Volume Rendering

Gurprit Singh

Philipp Slusalek  Karol Myszkowski
Overview

Volumetric Processes:
- Absorption
- Scattering
- Transmittance
- Phase Functions

Volumetric Rendering Equation
Volumetric Path Tracing
Woodcock Tracking
Fog

Brassai (Gyula Halasz) 1899-1984
Snow

Gurprit Singh
Surface or Volume?

Corona Renderer / Chaos Czech a.s. / Chaos Group
Defining Participating Media

Media properties are modeled as a probabilistic process

No need to consider individual interactions with particles (won't fit in the memory)
Defining Participating Media

Homoegeneous media:

- Infinite or bounded by a simple surface or simple shape

Krivanek et al. [2014]
Defining Participating Media

Heterogeneous media (spatially varying coefficients):
Defining Participating Media

Heterogeneous media (spatially varying coefficients):

- Procedurally e.g. using a noise function
Defining Participating Media

Heterogeneous media (spatially varying coefficients):

- Procedurally e.g. using a noise function
- Simulation + volume discretization, e.g., voxel grid
Radiance is the main quantity we are interested in for rendering.
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In \textbf{vaccum}, light transport radiance remains constant along rays between surfaces.
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\[ L_i(x, \bar{\omega}) = L_o(y, -\bar{\omega}) \]
Radiance

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In \textbf{vaccum}, light transport radiance remains constant along rays between surfaces

\[ L_i(x, \omega) = L_o(y, -\omega) \]

\[ y = r(x, \omega) \]
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Radiance is the main quantity we are interested in for rendering.

In **vaccum**, light transport radiance remains constant along rays between surfaces

\[ L_i(x, \bar{\omega}) = L_o(y, -\bar{\omega}) \]

\[ y = r(x, \bar{\omega}) \]

ray tracing function
In **participating media**, radiance may change along rays between surfaces

\[
L_i(x, \bar{\omega}) \neq L_o(y, -\bar{\omega})
\]

\[
y = r(x, \bar{\omega})
\]

ray tracing function
Volumetric Scattering Processes

Absorption

Scattering

Emission

Slide after Jan Novak
Participating Media
Participating Media
Participating Media
Participating Media
Participating Media
Participating Media
Participating Media
Finite distance Beam

How much light is gained or lost during the travel through this differential beam due to the interactions with the medium?
Differential Beam
Absorption

\[ L(x, \omega) \]

\[ dA \]

\[ dz \]

outgoing radiance

\[ \omega \]
Absorption

\[ \frac{dL(x, \vec{\omega})}{dz} = -\sigma_a L(x, \vec{\omega}) \]
Absorption

\[ \frac{dL(x, \omega)}{dz} = -\sigma_a L(x, \omega) \]

\( \sigma_a \): absorption coefficient \( m^{-1} \)
Absorption

Absorption described by medium's absorption cross-section $\sigma_a$
Absorption

Absorption described by medium's absorption cross-section $\sigma_a$

$$\sigma_a \in [0, \infty)$$

$$L(x, \bar{\omega})$$

$dA$

$dz$

outgoing radiance

$\bar{\omega}$
Absorption described by medium's absorption cross-section $\sigma_a$

$$\sigma_a \in [0, \infty)$$

It is the probability density that light is absorbed per unit distance travelled in the medium.

It can vary as a position and direction.
Out-Scattering

\[ L(x, \omega) \]

\[ dz \]

\[ x \]
Out-Scattering

\( L(x, \bar{\omega}) \)

\[ L(x, \bar{\omega}) \]

\[ d\bar{z} \]

\[ x \]

\[ t_3 \]
Out-Scattering

\[ \frac{dL(x, \bar{\omega})}{dz} = -\sigma_s L(x, \bar{\omega}) \]
Out-Scattering

The probability of an out-scattering event occurring per unit distance is given by the scattering coefficient

\[
\frac{dL(x, \bar{\omega})}{dz} = -\sigma_s L(x, \bar{\omega})
\]

\(\sigma_s\) : scattering coefficient
Attenuation / Extinction

Total reduction in radiance:
Attenuation / Extinction

Total reduction in radiance:

Out-scattering
Attenuation / Extinction

Total reduction in radiance:

Out-scattering

Absorption
Attenuation / Extinction

Total reduction in radiance:

\[ \sigma_a : \text{absorption coefficient} \]

\[ \sigma_s : \text{scattering coefficient} \]
Attenuation / Extinction

Total reduction in radiance:

\[ \sigma_t(x, \vec{w}) = \sigma_a(x, \vec{w}) + \sigma_s(x, \vec{w}) \]

\( \sigma_a \) : absorption coefficient
\( \sigma_s \) : scattering coefficient
\( \sigma_t \) : extinction coefficient
Albedo

\[ \alpha(x) = \frac{\sigma_s(x)}{\sigma_a(x) + \sigma_s(x)} = \frac{\sigma_s(x)}{\sigma_t(x)} \]

\( \sigma_s \): scattering coefficient

\( \sigma_t \): extinction coefficient
Albedo

\[ \alpha(x) = \frac{\sigma_s(x)}{\sigma_a(x) + \sigma_s(x)} = \frac{\sigma_s(x)}{\sigma_t(x)} \]

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Albedo

\[ \alpha(x) = \frac{\sigma_s(x)}{\sigma_a(x) + \sigma_s(x)} = \frac{\sigma_s(x)}{\sigma_t(x)} \]

\( \sigma_s : \) scattering coefficient

\( \sigma_t : \) extinction coefficient
Albedo

\[ \alpha(x) = \frac{\sigma_s(x)}{\sigma_t(x)} \]

The albedo is always between 0 and 1

It describes the probability of scattering (versus absorption) at a scattering event

\( \sigma_s \): scattering coefficient

\( \sigma_t \): extinction coefficient
Mean-free path

\[ \frac{1}{\sigma_t} \]

Mean free path gives the average distance travelled by the ray before interacting with a particle

\( \sigma_t \) : extinction coefficient
In-Scattering

dz

\[ x \]

\[ 13 \]
In-Scattering
In-Scattering

\[ \frac{dL(x, \bar{\omega})}{dz} = \sigma_s(x) L_s(x, \bar{\omega}) \]
In-Scattering

\[ \frac{dL(x, \omega)}{dz} = \sigma_s(x) L_s(x, \omega) \]

\( \sigma_s(x) \) : scattering coefficient
In-Scattering

\[ \frac{dL(x, \tilde{\omega})}{dz} = \sigma_s(x)L_s(x, \tilde{\omega}) \]

\( \sigma_s(x) \) : scattering coefficient

In-scattered radiance

\[ L_s(x, \tilde{\omega}) = \int_{S^2} f_p(\tilde{\omega}, \tilde{\omega}')L(x, \tilde{\omega}')d\tilde{\omega}' \]
Emission

\[ d\epsilon \]

