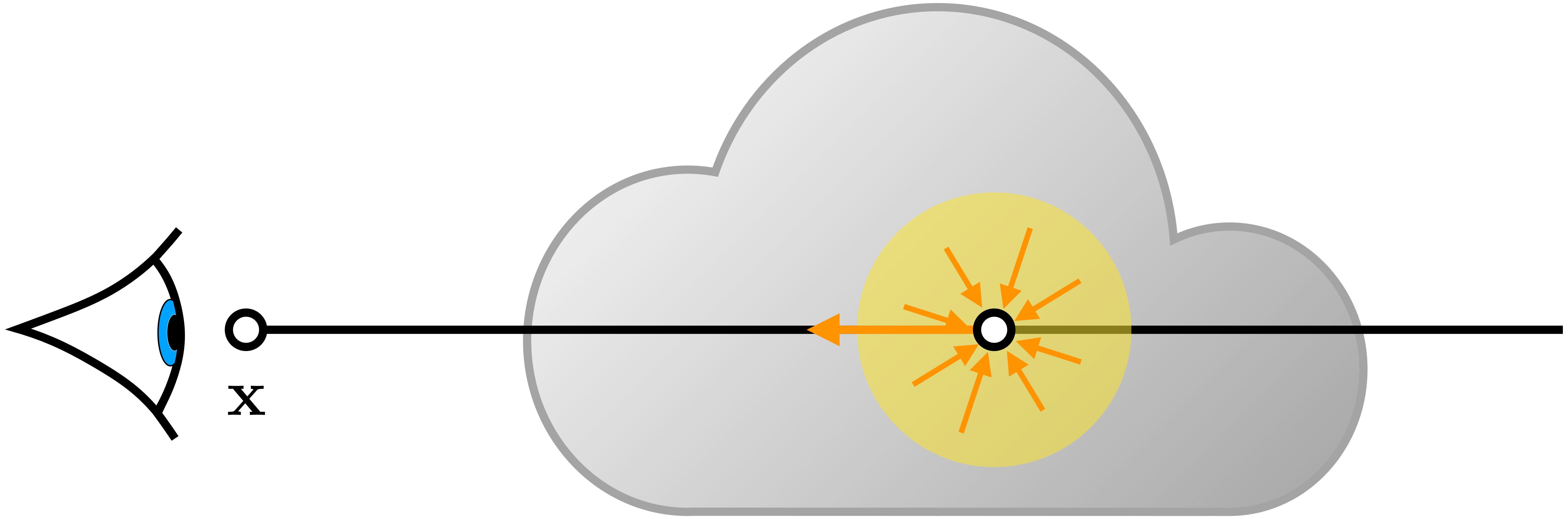
Ray Marching

$$L_s(\mathbf{x}_t, \vec{\omega}) = \boxed{\int_{S^2}} f_p(\mathbf{x}_t, \vec{\omega}, \vec{\omega}')L_i(\mathbf{x}_t, \vec{\omega}')d\vec{\omega}'$$



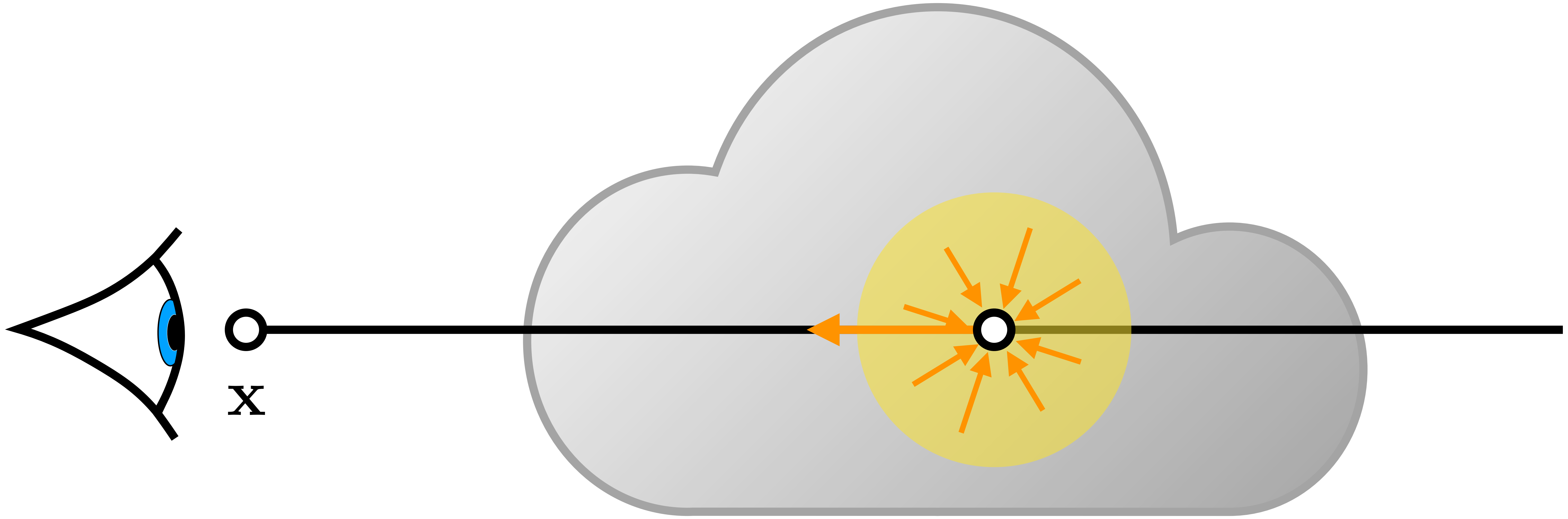
Ray Marching

$$L_s(\mathbf{x}_t, \vec{\omega}) \approx \frac{1}{M} \sum_{j=1}^{M} \frac{f_p(\mathbf{x}_t, \vec{\omega}, \vec{\omega}'_j) L_i(\mathbf{x}_t, \vec{\omega}'_j)}{p(\vec{\omega}'_j)}$$



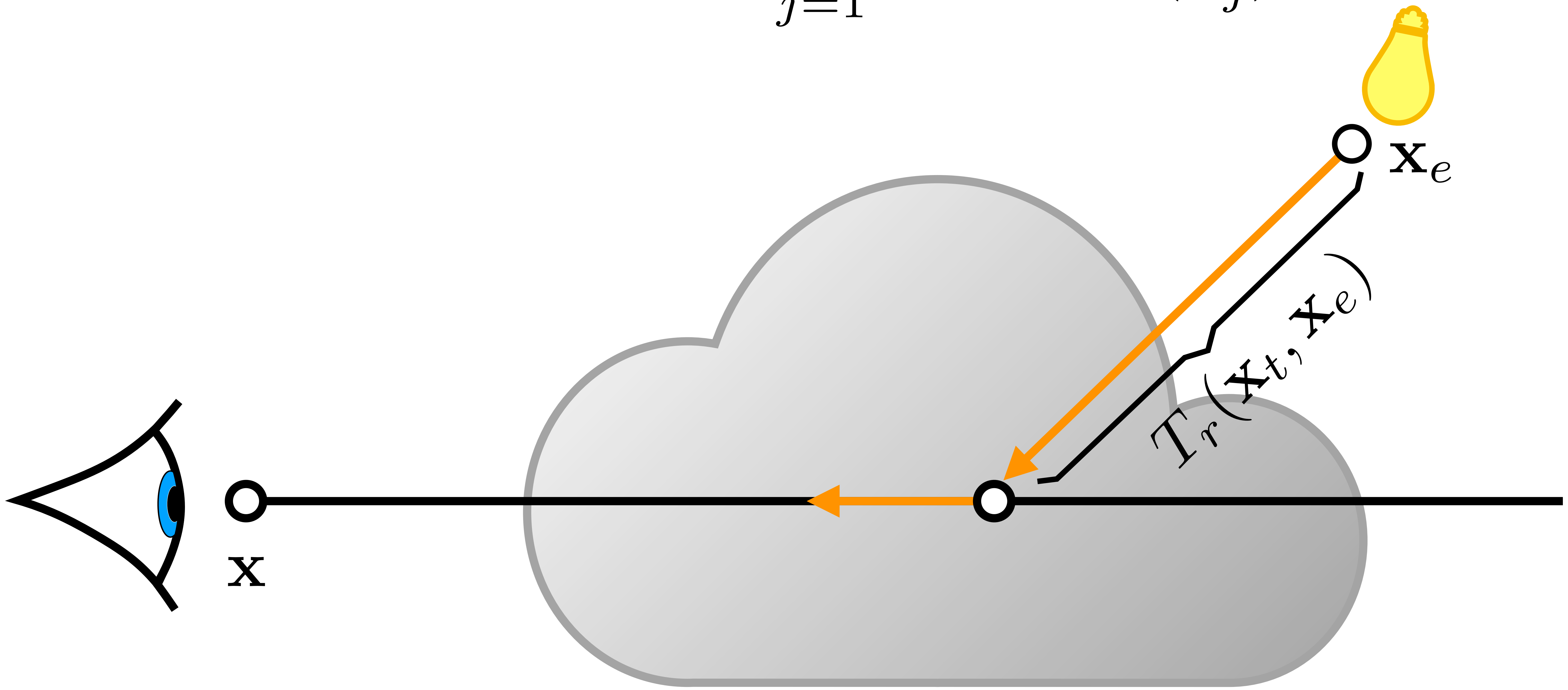
Ray Marching

$$L_s(\mathbf{x}_t, \vec{\omega}) \approx \frac{1}{M} \sum_{j=1}^{M} \frac{f_p(\mathbf{x}_t, \vec{\omega}, \vec{\omega}'_j) L_i(\mathbf{x}_t, \vec{\omega}'_j)}{p(\vec{\omega}'_j)}$$



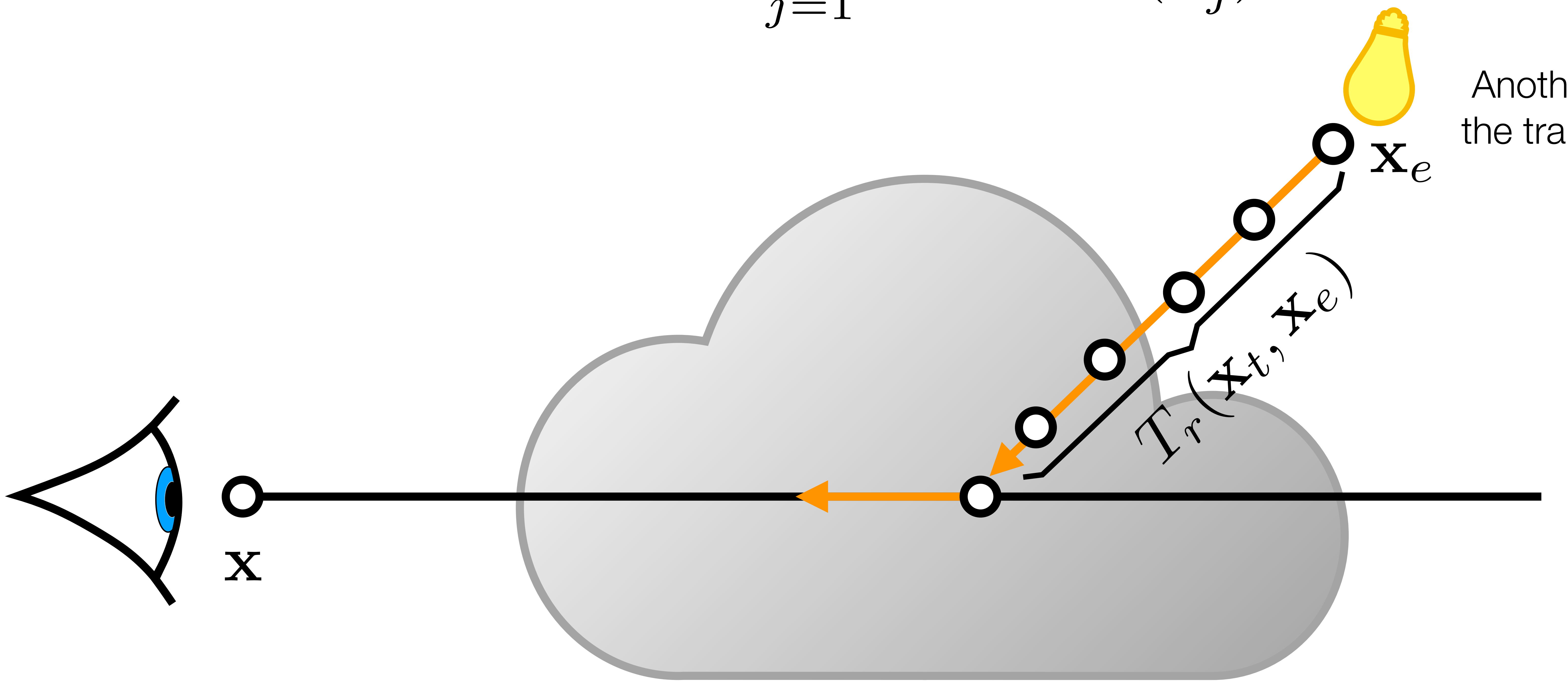
Ray Marching

$$L_s(\mathbf{x}_t, \vec{\omega}) \approx \frac{1}{M} \sum_{j=1}^{M} \frac{f_p(\mathbf{x}_t, \vec{\omega}, \vec{\omega}'_j) L_i(\mathbf{x}_t, \vec{\omega}'_j)}{p(\vec{\omega}'_j)}$$



Ray Marching

$$L_s(\mathbf{x}_t, \vec{\omega}) \approx \frac{1}{M} \sum_{j=1}^{M} \frac{f_p(\mathbf{x}_t, \vec{\omega}, \vec{\omega}'_j) L_i(\mathbf{x}_t, \vec{\omega}'_j)}{p(\vec{\omega}'_j)}$$

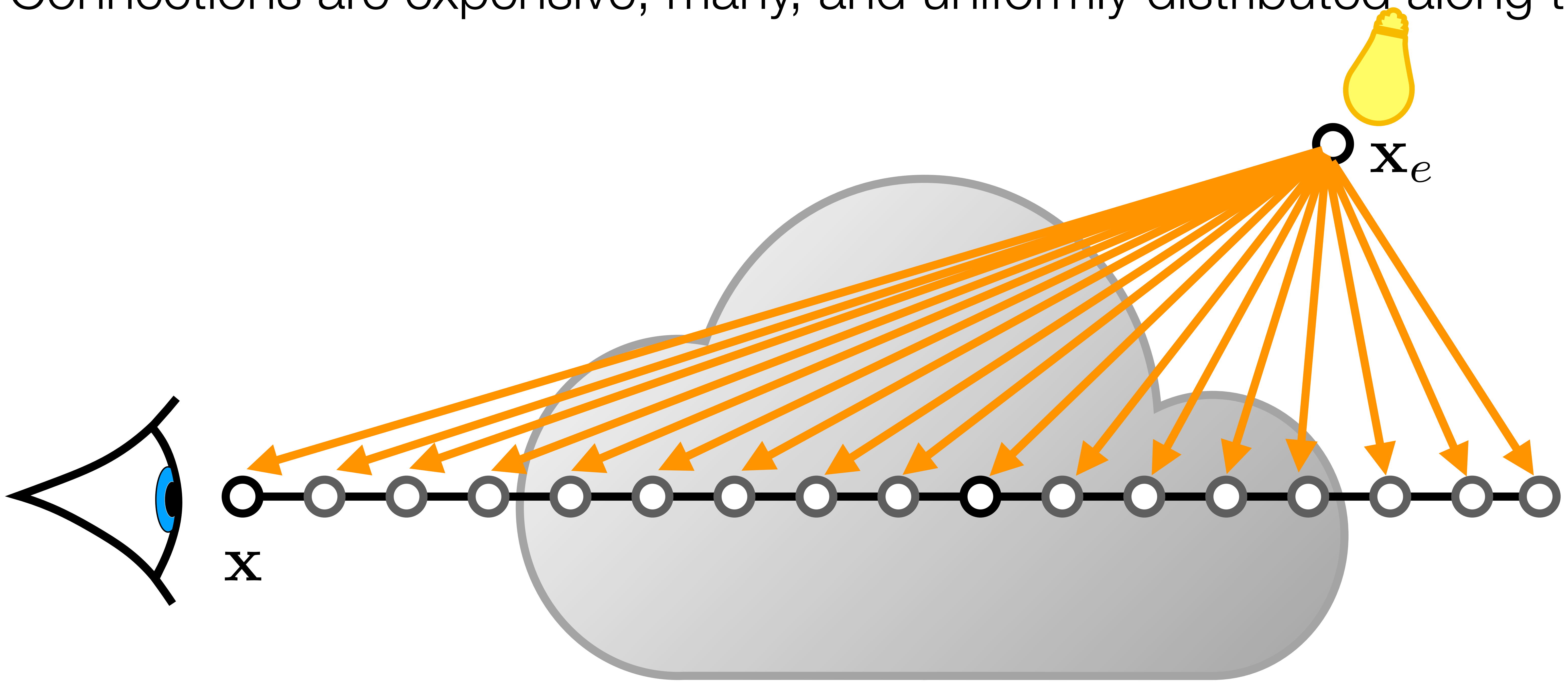


Another ray marching needed to estimate the transmittance along the connection ray (in the heterogeneous media)

Ray Marching in Heterogeneous Media

Marching towards the light source

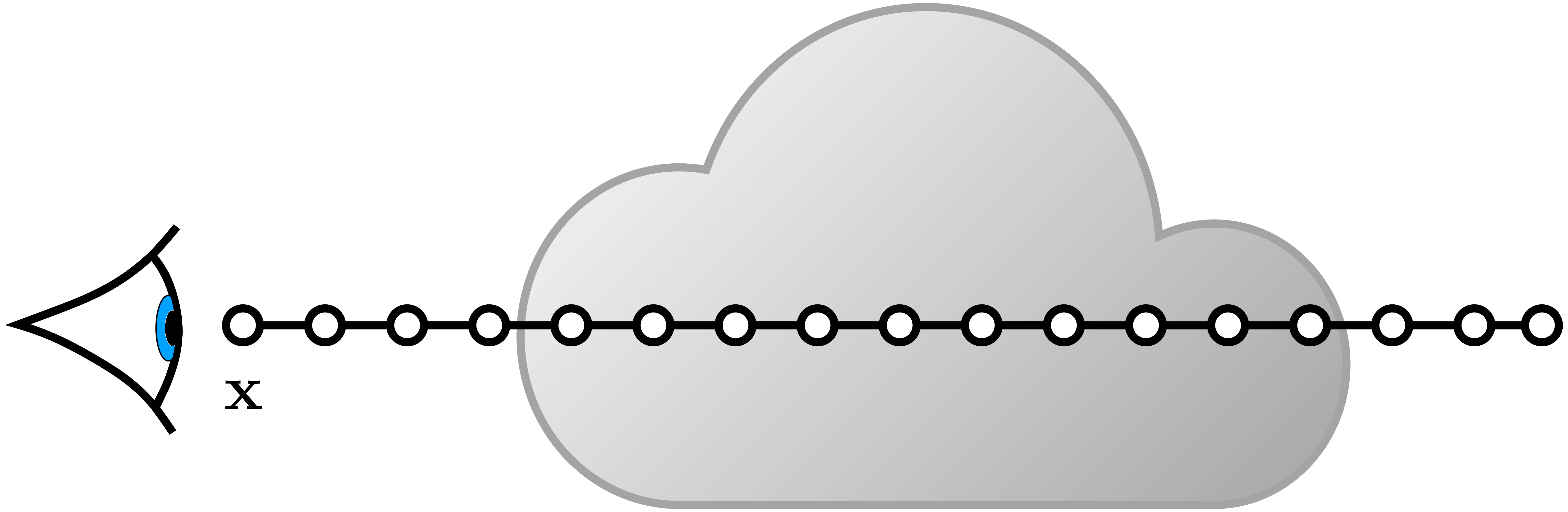
- Connections are expensive, many, and uniformly distributed along the primary ray



Decoupled Transmittance and in-scattering

1. Ray march and cache transmittance

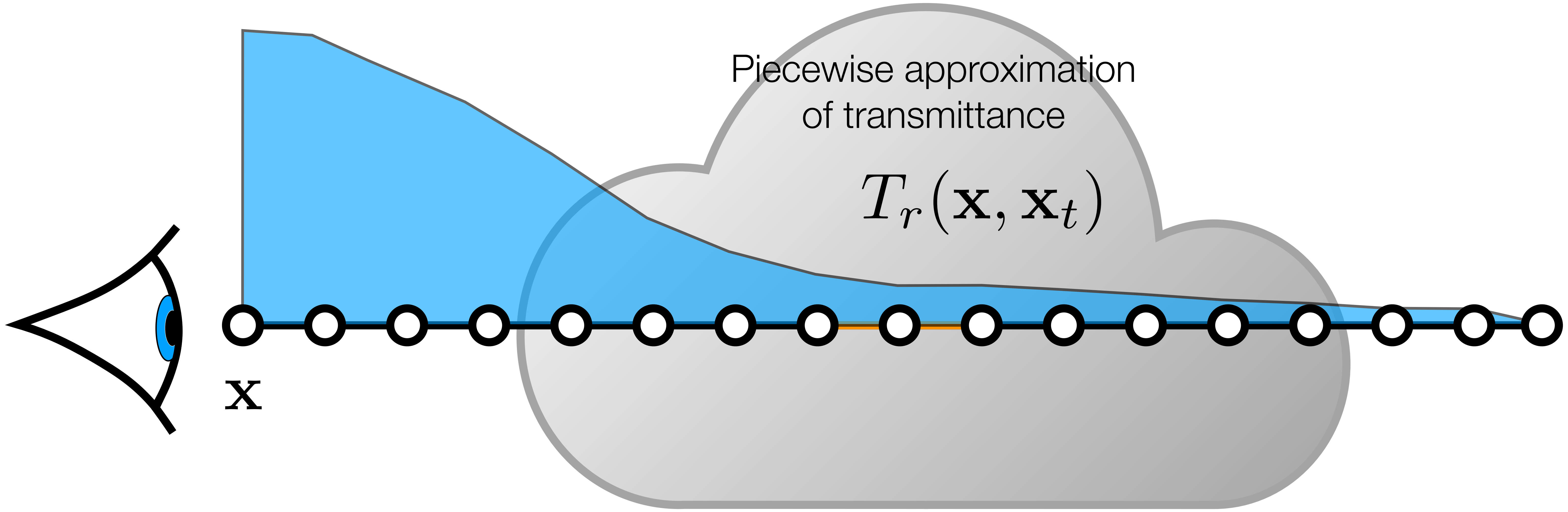
    - Choose step-size w.r.t. frequency content to accurately capture variations



Decoupled Transmittance and in-scattering

1. Ray march and cache transmittance

    - Choose step-size w.r.t. frequency content to accurately capture variations

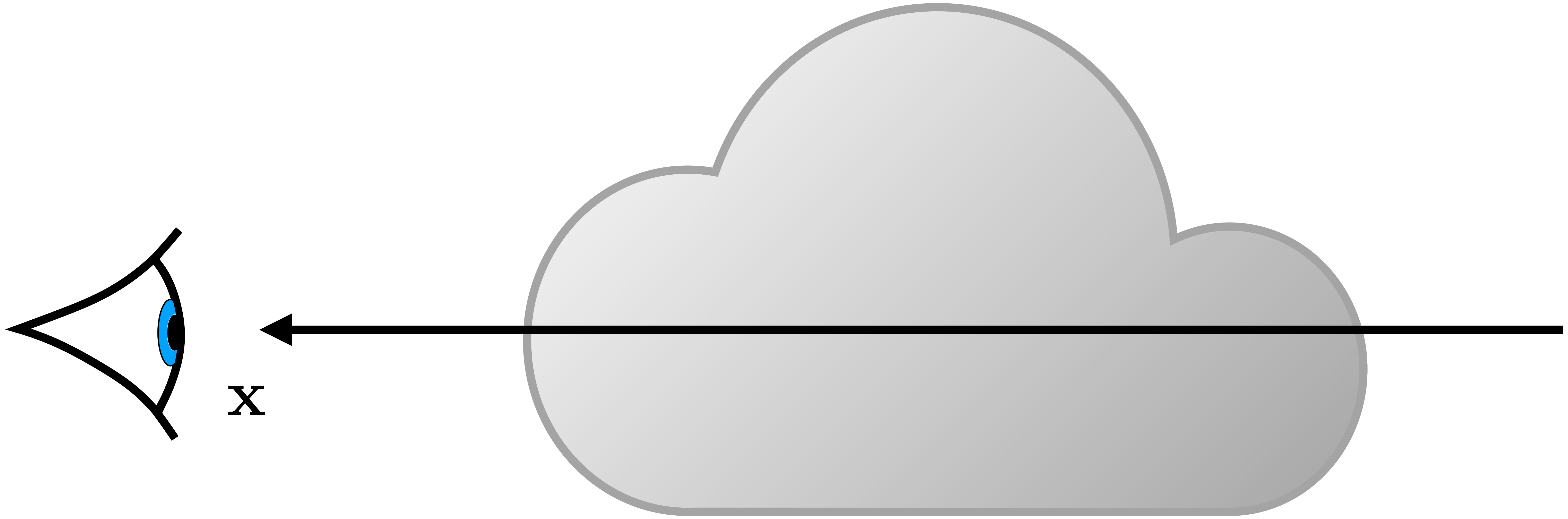

Piecewise approximation of transmittance

$T_r(\mathbf{x}, \mathbf{x}_t)$

Decoupled Transmittance and in-scattering

2. Estimate in-scattering using MC integration

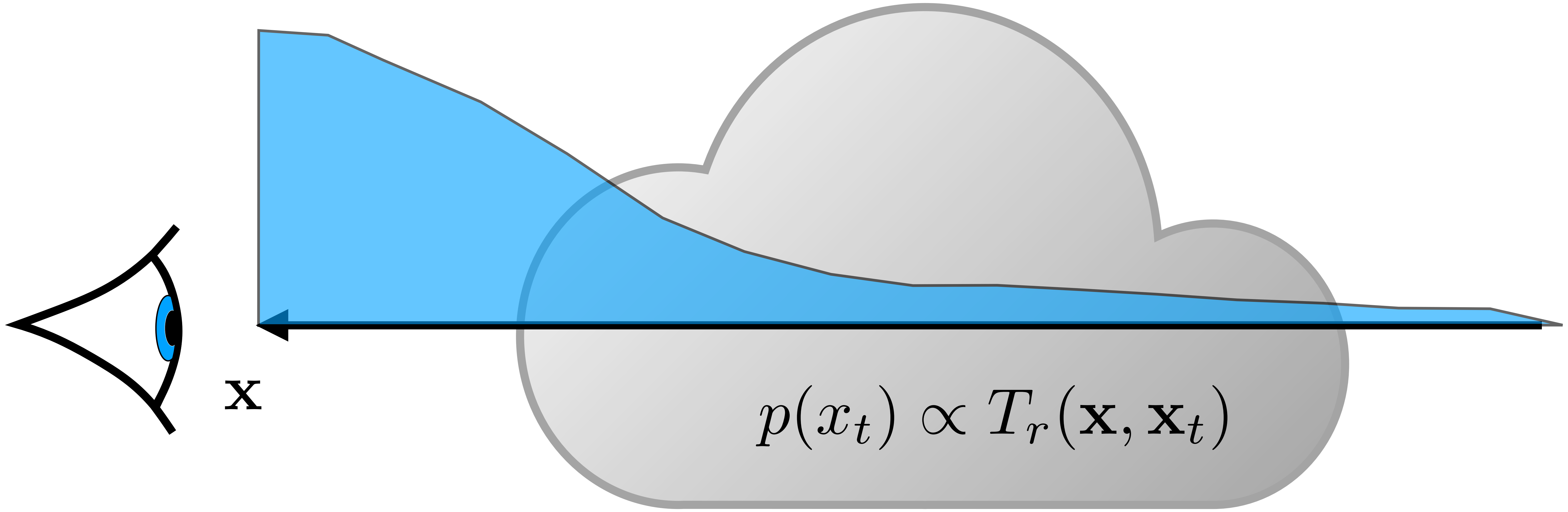
    - Distribute samples proportional to (part of) the integrand



Decoupled Transmittance and in-scattering

2. Estimate in-scattering using MC integration

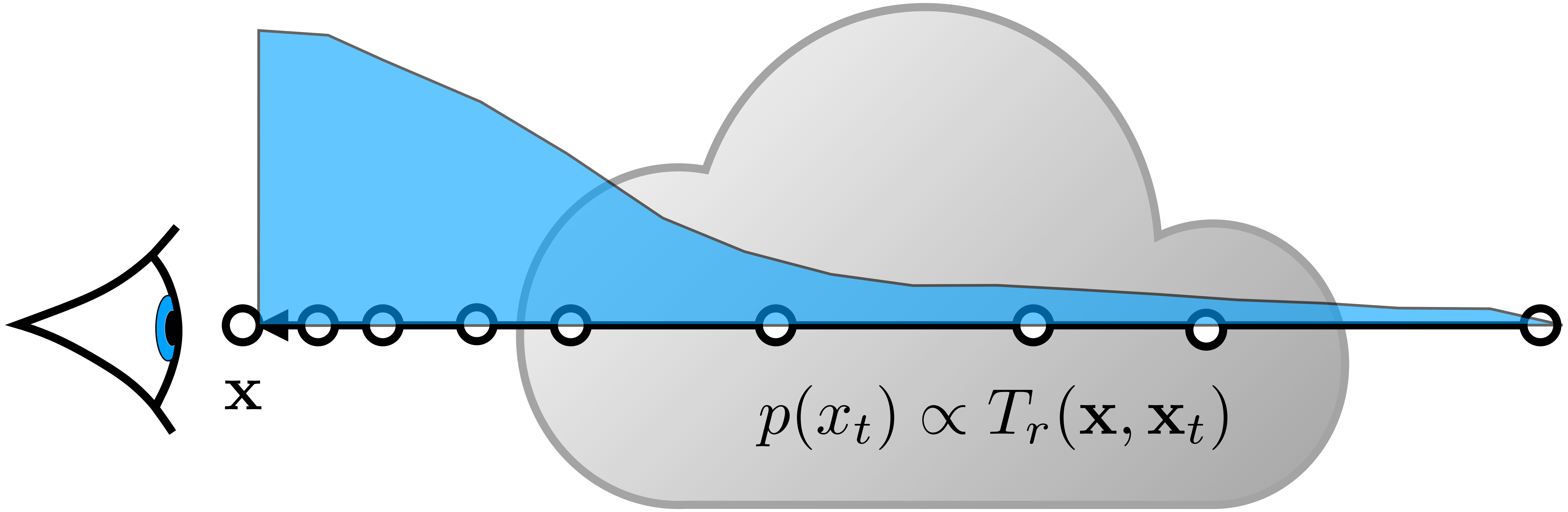
   - Distribute samples proportional to (part of) the integrand



Decoupled Transmittance and in-scattering

2. Estimate in-scattering using MC integration

  - Distribute samples proportional to (part of) the integrand

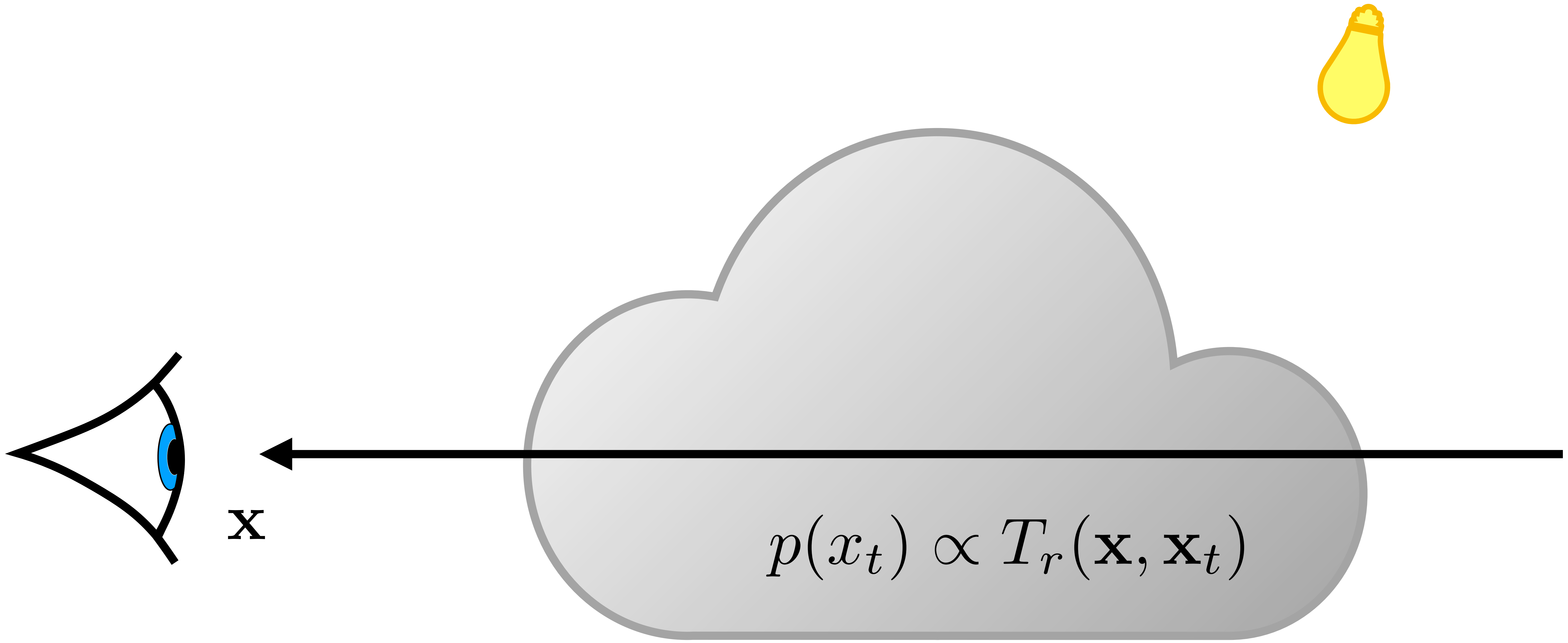


$$p(x_t) \propto T_r(\mathbf{x}, \mathbf{x}_t)$$

Decoupled Transmittance and in-scattering

2. Estimate in-scattering using MC integration

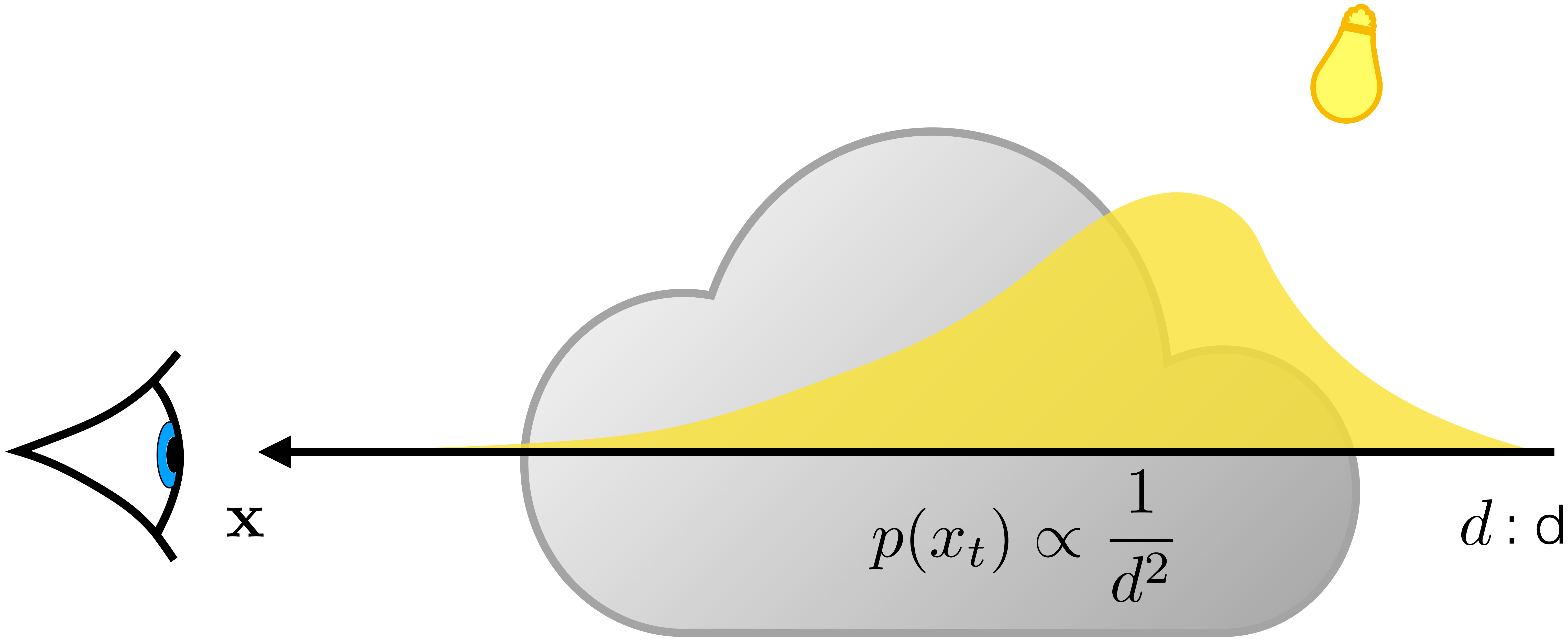
   - Distribute samples proportional to (part of) the integrand