Emitted radiance does not depend on the incoming light \( L_i \).
Emission

Emitted radiance does not depend on the incoming light $L_i$. 

\[ \text{d}z \]
Emission

\[
\frac{dL(x, \omega)}{dz} = \sigma_a(x) L_e(x, \omega)
\]

Emitted radiance does not depend on the incoming light \(L_i\).
Emission

\[ \frac{dL(x, \bar{\omega})}{dz} = \sigma_a(x) L_e(x, \bar{\omega}) \]

\[ L_e(x, \bar{\omega}): \text{emitted radiance} \]

Emitted radiance does not depend on the incoming light \( L_i \)
Emission

\[
\frac{dL(x, \bar{\omega})}{dz} = \sigma_a(x)L_e(x, \bar{\omega})
\]

\[L_e(x, \bar{\omega}) : \text{emitted radiance}\]

*sometimes modeled without the absorption coefficient term

Emitted radiance does not depend on the incoming light \(L_i\)
Emission

\[ \frac{dL(x, \bar{\omega})}{d\bar{z}} = \sigma_a(x)L_e(x, \bar{\omega}) \]

\( L_e(x, \bar{\omega}) \): emitted radiance

*sometimes modeled without the absorption coefficient term

Emitted radiance does not depend on the incoming light \( L_i \)

Here we made a choice to represent differential output radiance as a product of emitted radiance and absorption coefficient.
Radiative Transfer Equation
Radiative Transfer Equation (RTE)
Radiative Transfer Equation (RTE)

- Out-scattering
- Absorption
- In-scattering
- Emission
Radiative Transfer Equation (RTE)

\[ \frac{dL(x, \omega)}{dz} \]

- Out-scattering
- Absorption (Losses)
- In-scattering
- Emission (Gains)
Radiative Transfer Equation (RTE)

\[
\frac{dL(x, \tilde{\omega})}{dz} = -\sigma_s(x) L(x, \tilde{\omega}) - \sigma_a(x) L(x, \tilde{\omega})
\]

- **Losses**
  - Out-scattering
  - Absorption

- **Gains**
  - In-scattering
  - Emission

Where:

- \( \sigma_s(x) \) is the scattering cross-section
- \( \sigma_a(x) \) is the absorption cross-section
- \( L(x, \tilde{\omega}) \) is the radiative intensity
\[
\frac{dL(x, \omega)}{dz} = -\sigma_s(x)L(x, \omega) - \sigma_a(x)L(x, \omega) + \sigma_s(x)L_s(x, \omega) + \sigma_a(x)L_e(x, \omega)
\]

Out-scattering  
Absorption  
In-scattering  
Emission
Radiative Transfer Equation (RTE)

\[
\frac{dL(x, \omega)}{dz} = -\sigma_s(x)L(x, \omega) - \sigma_a(x)L(x, \omega) + \sigma_s(x)L_s(x, \omega) + \sigma_a(x)L_e(x, \omega)
\]

Out-scattering  Absorption

In-scattering  Emission

\[
\sigma_t(x, \omega) = \sigma_a(x, \omega) + \sigma_s(x, \omega)
\]
Radiative Transfer Equation (RTE)

\[
\frac{dL(x, \omega)}{dz} = -\sigma_t(x)L(x, \omega) + \sigma_s(x)L_s(x, \omega) + \sigma_a(x)L_e(x, \omega)
\]

- **Attenuation**
- **In-scattering**
- **Emission**
Radiative Transfer Equation (RTE)

\[
\frac{dL(x, \omega)}{dz} = -\sigma_t(x)L(x, \omega) + \sigma_s(x)L_s(x, \omega) + \sigma_a(x)L_e(x, \omega)
\]

Attenuation
In-scattering
Emission

What about a beam with finite-length \( z \)?
Extinction Along a Finite Beam

\[
\frac{dL(x, \omega)}{dz} = -\sigma_t(x) L(x, \omega)
\]
Extinction Along a Finite Beam

\[
\frac{dL(x, \omega)}{dz} = -\sigma_t(x)L(x, \omega)
\]

\[
\frac{dL(x, \omega)}{L(x, \omega)} = -\sigma_t(x)dz \quad \text{// Integrate along beam from 0 to } z
\]
Extinction Along a Finite Beam

\[
\frac{dL(x, \omega)}{dz} = -\sigma_t(x)L(x, \omega)
\]

\[
\frac{dL(x, \omega)}{L(x, \omega)} = -\sigma_t(x)dz \quad // \text{Integrate along beam from 0 to } z
\]

\[
\log_e L_z - \log_e L_0 = -\sigma_t(x)z
\]
Extinction Along a Finite Beam

\[
\frac{dL(x, \bar{\omega})}{dz} = -\sigma_t(x)L(x, \bar{\omega})
\]

\[
\frac{dL(x, \bar{\omega})}{L(x, \bar{\omega})} = -\sigma_t(x)dz \quad \text{// Integrate along beam from } 0 \text{ to } Z
\]

\[
\log_e L_z - \log_e L_0 = -\sigma_t(x)z
\]

\[
\log_e \left( \frac{L_z}{L_0} \right) = -\sigma_t z \quad \text{// Exponentiate}
\]
Extinction Along a Finite Beam

\[
\frac{dL(x, \bar{\omega})}{dz} = -\sigma_t(x)L(x, \bar{\omega})
\]

\[
\frac{dL(x, \bar{\omega})}{L(x, \bar{\omega})} = -\sigma_t(x)dz \quad \text{// Integrate along beam from 0 to } z
\]

\[
\log_e L_z - \log_e L_0 = -\sigma_t(x)z
\]

\[
\log_e \left( \frac{L_z}{L_0} \right) = -\sigma_t z \quad \text{// Exponentiate}
\]

\[
\frac{L_z}{L_0} = e^{-\sigma_t z}
\]
Beer-Lambert Law

The fraction refers to as the \textit{transmittance}

\[
\frac{L_z}{L_0} = e^{-\sigma_t z}
\]

Think of this as fractional visibility loss between two points
Beer-Lambert Law

The fraction refers to as the *transmittance*

\[
\frac{L_z}{L_0} = e^{-\sigma_t z}
\]

Think of this as fractional visibility loss between two points.

Radiance at distance 0
The fraction refers to as the *transmittance*

\[
\frac{L_z}{L_0} = e^{-\sigma_t z}
\]

Think of this as fractional visibility loss between two points.
Beer-Lambert Law

Expresses the remaining radiance after traveling a finite distance through the medium with constant extinction coefficient.