Decoupled Transmittance and in-scattering

2. Estimate in-scattering using MC integration

   - Distribute samples proportional to (part of) the integrand



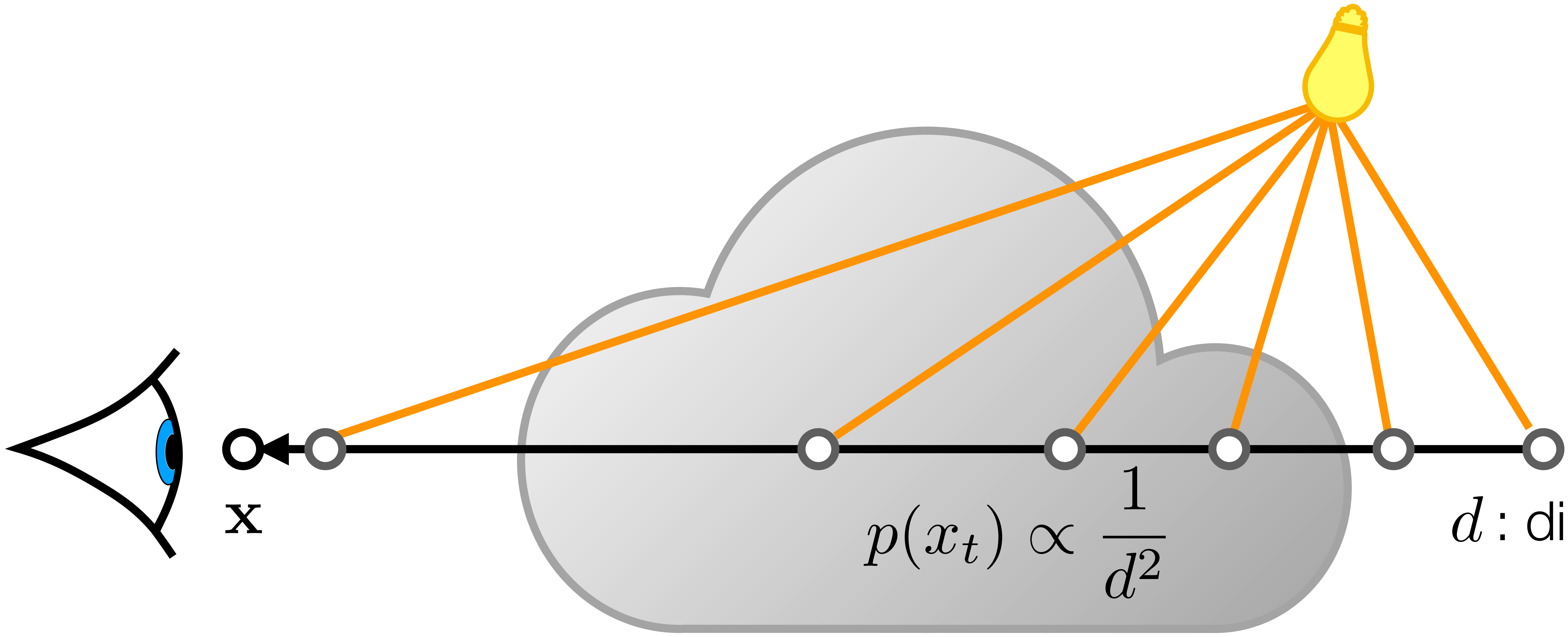
$$p(x_t) \propto \frac{1}{d^2}$$

d : distance to light

Decoupled Transmittance and in-scattering

2. Estimate in-scattering using MC integration

- Distribute samples proportional to (part of) the integrand



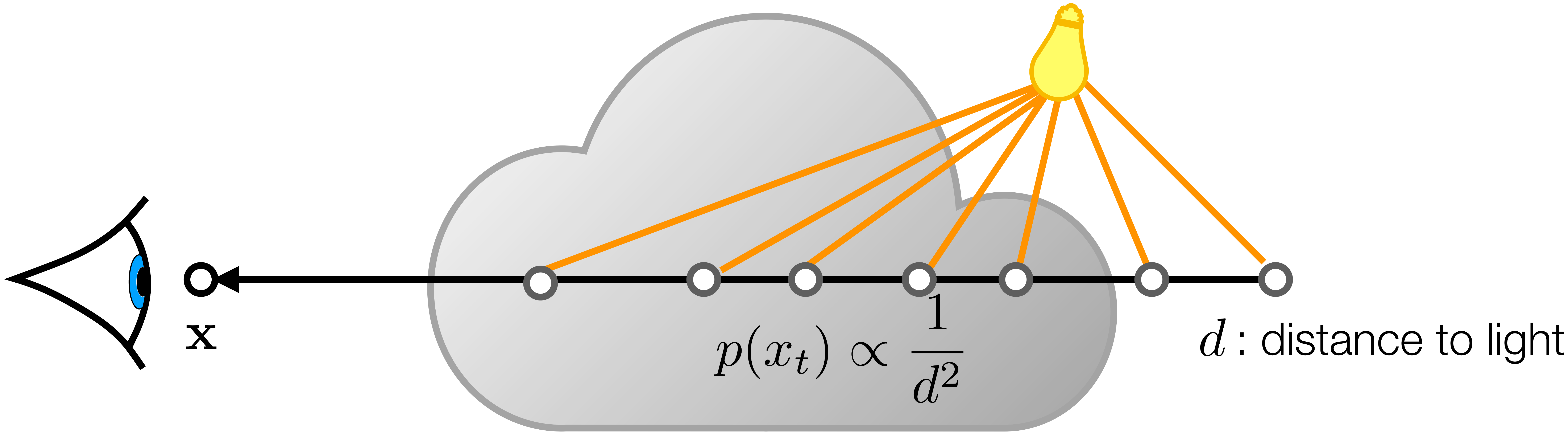
$$p(x_t) \propto \frac{1}{d^2}$$

d : distance to light

Decoupled Transmittance and in-scattering

2. Estimate in-scattering using MC integration

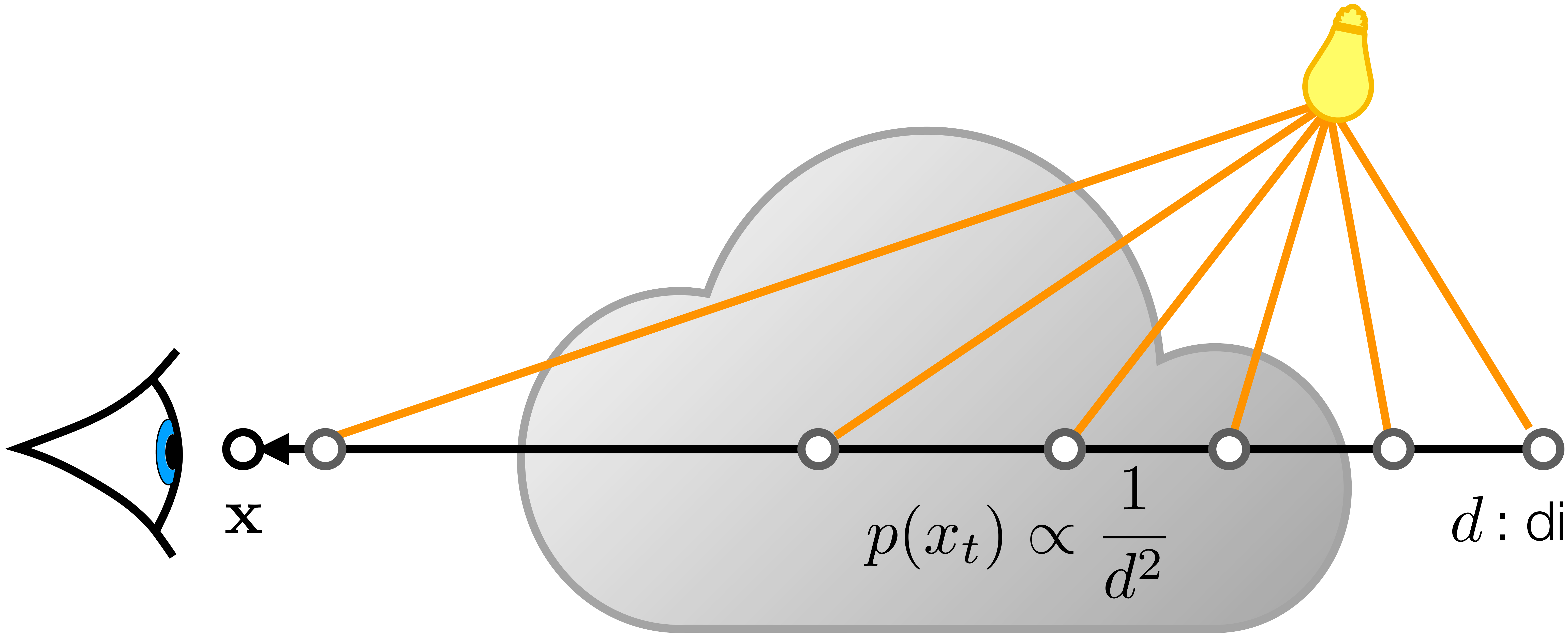
    - Distribute samples proportional to (part of) the integrand



Decoupled Transmittance and in-scattering

2. Estimate in-scattering using MC integration

- Distribute samples proportional to (part of) the integrand



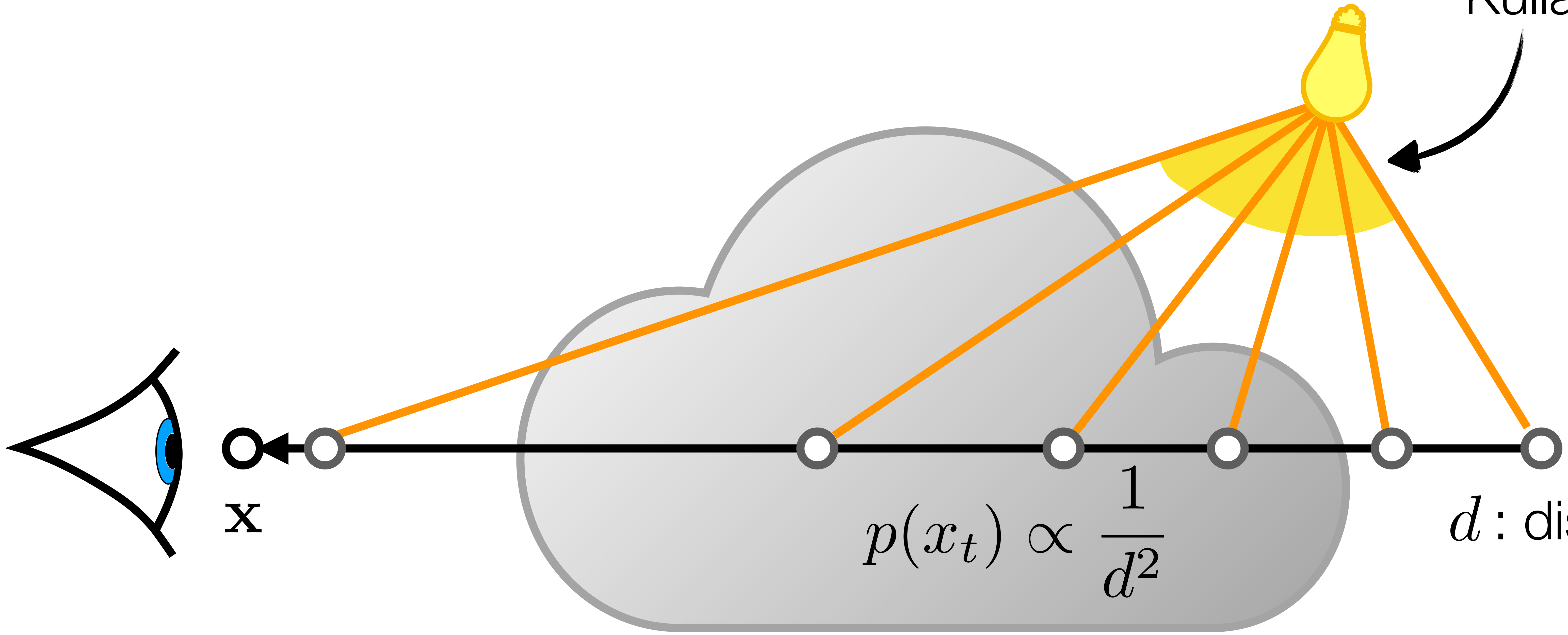
$p(x_t) \propto \frac{1}{d^2}$

d : distance to light

Decoupled Transmittance and in-scattering

2. Estimate in-scattering using MC integration

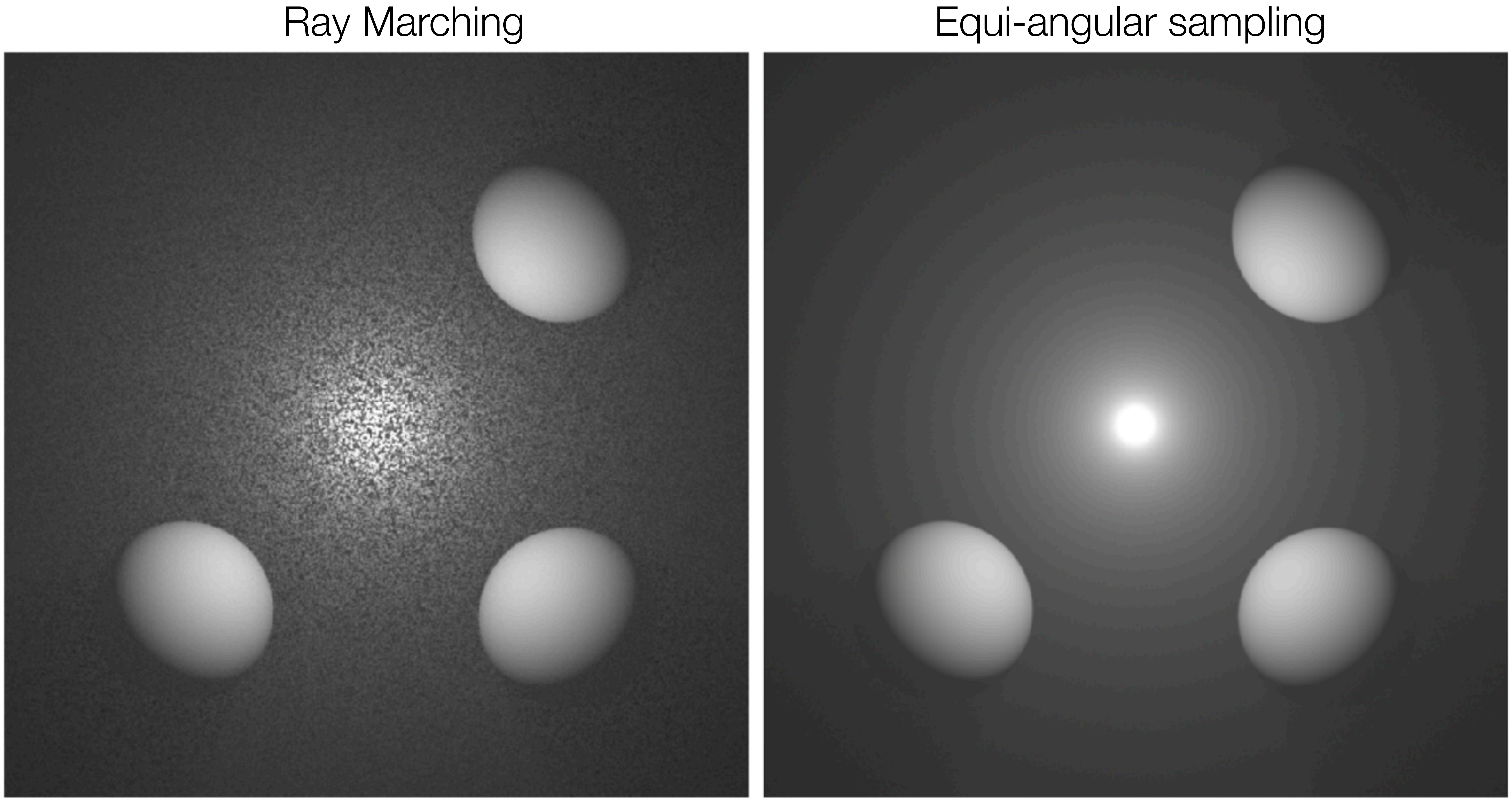
- Distribute samples proportional to (part of) the integrand



Equiangular sampling
Kulla and Fajardo [2012]

d : distance to light

Decoupled Transmittance and in-scattering



Ray Marching

Equi-angular sampling

Single scattering

Multiple scattering

Volumetric Path Tracing

Volumetric Path Tracing

Motivation

Same as with path tracing: avoid the exponential growth

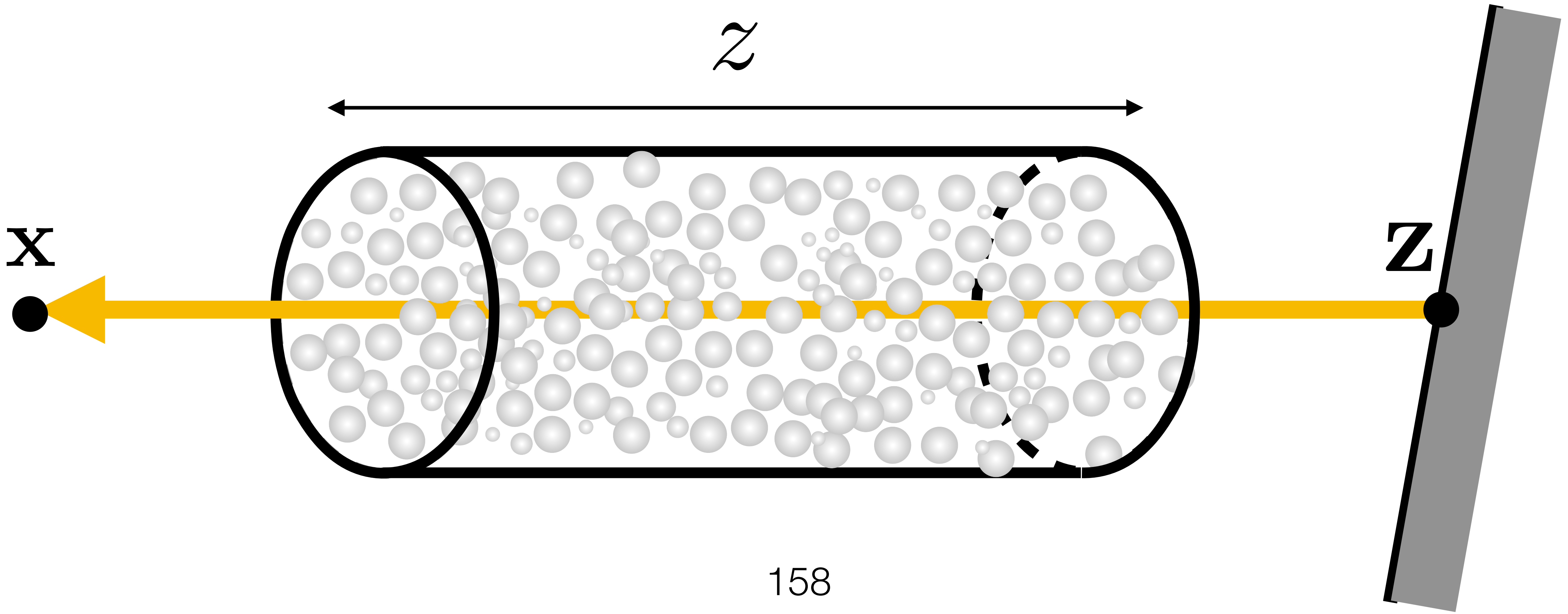
Paths can:

Reflect / Refract off surfaces

Scatter inside a volume

Volumetric Rendering Equation

$$
\begin{aligned}
L(\mathbf{x},\vec{\omega}) &= \int_0^z T_r(\mathbf{x},\mathbf{x}_t)\sigma_a(\mathbf{x}_t)L_e(\mathbf{x}_t,\vec{\omega})dt \\
&+ \int_0^z T_r(\mathbf{x},\mathbf{x}_t)\sigma_s(\mathbf{x}_t)L_s(\mathbf{x}_t,\omega)dt \\
&+ \ T_r(\mathbf{x},\mathbf{x}_z)L(\mathbf{x}_z,\vec{\omega})
\end{aligned}
$$



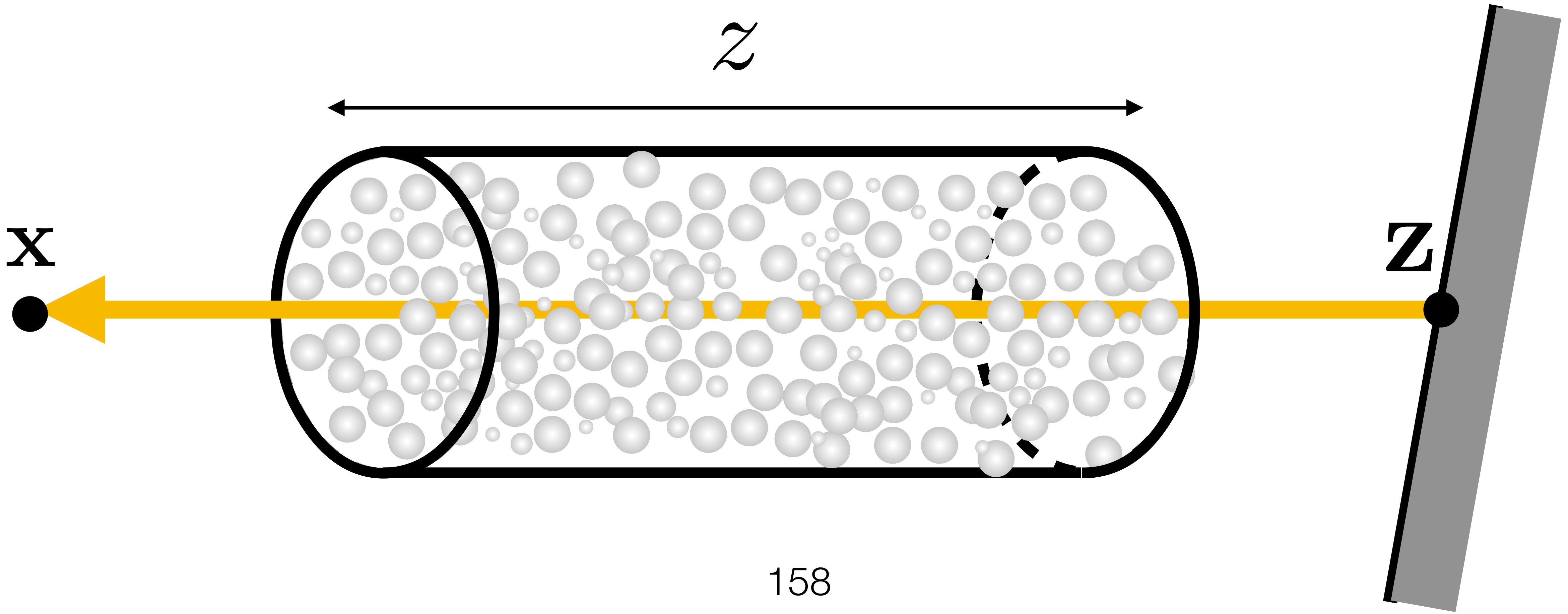
Volumetric Rendering Equation

$$L(\mathbf{x},\vec{\omega}) = \int_0^z T_r(\mathbf{x},\mathbf{x}_t)\sigma_a(\mathbf{x}_t)L_e(\mathbf{x}_t,\vec{\omega})dt$$

Accumulated emitted radiance

$$+ \int_0^z T_r(\mathbf{x},\mathbf{x}_t)\sigma_s(\mathbf{x}_t)L_s(\mathbf{x}_t,\omega)dt$$

$$+ \; T_r(\mathbf{x},\mathbf{x}_z)L(\mathbf{x}_z,\vec{\omega})$$



Volumetric Rendering Equation

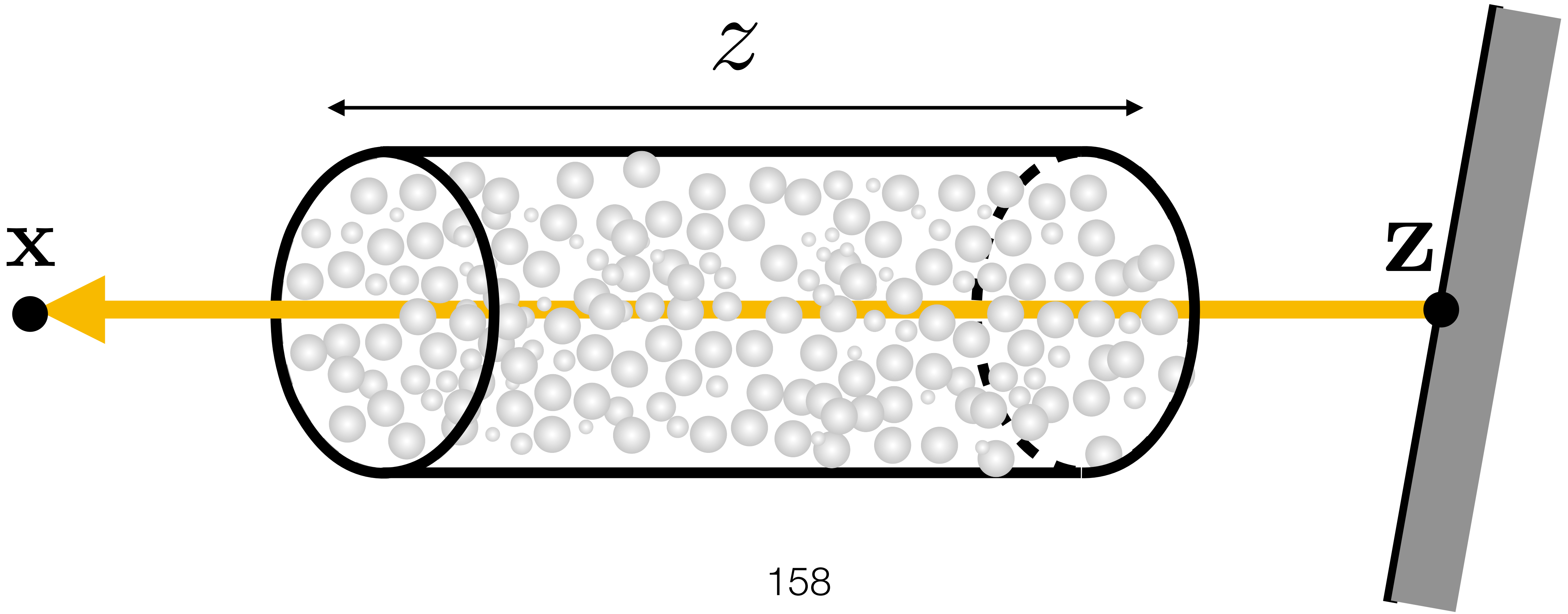
$$L(\mathbf{x},\vec{\omega}) = \int_0^z T_r(\mathbf{x},\mathbf{x}_t)\sigma_a(\mathbf{x}_t)L_e(\mathbf{x}_t,\vec{\omega})dt$$

Accumulated emitted radiance

$$+ \int_0^z T_r(\mathbf{x},\mathbf{x}_t)\sigma_s(\mathbf{x}_t)L_s(\mathbf{x}_t,\omega)dt$$

Accumulated in-scattered radiance

$$+ \ T_r(\mathbf{x},\mathbf{x}_z)L(\mathbf{x}_z,\vec{\omega})$$



Volumetric Rendering Equation

$$L(\mathbf{x},\vec{\omega}) = \int_0^z T_r(\mathbf{x},\mathbf{x}_t)\sigma_a(\mathbf{x}_t)L_e(\mathbf{x}_t,\vec{\omega})dt$$

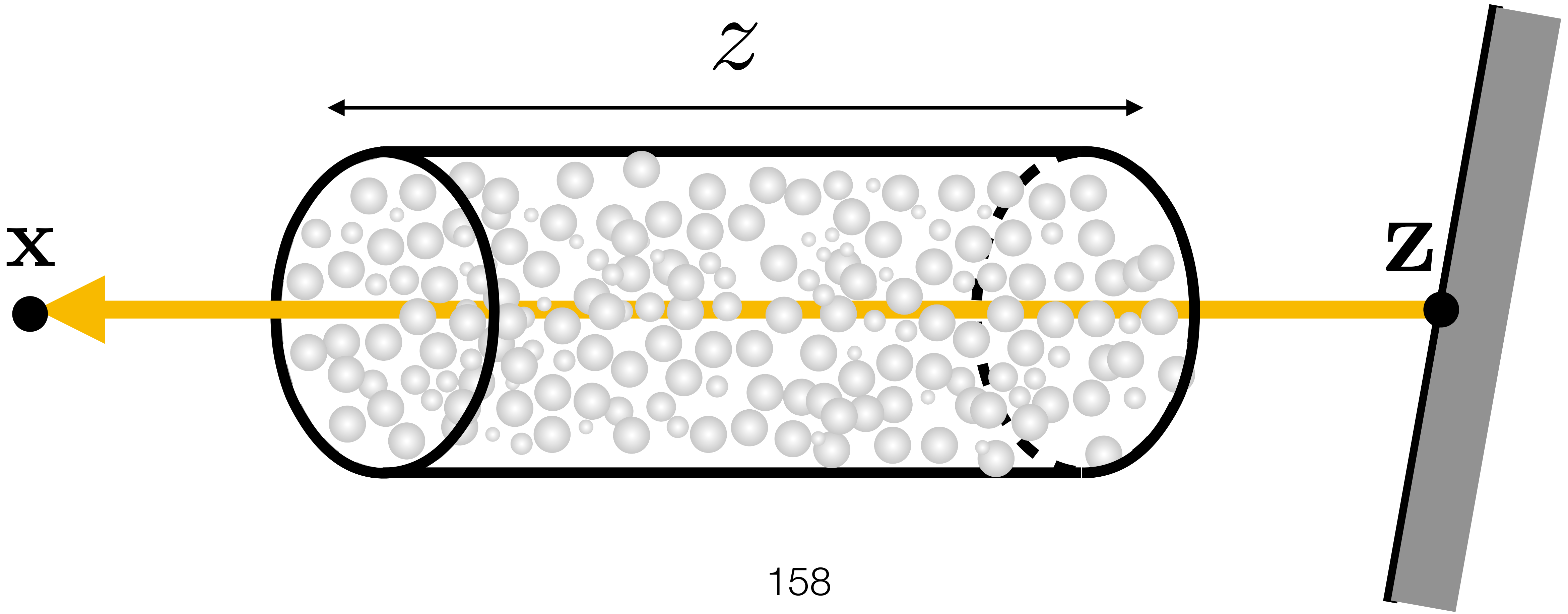
Accumulated emitted radiance

$$+\int_0^z T_r(\mathbf{x},\mathbf{x}_t)\sigma_s(\mathbf{x}_t)L_s(\mathbf{x}_t,\omega)dt$$

Accumulated in-scattered radiance

$$+\ T_r(\mathbf{x},\mathbf{x}_z)L(\mathbf{x}_z,\vec{\omega})$$

Attenuated background radiance



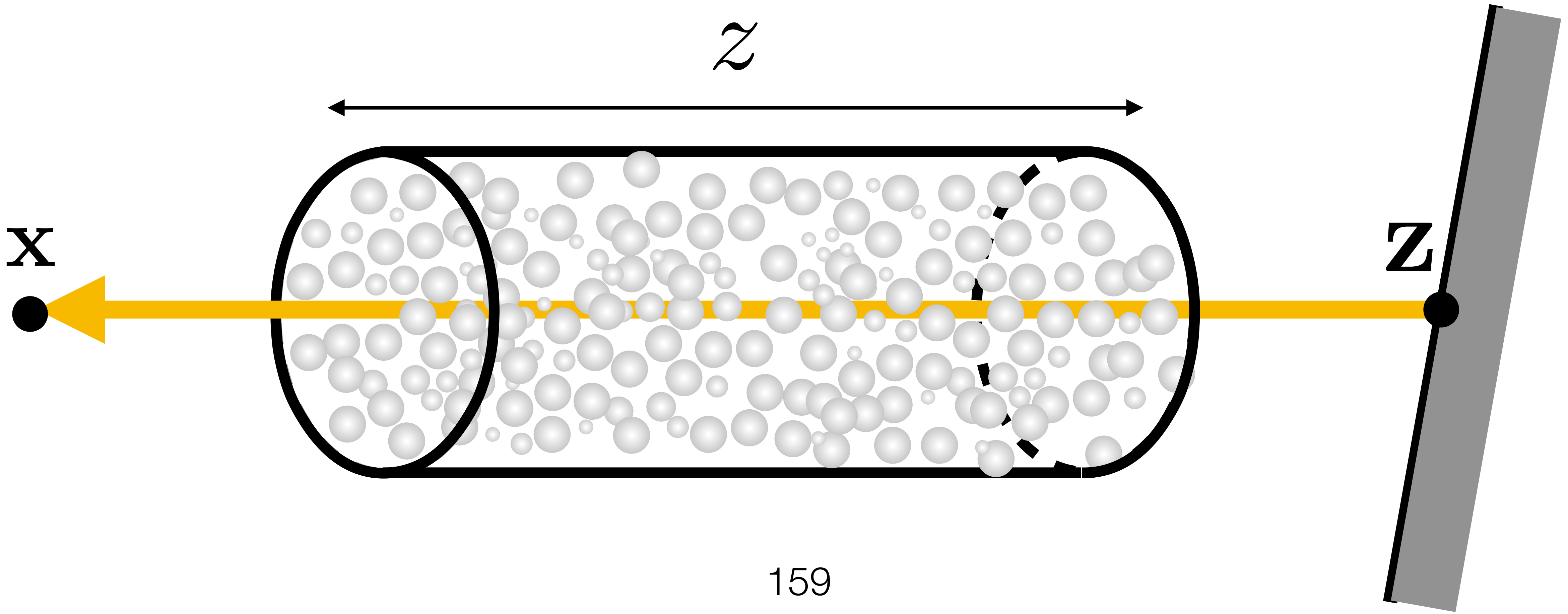
Volumetric Rendering Equation

Accumulated emitted + in-scattered radiance

$$L(\mathbf{x},\vec{\omega}) = \int_0^z T_r(\mathbf{x},\mathbf{x}_t)\Big[\sigma_a(\mathbf{x}_t)L_e(\mathbf{x}_t,\vec{\omega}) + \sigma_s(\mathbf{x}_t)L_s(\mathbf{x}_t,\vec{\omega})\Big]dt$$

$$+T_r(\mathbf{x},\mathbf{x}_z)L(\mathbf{x}_z,\vec{\omega})$$

Attenuated background radiance



Volumetric Rendering Equation

$$L(\mathbf{x},\vec{\omega}) = \int_0^z T_r(\mathbf{x},\mathbf{x}_t)\Big[\sigma_a(\mathbf{x}_t)L_e(\mathbf{x}_t,\vec{\omega}) + \sigma_s(\mathbf{x}_t)L_s(\mathbf{x}_t,\vec{\omega})\Big]dt$$

$$+T_r(\mathbf{x},\mathbf{x}_z)L(\mathbf{x}_z,\vec{\omega})$$

1-Sample Monte Carlo Estimator

$$\langle L(\mathbf{x},\vec{\omega})\rangle = \frac{T_r(\mathbf{x},\mathbf{x}_t)}{p(t)}\Big[\sigma_a(\mathbf{x}_t)L_e(\mathbf{x}_t,\vec{\omega}) + \sigma_s(\mathbf{x}_t)L_s(\mathbf{x}_t,\vec{\omega})\Big]$$

$$+\frac{T_r(\mathbf{x},\mathbf{x}_z)}{P(z)}L(\mathbf{x}_z,\vec{\omega})$$

1-Sample Monte Carlo Estimator

$$\langle L(\mathbf{x},\vec{\omega})\rangle = \frac{T_r(\mathbf{x},\mathbf{x}_t)}{p(t)}\Big[\sigma_a(\mathbf{x}_t)L_e(\mathbf{x}_t,\vec{\omega}) + \sigma_s(\mathbf{x}_t)L_s(\mathbf{x}_t,\vec{\omega})\Big] + \frac{T_r(\mathbf{x},\mathbf{x}_z)}{P(z)}L(\mathbf{x}_z,\vec{\omega})$$

$p(t)$ Probability density of distance t

$P(z)$ Probability of exceeding distance z

1-Sample Monte Carlo Estimator

$$\langle L(\mathbf{x},\vec{\omega})\rangle = \frac{T_r(\mathbf{x},\mathbf{x}_t)}{p(t)}\left[\sigma_a(\mathbf{x}_t)L_e(\mathbf{x}_t,\vec{\omega}) + \sigma_s(\mathbf{x}_t)\frac{f_p(\mathbf{x},\vec{\omega},\vec{\omega}_i)L(\mathbf{x}_t,\vec{\omega})}{p(\vec{\omega}_i)}\right]$$

$$+\frac{T_r(\mathbf{x},\mathbf{x}_z)}{P(z)}L(\mathbf{x}_z,\vec{\omega})$$

$p(t)$	Probability density of distance t
$P(z)$	Probability of exceeding distance z
$p(\vec{\omega}_i)$	Probability density of direction $\vec{\omega}_i$

Volumetric Path Tracing

1. Sample distance to next interaction

2. Scatter in the volume or bounce off a surface

Volumetric Path Tracing

1. Sample distance to next interaction

2. Scatter in the volume or bounce off a surface

Volumetric Path Tracing

1. Sample distance to next interaction

2. Scatter in the volume or bounce off a surface

Volumetric Path Tracing

1. Sample distance to next interaction

2. Scatter in the volume or bounce off a surface

Volumetric Path Tracing

1. Sample distance to next interaction

2. Scatter in the volume or bounce off a surface

Volumetric Path Tracing

1. Sample distance to next interaction

2. Scatter in the volume or bounce off a surface

Volumetric Path Tracing

1. Sample distance to next interaction

2. Scatter in the volume or bounce off a surface

Volumetric Path Tracing

1. Sample distance to next interaction

2. Scatter in the volume or bounce off a surface

Volumetric Path Tracing

1. Sample distance to next interaction

2. Scatter in the volume or bounce off a surface

Volumetric Path Tracing

1. Sample distance to next interaction

2. Scatter in the volume or bounce off a surface

Volumetric Path Tracing

1. Sample distance to next interaction

2. Scatter in the volume or bounce off a surface

Volumetric Path Tracing

1. Sample distance to next interaction

2. Scatter in the volume or bounce off a surface

Volumetric Path Tracing

1. Sample distance to next interaction

2. Scatter in the volume or bounce off a surface

Volumetric Path Tracing

1. Sample distance to next interaction

2. Scatter in the volume or bounce off a surface

Volumetric Path Tracing

1. Sample distance to next interaction

2. Scatter in the volume or bounce off a surface

Volumetric Path Tracing

1. Sample distance to next interaction

2. Scatter in the volume or bounce off a surface

Volumetric Path Tracing

1. Sample distance to next interaction

2. Scatter in the volume or bounce off a surface

Volumetric Path Tracing with NEE

Sampling the Phase Function

Isotropic:

Henyey-Greenstein:

Sampling the Phase Function

Isotropic: Uniform sphere sampling

Henyey-Greenstein:

Sampling the Phase Function

Isotropic: Uniform sphere sampling

Henyey-Greenstein: Using the inversion method we can derive

$$\cos\theta = \frac{1}{2g}\left(1 + g^2 - \left(\frac{1-g^2}{1-g+2g\xi_1}\right)^2\right)$$

$$\phi = 2\pi\xi_2$$

PDF is the value of the HG phase function

Free-path Sampling

Free-path or free-flight distance:

- Distance to the next interaction in the medium

- Dense media (e.g. milk): short mean-free path

- Thin media (e.g. atmosphere): long mean-free path

Free-path Sampling

Free-path or free-flight distance:

- Distance to the next interaction in the medium
- Dense media (e.g. milk): short mean-free path
- Thin media (e.g. atmosphere): long mean-free path

Ideally, we want to sample according to (part of) of the integrand:

$$p(\mathbf{x}_t | (\mathbf{x}, \vec{\omega})) \propto T_r(\mathbf{x}, \mathbf{x}_t)$$

Free-path Sampling

Free-path or free-flight distance:

- Distance to the next interaction in the medium
- Dense media (e.g. milk): short mean-free path
- Thin media (e.g. atmosphere): long mean-free path

Ideally, we want to sample according to (part of) of the integrand:

$$p(\mathbf{x}_t|(\mathbf{x},\vec{\omega})) \propto T_r(\mathbf{x},\mathbf{x}_t)$$

simplified notation

$$p(t) \propto T_r(t)$$

Free-path Sampling

Homogeneous media: $T_r(t) = e^{-\sigma_t t}$

Free-path Sampling

Homogeneous media: $T_r(t) = e^{-\sigma_t t}$

PDF: $p(t) \propto e^{-\sigma_t t}$

Free-path Sampling

Homogeneous media: $T_r(t) = e^{-\sigma_t t}$

PDF: $p(t) \propto e^{-\sigma_t t}$

$$p(t) = \frac{e^{-\sigma_t t}}{\int_0^\infty e^{-\sigma_t s} ds} = \sigma_t e^{-\sigma_t t}$$

Free-path Sampling

Homogeneous media: $T_r(t) = e^{-\sigma_t t}$

PDF: $p(t) \propto e^{-\sigma_t t}$

$$p(t) = \frac{e^{-\sigma_t t}}{\int_0^\infty e^{-\sigma_t s} ds} = \sigma_t e^{-\sigma_t t}$$

CDF: $P(t) = \int_0^t e^{-\sigma_t s} ds = 1 - e^{-\sigma_t t}$

Free-path Sampling

Homogeneous media: $T_r(t) = e^{-\sigma_t t}$

PDF: $p(t) \propto e^{-\sigma_t t}$

$$p(t) = \frac{e^{-\sigma_t t}}{\int_0^\infty e^{-\sigma_t s} ds} = \sigma_t e^{-\sigma_t t}$$

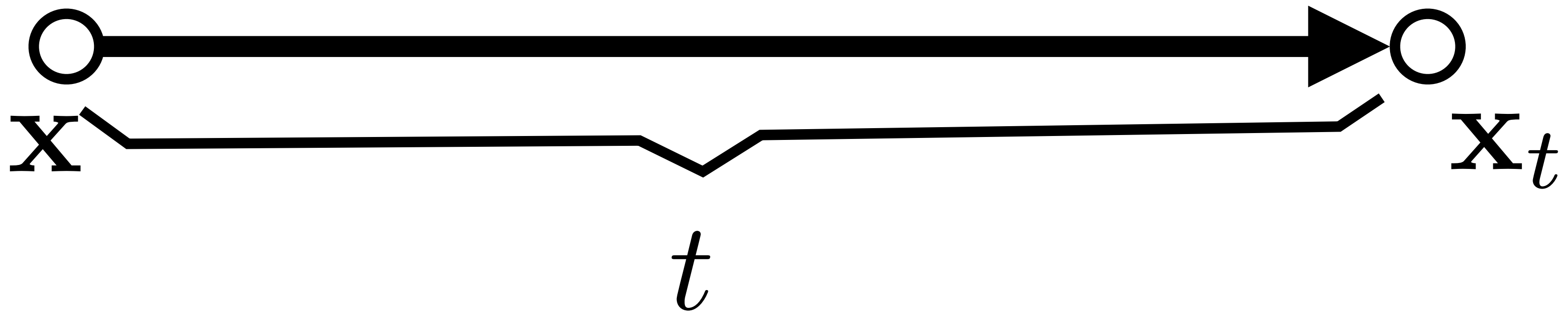
CDF: $P(t) = \int_0^t e^{-\sigma_t s} ds = 1 - e^{-\sigma_t t}$

Inverted CDF: $P^{-1}(\xi) = -\frac{log_e(1-\xi)}{\sigma_t}$

Free-path Sampling

Homogeneous media: $T_r(t) = e^{-\sigma_t t}$

Recipe:

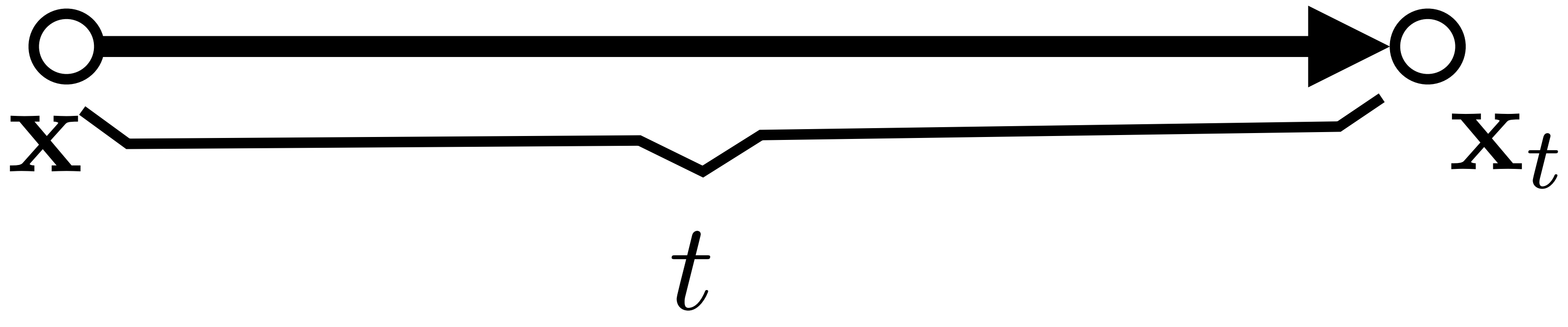


Free-path Sampling

Homogeneous media: $T_r(t) = e^{-\sigma_t t}$

Recipe:

Generate a random number ξ



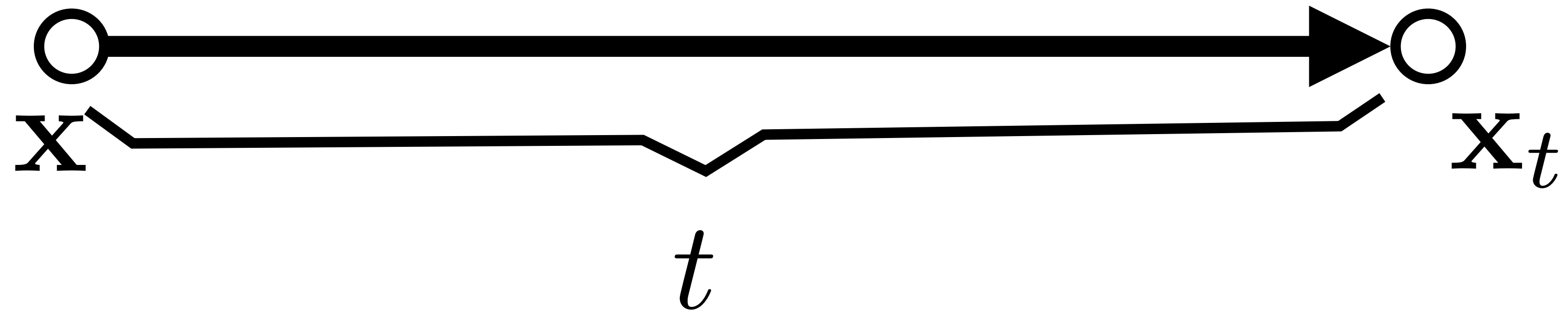
Free-path Sampling

Homogeneous media: $T_r(t) = e^{-\sigma_t t}$

Recipe:

Generate a random number ξ

Sample distance



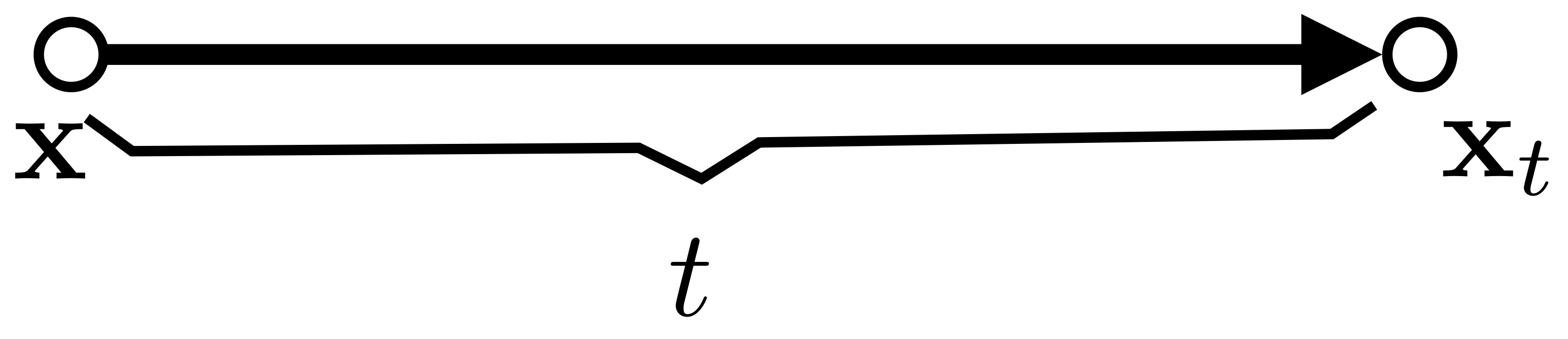
Free-path Sampling

Homogeneous media: $T_r(t) = e^{-\sigma_t t}$

Recipe:

Generate a random number ξ

Sample distance $t = -\frac{log_e(1-\xi)}{\sigma_t}$



Free-path Sampling

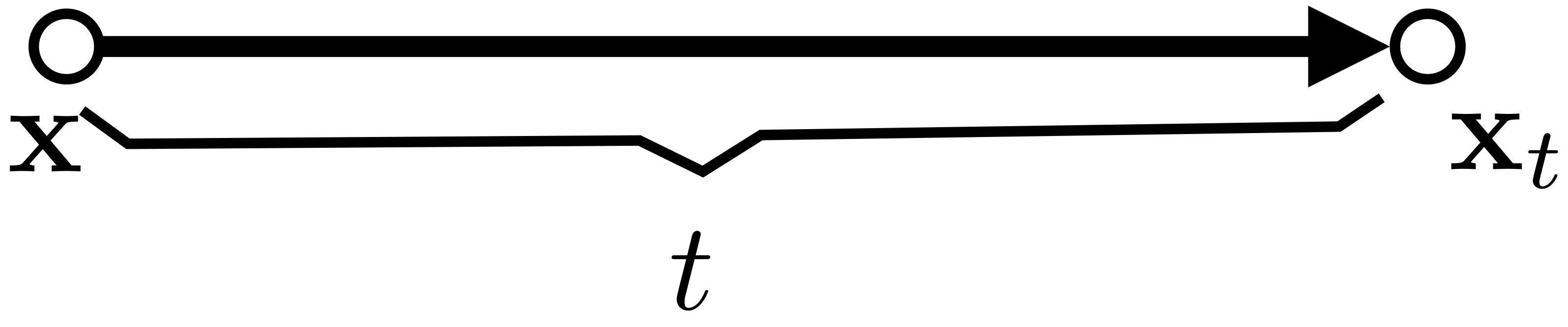
Homogeneous media: $T_r(t) = e^{-\sigma_t t}$

Recipe:

Generate a random number ξ

Sample distance $t = -\dfrac{log_e(1-\xi)}{\sigma_t}$

Compute PDF



Free-path Sampling

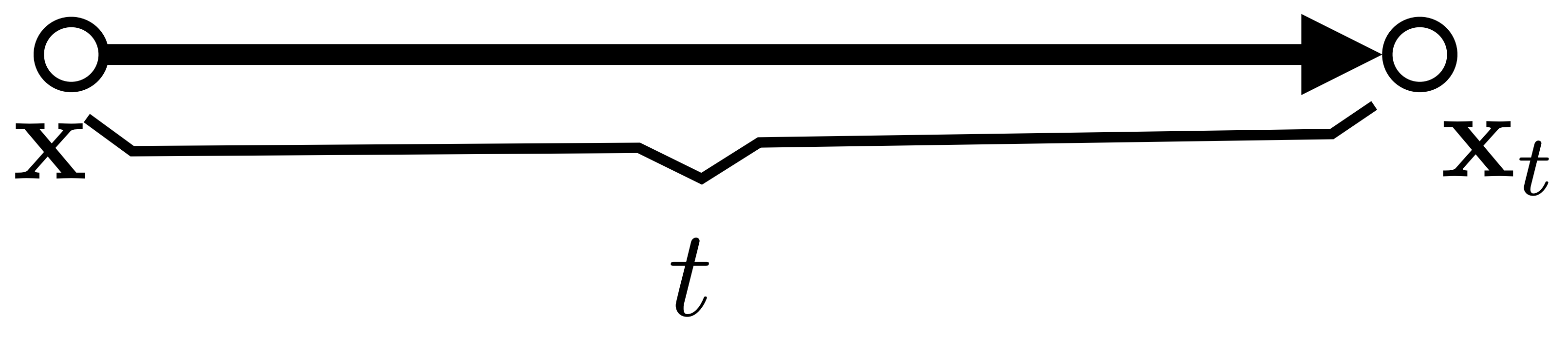
Homogeneous media: $T_r(t) = e^{-\sigma_t t}$

Recipe:

Generate a random number ξ

Sample distance $t = -\dfrac{log_e(1-\xi)}{\sigma_t}$

Compute PDF $p(t) = \sigma_t e^{-\sigma_t t}$



Free-path Sampling

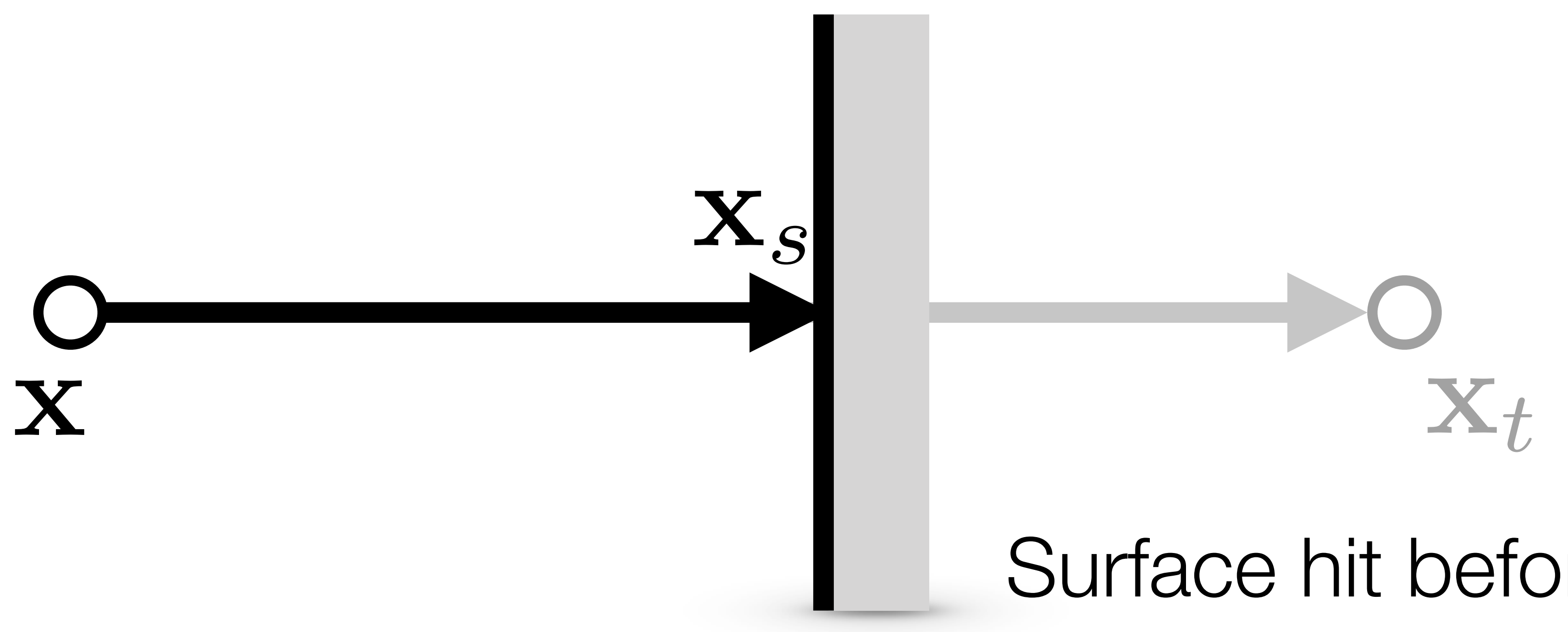
Homogeneous media: $T_r(t) = e^{-\sigma_t t}$

Recipe:

Generate a random number ξ

Sample distance $$t = -\frac{log_e(1-\xi)}{\sigma_t}$$

Compute PDF $p(t) = \sigma_t e^{-\sigma_t t}$



Surface hit before reaching t

Free-path Sampling

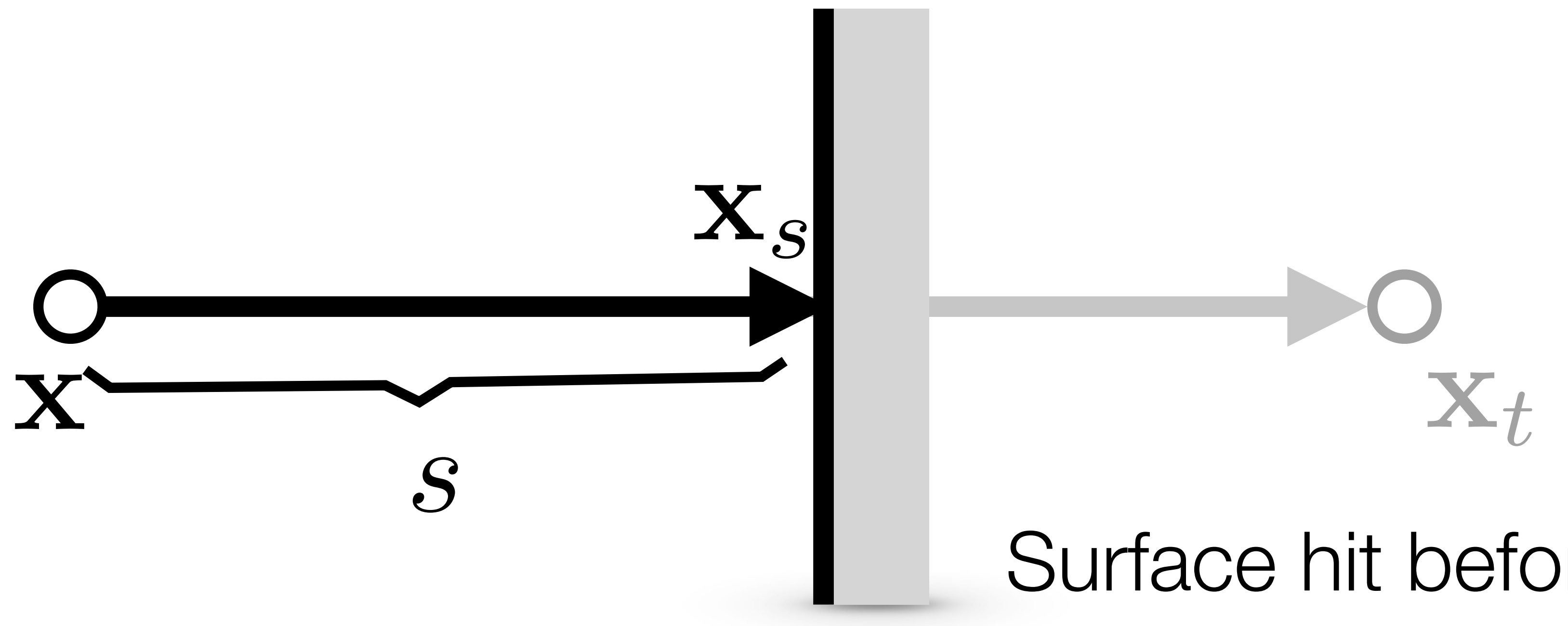
Homogeneous media: $T_r(t) = e^{-\sigma_t t}$

Recipe:

Generate a random number ξ

Sample distance $t = -\frac{log_e(1-\xi)}{\sigma_t}$

Compute PDF $p(t) = \sigma_t e^{-\sigma_t t}$



Surface hit before reaching t

Free-path Sampling

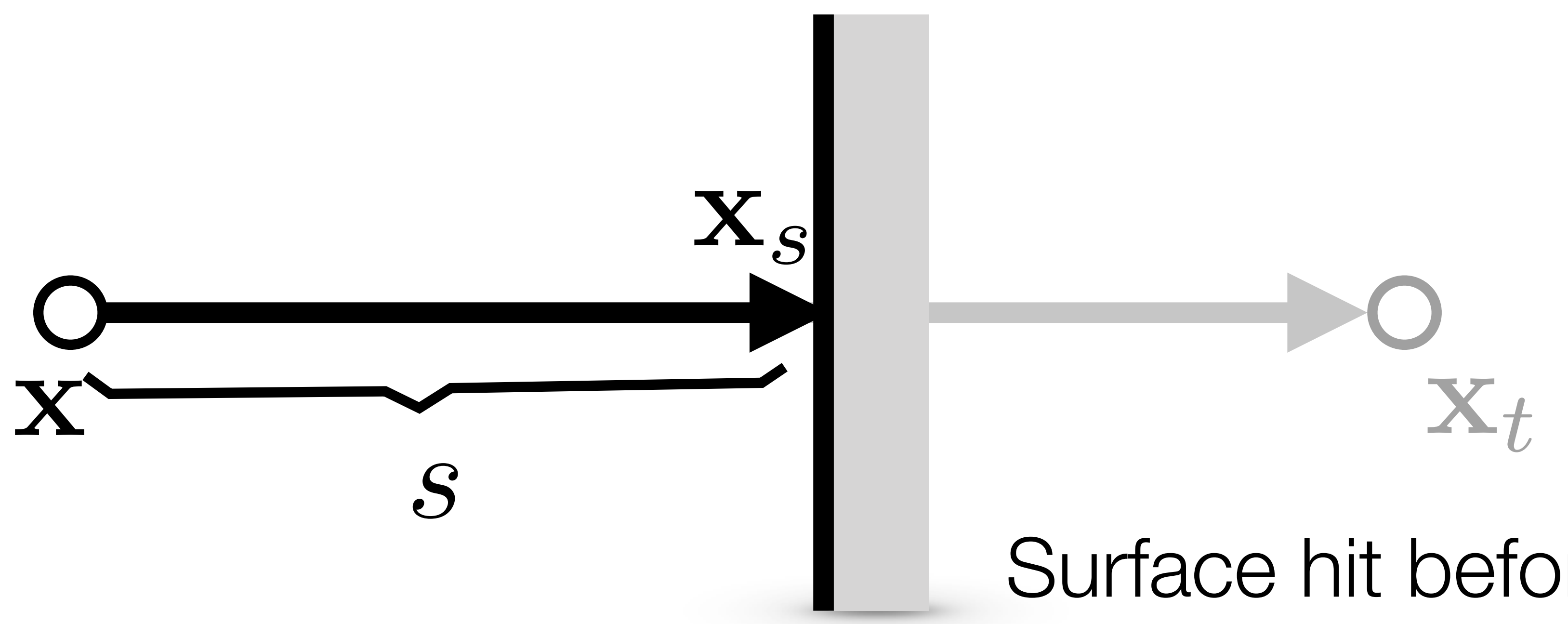
Homogeneous media: $T_r(t) = e^{-\sigma_t t}$

Recipe:

Generate a random number ξ

Sample distance $t = -\xcancel{\frac{log_e(1-\xi)}{\sigma_t}}$

Compute PDF $p(t) = \sigma_t e^{-\sigma_t t}$



Surface hit before reaching t

Free-path Sampling

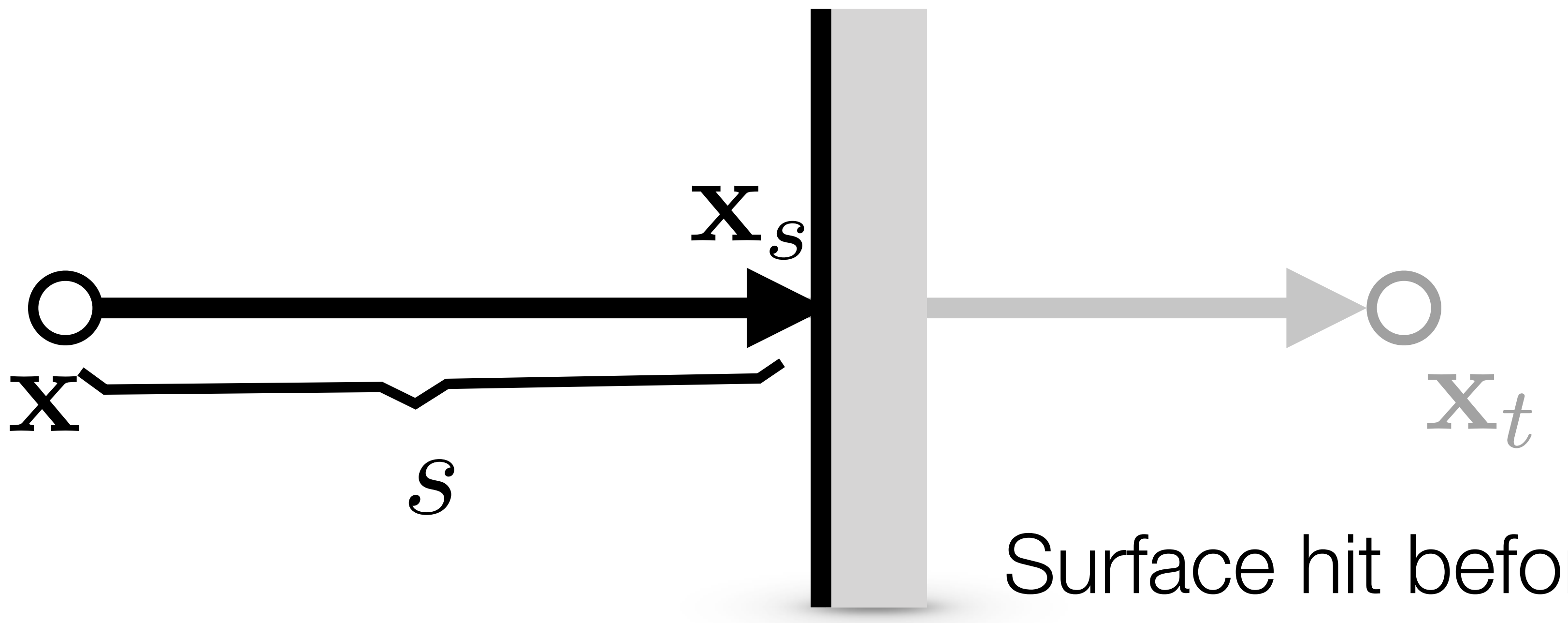
Homogeneous media: $T_r(t) = e^{-\sigma_t t}$

Recipe:

Generate a random number ξ

Sample distance $$t = -\xcancel{\frac{log_e(1-\xi)}{\sigma_t}} = s$$

Compute PDF $p(t) = \sigma_t e^{-\sigma_t t}$



Surface hit before reaching t

Free-path Sampling

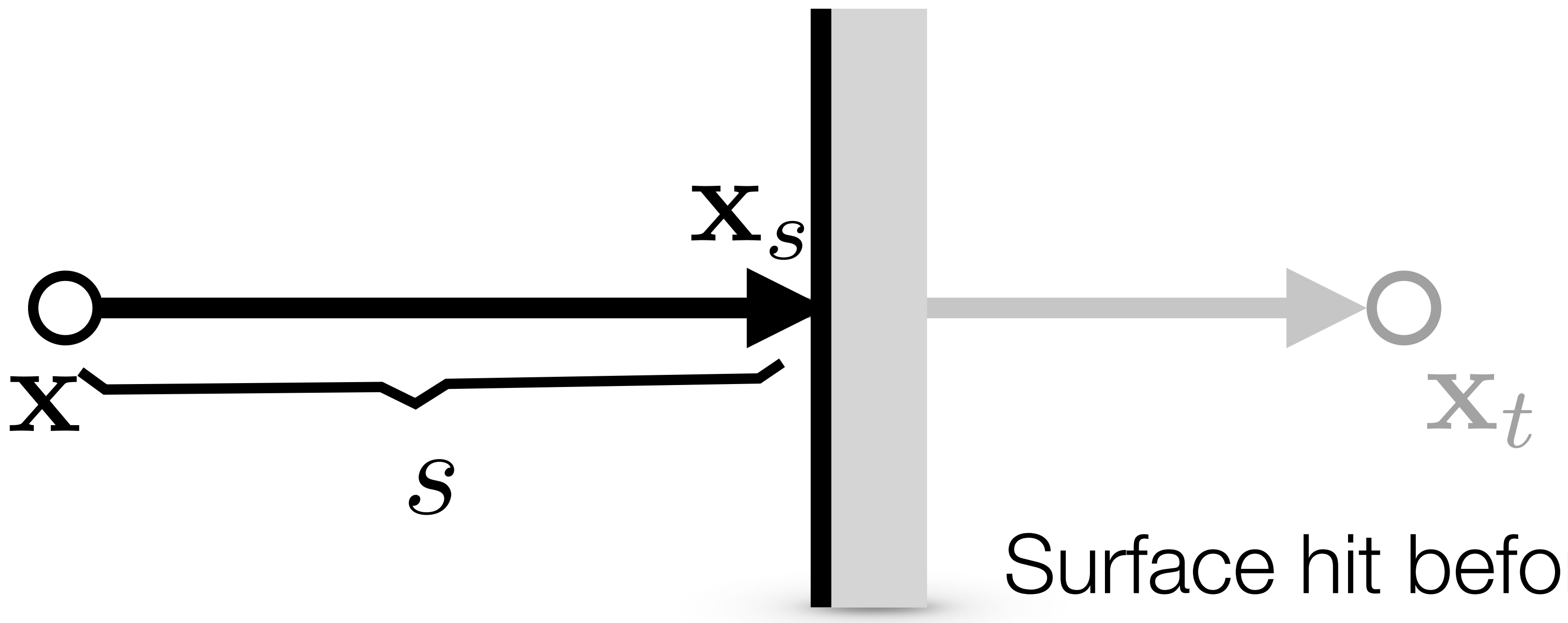
Homogeneous media: $T_r(t) = e^{-\sigma_t t}$

Recipe:

Generate a random number ξ

Sample distance $$t = -\xcancel{\frac{log_e(1-\xi)}{\sigma_t}} = s$$

Compute PDF $$p(t) = \xcancel{\sigma_t e}^{-\sigma_t t}$$



Surface hit before reaching t

Free-path Sampling

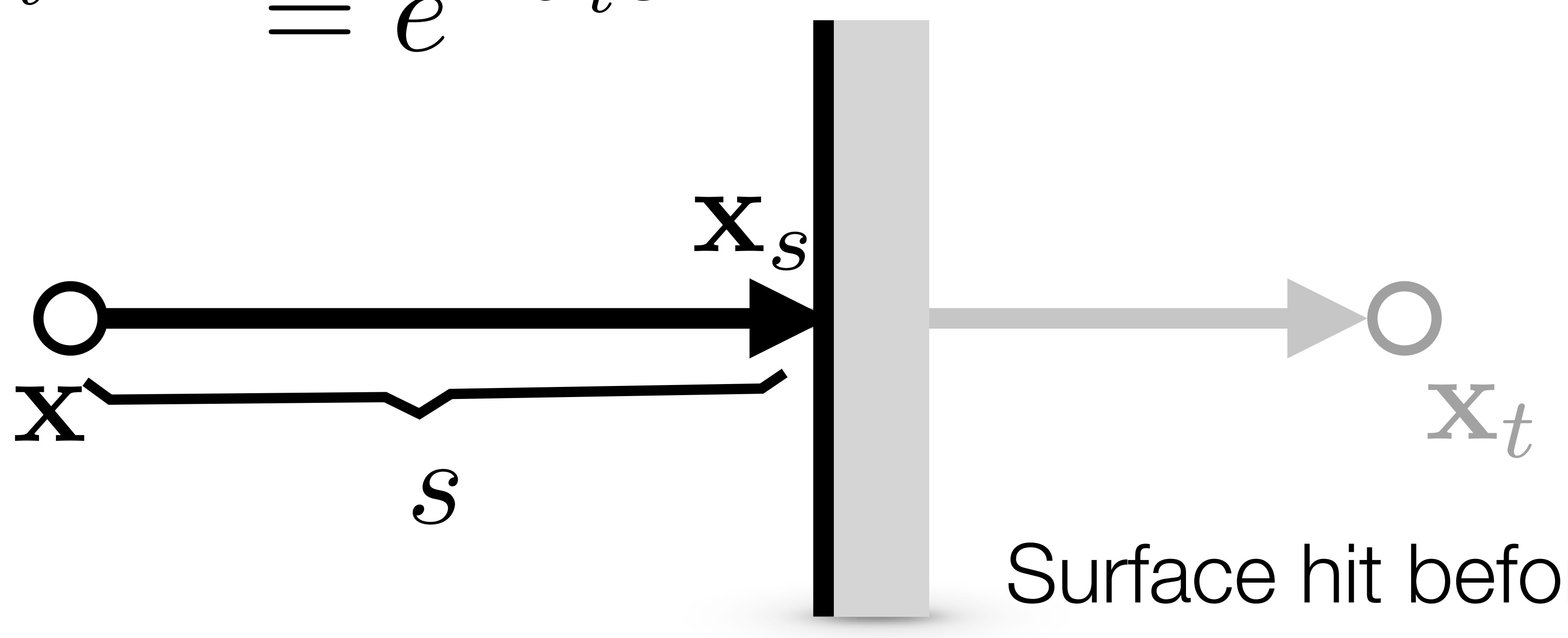
Homogeneous media: $T_r(t) = e^{-\sigma_t t}$

Recipe:

Generate a random number ξ

Sample distance $$t = -\xcancel{\frac{log_e(1-\xi)}{\sigma_t}} = s$$

Compute PDF $$p(t) = \xcancel{\sigma_t e}^{-\sigma_t t} = e^{-\sigma_t s}$$



Surface hit before reaching t

Free-path Sampling

Homogeneous media: $T_r(t) = e^{-\sigma_t t}$

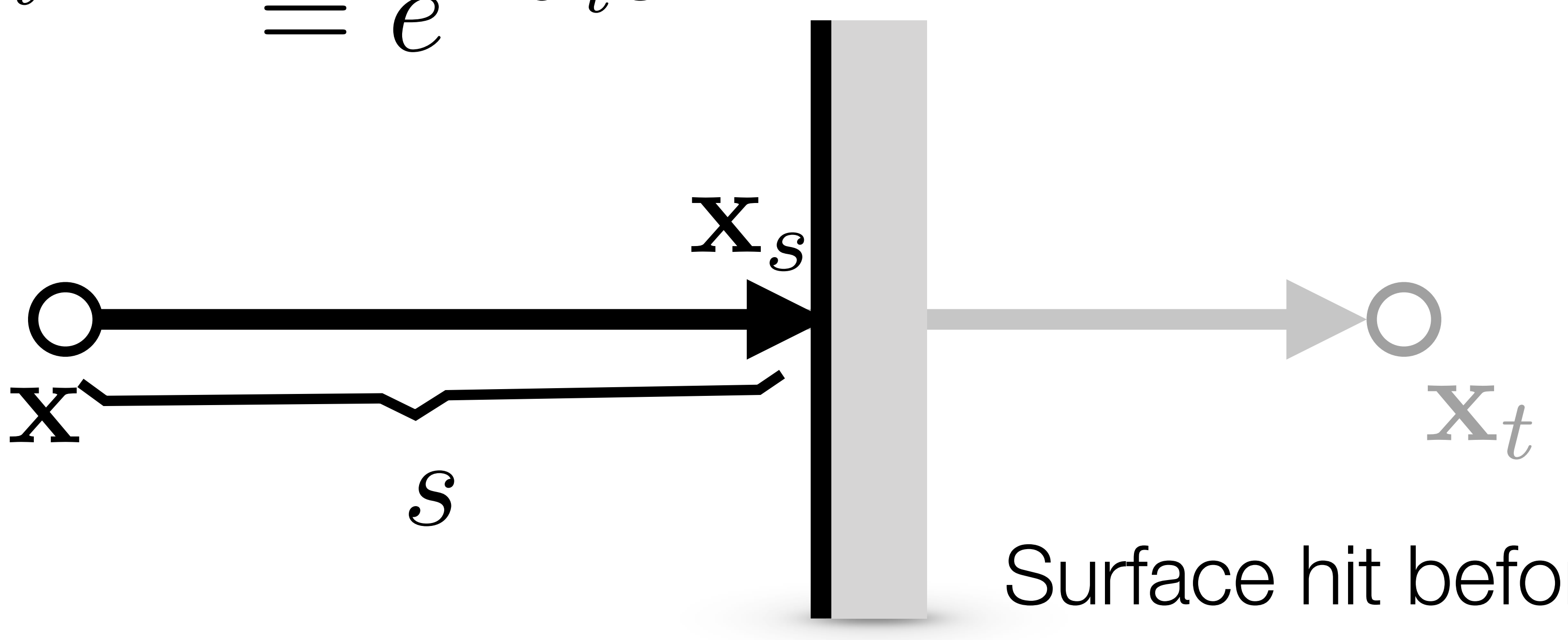
Recipe:

Generate a random number ξ

Sample distance $t = -\cancel{\frac{log_e(1-\xi)}{\sigma_t}} = s$

Compute PDF $p(t) = \cancel{\sigma_t e}^{-\sigma_t t} = e^{-\sigma_t s}$

Note: This is now a probability, not a probability density



Surface hit before reaching t

What about heterogeneous media?

Free-path Sampling

Heterogeneous medium: $T_r(t) = e^{\int_0^t -\sigma_t(s)ds}$

Free-path Sampling

Heterogeneous medium: $T_r(t) = e^{\int_0^t -\sigma_t(s)ds}$

- Closed form solutions exist but for only simple media

  e.g., linearly or exponentially varying extinction

Free-path Sampling

Heterogeneous medium: $T_r(t) = e^{\int_0^t -\sigma_t(s)ds}$

- Closed form solutions exist but for only simple media

e.g., linearly or exponentially varying extinction

- Other solutions:

  - Regular tracking (3D DDA)
  - Ray marching
  - Delta tracking

Free-path Sampling

How to sample the flight distance to the next interaction?

Free-path Sampling

How to sample the flight distance to the next interaction?

Random variable representing flight distance

$$T(t) = e^{-\int_0^t \mu_{\mathrm{t}}(s)\mathrm{d}s} = P(X > t)$$

Free-path Sampling

How to sample the flight distance to the next interaction?

Random variable representing flight distance

$$T(t) = e^{-\int_0^t \mu_{\mathrm{t}}(s)\mathrm{d}s} = P(X > t)$$

CDF

$$P(X \leq t) = F(t)$$

Free-path Sampling

How to sample the flight distance to the next interaction?

Random variable representing flight distance

$$T(t) = e^{-\int_0^t \mu_\mathrm{t}(s)\mathrm{d}s} = \boxed{P(X > t)}$$

CDF

$$\boxed{P(X \leq t)} = F(t)$$

Partition of unity

Free-path Sampling

How to sample the flight distance to the next interaction?

Random variable representing flight distance

$$T(t) = e^{-\int_0^t \mu_\mathrm{t}(s)\mathrm{d}s} = \boxed{\begin{array}{l} P(X > t) \\ \\ P(X \leq t) \end{array}} \begin{array}{l} \\ \\ = F(t) \end{array}$$

CDF

Partition of unity

$$F(t) = 1 - T(t)$$

Recipe for generating samples

Free-path Sampling

Cumulative distribution function (**CDF**)

$$F(t) = 1 - T(t) = 1 - e^{-\tau(t)}$$

Free-path Sampling

Cumulative distribution function (**CDF**)

$$F(t) = 1 - T(t) = 1 - e^{-\tau(t)}$$

Probability density function (**PDF**)

$$p(t) = \frac{\mathrm{d}F(t)}{\mathrm{d}t} = \frac{\mathrm{d}}{\mathrm{d}t}\left(1 - e^{-\tau(t)}\right) = \mu_{\mathrm{t}}(t)e^{-\tau(t)}$$

Free-path Sampling

Cumulative distribution function (**CDF**)

$$F(t) = 1 - T(t) = 1 - e^{-\tau(t)}$$

Probability density function (**PDF**)

$$p(t) = \frac{\mathrm{d}F(t)}{\mathrm{d}t} = \frac{\mathrm{d}}{\mathrm{d}t}\left(1 - e^{-\tau(t)}\right) = \mu_{\mathrm{t}}(t)e^{-\tau(t)}$$

Inverted cumulative distr. function (**CDF^{-1}**)

$$\xi = 1 - e^{-\tau(t)}$$

Solve for t

Free-path Sampling

Cumulative distribution function (**CDF**)

$$F(t) = 1 - T(t) = 1 - e^{-\tau(t)}$$

Probability density function (**PDF**)

$$p(t) = \frac{\mathrm{d}F(t)}{\mathrm{d}t} = \frac{\mathrm{d}}{\mathrm{d}t}\left(1 - e^{-\tau(t)}\right) = \mu_\mathrm{t}(t)e^{-\tau(t)}$$

Inverted cumulative distr. function (**CDF^{-1}**)

$$\xi = 1 - e^{-\tau(t)}$$

Solve for t

$$\tau(t) = -\ln(1 - \xi)$$

Free-path Sampling

Cumulative distribution function (**CDF**)

$$F(t) = 1 - T(t) = 1 - e^{-\tau(t)}$$

Probability density function (**PDF**)

$$p(t) = \frac{\mathrm{d}F(t)}{\mathrm{d}t} = \frac{\mathrm{d}}{\mathrm{d}t}\left(1 - e^{-\tau(t)}\right) = \mu_\mathrm{t}(t)e^{-\tau(t)}$$

Inverted cumulative distr. function (**CDF^{-1}**)

$$\xi = 1 - e^{-\tau(t)}$$

Solve for t

$$\int_0^t \mu_\mathrm{t}(s)\mathrm{d}s = -\ln(1-\xi)$$

Free-path Sampling

Cumulative distribution function (**CDF**)

$$F(t) = 1 - T(t) = 1 - e^{-\tau(t)}$$

Probability density function (**PDF**)

$$p(t) = \frac{\mathrm{d}F(t)}{\mathrm{d}t} = \frac{\mathrm{d}}{\mathrm{d}t}\left(1 - e^{-\tau(t)}\right) = \mu_\mathrm{t}(t)e^{-\tau(t)}$$

Inverted cumulative distr. function (**CDF^{-1}**)

$$\xi = 1 - e^{-\tau(t)}$$

Solve for t

$$\int_0^t \mu_\mathrm{t}(s)\mathrm{d}s = -\ln(1-\xi)$$

Approaches for finding t:

1) **ANALYTIC (closed-form CDF^{-1})**
2) **SEMI-ANALYTIC (regular tracking)**
3) **APPROXIMATE (ray marching)**

Free-path Sampling

Inverted cumulative distr. function (**CDF^{-1}**)

$$\int_0^t \mu_\mathrm{t}(s)\mathrm{d}s = -\ln(1-\xi)$$

Free-path Sampling

Inverted cumulative distr. function (**CDF^{-1}**)

$$\int_0^t \mu_{\mathrm{t}}(s)\mathrm{d}s = -\ln(1-\xi)$$

Some simple volumes permit closed-form solutions

Example: **homogeneous** medium ($\mu_{\mathrm{t}}(\mathbf{x}) = \mu_{\mathrm{t}}$)

Free-path Sampling

Inverted cumulative distr. function (**CDF^{-1}**)

$$\int_0^t \mu_{\mathrm{t}}(s)\mathrm{d}s = -\ln(1-\xi)$$

Some simple volumes permit closed-form solutions

Example: **homogeneous** medium ($\mu_{\mathrm{t}}(\mathbf{x}) = \mu_{\mathrm{t}}$)

Opt. thickness

$$\int_0^t \mu_{\mathrm{t}}(s)\mathrm{d}s = t\mu_{\mathrm{t}}$$

Free-path Sampling

Inverted cumulative distr. function (**CDF^{-1}**)

$$\int_0^t \mu_\mathrm{t}(s)\mathrm{d}s = -\ln(1-\xi)$$

Some simple volumes permit closed-form solutions

Example: **homogeneous** medium ($\mu_\mathrm{t}(\mathbf{x}) = \mu_\mathrm{t}$)

Opt. thickness

$$\int_0^t \mu_\mathrm{t}(s)\mathrm{d}s = t\mu_\mathrm{t}$$

$$\Rightarrow$$

Expression for t

$$t = -\frac{\ln(1-\xi)}{\mu_\mathrm{t}}$$

Free-path Sampling

Inverted cumulative distr. function (**CDF^{-1}**)

$$\int_0^t \mu_\mathrm{t}(s)\mathrm{d}s = -\ln(1-\xi)$$

Some simple volumes permit closed-form solutions

Example: **homogeneous** medium ($\mu_\mathrm{t}(\mathbf{x}) = \mu_\mathrm{t}$)

Opt. thickness

Inverted CDF

$$\int_0^t \mu_\mathrm{t}(s)\mathrm{d}s = t\mu_\mathrm{t} \quad \Rightarrow \quad F^{-1}(\xi) = -\frac{\ln(1-\xi)}{\mu_\mathrm{t}}$$

Analytic Approach

Inverted cumulative distr. function (**CDF^{-1}**)

$$\int_0^t \mu_{\mathrm{t}}(s)\mathrm{d}s = -\ln(1-\xi)$$

Homogeneous volume

Analytic Approach

Inverted cumulative distr. function (**CDF^{-1}**)

$$\int_0^t \mu_\mathrm{t}(s)\mathrm{d}s = -\ln(1-\xi)$$

Homogeneous volume

Sampling in homogeneous vol:

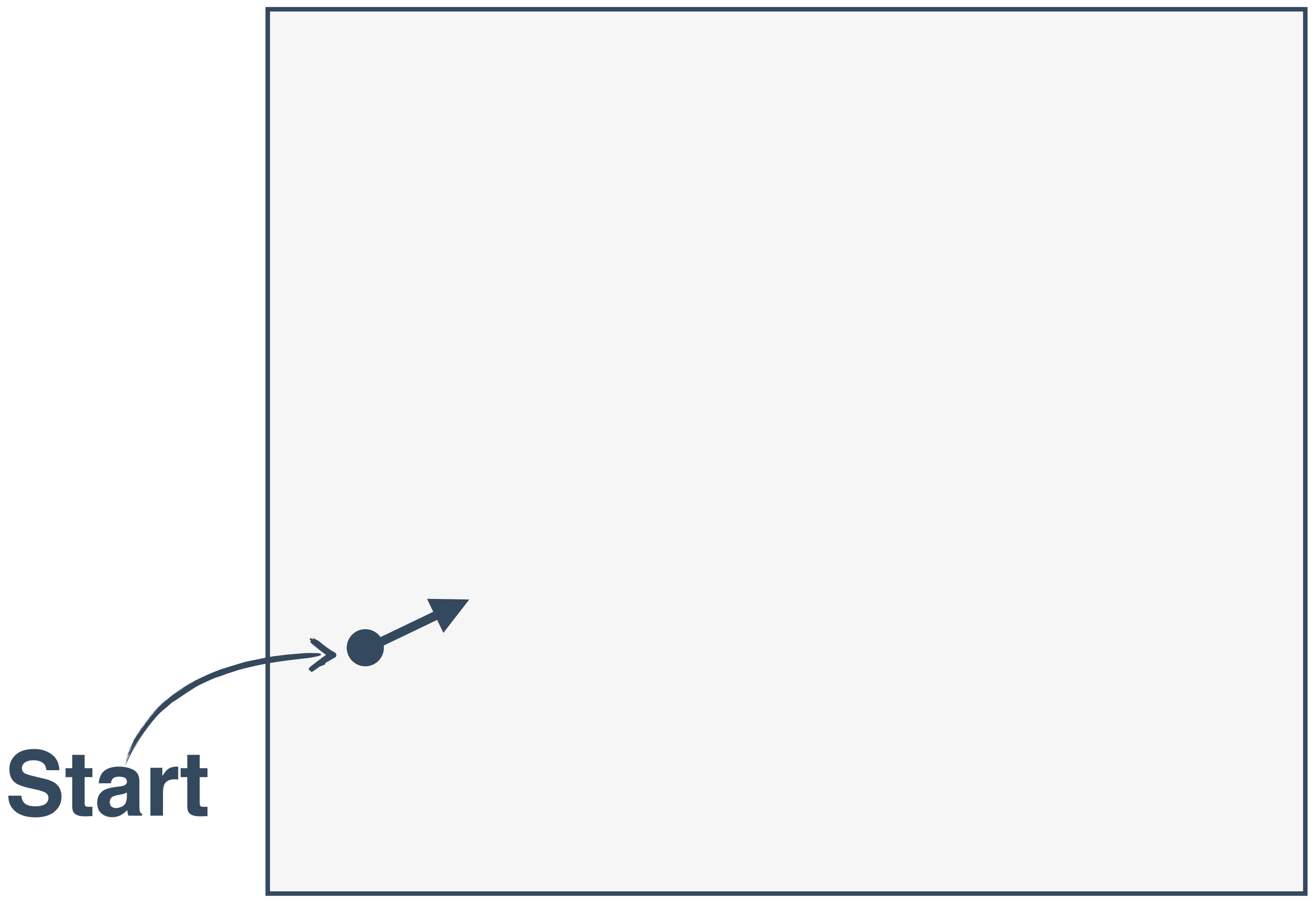
1) Draw a random number ξ
2) Set $t = -\frac{\ln(1-\xi)}{\mu_\mathrm{t}}$
3) Set $p(t) = \mu_\mathrm{t} e^{-t\mu_\mathrm{t}}$

Analytic Approach

Inverted cumulative distr. function (**CDF^{-1}**)

$$\int_0^t \mu_t(s)\mathrm{d}s = -\ln(1-\xi)$$

Homogeneous volume



Sampling in homogeneous vol:

1) Draw a random number ξ
2) Set $t = -\frac{\ln(1-\xi)}{\mu_t}$
3) Set $p(t) = \mu_t e^{-t\mu_t}$

Analytic Approach

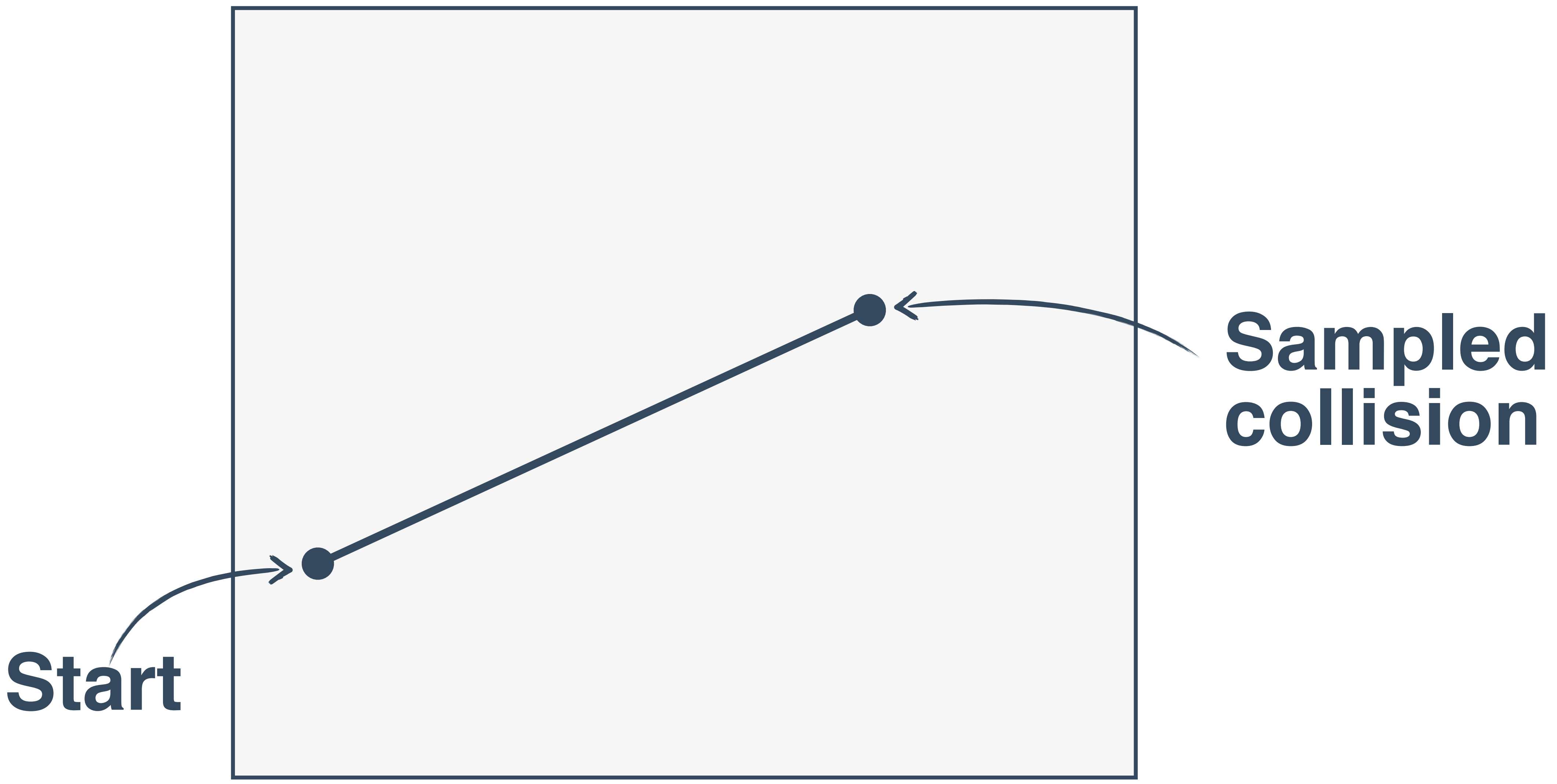
Inverted cumulative distr. function (**CDF^{-1}**)

$$\int_0^t \mu_\mathrm{t}(s)\mathrm{d}s = -\ln(1-\xi)$$

Sampling in homogeneous vol:

1) Draw a random number ξ
2) Set $t = -\frac{\ln(1-\xi)}{\mu_\mathrm{t}}$
3) Set $p(t) = \mu_\mathrm{t} e^{-t\mu_\mathrm{t}}$

Homogeneous volume



Regular Tracking (Semi-Analytic)

$$\int_0^t \mu_{\mathrm{t}}(s)\mathrm{d}s = -\ln(1-\xi)$$

Regular Tracking (Semi-Analytic)

For piecewise-simple (e.g. piecewise-constant), summation replaces integration

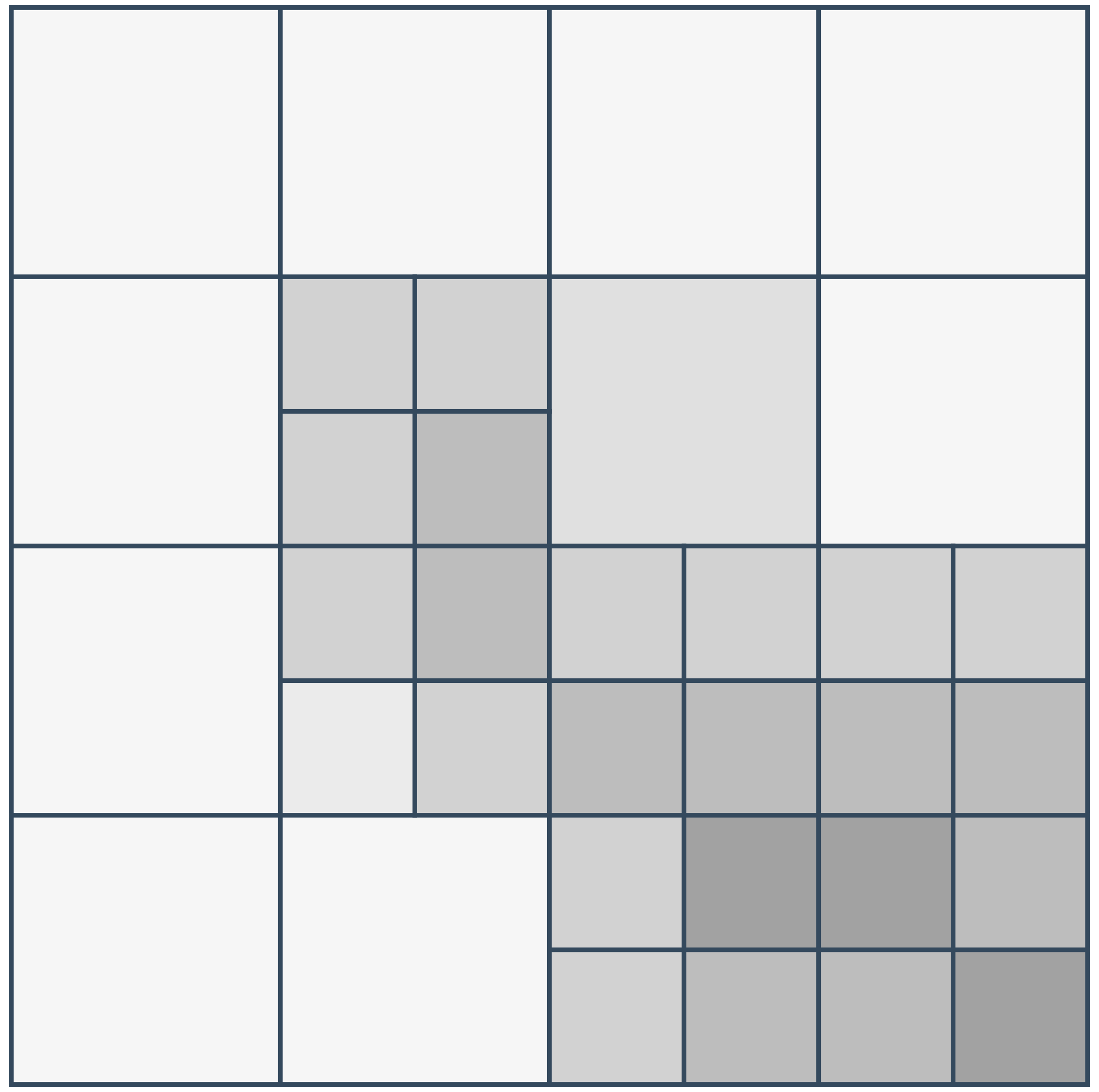
$$\int_0^t \mu_{\mathrm{t}}(s)\mathrm{d}s = -\ln(1-\xi)$$

Regular Tracking (Semi-Analytic)

For piecewise-simple (e.g. piecewise-constant), summation replaces integration

$$\int_0^t \mu_{\mathrm{t}}(s)\mathrm{d}s = -\ln(1-\xi)$$

(Hierarchical) voxel grid



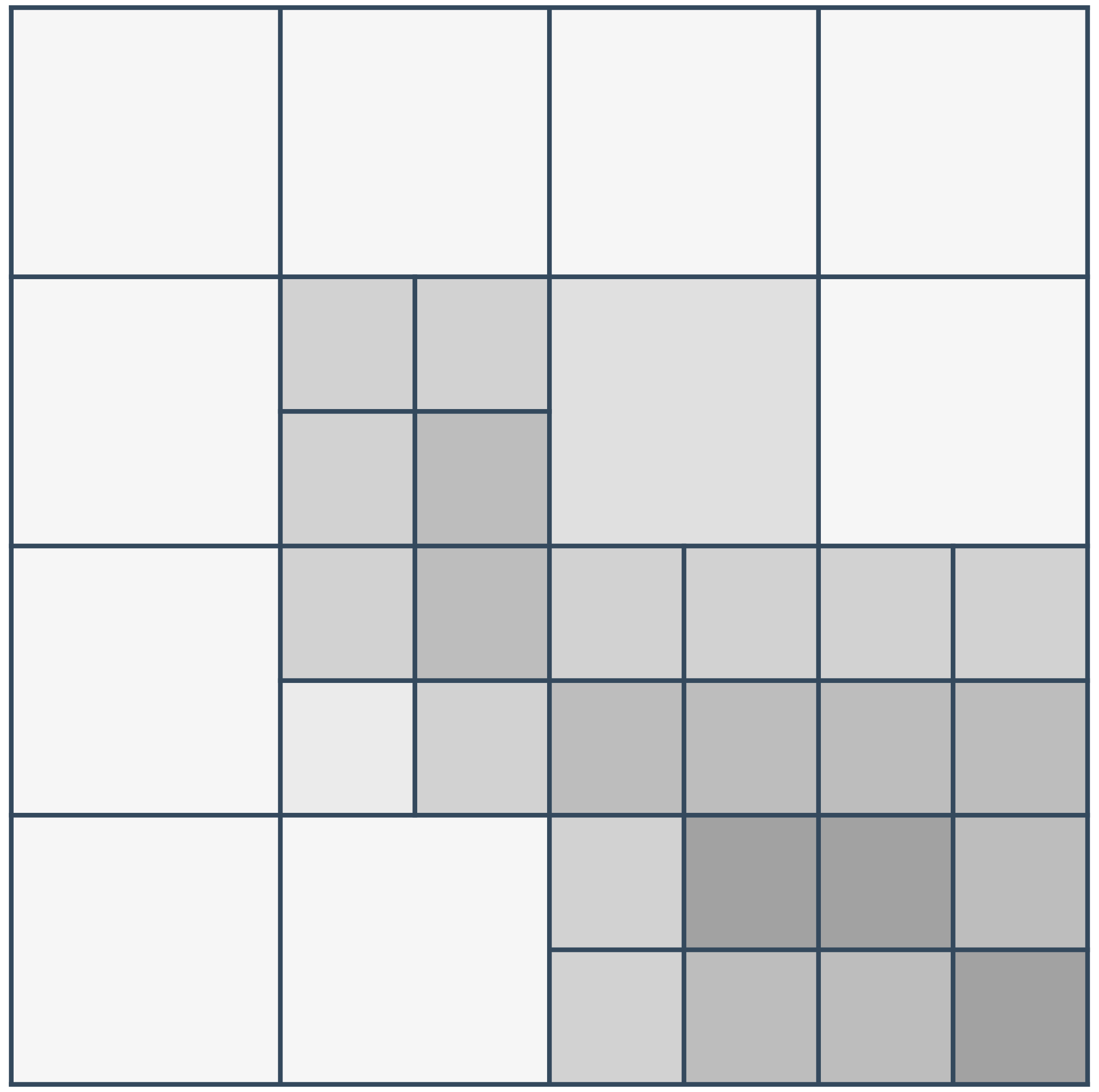
Regular Tracking (Semi-Analytic)

For piecewise-simple (e.g. piecewise-constant), summation replaces integration

$$\int_0^t \mu_{\mathrm{t}}(s)\mathrm{d}s = -\ln(1-\xi)$$

$$\sum_{i=1}^{k} \mu_{\mathrm{t},i}\,\Delta_i = -\ln(1-\xi)$$

(Hierarchical) voxel grid



Regular Tracking (Semi-Analytic)

For piecewise-simple (e.g. piecewise-constant), summation replaces integration

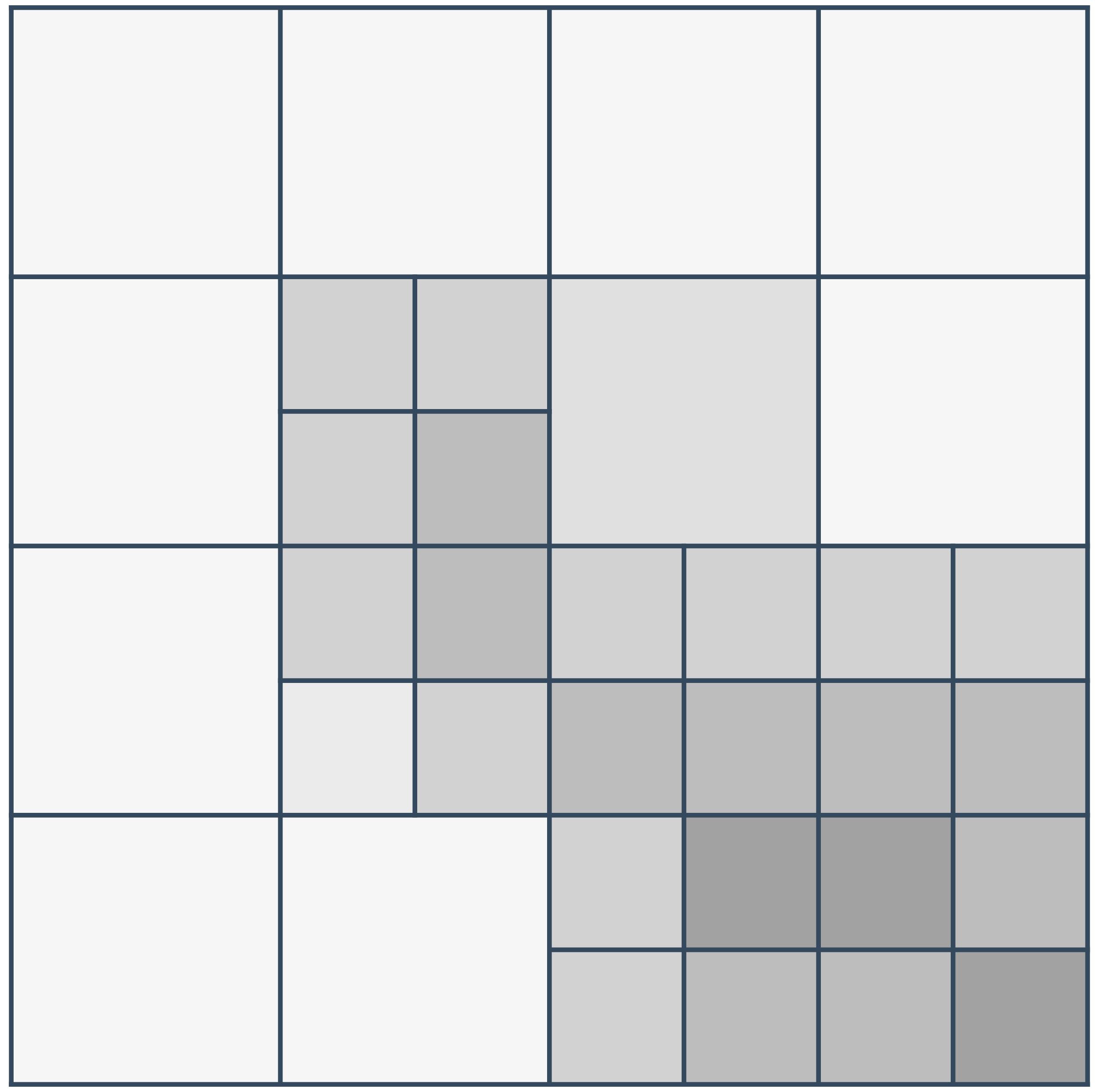
$$\int_0^t \mu_{\mathrm{t}}(s)\mathrm{d}s = -\ln(1-\xi)$$

$$\sum_{i=1}^{k} \mu_{\mathrm{t},i}\, \Delta_i = -\ln(1-\xi)$$

Regular tracking:

1) Draw a random number ξ
2) While LHS < RHS
   move to the next intersection
3) Find the exact location
   in the last segment analytically

(Hierarchical) voxel grid



Regular Tracking (Semi-Analytic)

For piecewise-simple (e.g. piecewise-constant), summation replaces integration

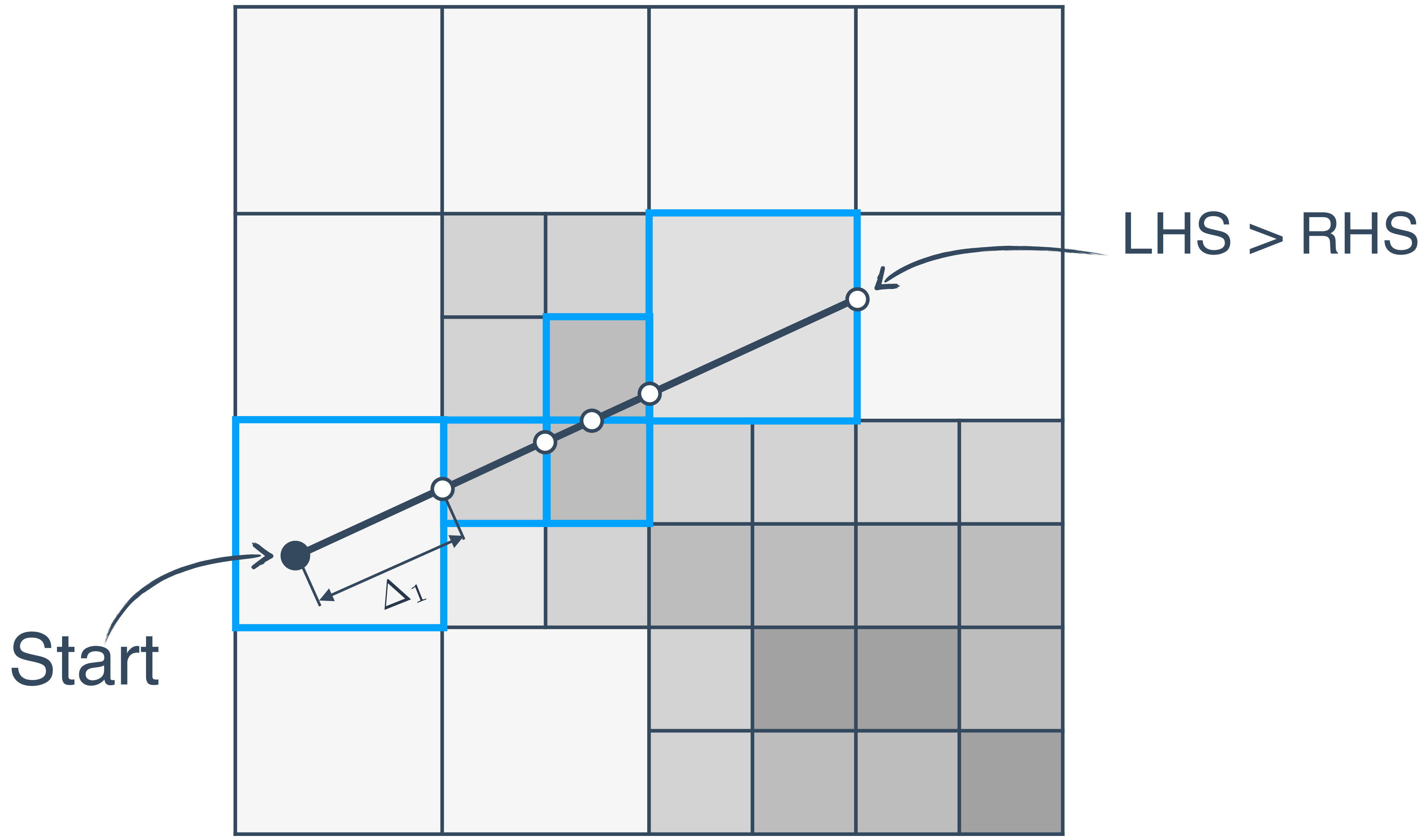
$$\int_0^t \mu_{\mathrm{t}}(s)\mathrm{d}s = -\ln(1-\xi)$$

$$\sum_{i=1}^{k} \mu_{\mathrm{t},i}\,\Delta_i = -\ln(1-\xi)$$

Regular tracking:

1) Draw a random number ξ
2) While LHS < RHS
   move to the next intersection
3) Find the exact location
   in the last segment analytically

(Hierarchical) voxel grid



Regular Tracking (Semi-Analytic)

For piecewise-simple (e.g. piecewise-constant), summation replaces integration

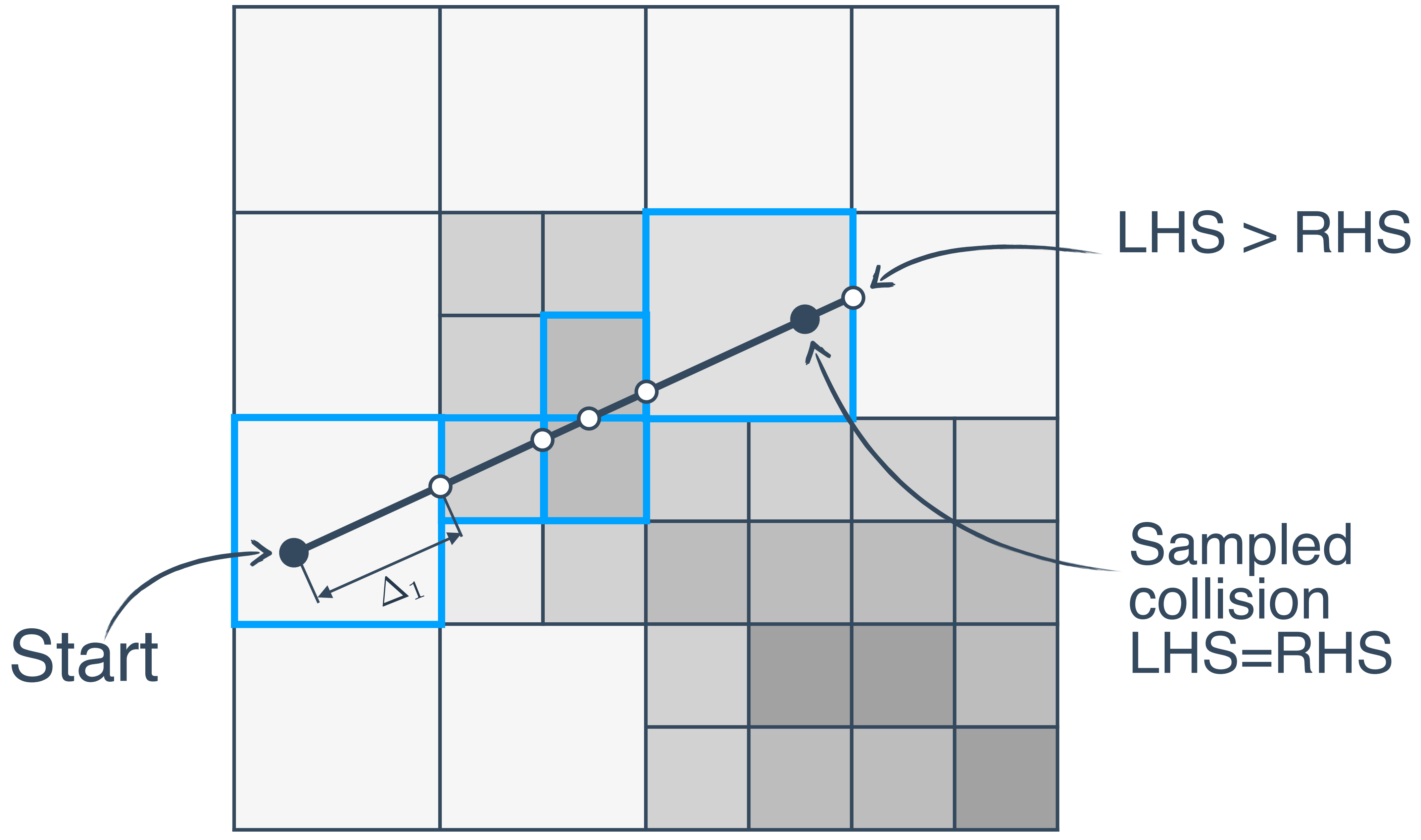
$$\int_0^t \mu_\mathrm{t}(s)\mathrm{d}s = -\ln(1-\xi)$$

$$\sum_{i=1}^{k} \mu_{\mathrm{t},i}\,\Delta_i = -\ln(1-\xi)$$

Regular tracking:

1) Draw a random number ξ
2) While LHS < RHS
   move to the next intersection
3) Find the exact location
   in the last segment analytically

(Hierarchical) voxel grid



Ray Marching

Find the collision distance approximately

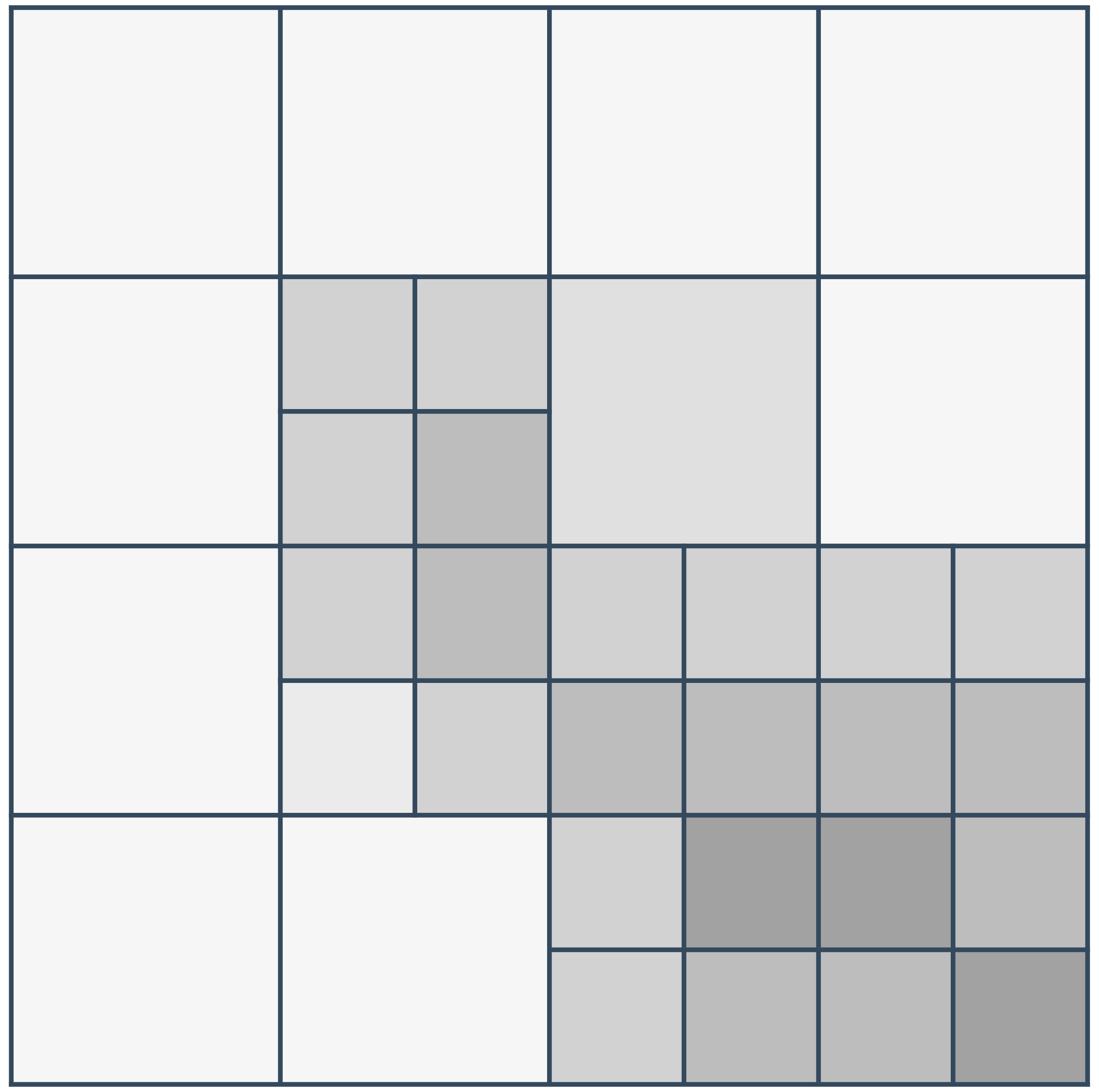
$$\int_0^t \mu_\mathrm{t}(s)\mathrm{d}s = -\ln(1-\xi)$$

$$\neq$$

$$\sum_{i=1}^{k} \mu_{\mathrm{t},i}\,\Delta \;\; = -\ln(1-\xi)$$

Constant step

(Hierarchical) voxel grid



Ray Marching

Find the collision distance approximately

$$\int_0^t \mu_\mathrm{t}(s)\mathrm{d}s = -\ln(1-\xi)$$

$$\nparallel$$

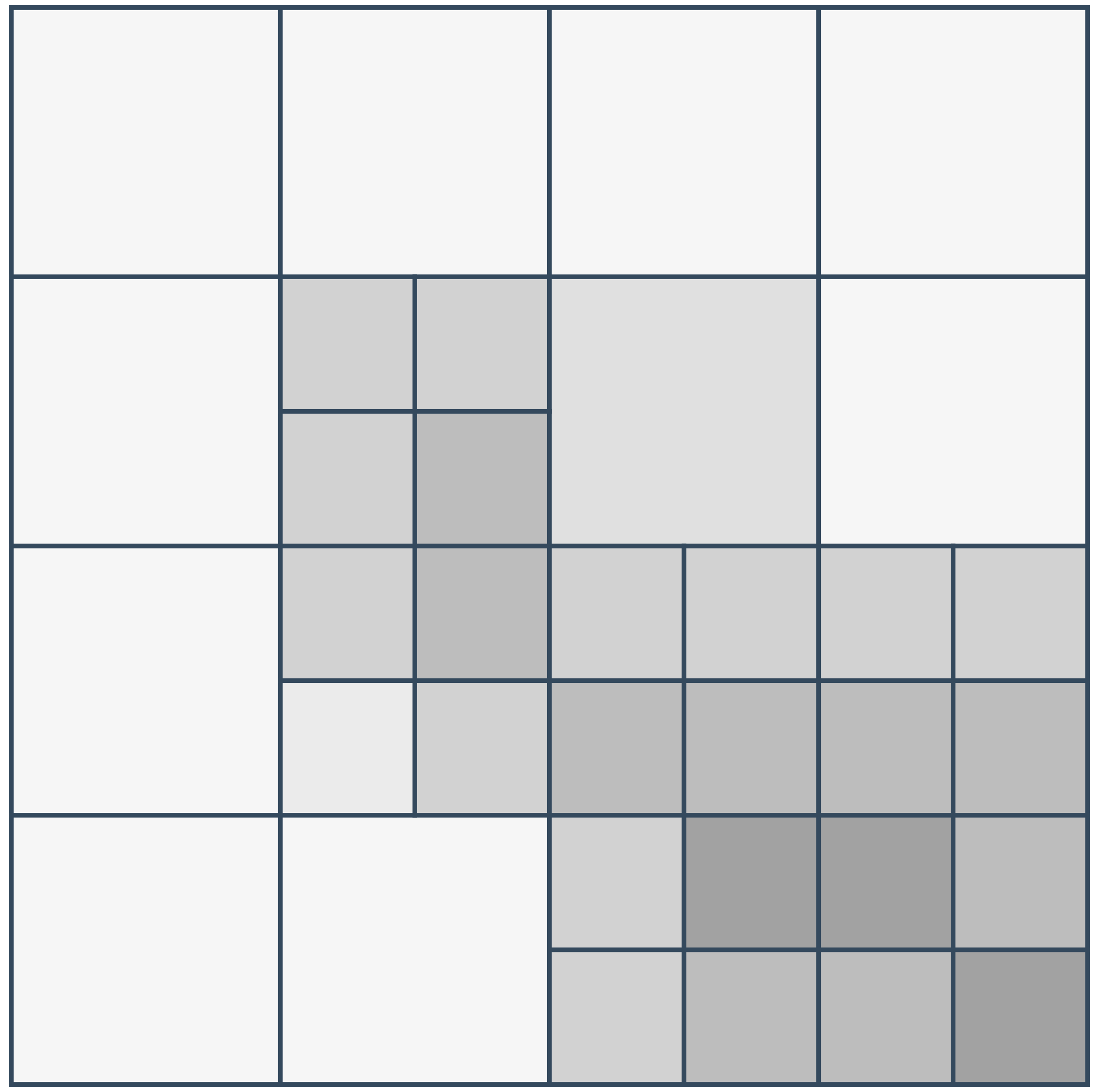
$$\sum_{i=1}^{k} \mu_{\mathrm{t},i}\,\Delta \;= -\ln(1-\xi)$$

Constant step

Ray marching:

1) Draw a random number ξ
2) While LHS < RHS
   make a (fixed-size) step
3) Find the exact location
   in the last segment analytically

(Hierarchical) voxel grid



Ray Marching

Find the collision distance approximately

$$\int_0^t \mu_\mathrm{t}(s)\mathrm{d}s = -\ln(1-\xi)$$

$$\neq$$

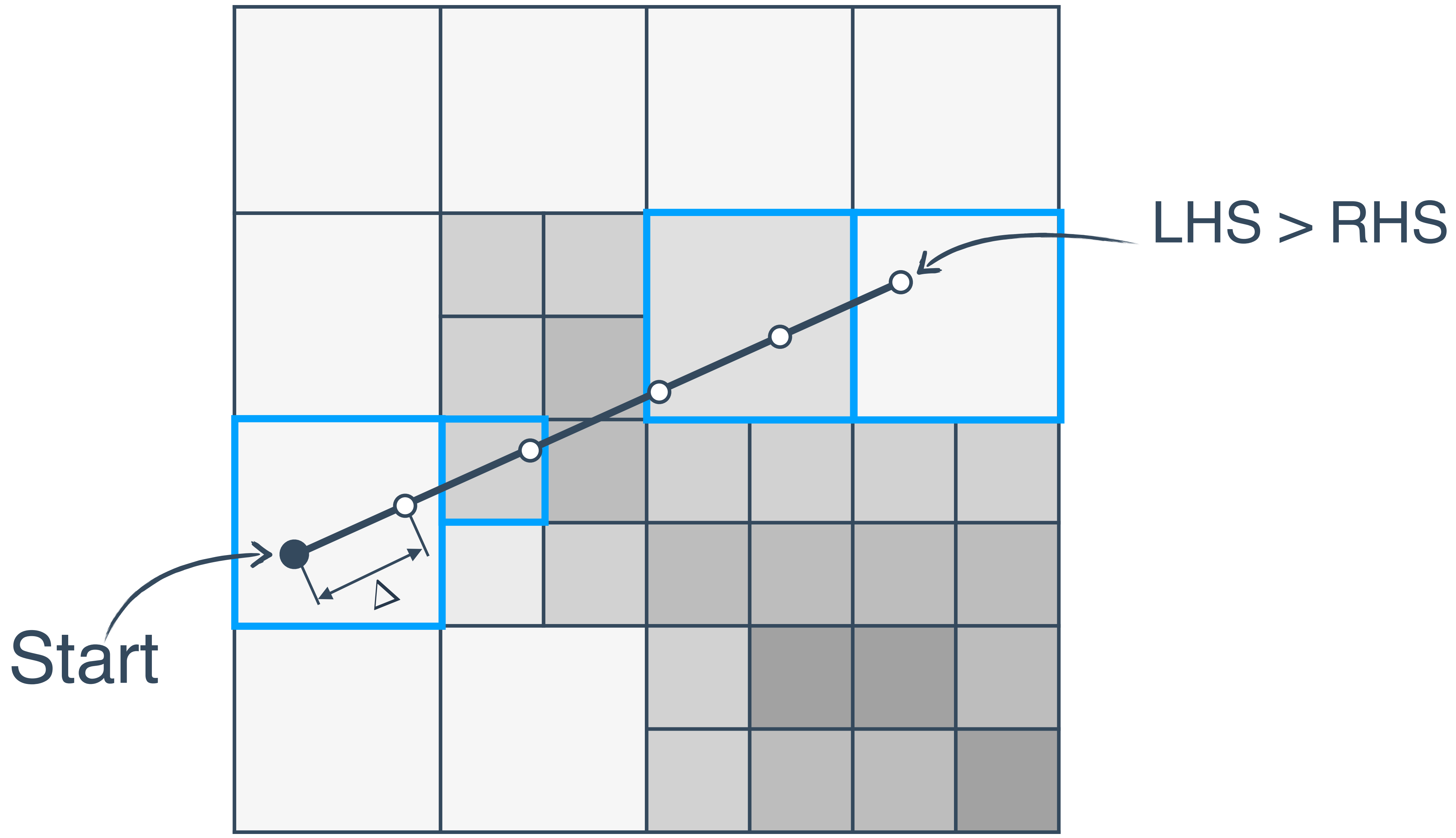
$$\sum_{i=1}^{k} \mu_{\mathrm{t},i}\,\Delta = -\ln(1-\xi)$$

Constant step

Ray marching:

1) Draw a random number ξ
2) While LHS < RHS
   make a (fixed-size) step
3) Find the exact location
   in the last segment analytically

(Hierarchical) voxel grid



Ray Marching

Find the collision distance approximately

$$\int_0^t \mu_\mathrm{t}(s)\mathrm{d}s = -\ln(1-\xi)$$

$$\nparallel$$

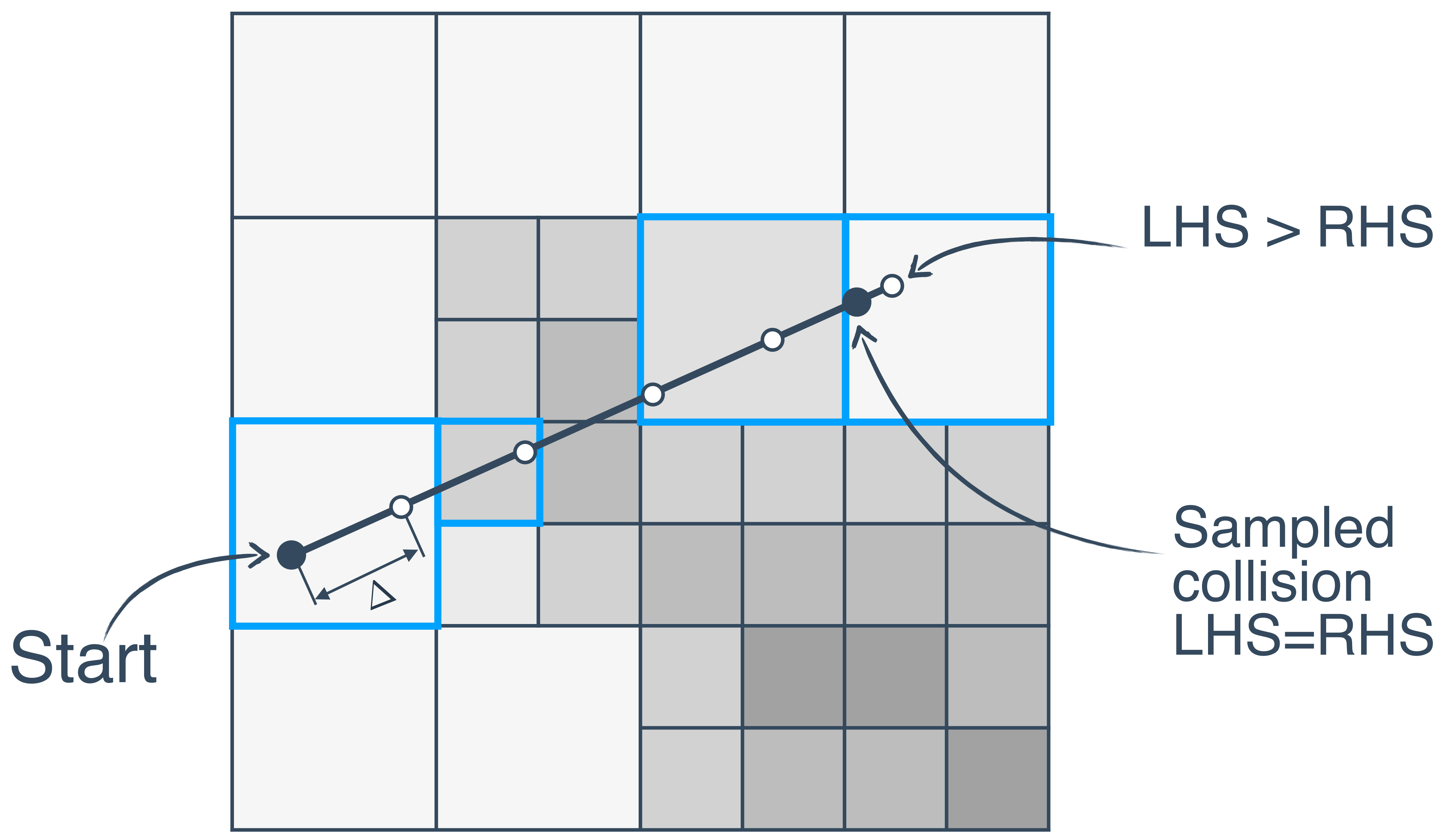
$$\sum_{i=1}^{k} \mu_{\mathrm{t},i}\, \Delta \;\; = -\ln(1-\xi)$$

Constant step

Ray marching:

1) Draw a random number ξ
2) While LHS < RHS
   make a (fixed-size) step
3) Find the exact location
   in the last segment analytically

(Hierarchical) voxel grid



Ray Marching

Find the collision distance approximately

$$\int_0^t \mu_\mathrm{t}(s)\mathrm{d}s = -\ln(1-\xi)$$

$$\nparallel$$

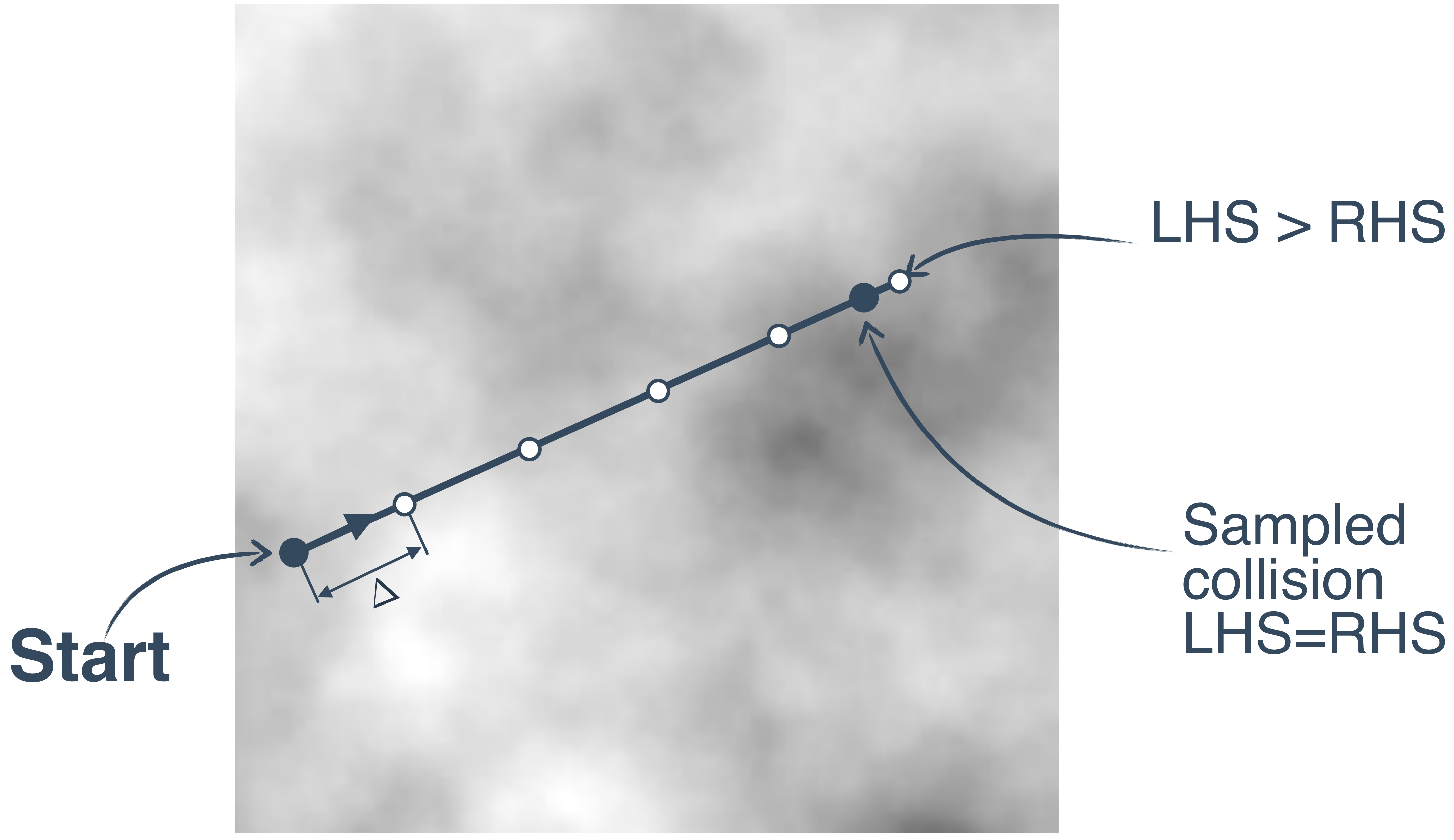
$$\sum_{i=1}^{k} \mu_{\mathrm{t},i}\,\Delta \;= -\ln(1-\xi)$$

Constant step

Ray marching:

1) Draw a random number ξ
2) While LHS < RHS
   make a (fixed-size) step
3) Find the exact location
   in the last segment analytically

General volume



Ray Marching

Find the collision distance approximately

$$\int_0^t \mu_{\mathrm{t}}(s)\mathrm{d}s = -\ln(1-\xi)$$

$$\nparallel$$

$$\sum_{i=1}^{k} \mu_{\mathrm{t},i}\,\Delta \;= -\ln(1-\xi)$$

Constant step

Ray marching:

1) Draw a random number ξ
2) While LHS < RHS
   make a (fixed-size) step
3) Find the exact location
   in the last segment analytically

General volume

Free-path Sampling

ANALYTIC CDF^{-1}	REGULAR TRACKING	RAY MARCHING

Free-path Sampling

ANALYTIC CDF^{-1}	REGULAR TRACKING	RAY MARCHING
▸ Efficient & simple, limited to few volumes		
▸ Simple volumes (e.g. homogeneous)		
▸ Unbiased		

Free-path Sampling

ANALYTIC CDF^{-1}	REGULAR TRACKING	RAY MARCHING
Efficient & simple, limited to few volumes	Iterative, inefficient if free paths cross many boundaries	
Simple volumes (e.g. homogeneous)	Piecewise-simple volumes	
Unbiased	Unbiased	

Free-path Sampling

ANALYTIC CDF^{-1}	REGULAR TRACKING	RAY MARCHING
Efficient & simple, limited to few volumes	Iterative, inefficient if free paths cross many boundaries	Iterative, inaccurate (or inefficient) for media with high frequencies
Simple volumes (e.g. homogeneous)	Piecewise-simple volumes	Any volume
Unbiased	Unbiased	Biased

Free-path Sampling

ANALYTIC CDF^{-1}	REGULAR TRACKING	RAY MARCHING
Efficient & simple, limited to few volumes	Iterative, inefficient if free paths cross many boundaries	Iterative, inaccurate (or inefficient) for media with high frequencies
Simple volumes (e.g. homogeneous)	Piecewise-simple volumes	Any volume
Unbiased	Unbiased	Biased

Common approach: sample optical thickness, find corresponding distance

Delta Tracking

a.k.a. Woodcock tracking, pseudo scattering, hole tracking, null-collision method,…

Physical Interpretation

Add **FICTITIOUS MATTER** to homogenize heterogeneous extinction

Physical Interpretation

Add **FICTITIOUS MATTER** to homogenize heterogeneous extinction

- albedo $\alpha(\mathbf{x}) = 1$

Physical Interpretation

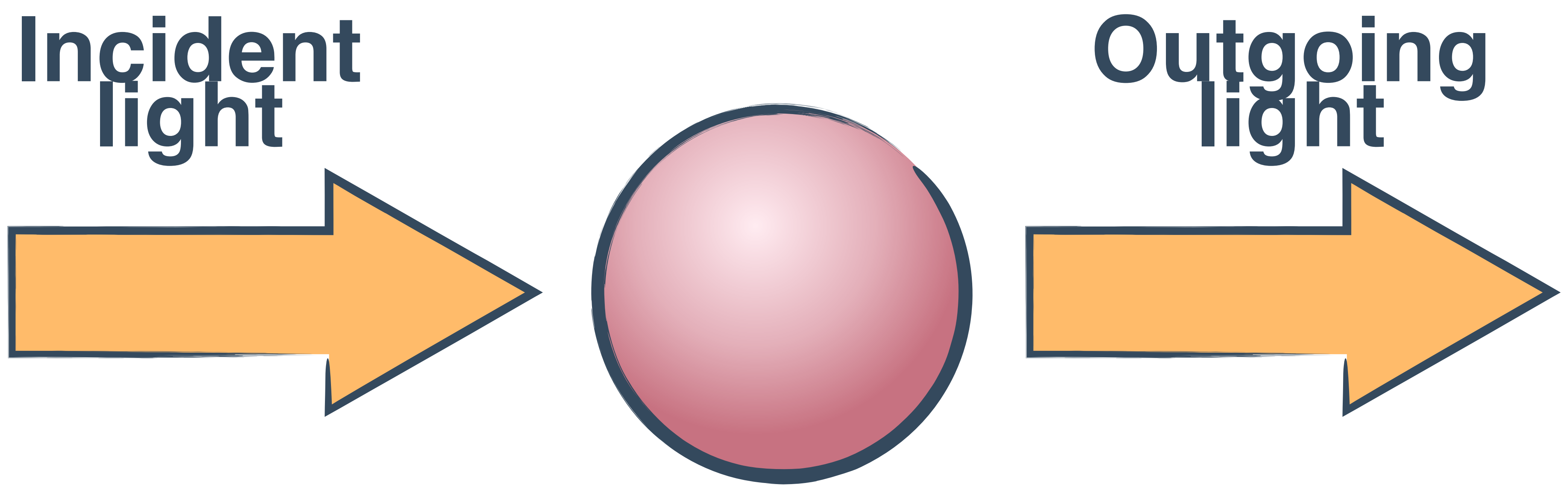
Add **FICTITIOUS MATTER** to homogenize heterogeneous extinction

- albedo $\alpha(\mathbf{x}) = 1$
- phase function $f_{\mathrm{p}}(\omega, \bar{\omega}) = \delta(\omega - \bar{\omega})$

Physical Interpretation

Add **FICTITIOUS MATTER** to homogenize heterogeneous extinction

- albedo $\alpha(\mathbf{x}) = 1$
- phase function $f_{\mathrm{p}}(\omega, \bar{\omega}) = \delta(\omega - \bar{\omega})$



Fictitious particle

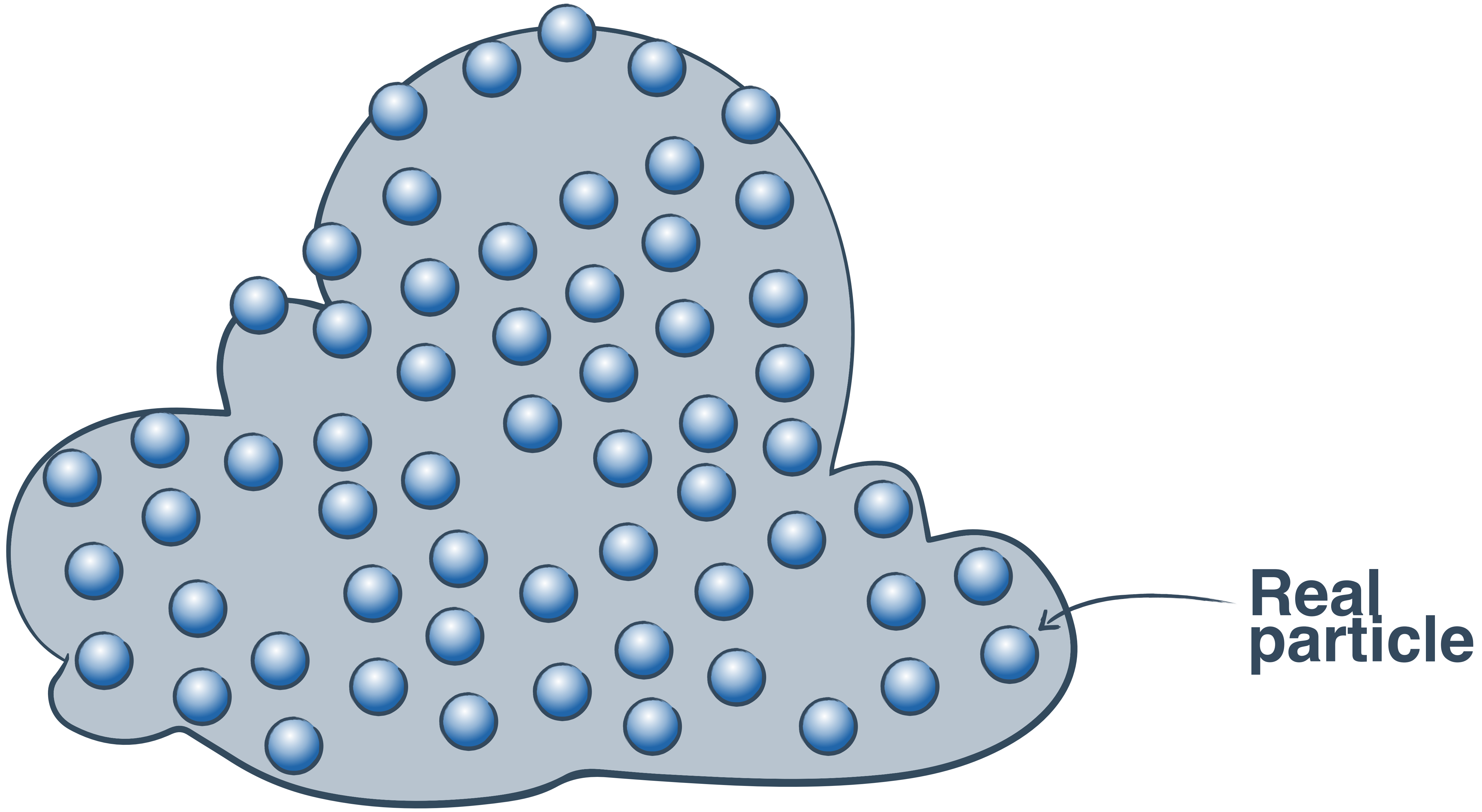
Presence of fictitious matter does not impact light transport