The fraction refers to as the *transmittance*.

\[
\frac{L_z}{L_0} = e^{-\sigma_t z}
\]

Think of this as fractional visibility loss between two points.
Beam Transmittance

$L_0(x, \vec{\omega})$

$\sigma_t$: extinction coefficient
Beam Transmittance

$L_0(x, \vec{\omega})$

$\sigma_t$ : extinction coefficient
Beam Transmittance

\( L_o(x, \omega) \)

\( \sigma_t \) : extinction coefficient
Beam Transmittance

\[ L_0(x, \omega) \]

\[ \sigma_t : \text{extinction coefficient} \]
Beam Transmittance

\[ L_o(\mathbf{x}, \omega) \]

\[ T_r(\mathbf{x} \rightarrow \mathbf{x'}) L_o(\mathbf{x}, \omega) \]

\( \sigma_t \): extinction coefficient
Beam Transmittance

\[ T_r(x \to y) = e^{- \int_0^{||x-y||} \sigma_t(t) \, dt} \]

\( \sigma_t \) : extinction coefficient
Beam Transmittance

\[ T_r(x \rightarrow y) = e^{-\int_0^{|x-y|} \sigma_t(t) \, dt} \]

\[ L_o(x, \omega) \]

\[ T_r(x \rightarrow x') L_o(x, \omega) \]

\( \sigma_t \): extinction coefficient
Beam Transmittance

\[ T_r(x \to y) = e^{-\int_0^{||x-y||} \sigma_t(t) \, dt} \]

\( L_o(x, \omega) \)

\( x \)

\( y \)

\( \sigma_t : \text{extinction coefficient} \)
Beam Transmittance: \textbf{Multiplicative}

\[ L_0(x, \omega) \]

\( x \)

\( x'' \)

\( \sigma_t : \text{extinction coefficient} \)
Beam Transmittance: Multiplicative

\[ L_0(x, \omega) \]

\[ \sigma_t : \text{extinction coefficient} \]
Beam Transmittance: **Multiplicative**

\[ T_r(x \rightarrow x') \]

\[ \sigma_t : \text{extinction coefficient} \]
Beam Transmittance: \textbf{Multiplicative}

\[ T_r(x \rightarrow x') \times T_r(x' \rightarrow x'') \]

\[ \sigma_t : \text{extinction coefficient} \]
Beam Transmittance: **Multiplicative**

\[
T_r(x \rightarrow x''') = T_r(x \rightarrow x') T_r(x' \rightarrow x''')
\]

\[
L_o(x, \vec{\omega})
\]

\[
T_r(x \rightarrow x')
\]

\[
T_r(x' \rightarrow x''')
\]

\[
\sigma_t : \text{extinction coefficient}
\]
Beam Transmittance

In Homogeneous medium $\sigma_t$ is a constant:
Beam Transmittance

In Homogeneous medium $\sigma_t$ is a constant:

$$T_r(x \rightarrow y) = e^{-\sigma_t ||x-y||}$$
Beam Transmittance

In Homogeneous medium $\sigma_t$ is a constant:

$$T_r(x \rightarrow y) = e^{-\sigma_t ||x-y||}$$

In Heterogeneous medium (spatially varying $\sigma_t$):
Beam Transmittance

In Homogeneous medium $\sigma_t$ is a constant:

$$T_r(x \rightarrow y) = e^{-\sigma_t||x-y||}$$

In Heterogeneous medium (spatially varying $\sigma_t$):

$$T_r(x \rightarrow y) = e^{-\int_0^{||x-y||} \sigma_t(t) dt}$$
Beam Transmittance

In Homogeneous medium $\sigma_t$ is a constant:

\[ T_r(x \rightarrow y) = e^{-\sigma_t||x-y||} \]

In Heterogeneous medium (spatially varying $\sigma_t$):

\[ T_r(x \rightarrow y) = e^{-\int_0^{||x-y||} \sigma_t(t) dt} \]

Optical thickness
Radiative Transfer Equation (RTE)

\[
\frac{dL(x, \bar{\omega})}{dz} = -\sigma_t(x)L(x, \bar{\omega}) + \sigma_s(x)L_s(x, \bar{\omega}) + \sigma_a(x)L_e(x, \bar{\omega})
\]
Radiative Transfer Equation (RTE)

\[
\frac{dL(x, \omega)}{dz} = -\sigma_t(x)L(x, \omega) + \sigma_s(x)L_s(x, \omega) + \sigma_a(x)L_e(x, \omega)
\]

What about a beam with finite-length \(z\)?
Volumetric Rendering Equation

\[ L(x, \omega) = T_r(x, x_z) L(x_z, \omega) \]
Volumetric Rendering Equation

\[ L(x, \bar{\omega}) = T_r(x, x_z) L(x_z, \bar{\omega}) \]

Reduced (background) surface radiance
Volumetric Rendering Equation

\[ L(x, \omega) = T_r(x, x_z) L(x_z, \omega) \]

\[ + \int_0^z T_r(x, x_t) \sigma_a(x_t) L_e(x_t, \omega) dt \]

Accumulated emitted radiance
Volumetric Rendering Equation

\[
L(x, \omega) = T_r(x, x_z)L(x_z, \omega)
\]

\[
+ \int_0^z T_r(x, x_t)\sigma_a(x_t)L_e(x_t, \omega)\,dt
\]

\[
+ \int_0^z T_r(x, x_t)\sigma_s(x_t)L_s(x_t, \omega)\,dt
\]

Accumulated in-scattered radiance
Volumetric Rendering Equation

\[
L(x, \omega) = T_r(x, x_z)L(x_z, \omega) \\
+ \int_0^z T_r(x, x_t)\sigma_a(x_t)L_e(x_t, \omega)dt \\
+ \int_0^z T_r(x, x_t)\sigma_s(x_t)\int_{S^2} f_p(x_t, \omega', \omega)L_i(x_t, \omega')d\omega' dt
\]

\[
T_r(x, x_t)
\]
Volumetric Rendering Equation

\[ L(x, \bar{\omega}) = T_r(x, x_z) L(x_z, \bar{\omega}) \]

\[ + \int_0^z T_r(x, x_t) \sigma_a(x_t) L_e(x_t, \bar{\omega}) dt \]

\[ + \int_0^z T_r(x, x_t) \sigma_s(x_t) \int_{S^2} f_p(x_t, \bar{\omega}', \bar{\omega}) L_i(x_t, \bar{\omega}') d\omega' dt \]
Scattering in Media
Phase Functions

It describes the angular distribution of scattered radiation at a point;
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It is the volumetric analog to the BSDF, but it is different from the BSDF.
Phase Functions

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It is the volumetric analog to the BSDF, but it is different from the BSDF.

It has a normalization constant:

$$\int_{S^2} f_p(\vec{\omega}, \vec{\omega}') \, d\vec{\omega}' = 1 \quad \forall \vec{\omega}$$
Phase Functions

It describes the angular distribution of scattered radiation at a point;

It is the volumetric analog to the BSDF, but it is different from the BSDF.