Physical Interpretation

HOMOGENIZATION

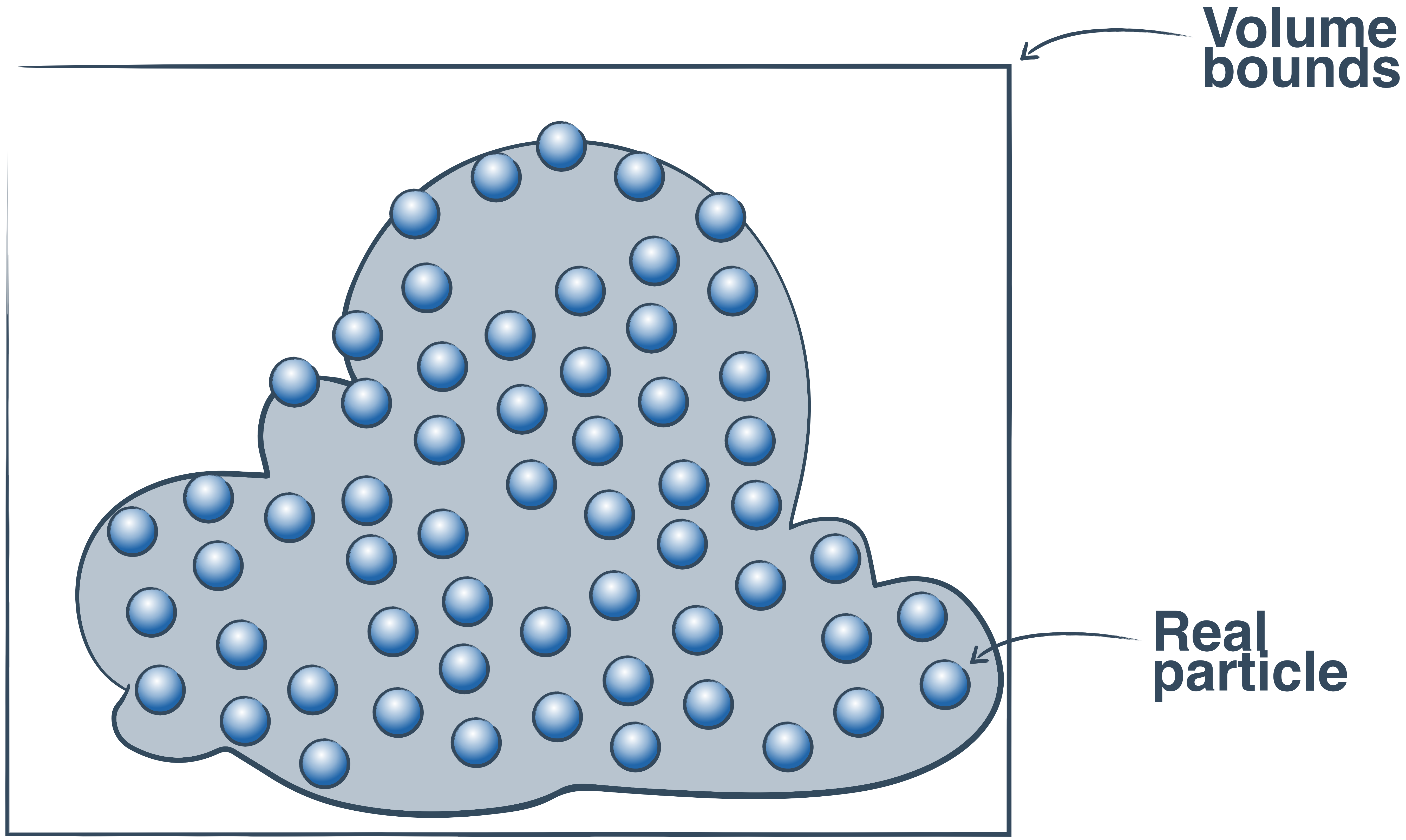
Physical Interpretation

HOMOGENIZATION



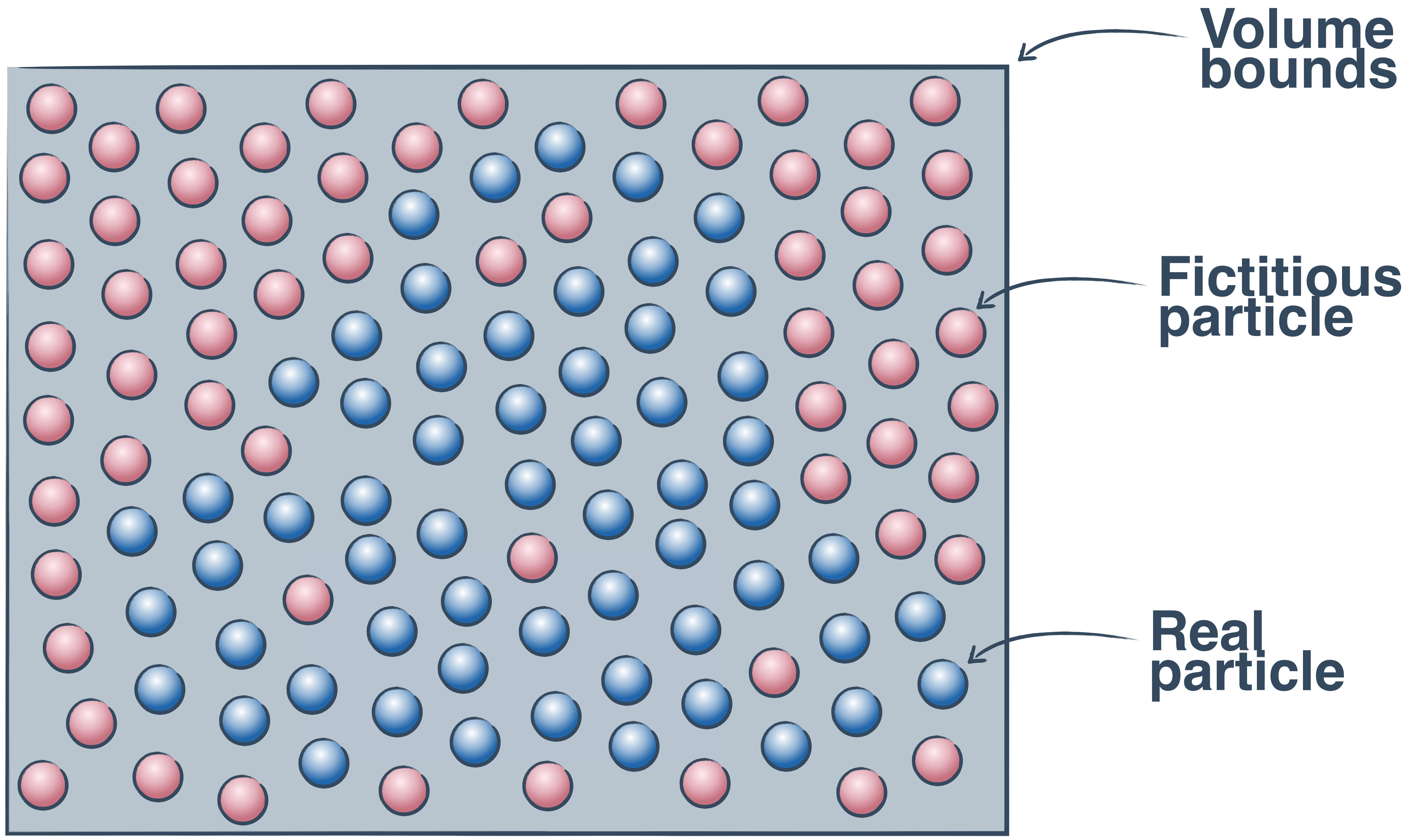
Physical Interpretation

HOMOGENIZATION



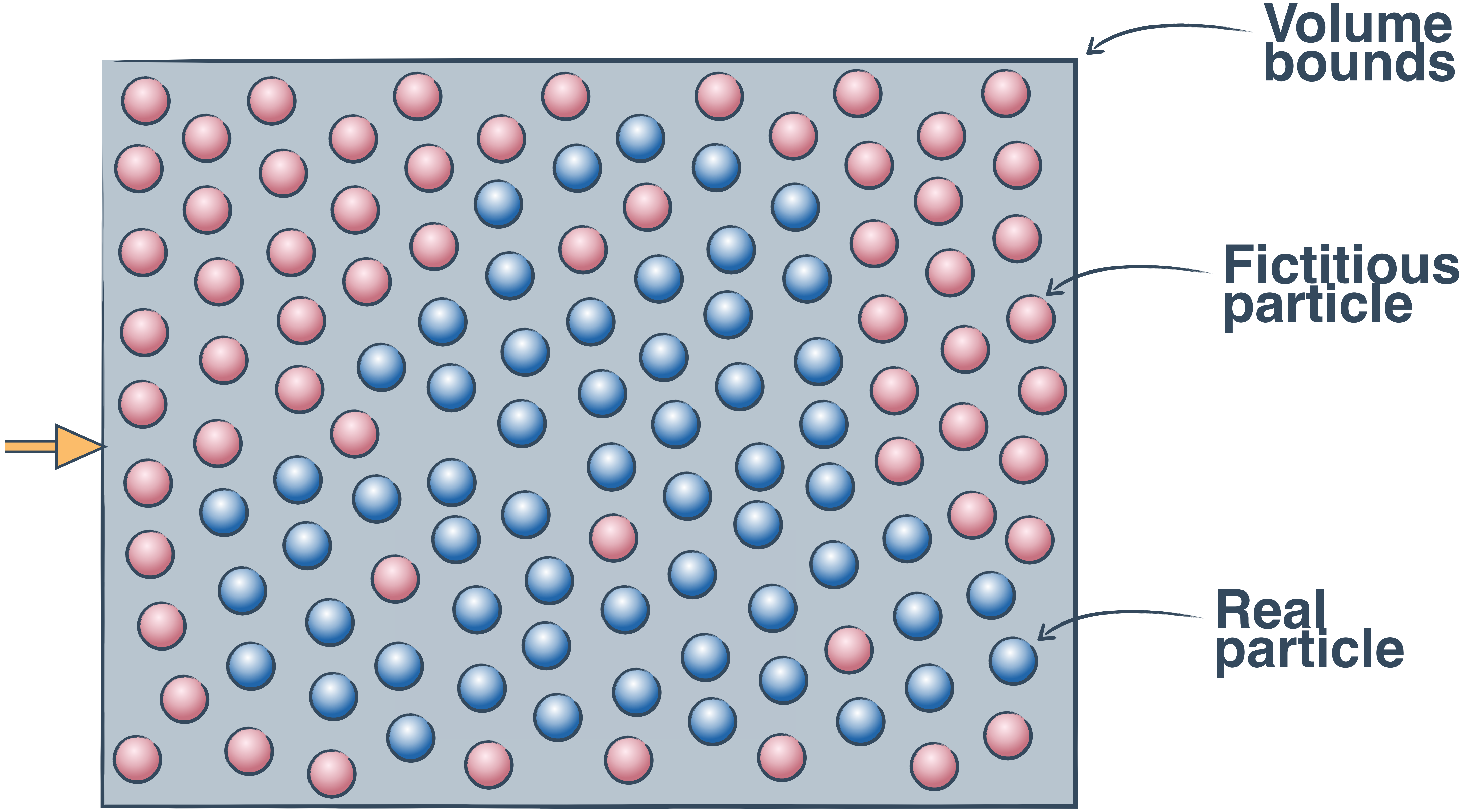
Physical Interpretation

HOMOGENIZATION



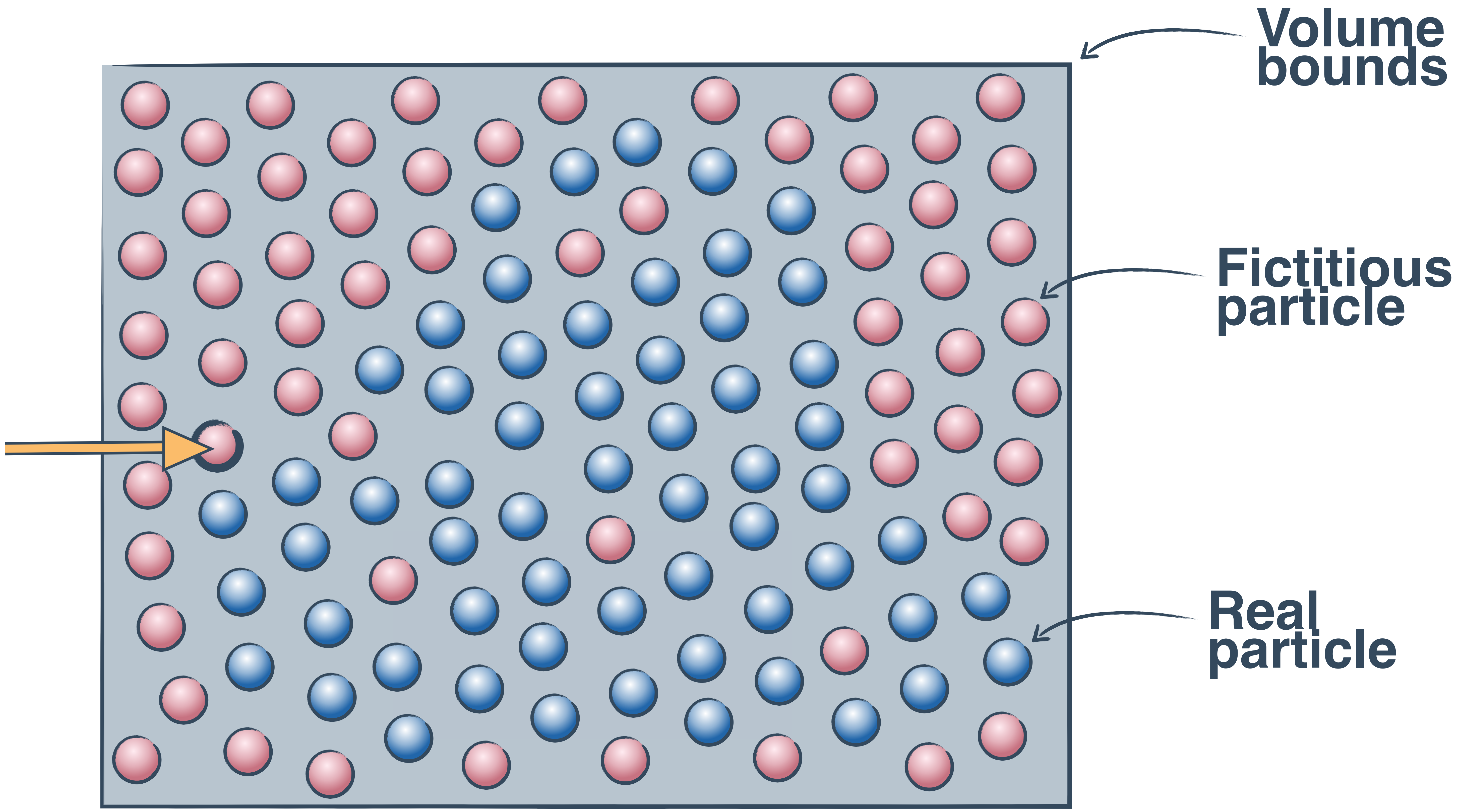
Physical Interpretation

HOMOGENIZATION



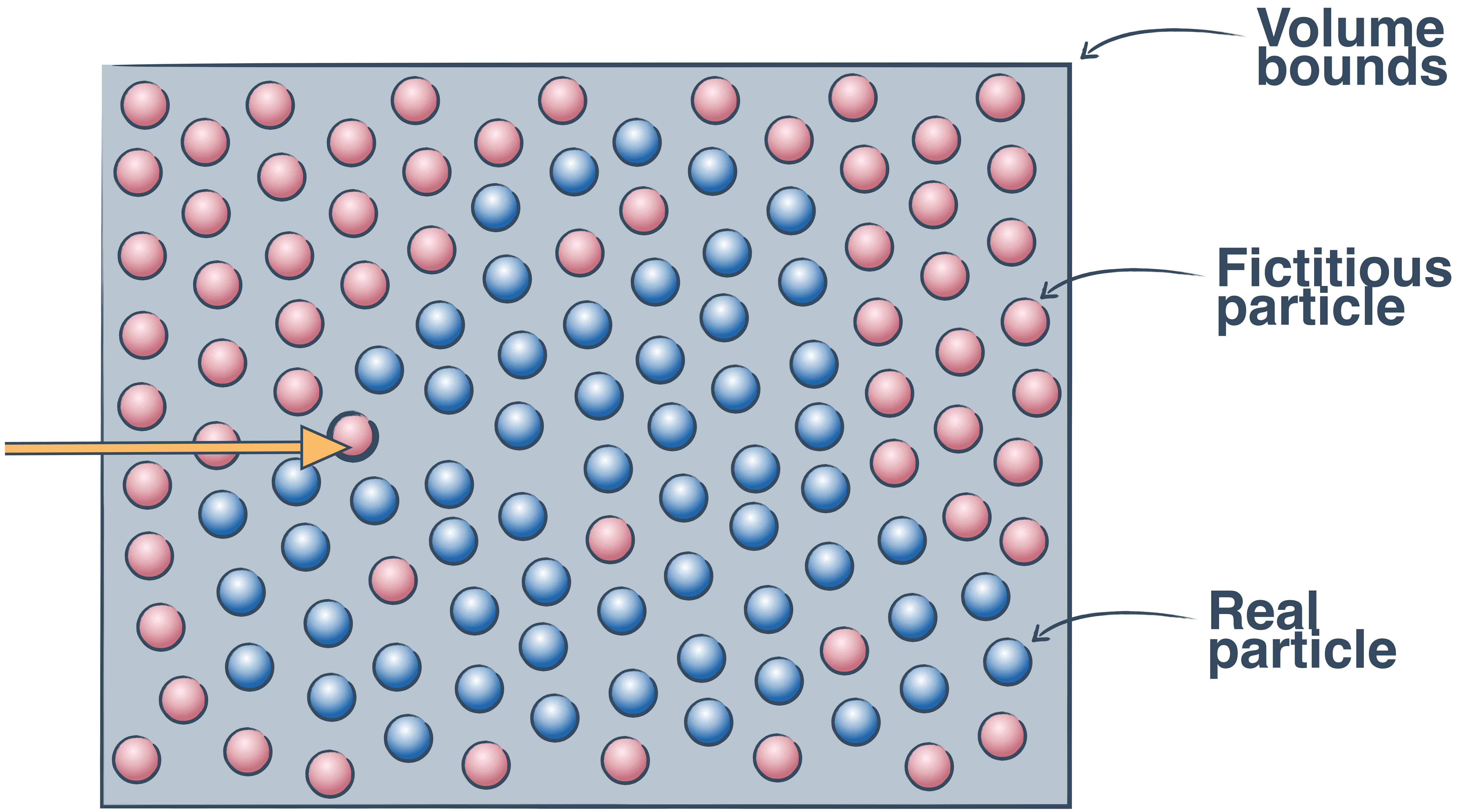
Physical Interpretation

HOMOGENIZATION



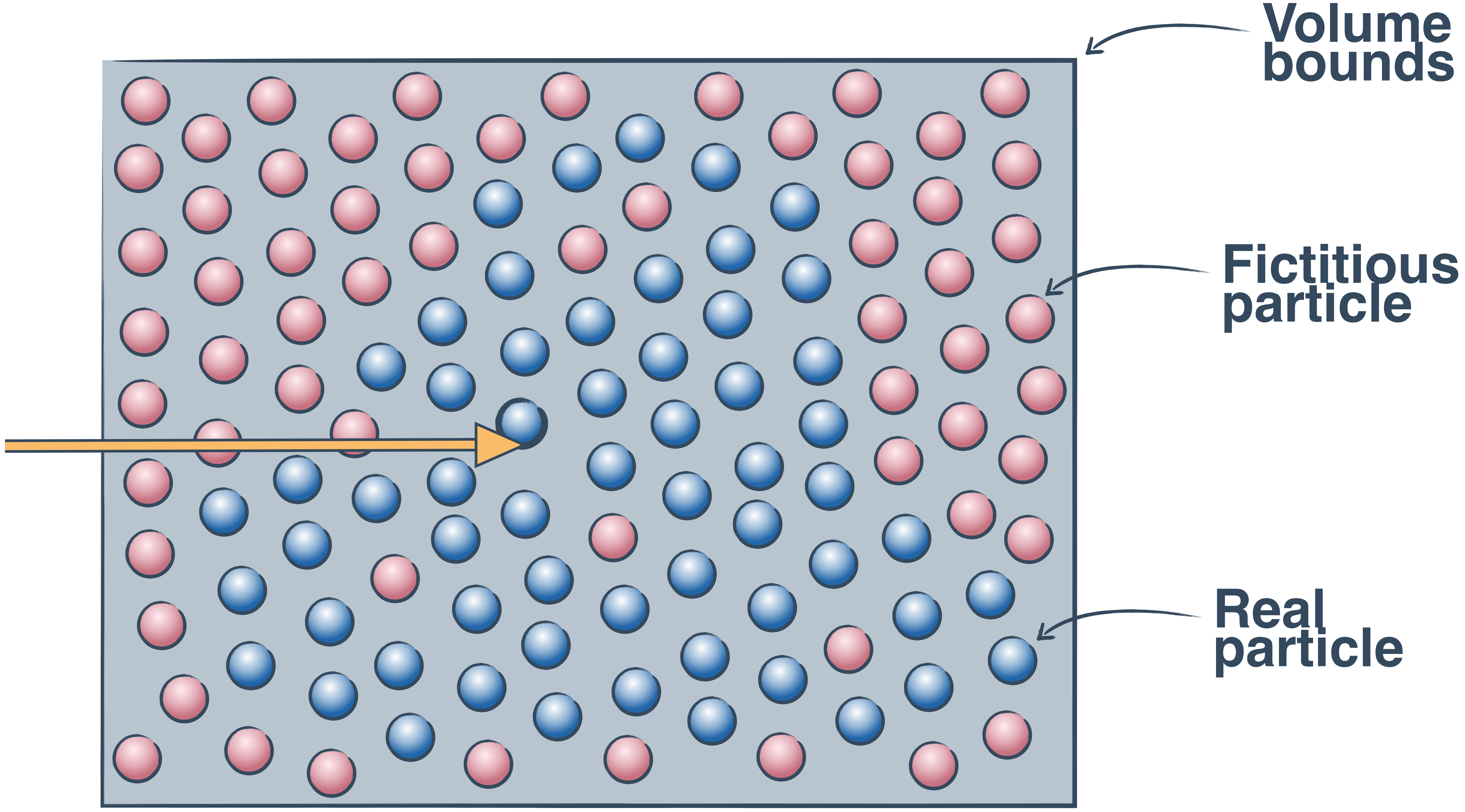
Physical Interpretation

HOMOGENIZATION



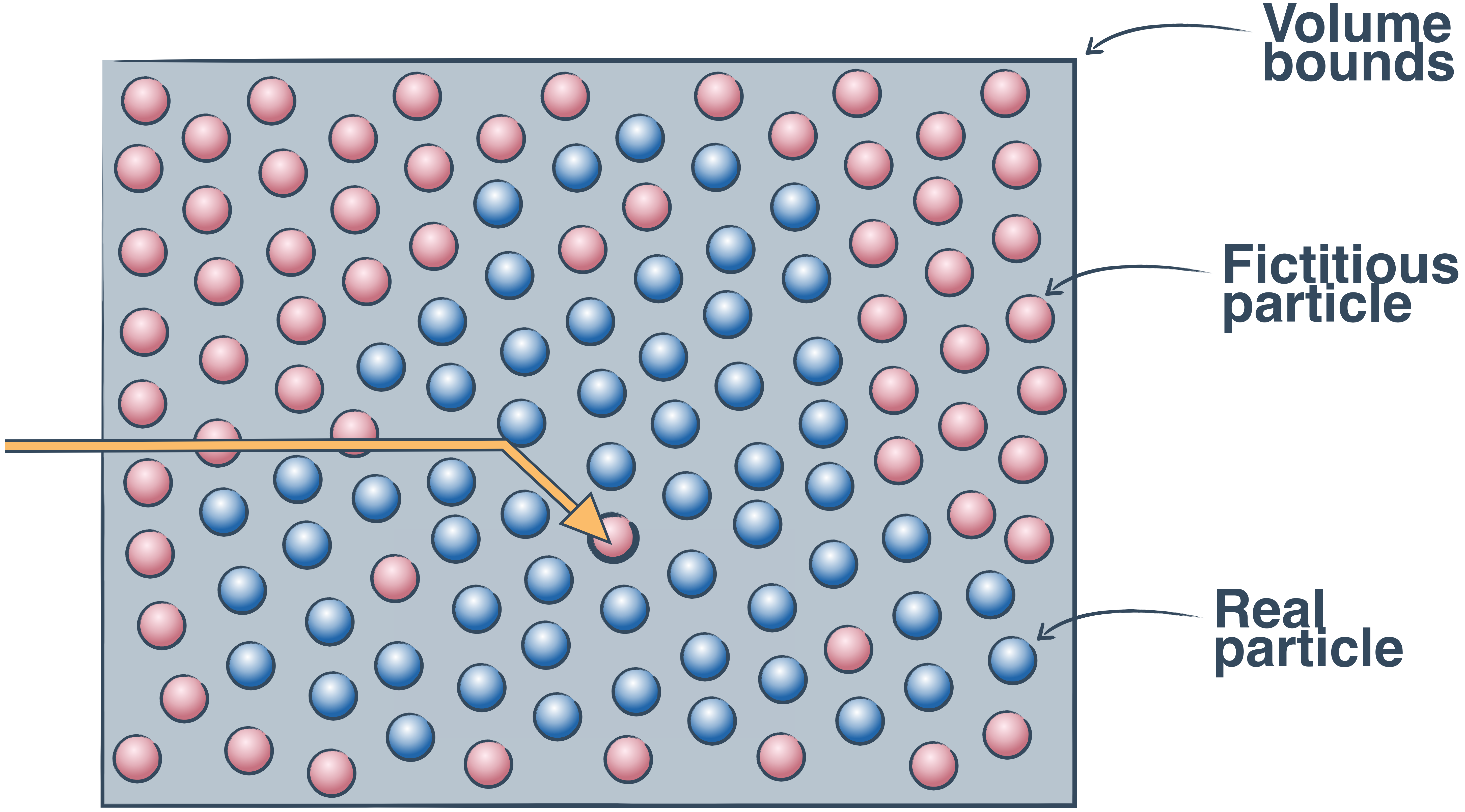
Physical Interpretation

HOMOGENIZATION



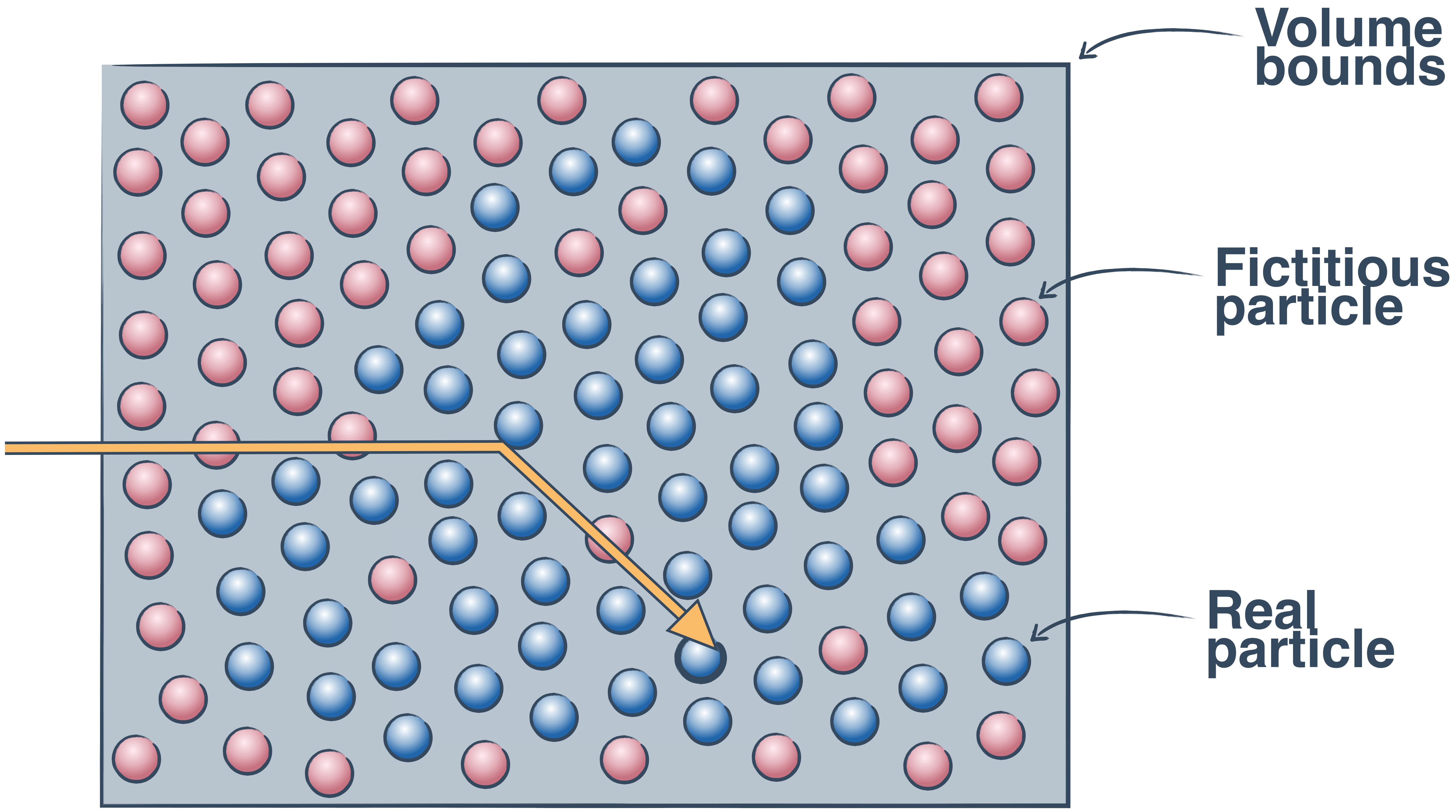
Physical Interpretation

HOMOGENIZATION



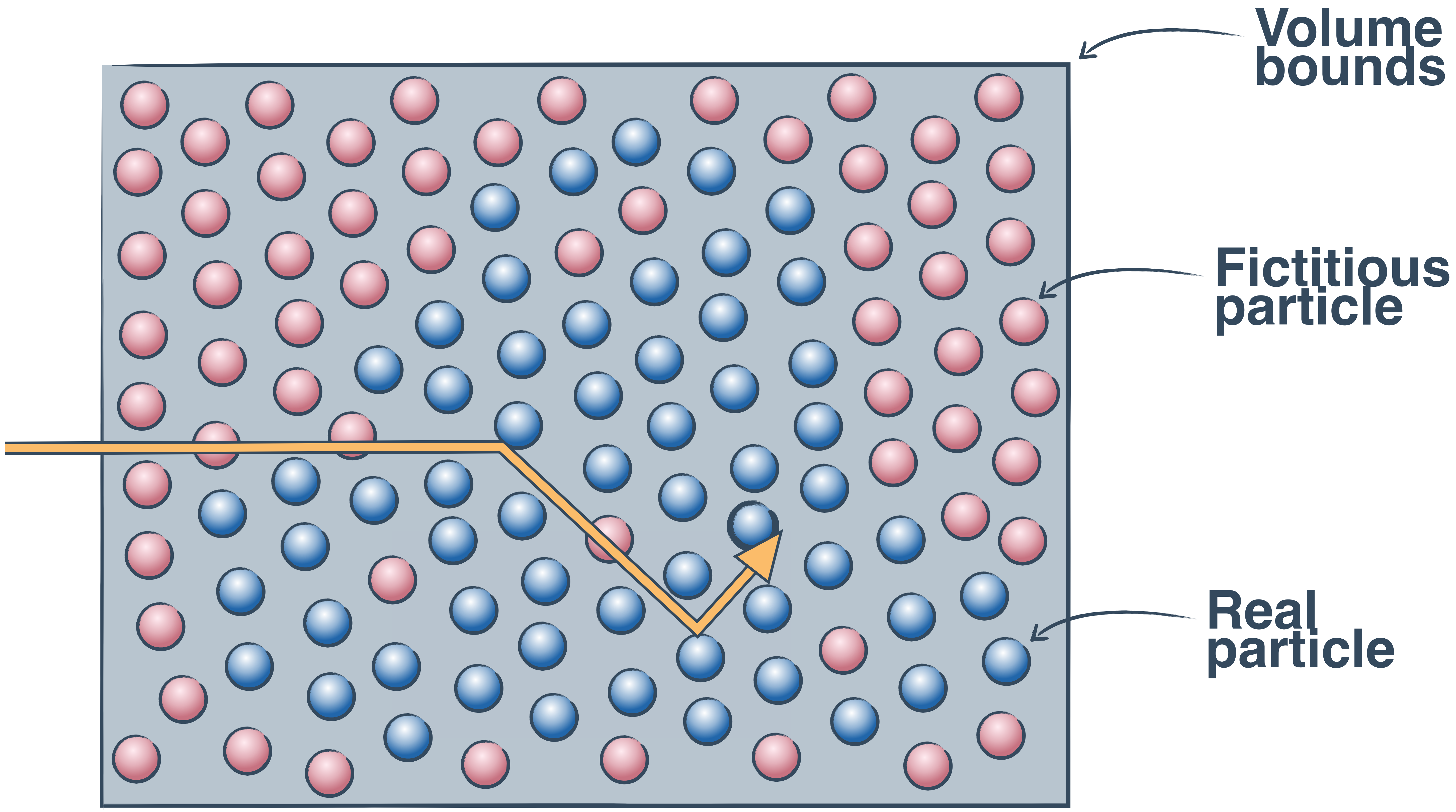
Physical Interpretation

HOMOGENIZATION



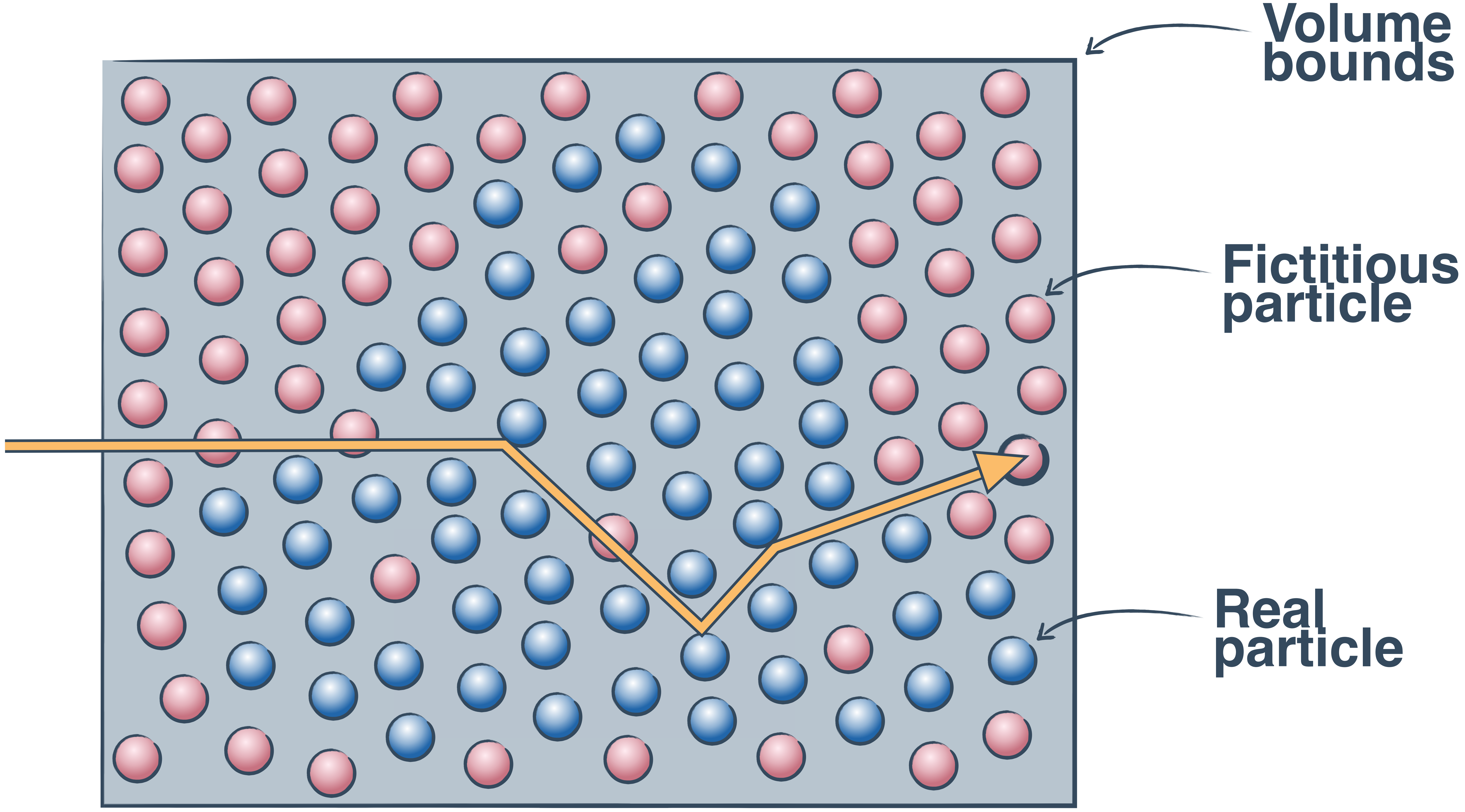
Physical Interpretation

HOMOGENIZATION



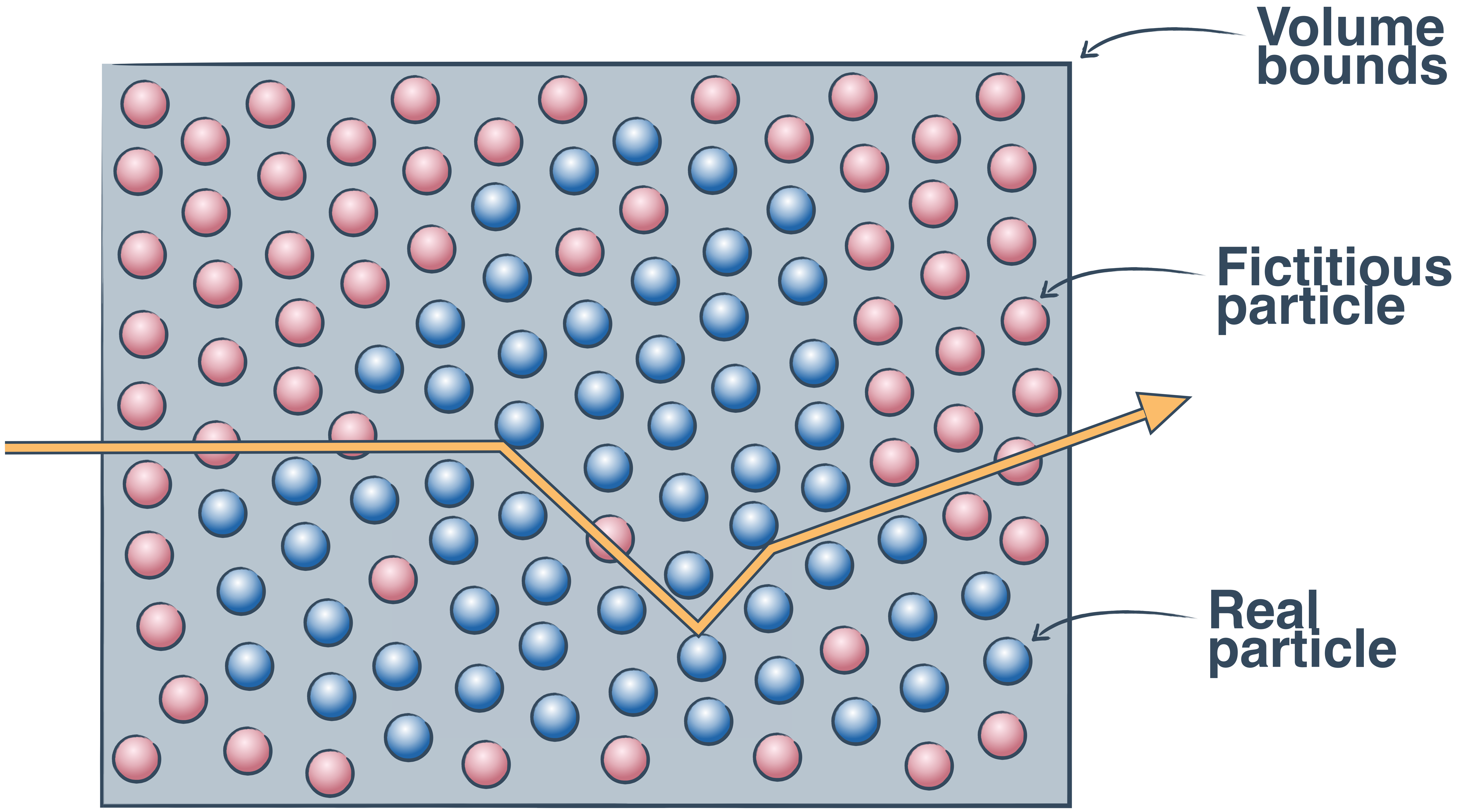
Physical Interpretation

HOMOGENIZATION



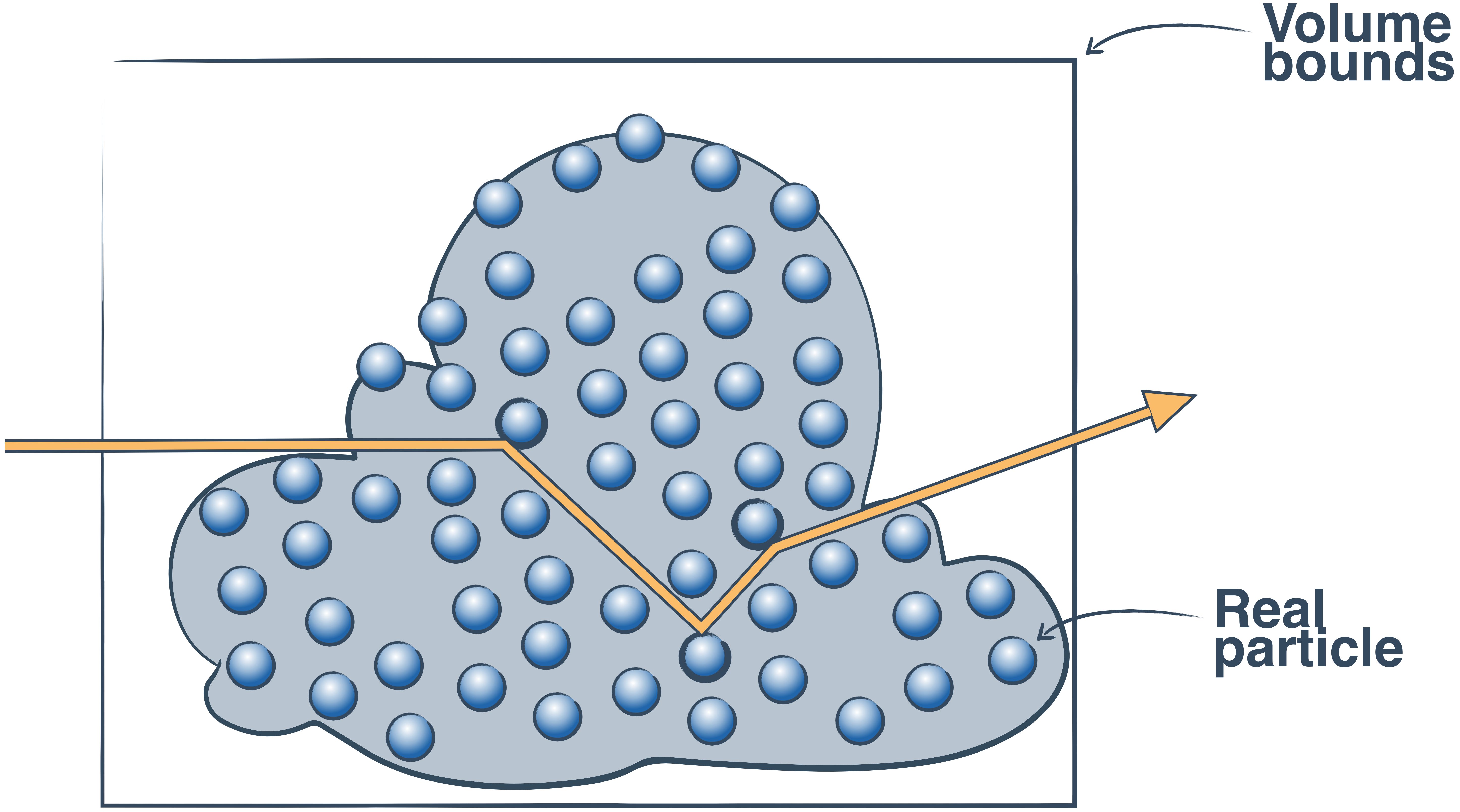
Physical Interpretation

HOMOGENIZATION

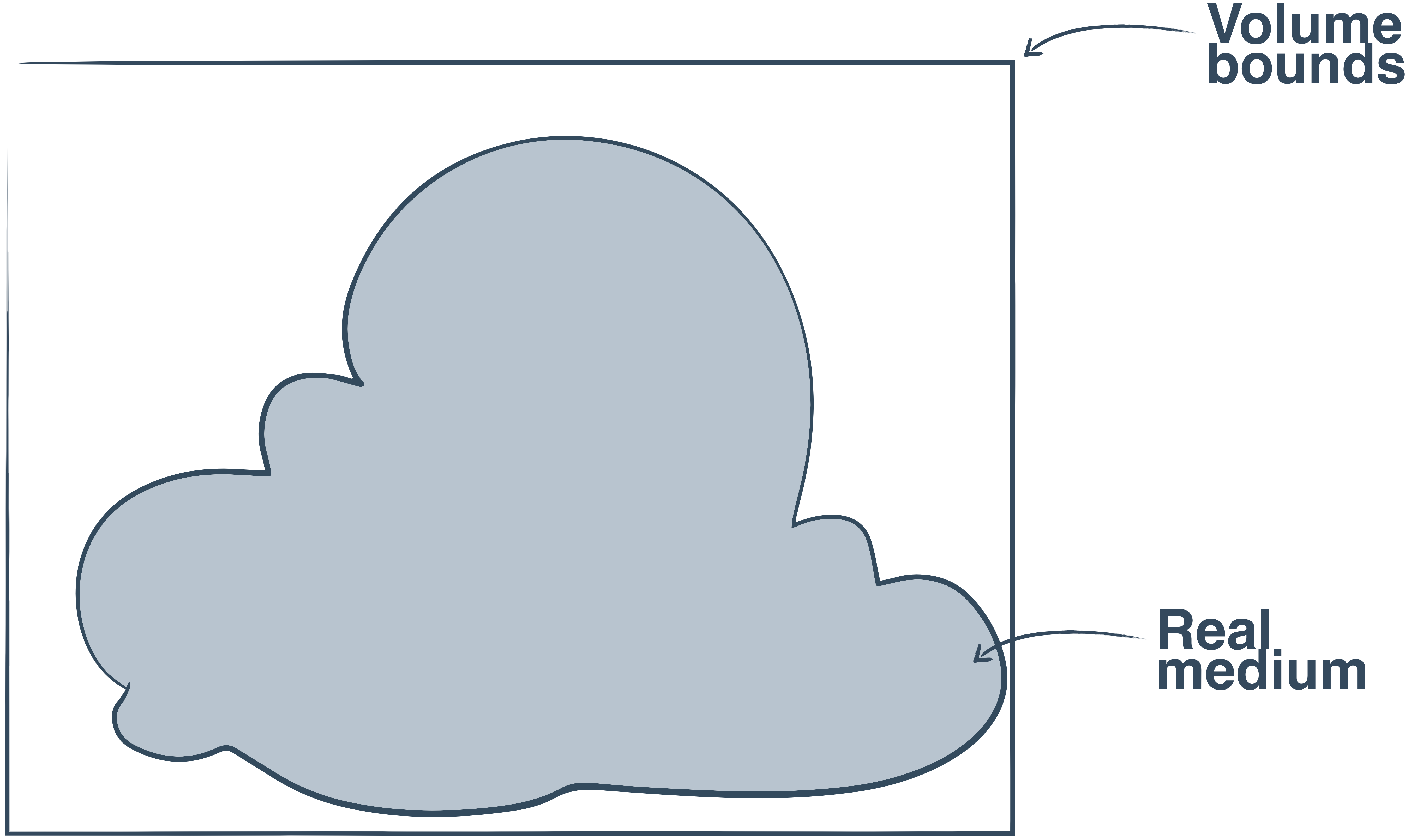


Physical Interpretation

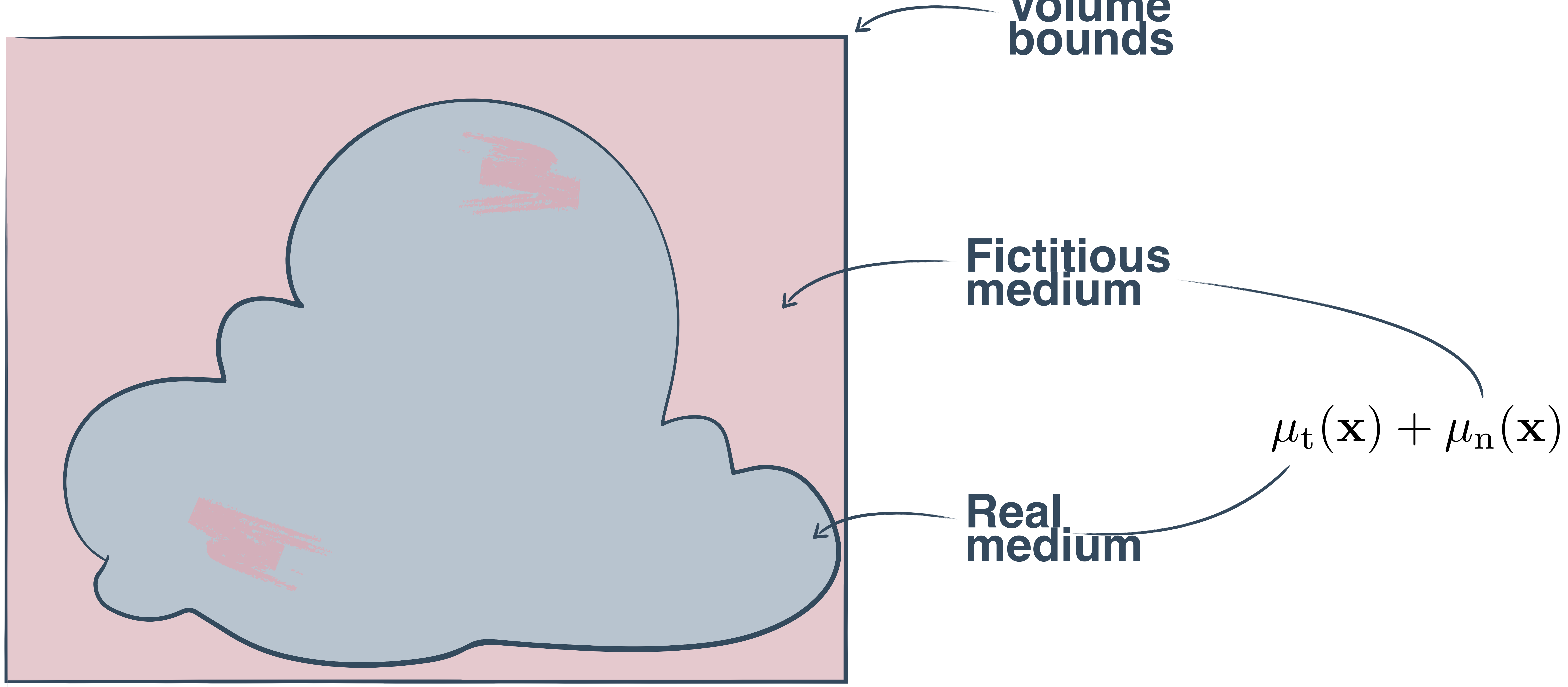
HOMOGENIZATION



Stochastic Sampling

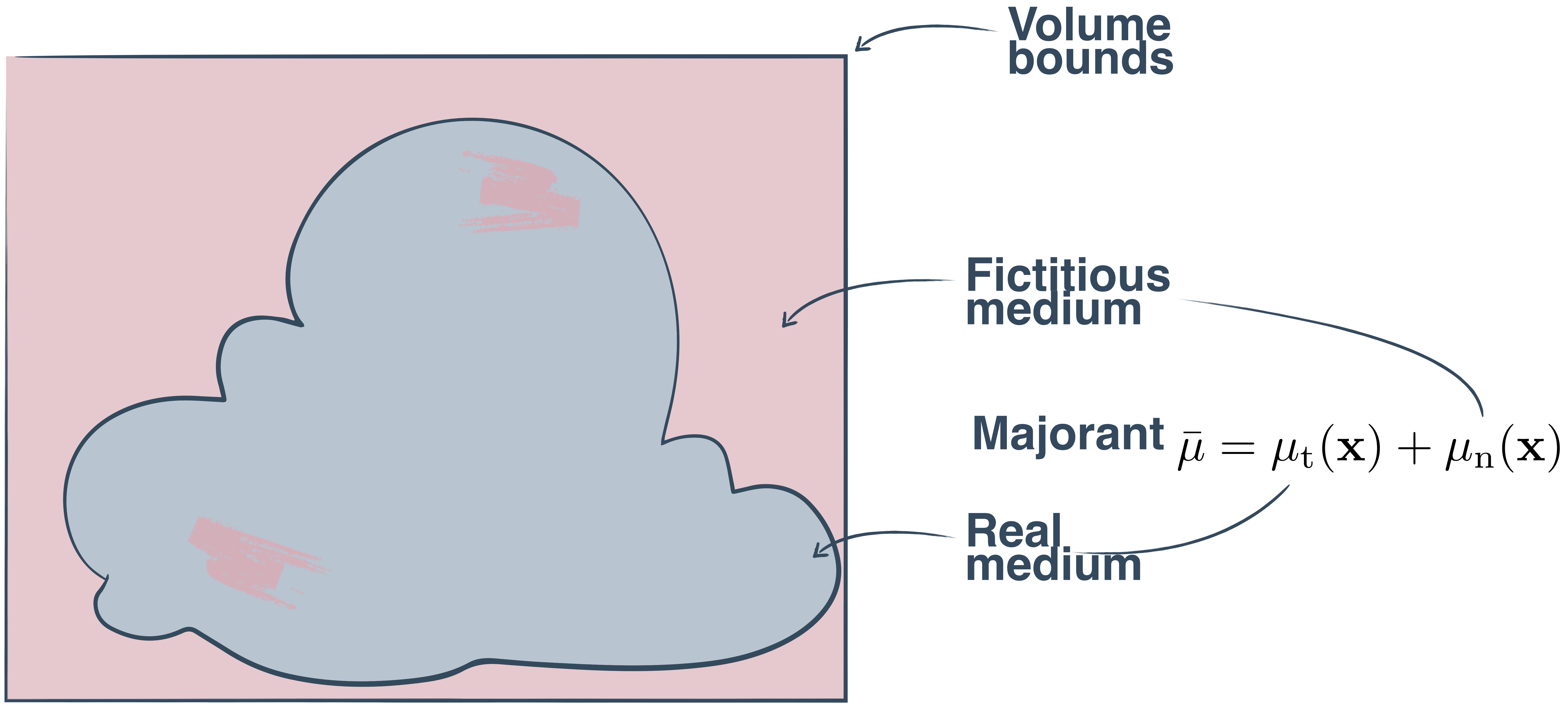


Stochastic Sampling

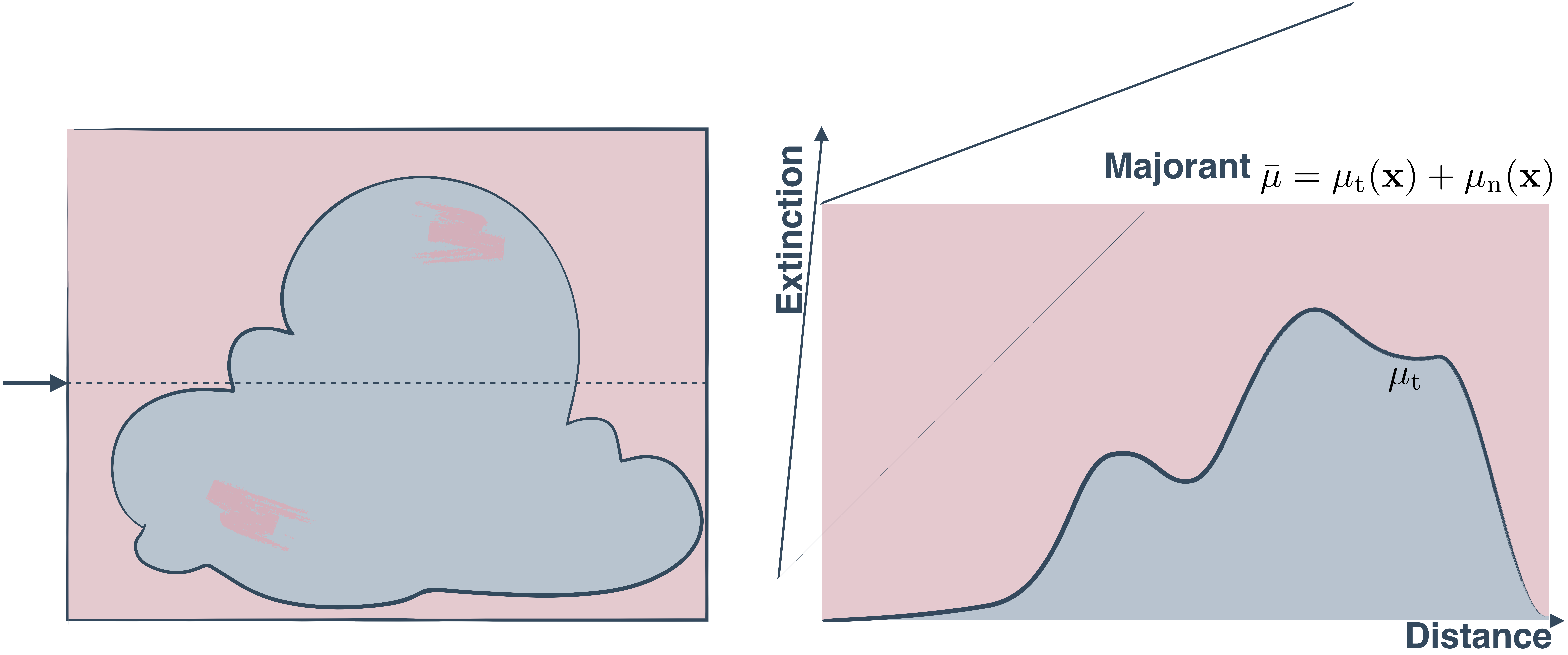


$$\mu_t(\mathbf{x}) + \mu_n(\mathbf{x})$$

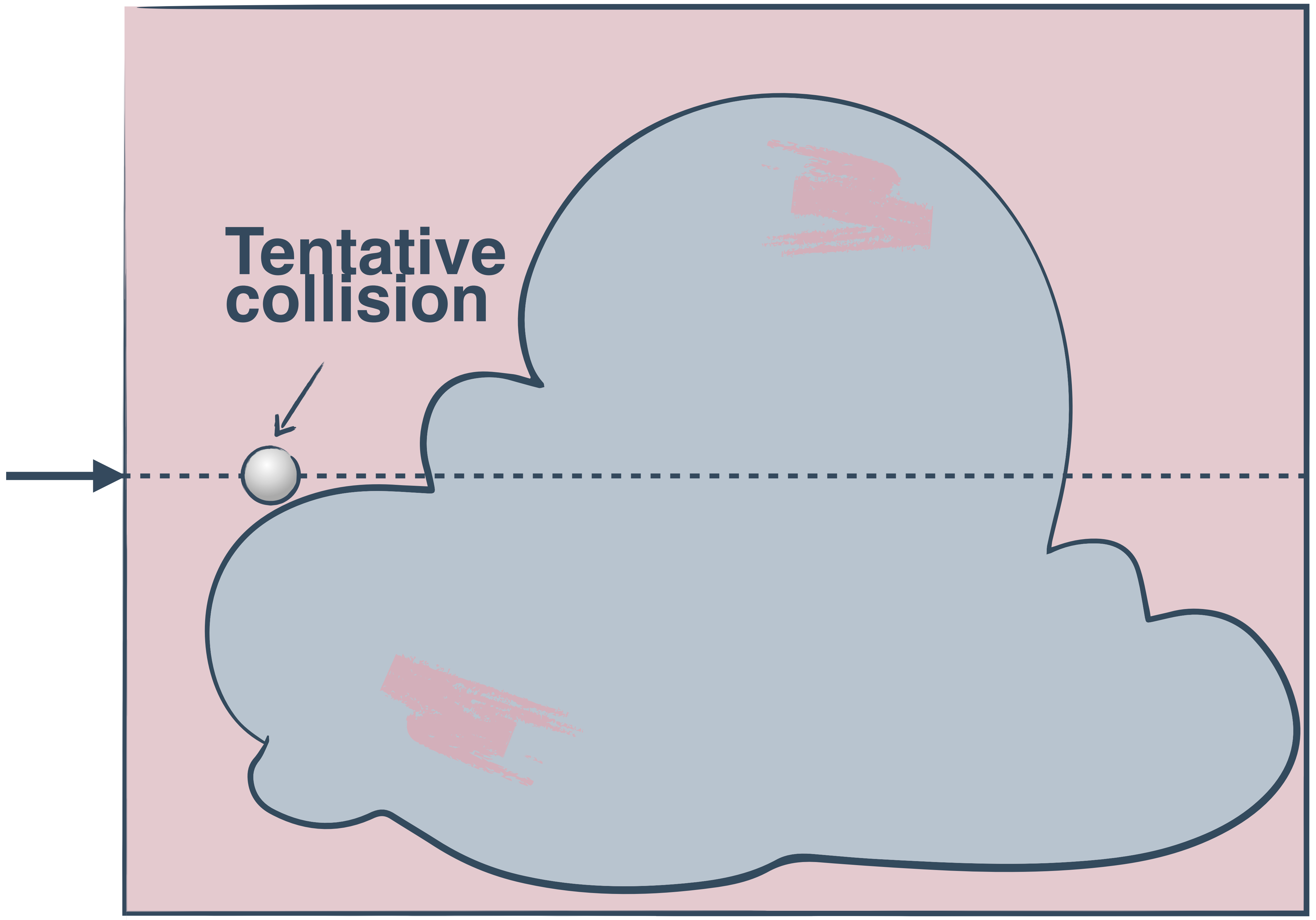
Stochastic Sampling

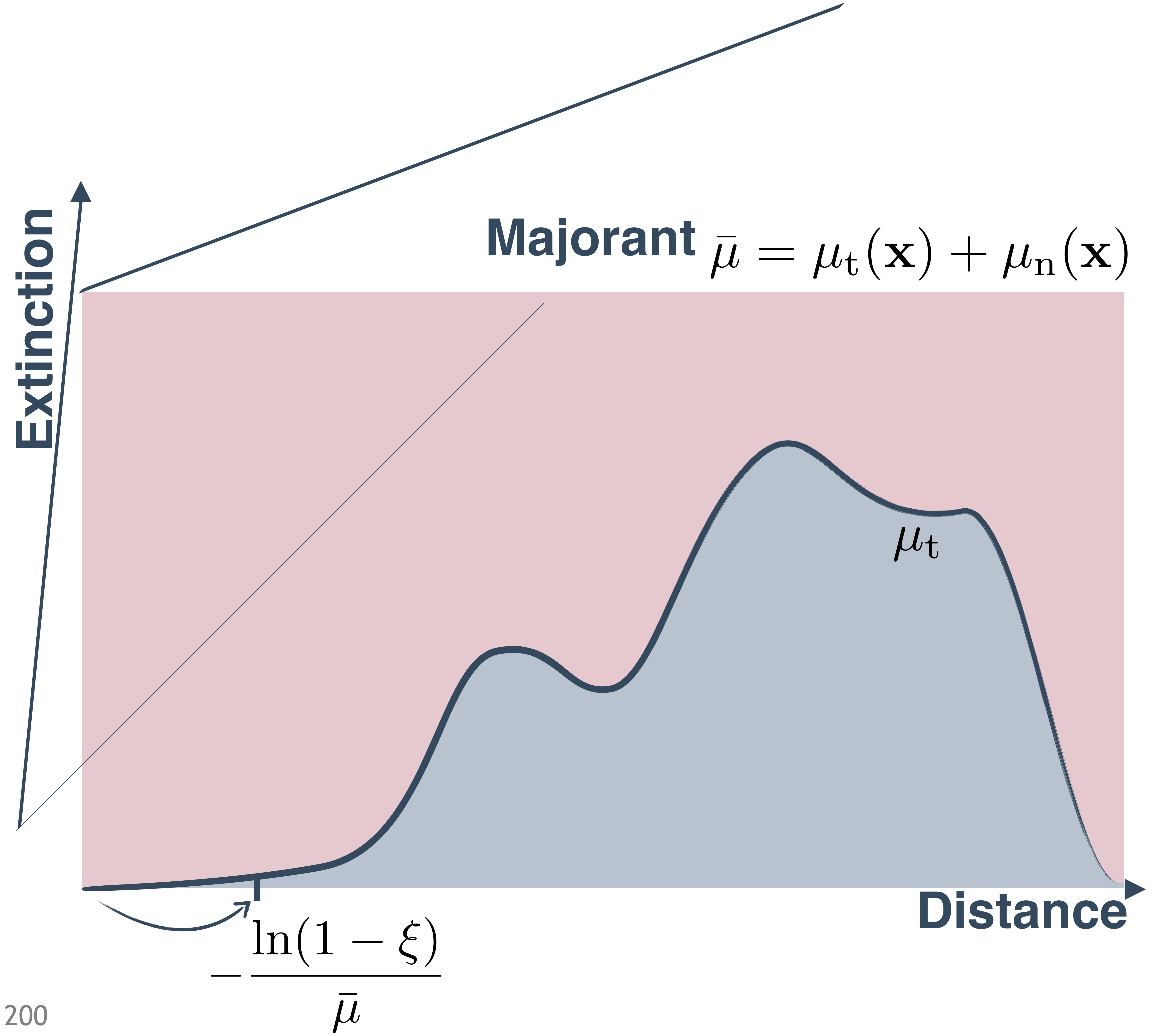


Stochastic Sampling

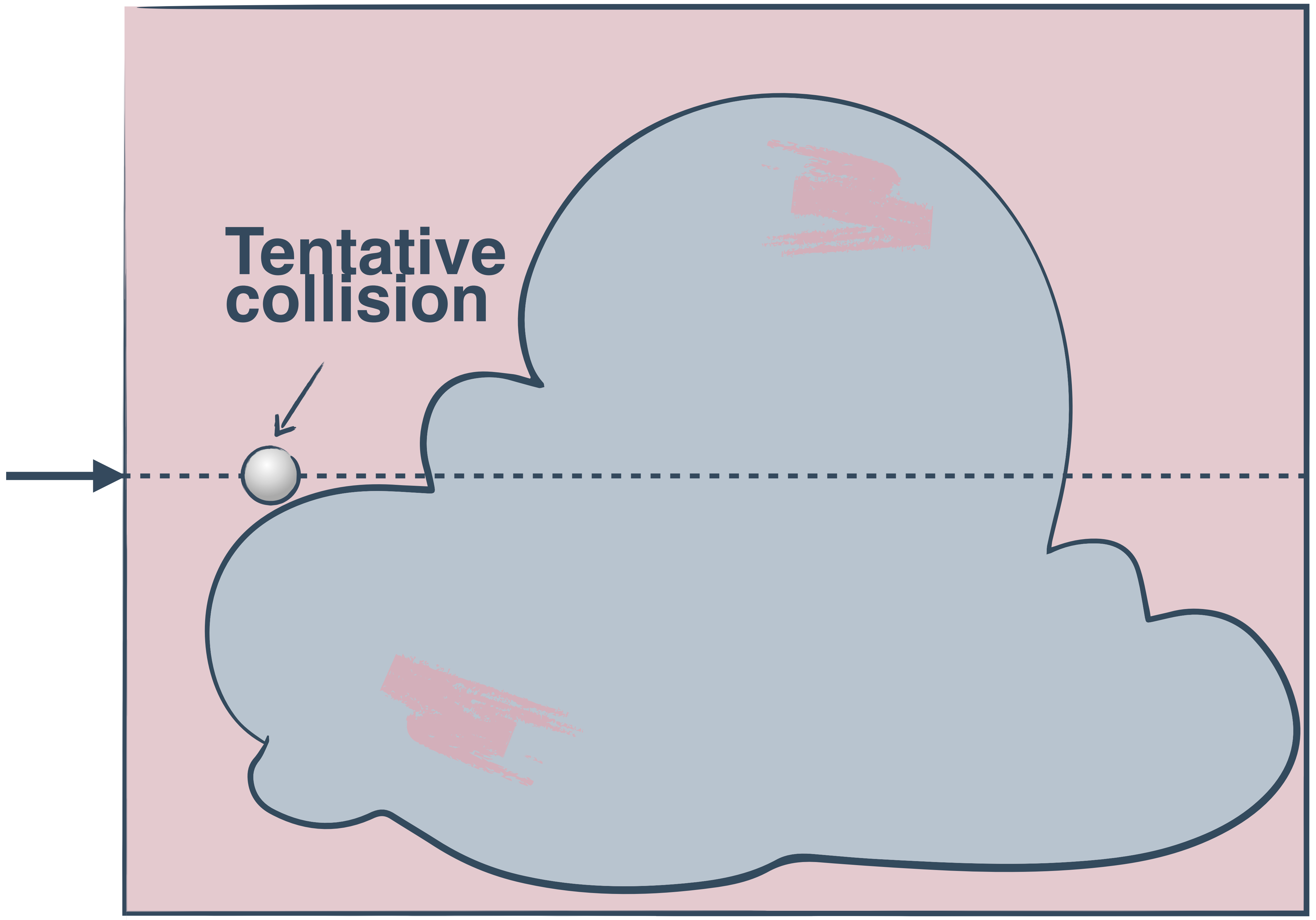


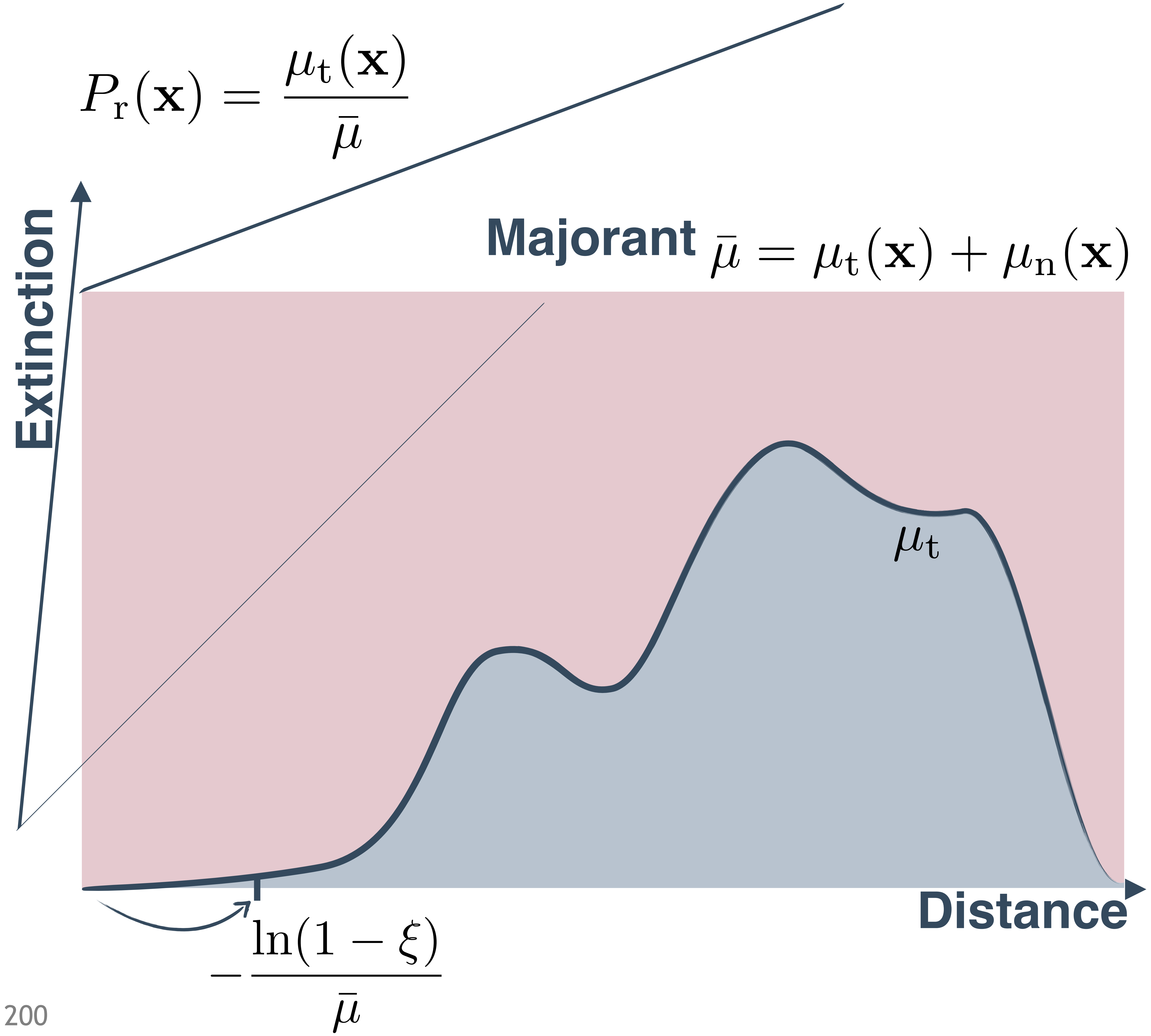
Stochastic Sampling





Stochastic Sampling



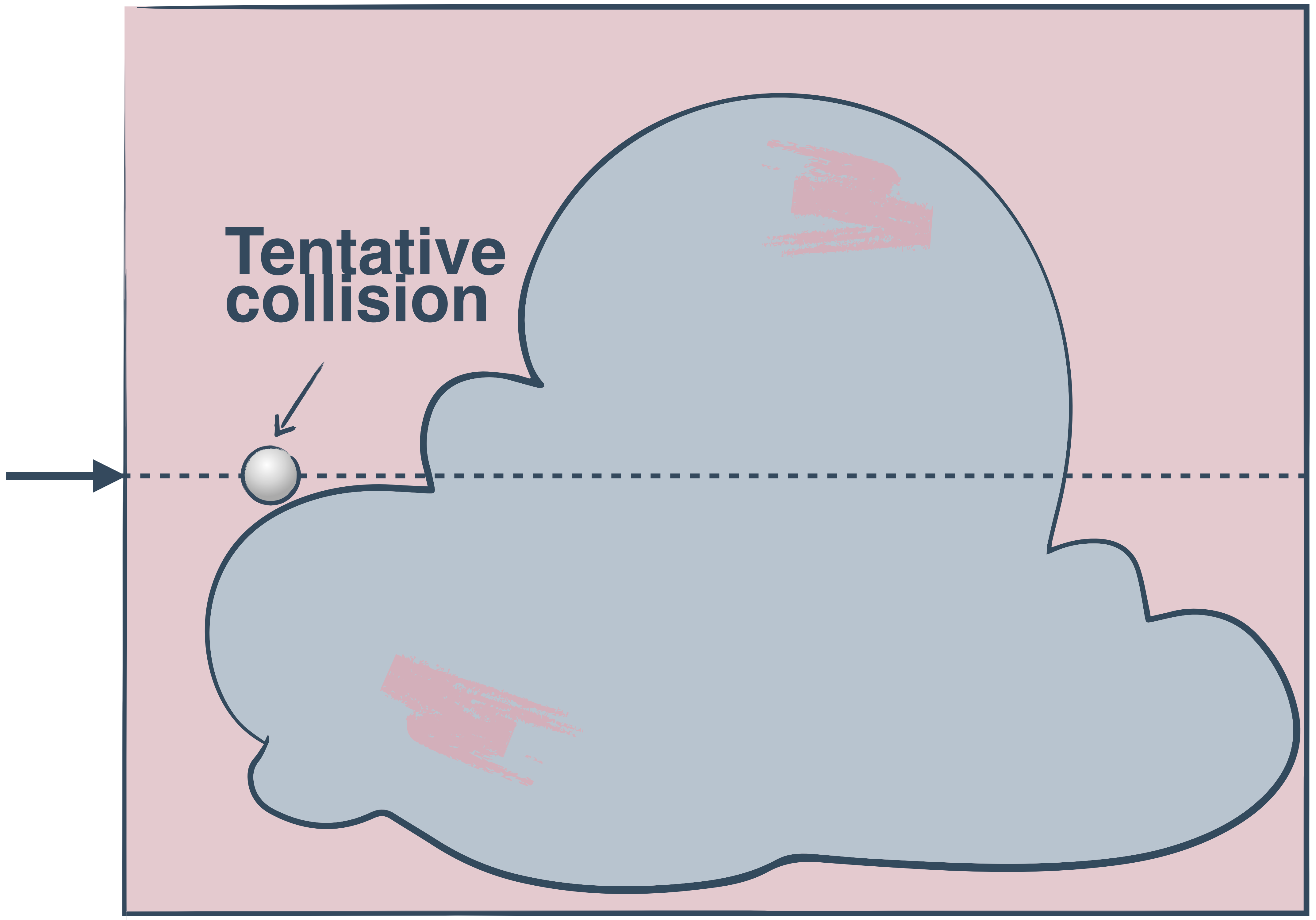


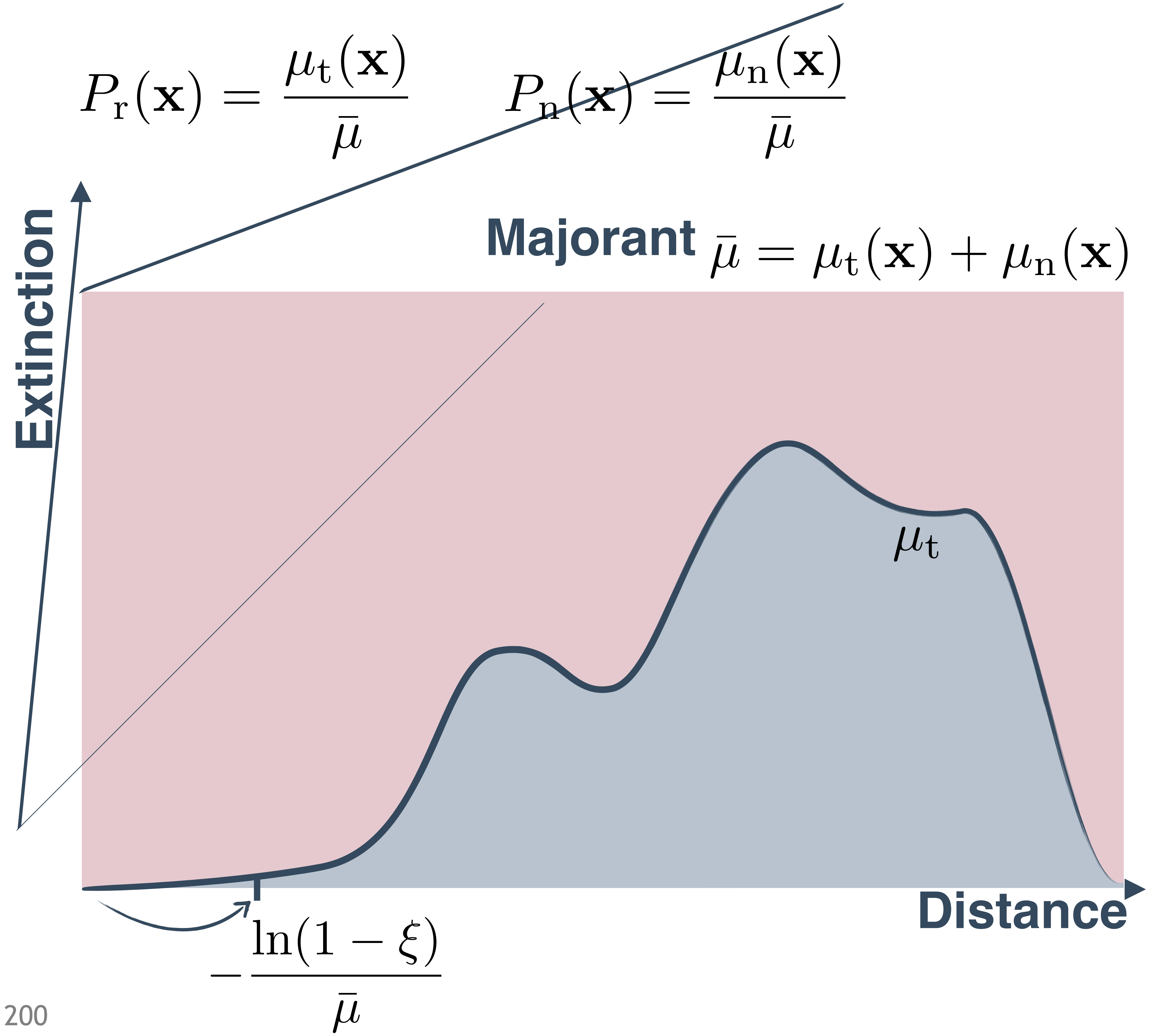
$$P_r(\mathbf{x}) = \frac{\mu_t(\mathbf{x})}{\bar{\mu}}$$

Majorant $\bar{\mu} = \mu_t(\mathbf{x}) + \mu_n(\mathbf{x})$

$$-\frac{\ln(1-\xi)}{\bar{\mu}}$$

Stochastic Sampling



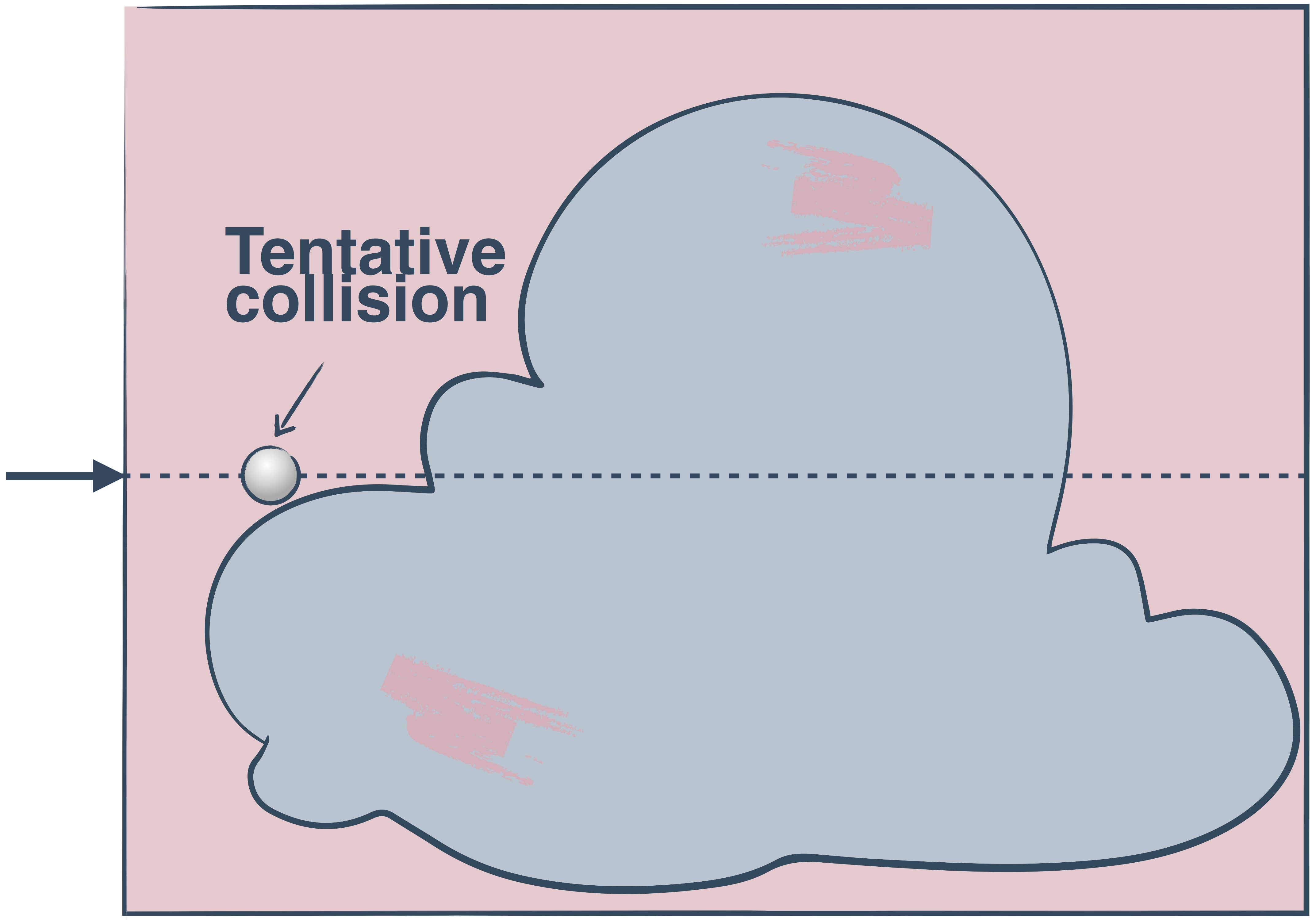


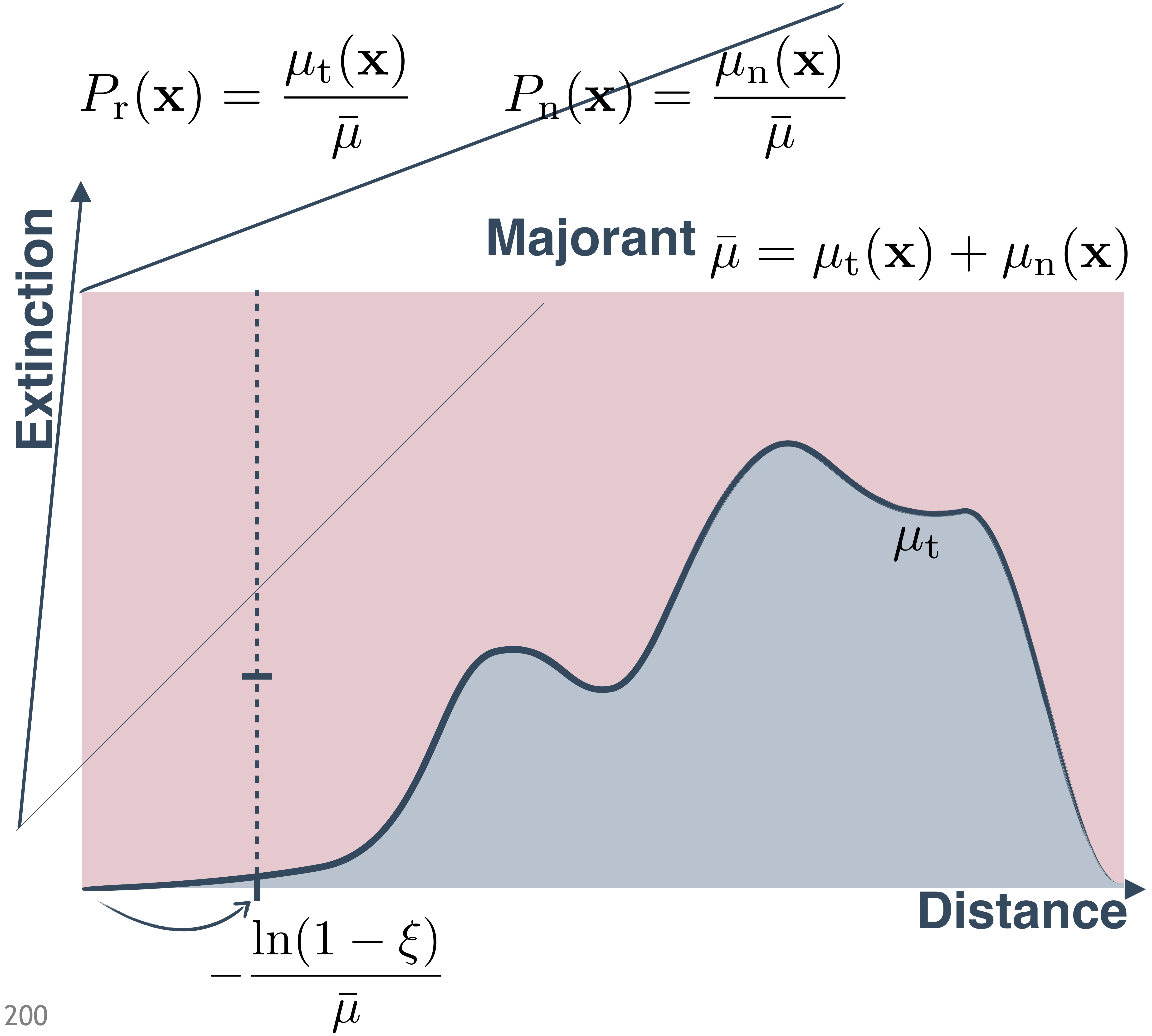
$$P_{\mathrm{r}}(\mathbf{x}) = \frac{\mu_{\mathrm{t}}(\mathbf{x})}{\bar{\mu}} \qquad P_{\mathrm{n}}(\mathbf{x}) = \frac{\mu_{\mathrm{n}}(\mathbf{x})}{\bar{\mu}}$$

Majorant $\bar{\mu} = \mu_{\mathrm{t}}(\mathbf{x}) + \mu_{\mathrm{n}}(\mathbf{x})$

$$-\frac{\ln(1-\xi)}{\bar{\mu}}$$

Stochastic Sampling





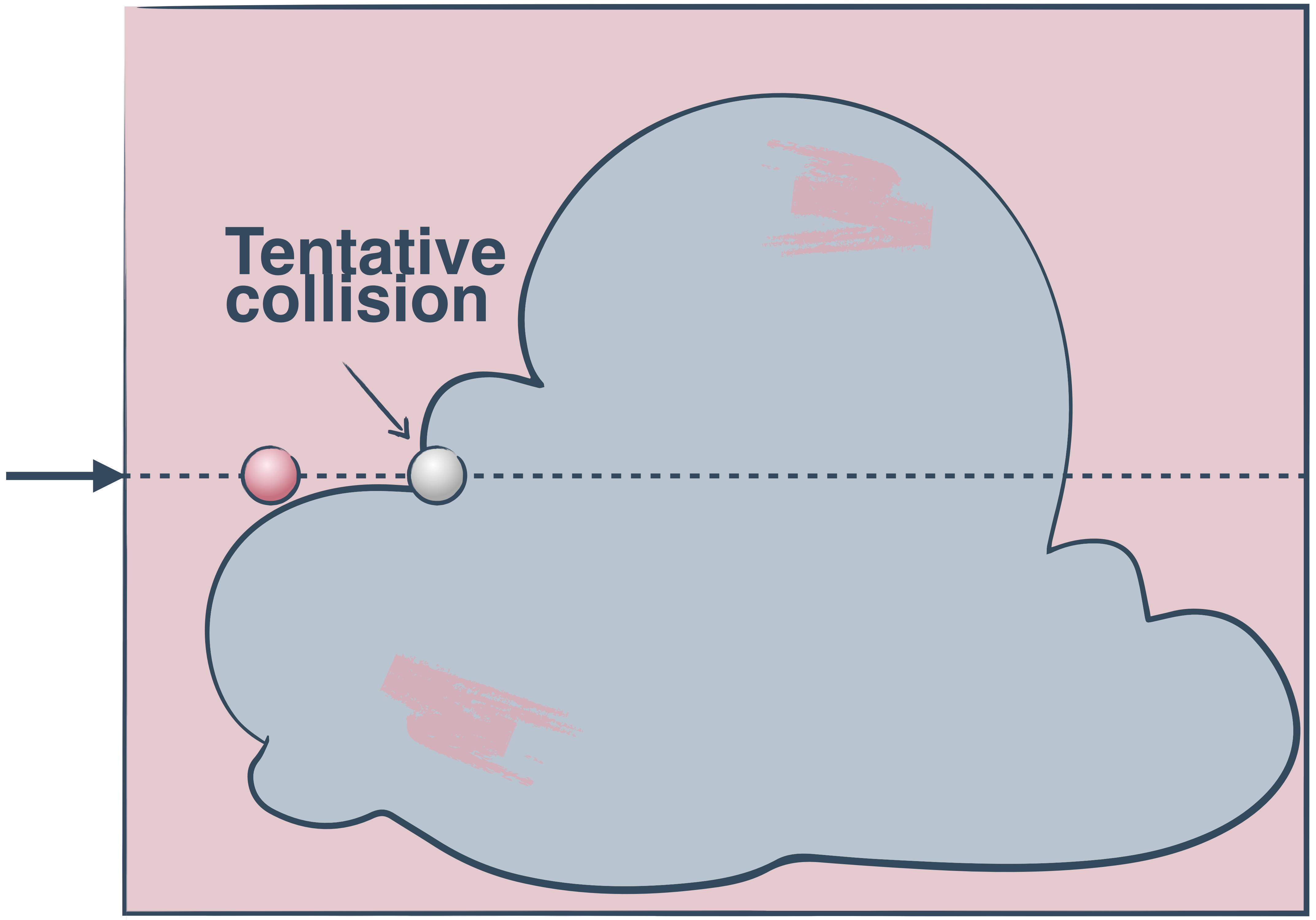
$$P_\mathrm{r}(\mathbf{x}) = \frac{\mu_\mathrm{t}(\mathbf{x})}{\bar{\mu}} \qquad P_\mathrm{n}(\mathbf{x}) = \frac{\mu_\mathrm{n}(\mathbf{x})}{\bar{\mu}}$$

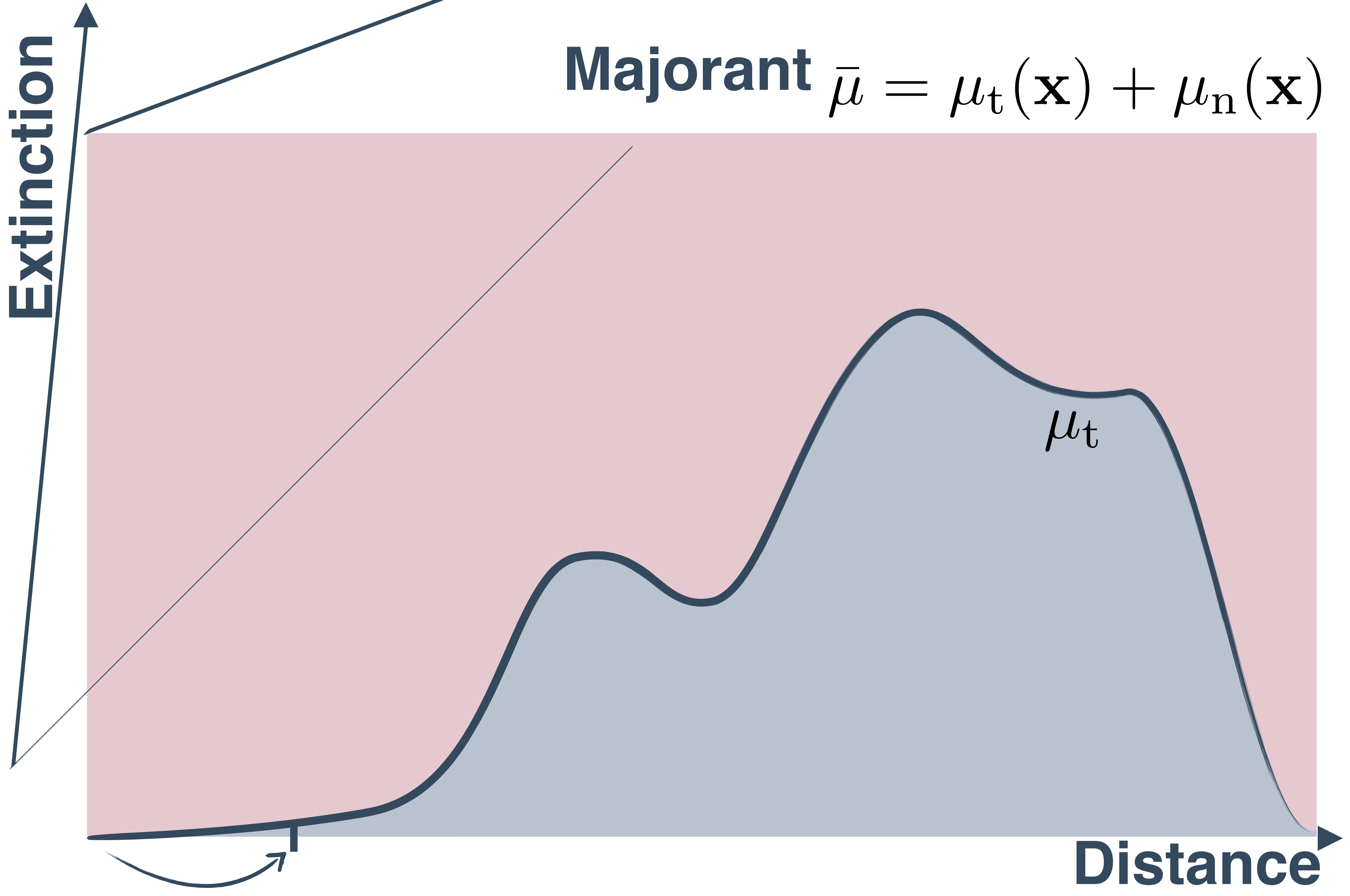
Majorant $\bar{\mu} = \mu_\mathrm{t}(\mathbf{x}) + \mu_\mathrm{n}(\mathbf{x})$

$$-\frac{\ln(1-\xi)}{\bar{\mu}}$$

Stochastic Sampling

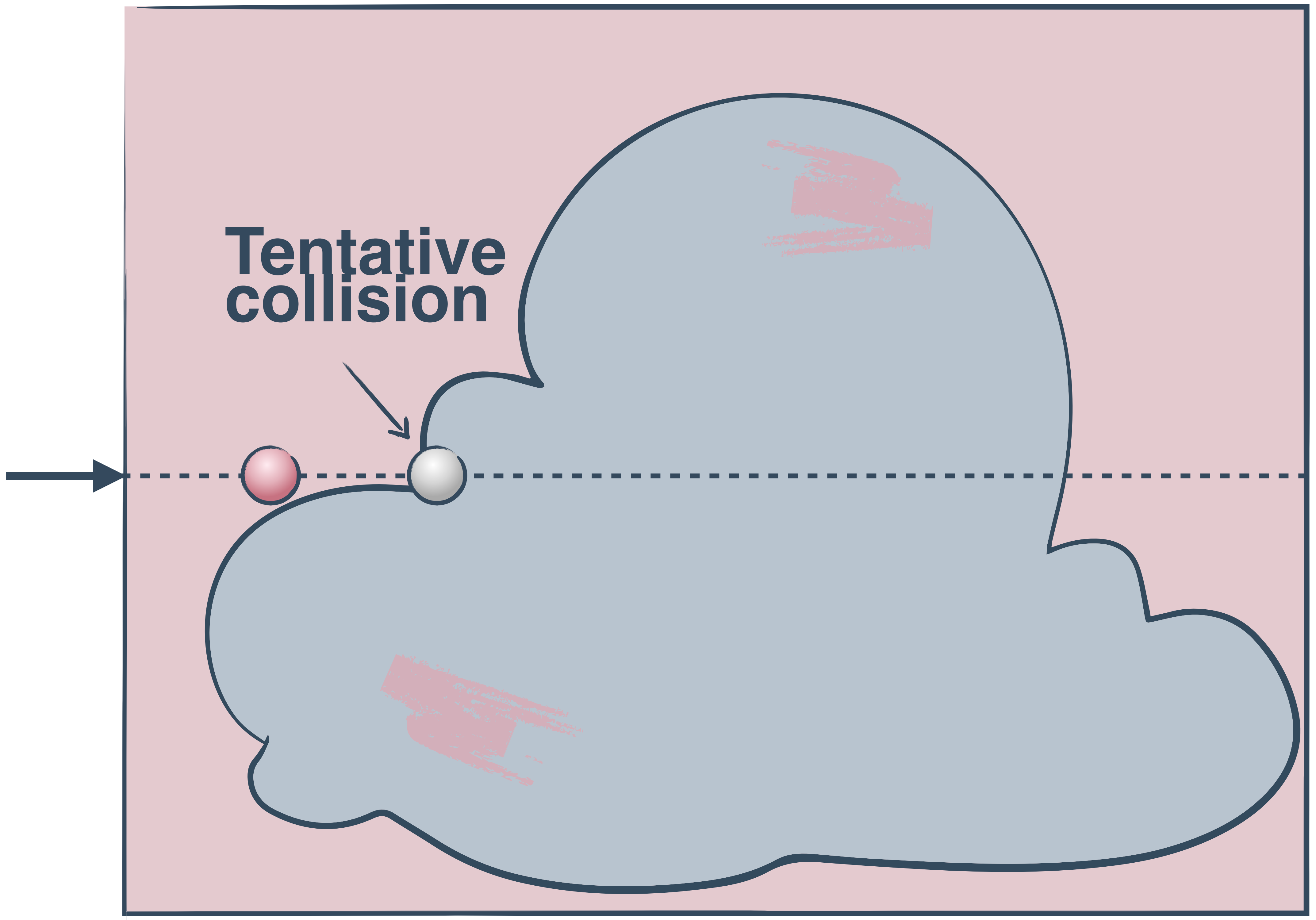
$$P_\mathrm{r}(\mathbf{x}) = \frac{\mu_\mathrm{t}(\mathbf{x})}{\bar{\mu}} \qquad P_\mathrm{n}(\mathbf{x}) = \frac{\mu_\mathrm{n}(\mathbf{x})}{\bar{\mu}}$$

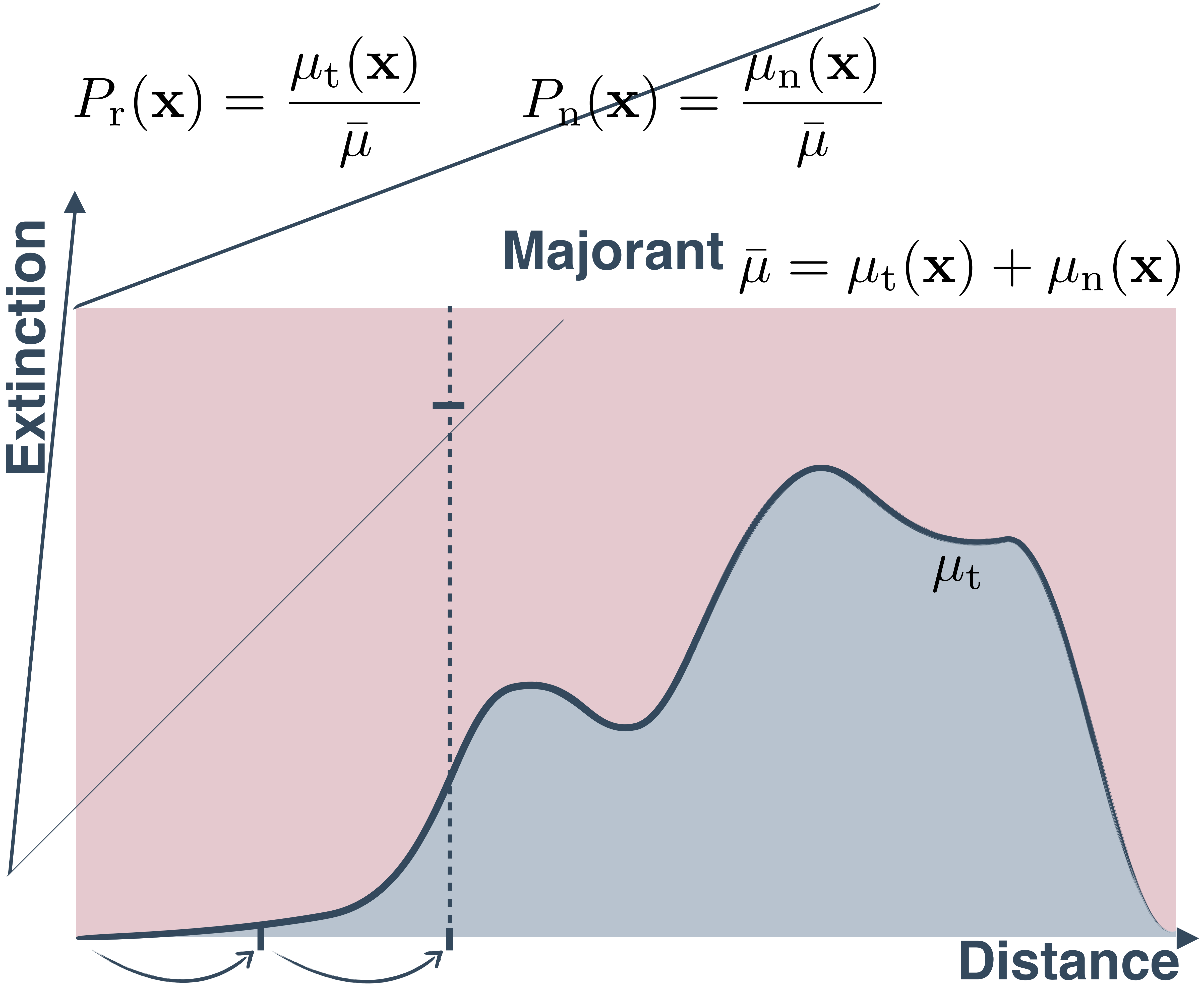




Stochastic Sampling

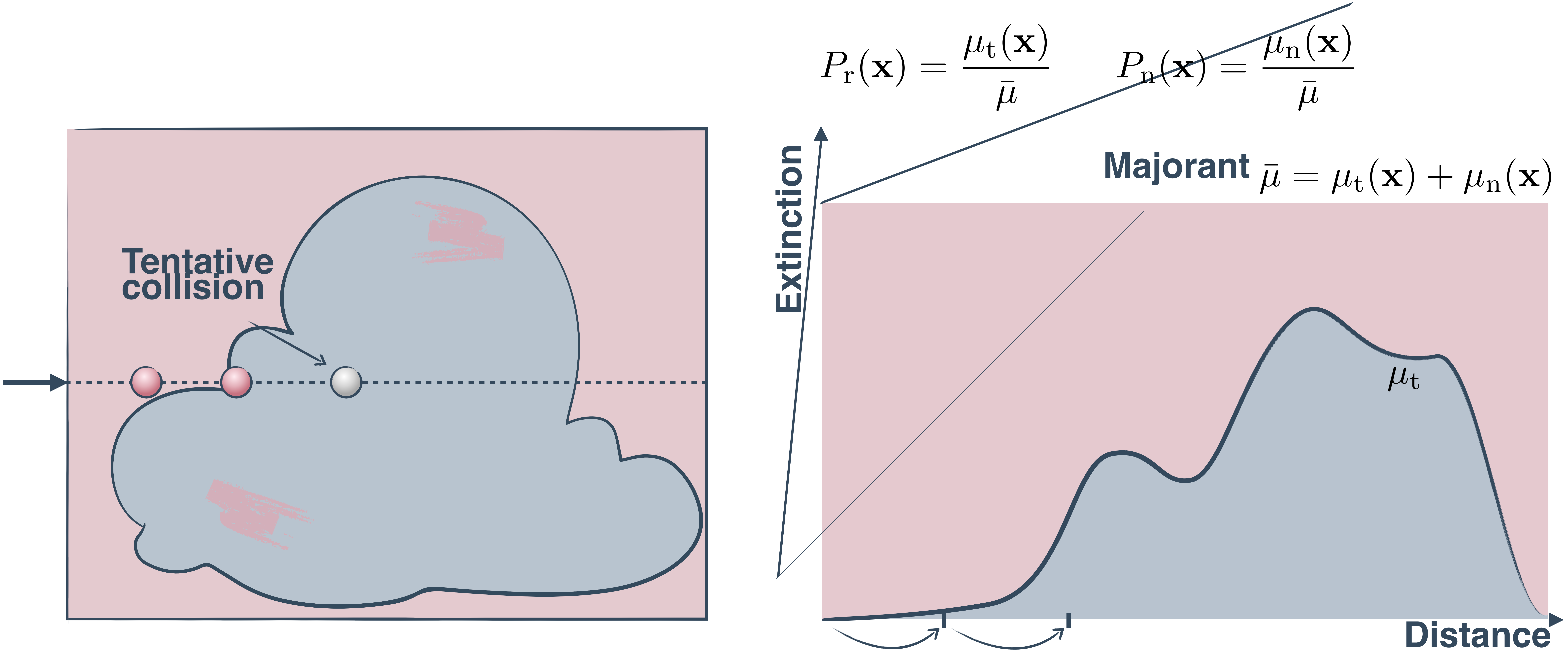
$$P_{\mathrm{r}}(\mathbf{x}) = \frac{\mu_{\mathrm{t}}(\mathbf{x})}{\bar{\mu}} \qquad P_{\mathrm{n}}(\mathbf{x}) = \frac{\mu_{\mathrm{n}}(\mathbf{x})}{\bar{\mu}}$$