It has a normalization constant:

$$\int_{S^2} f_p(\bar{\omega}, \bar{\omega}') d\bar{\omega}' = 1 \quad \forall \bar{\omega}$$

This constraint means that phase functions actually define probability distributions for scattering in a particular direction.
Phase Functions

Isotropic:

\[ f_p(\vec{\omega}_o, \vec{\omega}_i) = \frac{1}{4\pi} \]

Uniform scattering, analogous to Lambertian BRDF
Phase Functions

\[ \begin{align*}
\omega' & \quad \theta \\
-\overrightarrow{\omega} & \quad \overrightarrow{\omega} 
\end{align*} \]
Quantifying anisotropy by

\[ g = \int_{S^2} f_p(x, \omega, \omega') \cos \theta d\omega' \]
Phase Functions

Quantifying anisotropy by

\[ g = \int_{S^2} f_p(x, \omega, \omega') \cos \theta d\omega' \]

where

\[ \cos \theta = -\omega \cdot \omega' \]

\( g \) is the asymmetry parameter
Phase Functions

Quantifying anisotropy by

\[ g = \int_{S^2} f_p(x, \bar{\omega}, \bar{\omega}') \cos \theta d\bar{\omega}' \]

where

\[ \cos \theta = -\bar{\omega} \cdot \bar{\omega}' \]

\[ g = 0 : \text{isotropic scattering (on average)} \]
\[ g > 0 : \text{forward scattering} \]
\[ g < 0 : \text{backward scattering} \]

\( g \) is the asymmetry parameter
Henyey-Greenstein Phase Function

\[ f_p(\theta) = \frac{1}{4\pi} \frac{1 - g^2}{(1 + g^2 + 2g(\cos \theta))^{3/2}} \]

\[ g \in [-1, 1] \]
Henyey-Greenstein Phase Function

\[ f_p(\theta) = \frac{1}{4\pi} \frac{1 - g^2}{(1 + g^2 + 2g(\cos \theta))^{3/2}} \]

\[ g \in [-1, 1] \]

\[ g = 0 \]
Henyey-Greenstein Phase Function

\[ f_p(\theta) = \frac{1}{4\pi} \frac{1 - g^2}{(1 + g^2 + 2g\cos\theta)^{3/2}} \quad g \in [-1, 1] \]

\[ g = 0 \]

\[ g > 0 \]
Henyey-Greenstein Phase Function

\[
 f_p(\theta) = \frac{1}{4\pi} \frac{1 - g^2}{(1 + g^2 + 2g(\cos \theta))^{3/2}} \quad g \in [-1, 1]
\]
Henyey-Greenstein Phase Function

\[ f_p(\theta) = \frac{1}{4\pi} \frac{1 - g^2}{(1 + g^2 + 2g \cos \theta)^{3/2}} \]

- \( g = -0.5 \)
- \( g = 0 \)
- \( g = 0.8 \)
Henyey-Greenstein Phase Function

\[ g = -0.7 \]

Strong backward scattering

\[ g = 0.7 \]

Strong forward scattering

PBRTv3 [2016]
Schlick's Phase Function

Empirical Phase Function

Faster approximation to HG

\[ f_p(\theta) = \frac{1}{4\pi} \frac{1 - k^2}{(1 - k \cos \theta)^2} \]

\[ k = 1.55g - 0.55g^3 \]
Schlick's Phase Function

Empirical Phase Function

Faster approximation to HG

\[ g = -0.5 \quad k = -0.706 \]

\[ g = 0 \quad k = 0 \]

\[ g = 0.8 \quad k = 0.96 \]
Rainbows
Lorenz-Mie Scattering

For large-size particles (scatterers), we cannot ignore the wave nature of light.

Solution to Maxwell's equations for scattering from many spherical dielectric particles explains many phenomena.

Complicated: solution is an infinite analytic series.
Lorenz-Mie Scattering

Sphere diameter = 1µm

Sphere diameter = 10µm

Sphere diameter = 100µm

Linear plot
Lorenz-Mie Scattering

Sphere diameter = 1 µm  
Sphere diameter = 10 µm  
Sphere diameter = 100 µm

Linear plot

Log plot
Lorenz-Mie Approximations

\[ f_p^{\text{hazy}}(\theta) = \frac{1}{4\pi} \left( 5 + \left( \frac{1 + \cos \theta}{2} \right)^8 \right) \]

Hazy atmosphere
Lorenz-Mie Approximations

For a hazy atmosphere:

\[
 f_p^{\text{hazy}}(\theta) = \frac{1}{4\pi} \left( 5 + \left( \frac{1 + \cos \theta}{2} \right)^8 \right)
\]

For a murky atmosphere:

\[
 f_p^{\text{murky}}(\theta) = \frac{1}{4\pi} \left( 17 + \left( \frac{1 + \cos \theta}{2} \right)^{32} \right)
\]
Why is the Sky Blue?
Why is the Sunset Red?
Why is the Sunset Red?
Rayleigh Scattering
Rayleigh Scattering

Approximation of Lorenz-Mie for tiny particles (scatterers) that are typically smaller than 1/10th the wavelength of visible light

Used for atmospheric scattering, gasses, transparent solids

Highly wavelength dependent
Rayleigh Phase Function

$$f_{p}^{\text{Rayleigh}}(\theta) = \frac{3}{16\pi} \left( 1 + \cos^2 \theta \right)$$

Scattering at right angles is half as likely as scattering forward or backward.
Rayleigh Scattering

\[ \beta_s^{\text{Rayleigh}}(\lambda, d, \eta, \rho) = \rho \frac{2\pi^5 d^6}{3\lambda^4} \left( \frac{\eta^2 - 1}{\eta^2 + 2} \right)^2 \]
Rayleigh Scattering

\[ \beta_{\text{Rayleigh}}(\lambda, d, \eta, \rho) = \rho \frac{2\pi^5 d^6}{3\lambda^4} \left( \frac{\eta^2 - 1}{\eta^2 + 2} \right)^2 \]
Rayleigh Scattering

\[ \beta_{\text{Rayleigh}}(\lambda, d, \eta, \rho) = \rho \frac{2\pi^5 d^6}{3\lambda^4} \left(\frac{\eta^2 - 1}{\eta^2 + 2}\right)^2 \]

Wavelength

Diameter of particles
Rayleigh Scattering

\[ \beta_{s, \text{Rayleigh}}(\lambda, d, \eta, \rho) = \rho \frac{2\pi^5 d^6}{3\lambda^4} \left( \frac{\eta^2 - 1}{\eta^2 + 2} \right)^2 \]

- Wavelength
- Index of refraction
- Diameter of particles
Rayleigh Scattering

\[ \beta_{Rayleigh}^{\lambda, d, \eta, \rho} = \rho \frac{2\pi^5 d^6}{3 \lambda^4} \left( \frac{\eta^2 - 1}{\eta^2 + 2} \right)^2 \]

- Wavelength
- Index of refraction
- Density of particles
- Diameter of particles
Recap: Phase Functions
Recap: Phase Functions

Isotropic

\[ L(y, \omega) = \frac{1}{2} f_p(\omega) \]

\[ \omega \]
Recap: Phase Functions

Isotropic

\[ L_s(y, \omega) = \frac{1}{2} \int f_p(\hat{\omega}, \bar{\omega}) L(y, \bar{\omega}) d\bar{\omega} \]

Henyey-Greenstein

\[ L_s(y, \omega) = \frac{1}{2} \int f_p(\hat{\omega}, \bar{\omega}) L(y, \bar{\omega}) d\bar{\omega} \]
Recap: Phase Functions

- **Isotropic**
  \[ L_s(y, \omega) = \sum S_2 f_p(\omega) L(y, \bar{\omega}) d\bar{\omega} \]

- **Henyey-Greenstein**
  \[ L_s(y, \omega) = \sum S_2 f_p(\omega) L(y, \bar{\omega}) d\bar{\omega} \]

- **Rayleigh**
  \[ L_s(y, \omega) = \sum S_2 f_p(\omega) L(y, \bar{\omega}) d\bar{\omega} \]
Recap: Phase Functions

Isotropic

Henyey-Greenstein

Rayleigh

Lorenz-Mie small particles

Lorenz-Mie large particles
Anisotropy: Phase Function vs. Medium

Isotropic Medium
## Anisotropy: Phase Function vs. Medium

<table>
<thead>
<tr>
<th>Isotropic Medium</th>
<th></th>
</tr>
</thead>
<tbody>
<tr>
<td>Isotropic phase function</td>
<td>Anisotropic phase function</td>
</tr>
</tbody>
</table>
Anisotropy: Phase Function vs. Medium

Isotropic Medium

| Isotropic phase function | Anisotropic phase function |

- Isotropic phase function
- Anisotropic phase function
Anisotropy: Phase Function vs. Medium

Isotropic Medium

Isotropic phase function

Anisotropic phase function

Slide after Jan Novak
Anisotropy: Phase Function vs. Medium

Isotropic Medium
- Isotropic phase function

Anisotropic Medium
- Anisotropic phase function

Slide after Jan Novak
Recap: Media Properties

Given:

- Absorption coefficient: $\sigma_a(x)$ $[m^{-1}]$
- Scattering coefficient: $\sigma_s(x)$ $[m^{-1}]$
- Phase function: $f_p(x, \bar{\omega}, \bar{\omega}')$ $[sr^{-1}]$
Recap: Media Properties

Given:

- Absorption coefficient \( \sigma_a(x) \) \([m^{-1}]\)
- Scattering coefficient \( \sigma_s(x) \) \([m^{-1}]\)
- Phase function \( f_p(x, \vec{\omega}, \vec{\omega}') \) \([sr^{-1}]\)

Derived:

- Extinction coefficient \( \sigma_t(x) = \sigma_a(x) + \sigma_s(x) \) \([m^{-1}]\)
- Albedo \( \alpha(x) = \sigma_s(x)/\sigma_t(x) \) \([None]\)
- Mean-free path \( 1/\sigma_t(x) \) \([m]\)
- Transmittance \( T_r(x, y) = e^{-\int_{0}^{||x-y||} \sigma_t(t)dt} \) \([None]\)
For Homogeneous Isotropic Medium

Given:

- Absorption coefficient \( \sigma_a \) [\( m^{-1} \)]
- Scattering coefficient \( \sigma_s \) [\( m^{-1} \)]
- Phase function \( \frac{1}{4\pi} \) [\( sr^{-1} \)]

Derived:

- Extinction coefficient \( \sigma_t = \sigma_a + \sigma_s \) [\( m^{-1} \)]
- Albedo \( \alpha = \frac{\sigma_s}{\sigma_t} \) [None]
- Mean-free path \( \frac{1}{\sigma_t} \) [\( m \)]
- Transmittance \( T_r(x, y) = e^{\sigma_t ||x-y||} \) [None]
Solving the Volumetric Rendering Equation
Complexity

Homogeneous vs. Heterogeneous

Scattering
- none
- fake
- single scattering
- multiple scattering
Volumetric Rendering Equation

\[ L(x, \bar{\omega}) = T_r(x, x_z) L(x_z, \bar{\omega}) \]

\[ + \int_0^z T_r(x, x_t) \sigma_a(x_t) L_e(x_t, \bar{\omega}) dt \]

\[ + \int_0^z T_r(x, x_t) \sigma_s(x_t) \int_{S^2} f_p(x_t, \bar{\omega}', \bar{\omega}) L_i(x_t, \bar{\omega}') d\omega' dt \]
Volumetric Rendering Equation

\[ L(\mathbf{x}, \mathbf{\omega}) = T_r(\mathbf{x}, \mathbf{x}_z) L(\mathbf{x}_z, \mathbf{\omega}) \]

Attenuated background radiance

+ \int_0^z T_r(\mathbf{x}, \mathbf{x}_t) \sigma_a(\mathbf{x}_t) L_e(\mathbf{x}_t, \mathbf{\omega}) dt

+ \int_0^z T_r(\mathbf{x}, \mathbf{x}_t) \sigma_s(\mathbf{x}_t) \int_{S^2} f_p(\mathbf{x}_t, \mathbf{\omega}', \mathbf{\omega}) L_i(\mathbf{x}_t, \mathbf{\omega}') d\omega' dt

Attenuated background radiance
Volumetric Rendering Equation

\[ L(x, \bar{\omega}) = T_r(x, x_z) L(x_z, \bar{\omega}) \]

Attenuated background radiance

\[ + \int_0^z T_r(x, x_t) \sigma_a(x_t) L_e(x_t, \bar{\omega}) dt \]
Accumulated emitted radiance

\[ + \int_0^z T_r(x, x_t) \sigma_s(x_t) \int_{S^2} f_p(x_t, \bar{\omega}', \bar{\omega}) L_i(x_t, \bar{\omega}') d\omega' dt \]

Accumulated emitted radiance
Volumetric Rendering Equation

\[ L(x, \bar{\omega}) = T_r(x, x_z) L(x_z, \bar{\omega}) \]

Attenuated background radiance

\[ + \int_{0}^{z} T_r(x, x_t) \sigma_a(x_t) L_e(x_t, \bar{\omega}) dt \]

Accumulated emitted radiance

\[ + \int_{0}^{z} T_r(x, x_t) \sigma_s(x_t) \int_{S^2} f_p(x_t, \bar{\omega}', \bar{\omega}) L_i(x_t, \bar{\omega}') d\omega' dt \]