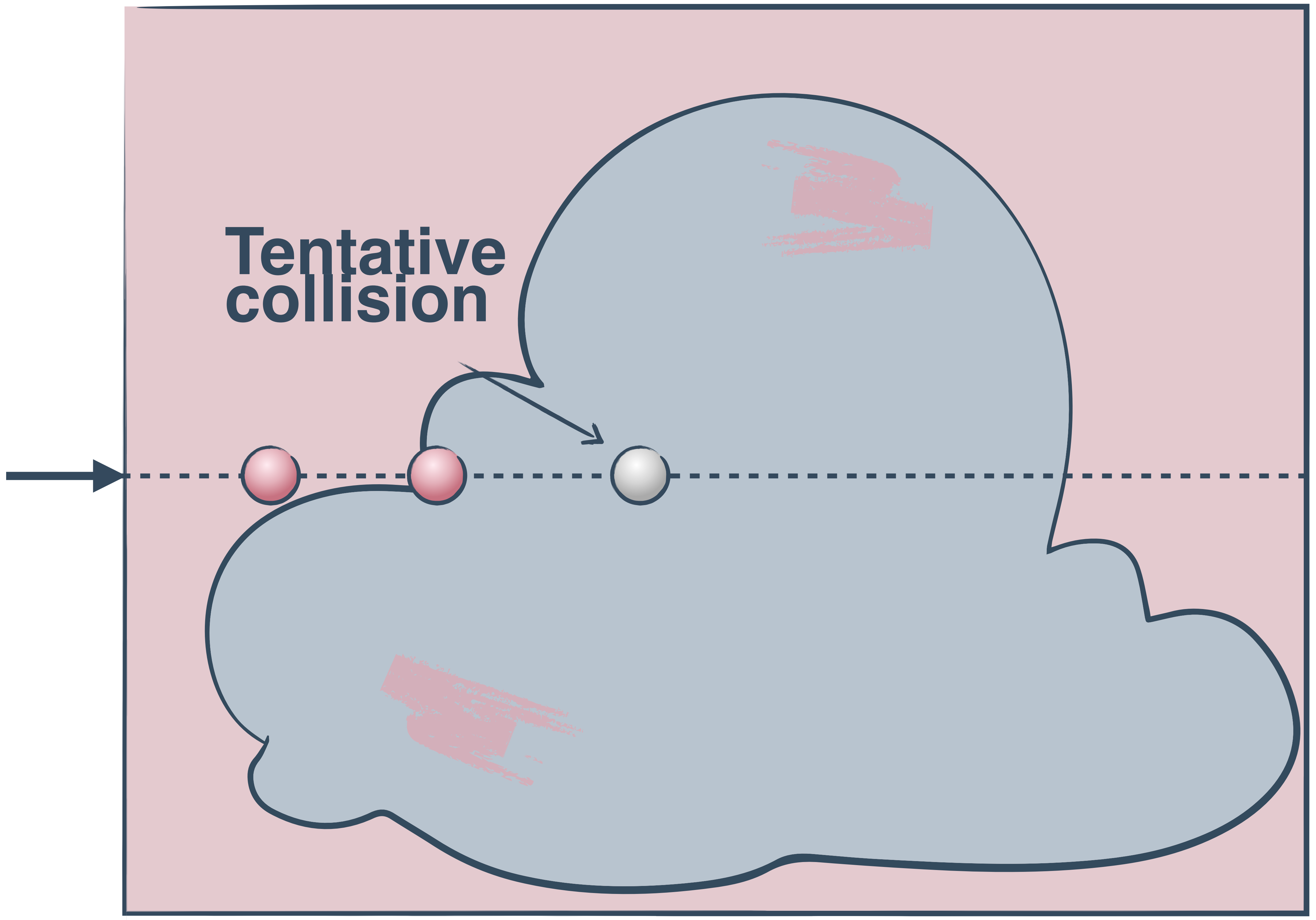
Stochastic Sampling

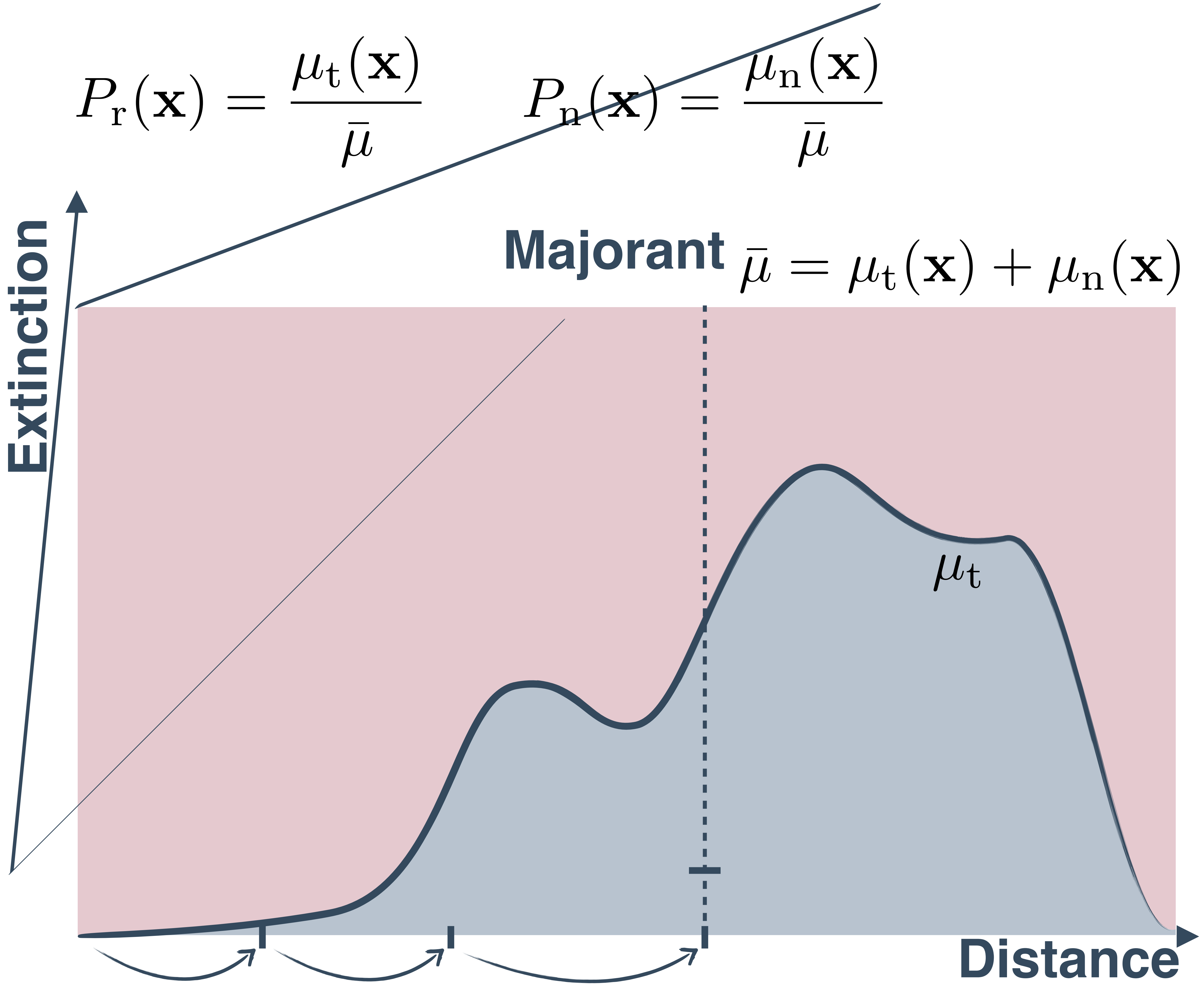
$$P_{\mathrm{r}}(\mathbf{x}) = \frac{\mu_{\mathrm{t}}(\mathbf{x})}{\bar{\mu}} \qquad P_{\mathrm{n}}(\mathbf{x}) = \frac{\mu_{\mathrm{n}}(\mathbf{x})}{\bar{\mu}}$$



Stochastic Sampling

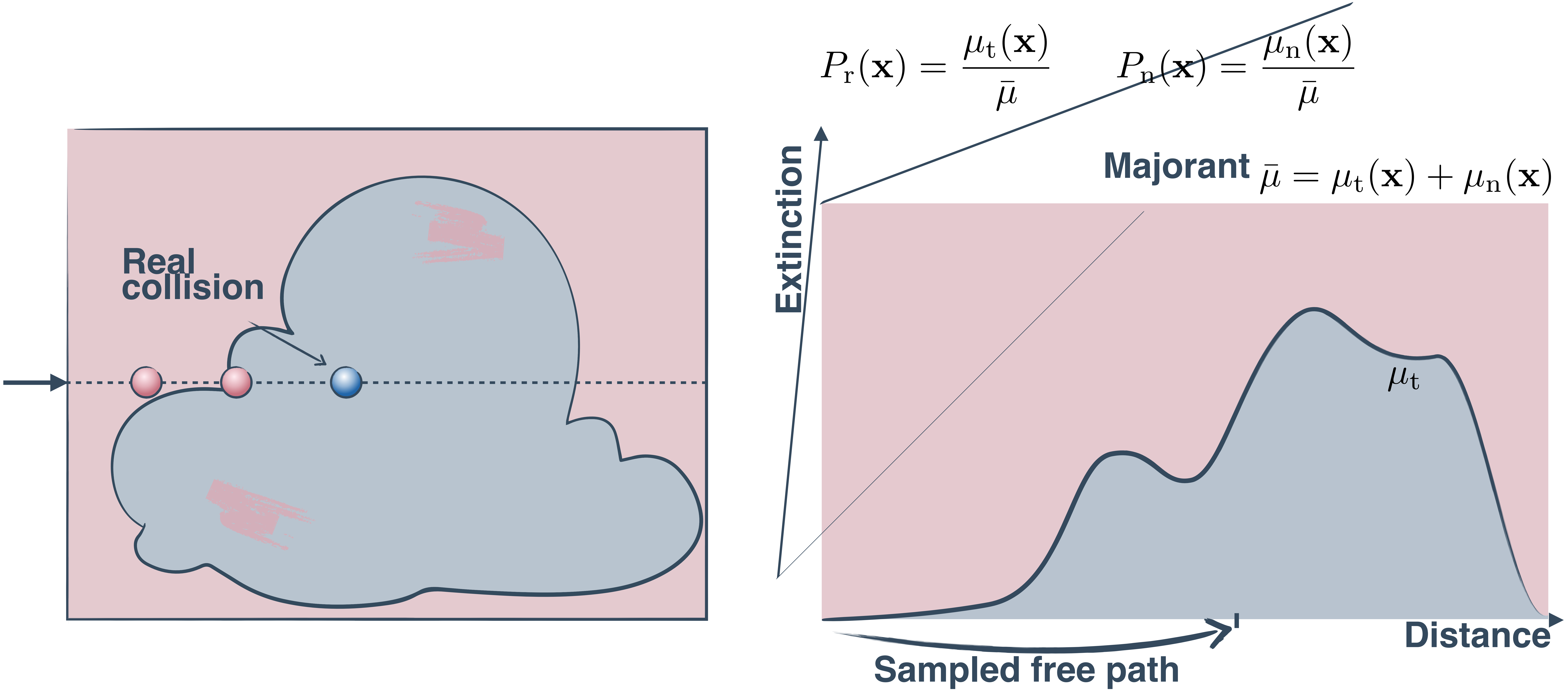
$$P_{\mathrm{r}}(\mathbf{x}) = \frac{\mu_{\mathrm{t}}(\mathbf{x})}{\bar{\mu}} \qquad P_{\mathrm{n}}(\mathbf{x}) = \frac{\mu_{\mathrm{n}}(\mathbf{x})}{\bar{\mu}}$$



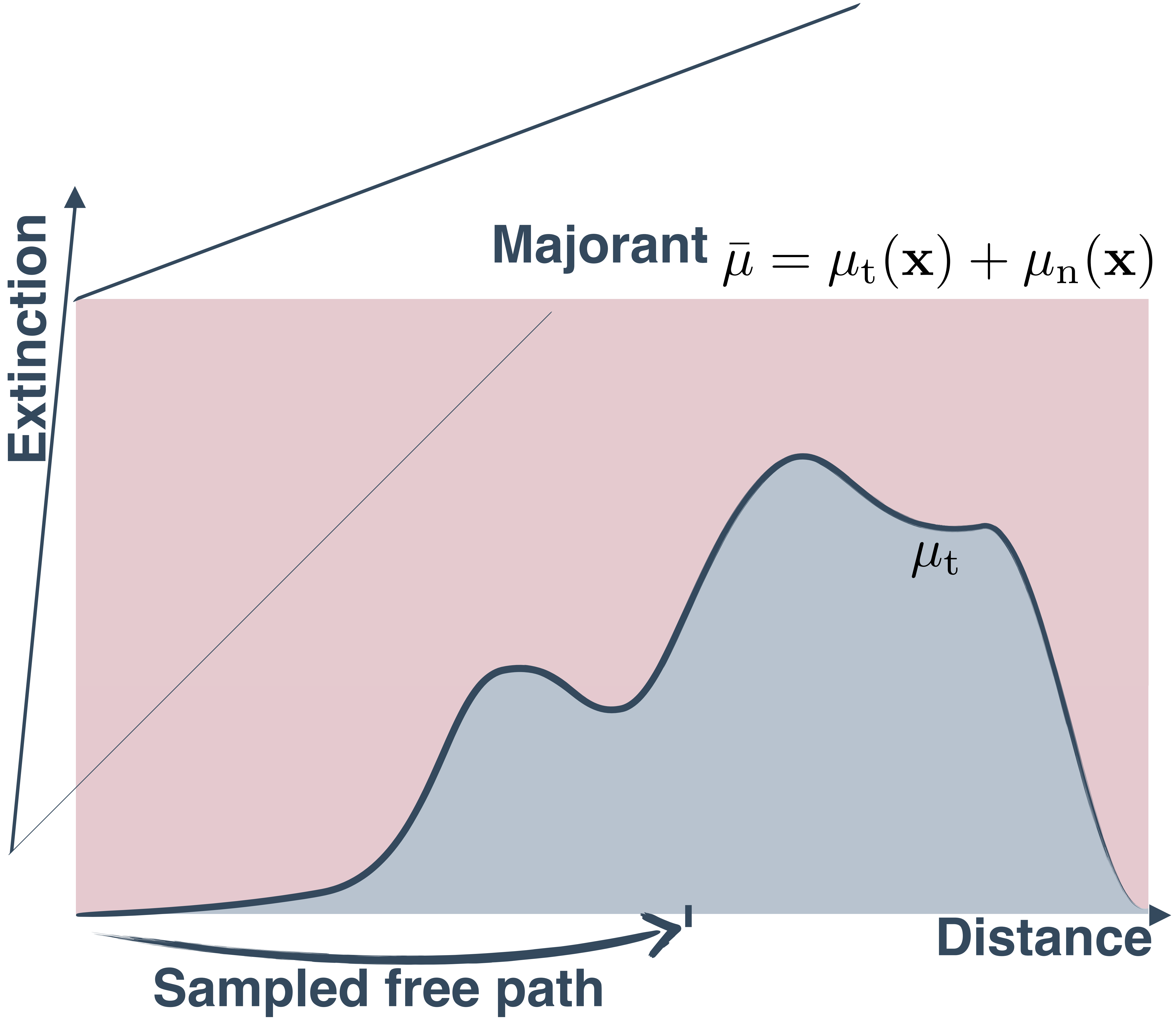


Stochastic Sampling

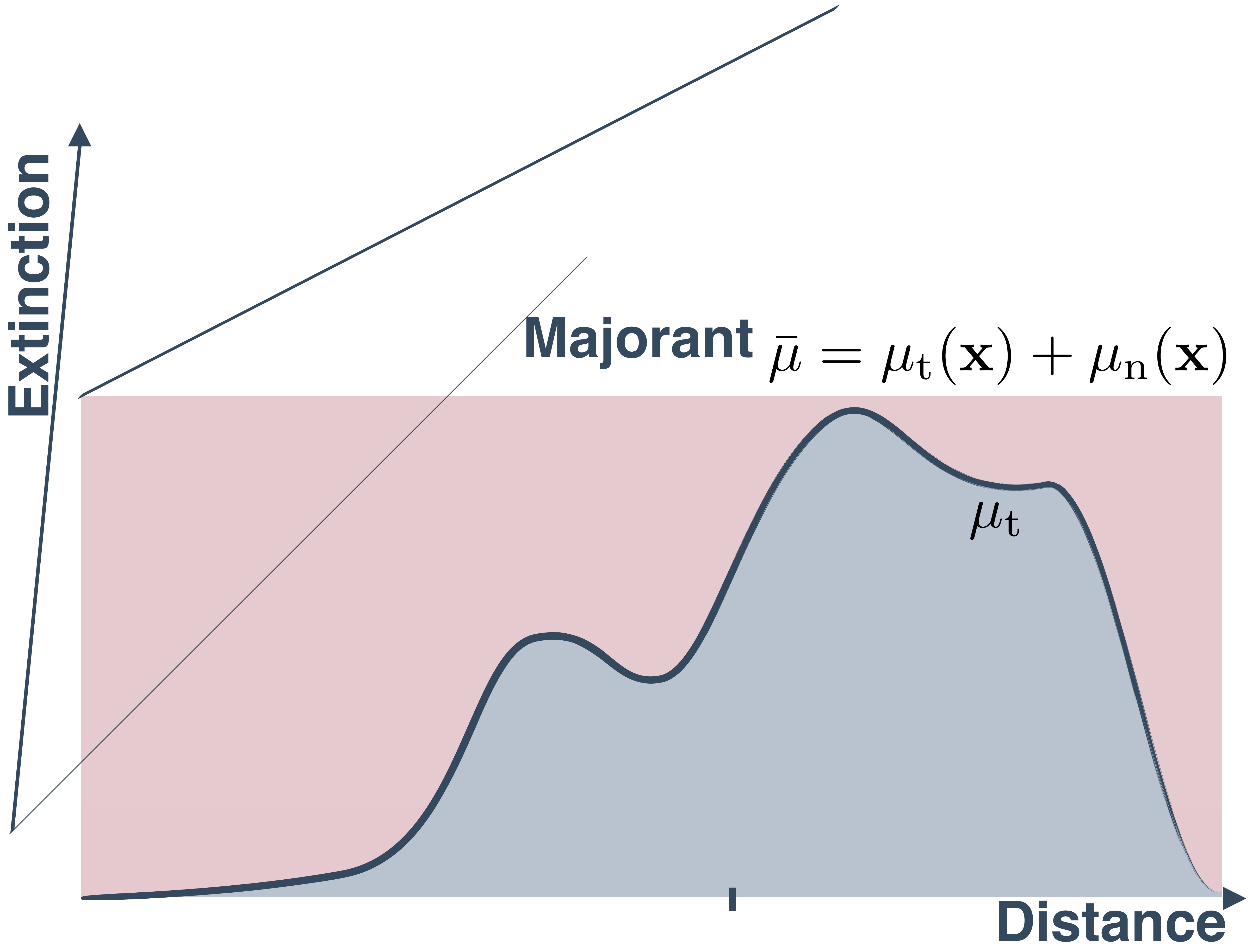
$$P_{\mathrm{r}}(\mathbf{x}) = \frac{\mu_{\mathrm{t}}(\mathbf{x})}{\bar{\mu}} \qquad P_{\mathrm{n}}(\mathbf{x}) = \frac{\mu_{\mathrm{n}}(\mathbf{x})}{\bar{\mu}}$$



Impact of Majorant

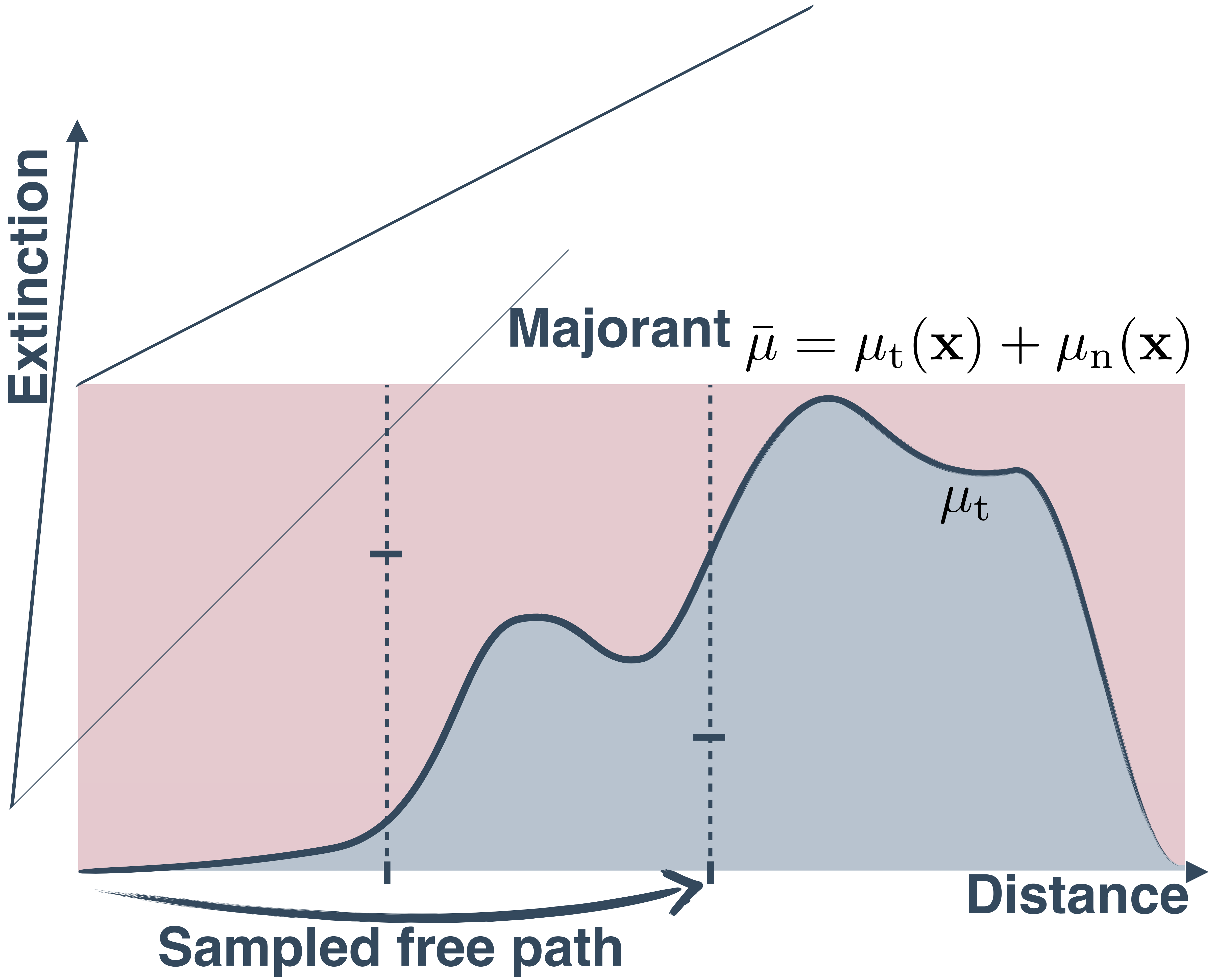


Impact of Majorant

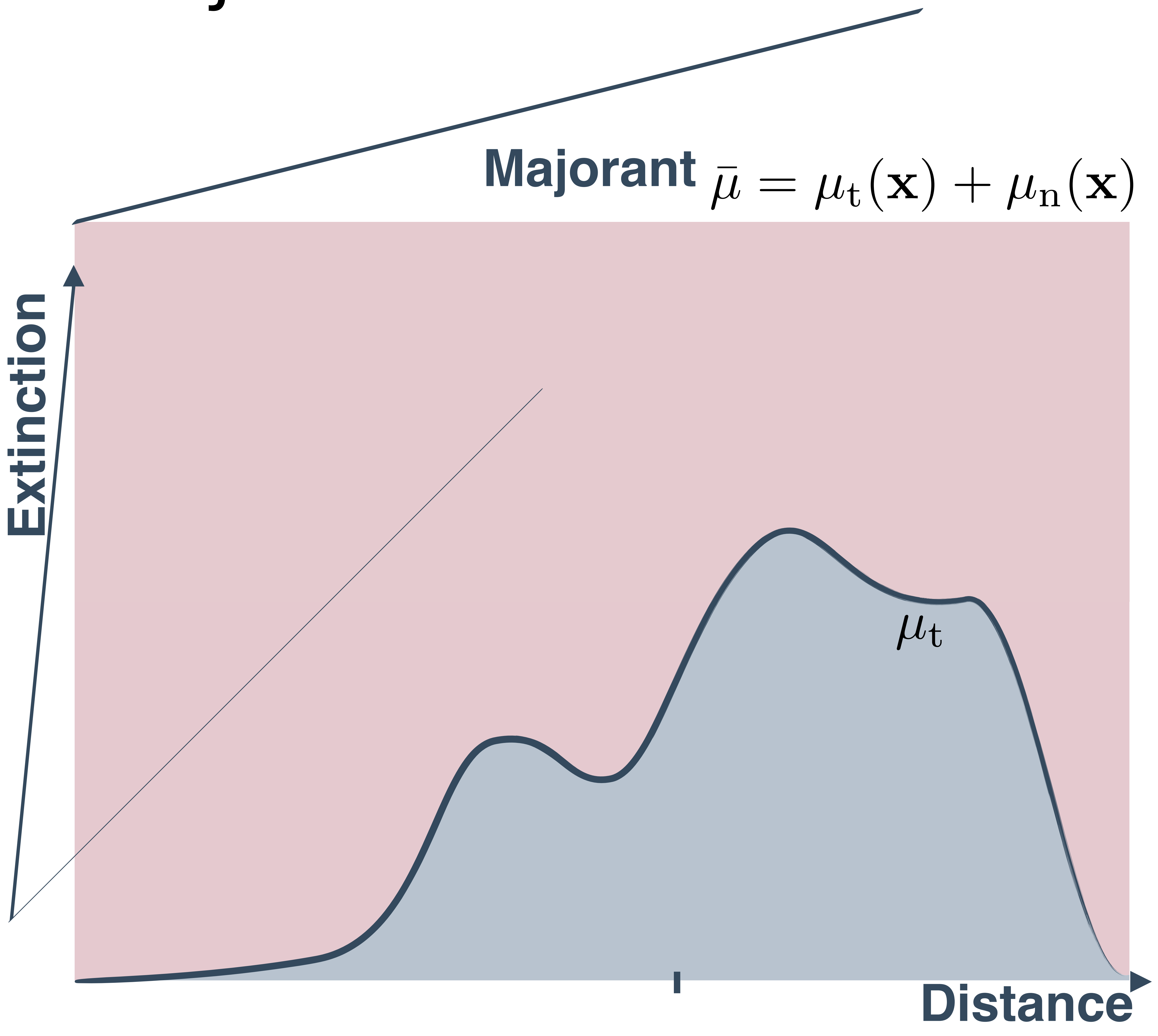


Impact of Majorant

Tight majorant = GOOD
(few rejected collisions)

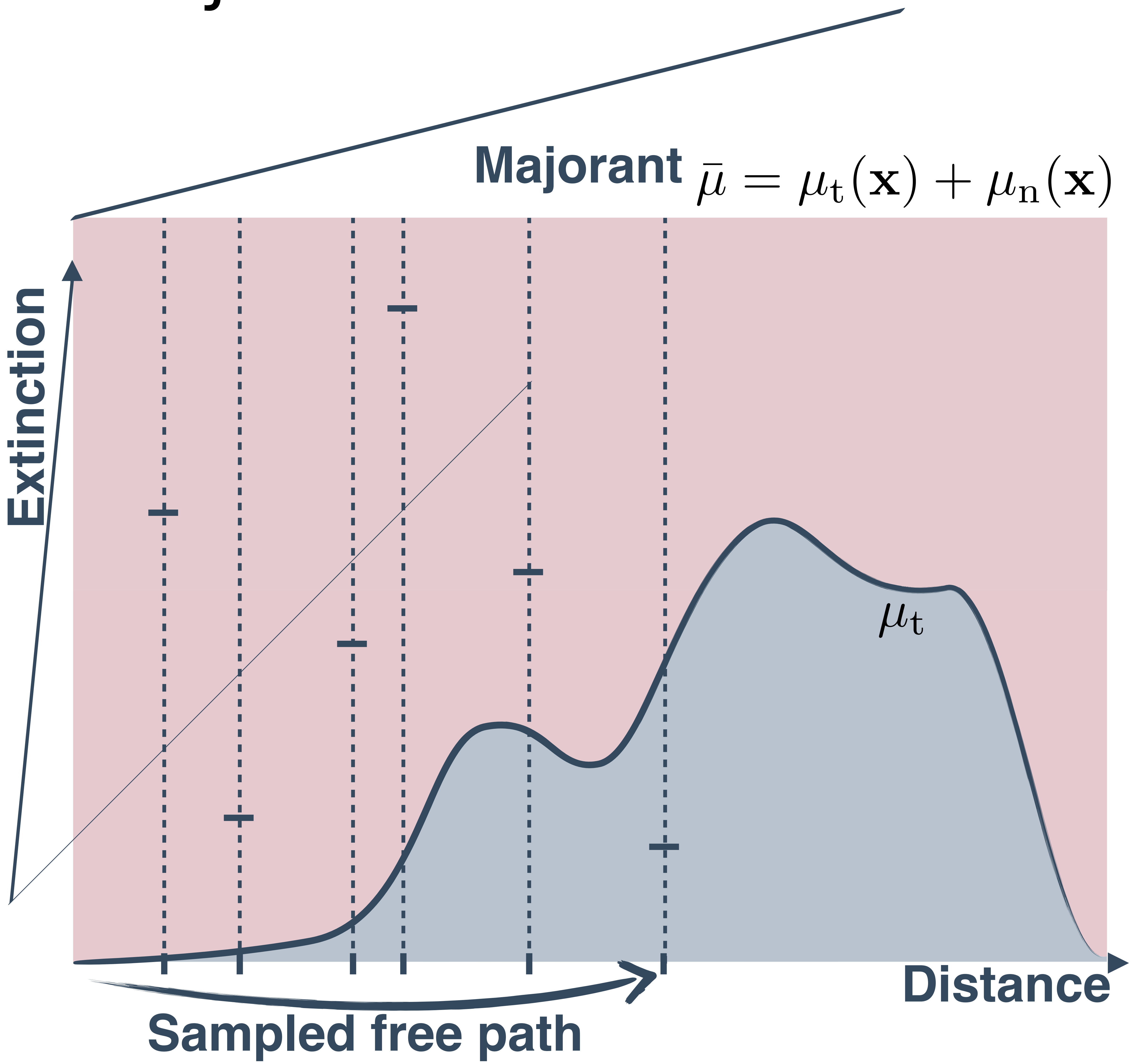


Impact of Majorant

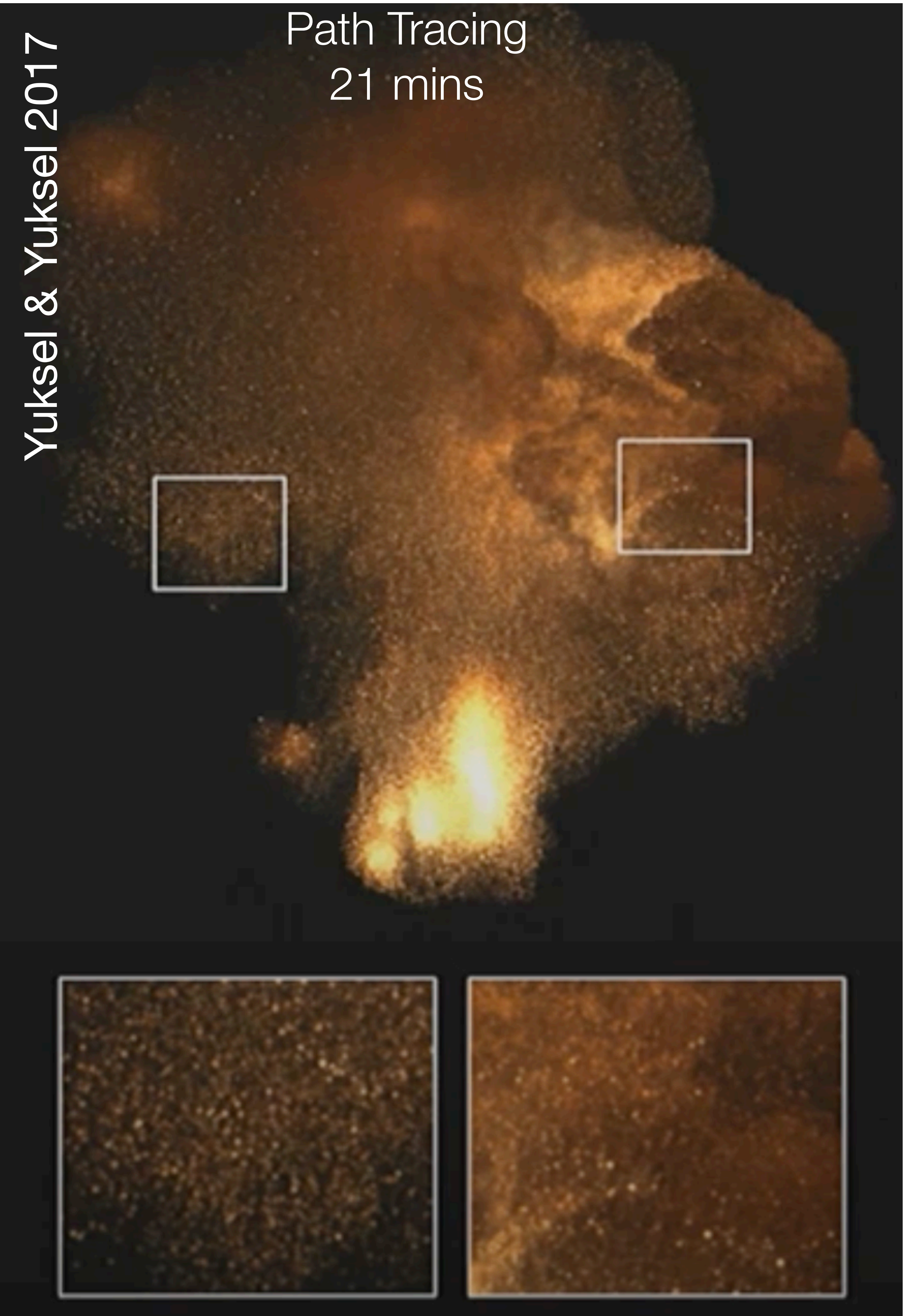


Impact of Majorant

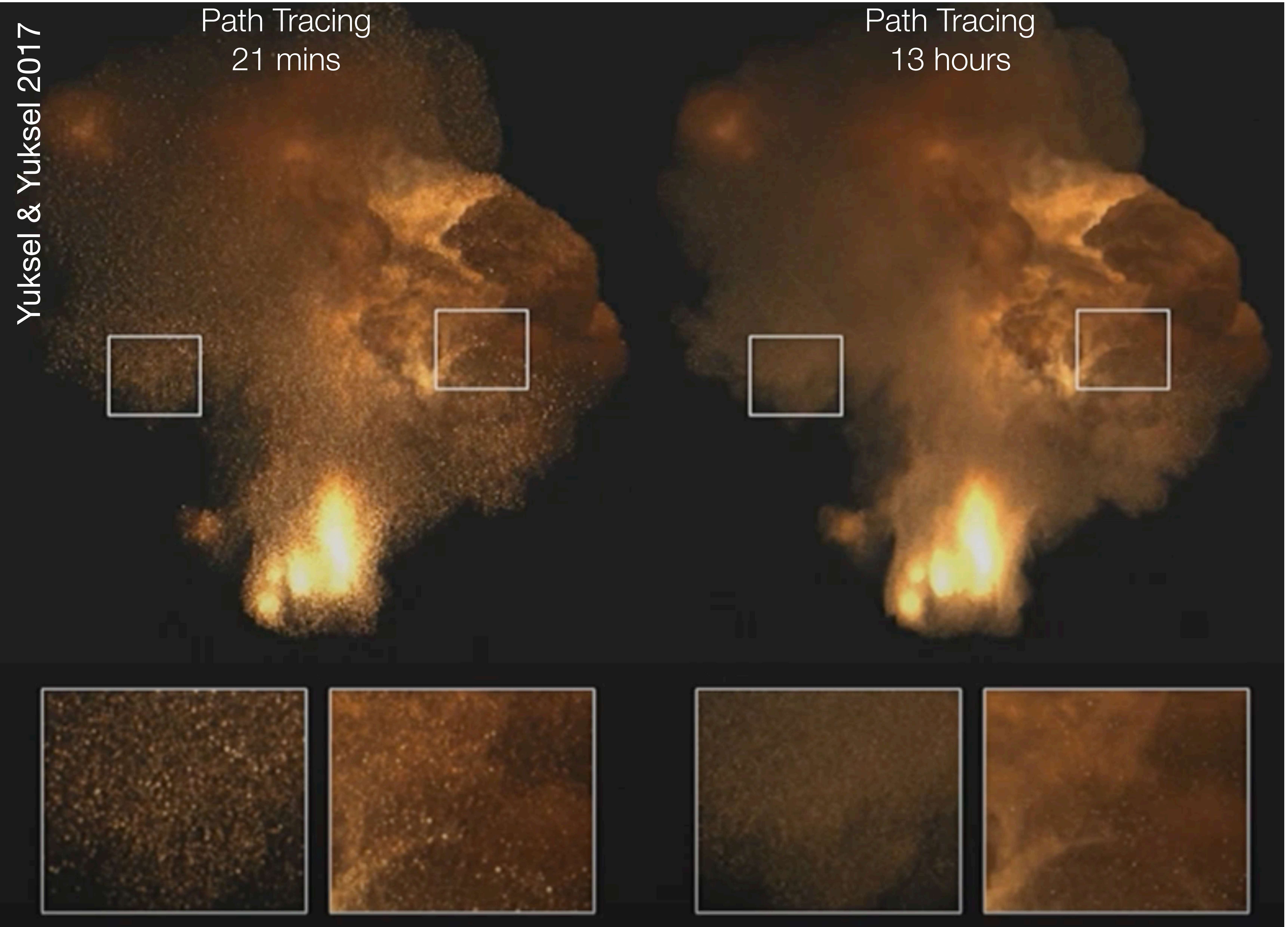
Loose majorant = BAD
(many expensive rejected



Next Lecture: Many-Light Methods



Next Lecture: Many-Light Methods



Yuksel & Yuksel 2017

Next Lecture: Many-Light Methods

Yuksel & Yuksel 2017

Acknowledgements

Slides material borrowed from multiple resources.

Special thanks to Wojciech Jarosz and Jan Novak et al. for making their lectures and SIGGRAPH 2018 slides available online