Accumulated in-scattered radiance
Heterogeneous/Homogeneous media
Homogeneous media
Participating Media: Heterogeneous
Participating Media: Heterogeneous
Participating Media: Heterogeneous

\[ L(x, \omega) = \int_0^s T_r(x \leftrightarrow x_t) \sigma_s(x_t) L_i(x_t, \omega) dt + T_r(x \leftrightarrow x_s) L(x_s, \omega) \]
Participating Media: Heterogeneous

\[ L(x, \bar{\omega}) = \int_{0}^{s} T_r(x \leftrightarrow x_t) \sigma_s(x_t) L_i(x_t, \bar{\omega}) dt + T_r(x \leftrightarrow x_s) L(x_s, \bar{\omega}) \]
Participating Media: Homogeneous

\[ L(x, \bar{\omega}) = \int_{0}^{S} T_{r}(x \leftrightarrow x_{t})\sigma_{s}(x_{t})L_{i}(x_{t}, \bar{\omega})dt + T_{r}(x \leftrightarrow x_{s})L(x_{s}, \bar{\omega}) \]
Participating Media: Homogeneous

\[ L(x, \tilde{\omega}) = \int_{0}^{s} T_{r}(x \leftrightarrow x_{t})\sigma_{s}(x_{t})L_{i}(x_{t}, \tilde{\omega})dt + T_{r}(x \leftrightarrow x_{s})L(x_{s}, \tilde{\omega}) \]

\[ L(x, \tilde{\omega}) = \sigma_{s} \int_{0}^{s} T_{r}(x \leftrightarrow x_{t})L_{i}(x_{t}, \tilde{\omega})dt + T_{r}(x \leftrightarrow x_{s})L(x_{s}, \tilde{\omega}) \]
Participating Media: Homogeneous

\[ L(x, \tilde{\omega}) = \int_0^s T_r(x \leftrightarrow x_t) \sigma_s(x_t) L_i(x_t, \tilde{\omega}) dt + T_r(x \leftrightarrow x_s) L(x_s, \tilde{\omega}) \]

\[ L(x, \tilde{\omega}) = \sigma_s \int_0^s T_r(x \leftrightarrow x_t) L_i(x_t, \tilde{\omega}) dt + T_r(x \leftrightarrow x_s) L(x_s, \tilde{\omega}) \]
Participating Media: Homogeneous

\[
L(x, \bar{\omega}) = \int_{0}^{s} T_r(x \leftrightarrow x_t) \sigma_s(x_t) L_i(x_t, \bar{\omega}) dt + T_r(x \leftrightarrow x_s) L(x_s, \bar{\omega})
\]

\[
L(x, \bar{\omega}) = \sigma_s \int_{0}^{s} T_r(x \leftrightarrow x_t) L_i(x_t, \bar{\omega}) dt + T_r(x \leftrightarrow x_s) L(x_s, \bar{\omega})
\]

\[
L(x, \bar{\omega}) = \sigma_s \int_{0}^{s} e^{-t\sigma_t} L_i(x_t, \bar{\omega}) dt + e^{-s\sigma_t} L(x_s, \bar{\omega})
\]
Participating Media: Homogeneous

\[ L(x, \vec{\omega}) = \int_0^s T_r(x \leftrightarrow x_t) \sigma_s(x_t) L_i(x_t, \vec{\omega}) dt + T_r(x \leftrightarrow x_s) L(x_s, \vec{\omega}) \]

\[ L(x, \vec{\omega}) = \sigma_s \int_0^s T_r(x \leftrightarrow x_t) L_i(x_t, \vec{\omega}) dt + T_r(x \leftrightarrow x_s) L(x_s, \vec{\omega}) \]

\[ L(x, \vec{\omega}) = \sigma_s \int_0^s e^{-t \sigma_t} L_i(x_t, \vec{\omega}) dt + e^{-s \sigma_t} L(x_s, \vec{\omega}) \]
Participating Media: Homogeneous

\[ L(x, \omega) = \sigma_s \int_0^s e^{-t \sigma_t} L_i(x_t, \omega) \, dt + e^{-s \sigma_t} L(x_s, \omega) \]
Homogeneous Ambient Media

\[ L(x, \omega) = \sigma_s \int_0^s e^{-t\sigma_t} L_i(x_t, \omega) \, dt + e^{-s\sigma_t} L(x_s, \omega) \]
Homogeneous Ambient Media

Assume in-scattered radiance is an ambient constant

\[ L(x, \bar{\omega}) = \sigma_s \int_0^s e^{-t\sigma_t} L_i(x_t, \bar{\omega}) dt + e^{-s\sigma_t} L(x_s, \bar{\omega}) \]
Homogeneous Ambient Media

Assume in-scattered radiance is an ambient constant

\[ L(x, \omega) = \sigma_s \int_0^s e^{-t\sigma_t} L_i(x_t, \omega) dt + e^{-s\sigma_t} L(x_s, \omega) \]

\[ L(x, \omega) = \sigma_s L_i \int_0^s e^{-t\sigma_t} dt + e^{-s\sigma_t} L(x_s, \omega) \]
Homogeneous Ambient Media

Assume in-scattered radiance is an ambient constant

\[
L(x, \omega) = \sigma_s \int_0^s e^{-\sigma_t} L_i(x_t, \omega) dt + e^{-s\sigma_t} L(x_s, \omega)
\]

\[
L(x, \omega) = \sigma_s L_i \int_0^s e^{-\sigma_t} dt + e^{-s\sigma_t} L(x_s, \omega)
\]
Homogeneous Ambient Media

Assume in-scattered radiance is an ambient constant

\[
L(x, \bar{\omega}) = \sigma_s \int_0^s e^{-t\sigma_t} L_i(x_t, \bar{\omega}) dt + e^{-s\sigma_t} L(x_s, \bar{\omega})
\]

\[
L(x, \bar{\omega}) = \sigma_s L_i \int_0^s e^{-t\sigma_t} dt + e^{-s\sigma_t} L(x_s, \bar{\omega})
\]

\[
L(x, \bar{\omega}) = \sigma_s L_i \frac{1 - e^{-s\sigma_t}}{\sigma_t} + e^{-s\sigma_t} L(x_s, \bar{\omega})
\]
Homogeneous Ambient Media

Assume in-scattered radiance is an ambient constant

\[
L(x, \bar{\omega}) = \sigma_s \int_0^s e^{-t\sigma_t} L_i(x_t, \bar{\omega}) dt + e^{-s\sigma_t} L(x_s, \bar{\omega})
\]

\[
L(x, \bar{\omega}) = \sigma_s L_i \int_0^s e^{-t\sigma_t} dt + e^{-s\sigma_t} L(x_s, \bar{\omega})
\]

\[
L(x, \bar{\omega}) = \sigma_s L_i \frac{1 - e^{-s\sigma_t}}{\sigma_t} + e^{-s\sigma_t} L(x_s, \bar{\omega})
\]

\[
L(x, \bar{\omega}) = \text{lerp} \left( \frac{\sigma_s}{\sigma_t} L_i, L(x_s, \bar{\omega}), e^{-s\sigma_t} \right)
\]
Homogeneous Ambient Media

Fog

Clear Day
Fog
Volumetric Rendering Equation

\[ L(x, \vec{\omega}) = T_r(x, x_z) L(x_z, \vec{\omega}) \]

\[ + \int_0^z T_r(x, x_t) \sigma_a(x_t) L_e(x_t, \vec{\omega}) dt \]

\[ + \int_0^z T_r(x, x_t) \sigma_s(x_t) \int_{S^2} f_p(x_t, \vec{\omega}', \vec{\omega}) L_i(x_t, \vec{\omega}') d\omega' dt \]

Accumulated in-scattered radiance
In-scattered Radiance

\[ L(x, \omega) = \int_0^Z T_r(x, x_t) \sigma_s(x_t) \int_{S^2} f_p(x_t, \omega', \bar{\omega}) L_i(x_t, \bar{\omega}') d\omega' dt \]
In-scattered Radiance

\[ L(x, \omega) = \int_0^z T_r(x, x_t)\sigma_s(x_t) \int_{S^2} f_p(x_t, \omega', \bar{\omega}) L_i(x_t, \bar{\omega}') d\omega' dt \]
In-scattered Radiance

\[ L(x, \omega) = \int_0^Z T_r(x, x_t) \sigma_s(x_t) \int_{S^2} f_p(x_t, \omega', \omega) L_i(x_t, \omega') d\omega' dt \]

\[ L_s(x, \omega) = \int_{S^2} f_p(x_t, \omega', \omega) L_i(x_t, \omega') d\omega' dt \]
In-scattered Radiance

\[ L(x, \omega) = \int_0^Z T_r(x, x_t) \sigma_s(x_t) \int_{S^2} f_p(x_t, \omega', \omega) L_i(x_t, \omega') d\omega' dt \]

\[ L_s(x, \omega) = \int_{S^2} f_p(x_t, \omega', \omega) L_i(x_t, \omega') d\omega' dt \]

Single scattering \( L_i \) arrives directly from a light source (direct illumination)
In-scattered Radiance

\[ L(x, \omega) = \int_0^z T_r(x, x_t) \sigma_s(x_t) \int_{S^2} f_p(x_t, \omega', \omega) L_i(x_t, \omega') d\omega' dt \]

\[ L_s(x, \omega) = \int_{S^2} f_p(x_t, \omega', \omega) L_i(x_t, \omega') d\omega' dt \]

Single scattering  
\( L_i \) arrives directly from a light source (direct illumination)

\[ L_i(x, \omega) = T_r(x, r(x, \omega)) L_e(r(x, \omega), -\omega) \]
In-scattered Radiance

\[ L(x, \omega) = \int_0^z T_r(x, x_t) \sigma_s(x_t) \int_{S^2} f_p(x_t, \omega', \bar{\omega}) L_i(x_t, \bar{\omega}') d\omega' dt \]

\[ L_s(x, \omega) = \int_{S^2} f_p(x_t, \omega', \bar{\omega}) L_i(x_t, \bar{\omega}') d\omega' dt \]

Single scattering

\[ L_i \] arrives directly from a light source (direct illumination)

\[ L_i(x, \bar{\omega}) = T_r(x, r(x, \bar{\omega})) L_e(r(x, \bar{\omega}), -\bar{\omega}) \]

Multiple scattering

arrives through multiple bounces (indirect illumination)
Single Scattering

\[ L(x, \tilde{\omega}) = \int_{0}^{\tilde{z}} T_r(x, x_t) \sigma_s(x_t) \int_{S^2} f_p(x, \tilde{\omega}, \tilde{\omega}') T_r(x_t, x_e) L_e(x_e, -\tilde{\omega}) d\tilde{\omega}' dt \]
Single Scattering

\[ L(x, \vec{\omega}) = \int_0^\tilde{z} T_r(x, x_t) \sigma_s(x_t) \int_{S^2} f_p(x, \vec{\omega}, \vec{\omega}') T_r(x_t, x_e) L_e(x_e, -\vec{\omega}) d\vec{\omega}' dt \]
Single Scattering

\[ L(x, \omega) = \int_0^Z T_r(x, x_t) \sigma_s(x_t) \int_{S^2} f_p(x, \omega, \omega') T_r(x_t, x_e) L_e(x_e, -\omega) d\omega' dt \]
Single Scattering

\[ L(x, \bar{\omega}) = \int_0^z T_r(x, x_t) \sigma_s(x_t) \int_{S^2} f_p(x, \bar{\omega}, \bar{\omega}') T_r(x_t, x_e) L_e(x_e, -\bar{\omega}) d\bar{\omega}' dt \]
Single Scattering

\[ L(\mathbf{x}, \bar{\omega}) = \int_{0}^{z} T_r(\mathbf{x}, \mathbf{x}_t) \sigma_s(\mathbf{x}_t) \int_{S^2} f_p(\mathbf{x}, \bar{\omega}, \bar{\omega}') T_r(\mathbf{x}_t, \mathbf{x}_e) L_e(\mathbf{x}_e, -\bar{\omega}) d\bar{\omega}' dt \]
Single Scattering

\[ L(\mathbf{x}, \vec{\omega}) = \int_0^{z} T_r(\mathbf{x}, x_t) \sigma_s(x_t) \int_{S^2} f_p(\mathbf{x}, \vec{\omega}, \vec{\omega}') T_r(x_t, x_e) L_e(x_e, -\vec{\omega}) d\vec{\omega}' dt \]
Single Scattering

\[ L(x, \vec{\omega}) = \int_{0}^{\infty} T_r(x, x_t) \sigma_s(x_t) \int_{S^2} f_p(x, \vec{\omega}, \vec{\omega}') T_r(x_t, x_e) L_e(x_e, -\vec{\omega}) d\vec{\omega}' dt \]
Single Scattering

\[
L(x, \bar{\omega}) = \int_{0}^{\tilde{z}} T_r(x, x_t) \sigma_s(x_t) \int_{S^2} f_p(x, \bar{\omega}, \bar{\omega}') T_r(x_t, x_e) L_e(x_e, -\bar{\omega}) d\bar{\omega}' dt
\]
Single Scattering

\[ L(\mathbf{x}, \bar{\omega}) = \int_{0}^{z} T_r(\mathbf{x}, \mathbf{x}_t) \sigma_s(\mathbf{x}_t) \int_{S^2} f_p(\mathbf{x}, \bar{\omega}, \bar{\omega}') T_r(\mathbf{x}_t, \mathbf{x}_e) L_e(\mathbf{x}_e, -\bar{\omega}) d\bar{\omega}' dt \]
Single Scattering

\[ L(x, \bar{\omega}) = \int_{0}^{z} T_{r}(x, x_{t}) \sigma_{s}(x_{t}) \int_{S^2} f_{p}(x, \bar{\omega}, \bar{\omega}') T_{r}(x_{t}, x_{e}) L_{e}(x_{e}, -\bar{\omega}) d\bar{\omega}' dt \]
Single Scattering

\[ L(x, \omega) = \int_0^Z T_r(x, x_t) \sigma_s(x_t) \int_{S^2} f_p(x, \omega, \omega') T_r(x_t, x_e) L_e(x_e, -\omega) d\omega' dt \]
Single Scattering

\[ L(x, \bar{\omega}) = \int_0^\infty T_r(x, x_t) \sigma_s(x_t) \int_{S^2} f_p(x, \bar{\omega}, \bar{\omega}') T_r(x_t, x_e) L_e(x_e, -\bar{\omega}) d\bar{\omega}' dt \]

Semi-analytic solutions

Sun et al. [2005]

Pegoraro et al. [2009, 2010]
Single Scattering

\[ L(\mathbf{x}, \mathbf{\omega}) = \int_0^z T_r(\mathbf{x}, \mathbf{x}_t) \sigma_s(\mathbf{x}_t) \int_{S^2} f_p(\mathbf{x}, \mathbf{\omega}, \mathbf{\omega}') T_r(\mathbf{x}_t, \mathbf{x}_e)L_e(\mathbf{x}_e, -\mathbf{\omega}) d\mathbf{\omega}' dt \]

Semi-analytic solutions

Sun et al. [2005]

Pegoraro et al. [2009, 2010]

Numerical solutions

Ray marching

Equiangular sampling
Analytic Single Scattering

\[ L(x, \tilde{\omega}) = \int_{0}^{Z} T_r(x, x_t) \sigma_s(x_t) \int_{S^2} f_p(x, \tilde{\omega}, \tilde{\omega}') T_r(x_t, x_e) L_e(x_e, -\tilde{\omega}) d\tilde{\omega}' dt \]

Assumptions:
Analytic Single Scattering

\[
L(\mathbf{x}, \bar{\omega}) = \int_{0}^{\mathcal{Z}} T_r(\mathbf{x}, \mathbf{x}_t) \sigma_s(\mathbf{x}_t) \int_{S^2} f_p(\mathbf{x}, \bar{\omega}, \bar{\omega}') T_r(\mathbf{x}_t, \mathbf{x}_e) L_e(\mathbf{x}_e, -\bar{\omega}) d\bar{\omega}' dt
\]

Assumptions:

Homogeneous
Analytic Single Scattering

\[ L(x, \omega) = \int_0^\infty T_r(x, x_t) \sigma_s(x_t) \int_{S^2} f_p(x, \omega, \omega') T_r(x_t, x_e) L_e(x_e, -\omega) d\omega' dt \]

Assumptions:

- Homogeneous
- Point or spot light
Analytic Single Scattering

\[
L(x, \bar{\omega}) = \int_0^Z T_r(x, x_t) \sigma_s(x_t) \int_{S^2} f_p(x, \bar{\omega}, \bar{\omega}') T_r(x_t, x_e) L_e(x_e, -\bar{\omega}) d\bar{\omega}' dt
\]

Assumptions:

- Homogeneous
- Point or spot light
- Relatively simple phase function
Analytic Single Scattering

\[ L(x, \omega) = \int_{0}^{Z} T_r(x, x_t) \sigma_s(x_t) \int_{S^2} f_p(x, \omega, \omega') T_r(x_t, x_e) L_e(x_e, -\omega) d\omega' dt \]

Assumptions:

- Homogeneous
- Point or spot light
- Relatively simple phase function
- No occlusion
Analytic Single Scattering

\[ L(x, \bar{\omega}) = \int_0^Z T_r(x, x_t) \sigma_s(x_t) \int_{S^2} f_p(x, \bar{\omega}, \bar{\omega}') T_r(x_t, x_e) L_e(x_e, -\bar{\omega}) d\bar{\omega}' dt \]

Assumptions:

- Homogeneous
- Point or spot light
- Relatively simple phase function
- No occlusion
Analytic Single Scattering

\[ L(x, \bar{\omega}) = \int_0^Z T_r(x, x_t) \sigma_s(x_t) \int_{S^2} f_p(x, \bar{\omega}, \bar{\omega}') T_r(x_t, x_e) L_e(x_e, -\bar{\omega}) d\bar{\omega}' dt \]

Assumptions:

Homogeneous
Point or spot light
Relatively simple phase function
No occlusion

\[ L(x, \bar{\omega}) = \Phi \frac{1}{4\pi} \frac{1}{4\pi} \int_0^Z e^{-\sigma_t ||x, x_t||} \frac{e^{-\sigma_t ||x_t, x_p||}}{e^{-\sigma_t ||x_t, x_p||^2}} dt \]
OpenGL Fog
Analytic Single Scattering
Analytic Single Scattering
Analytic Single Scattering

\[ L_m(x_a, x_b, \hat{b}) = \frac{K_s}{\hbar} e^{\kappa_i(x_a-x_b)} \sum_{n=0}^{N-1} c(n) \sum_{k=0}^{2n} d(n,k) \int_{v_a}^{v_b} \frac{e^{-Hv}}{(v^2 + 1)^{n+1}} v^k dv \]

\[
\int \frac{e^{av}}{(v^2 + 1)^m} v^n dv = \frac{1}{2^{m-1}} \sum_{i=0}^{m-1} \frac{1}{2i} \left( \begin{array}{c} m-1+l \\ m-1 \end{array} \right) \sum_{k=0}^{\min\{m-1-l,n\}} \left( \begin{array}{c} n \\ k \end{array} \right) \frac{a^{m-1-l-k}}{(m-1-l-k)!} E(a, v, m-n-l+k) \\
- e^{av} \sum_{j=1}^{m-1-l-k} \frac{(j-1)!}{(m-1-l-k)!} \frac{a^{m-1-l-k-j}}{(v^2 + 1)^j} \sum_{i=(m-n-l+k-j) \mod 2}^{\leq j} \left( \begin{array}{c} n \\ i \end{array} \right) \left( \begin{array}{c} j \\ v \end{array} \right) \\
+ \frac{e^{av} \leq n-m+l}{a} \sum_{k=0}^{\leq j} \left( \begin{array}{c} n \\ k \end{array} \right) \sum_{j=0}^{m+l-k} \frac{(n-m+l-k)!}{j!} \frac{1}{(-a)^{n-m-l-k-j}} \sum_{i=(-m+l+k-j) \mod 2}^{\leq j} \left( \begin{array}{c} m+l+k \\ j \end{array} \right) \left( \begin{array}{c} j \\ i \end{array} \right) \right)
\]

No shadows, implementation nightmare, computationally intensive,...

Let’s try brute force!
Ray Marching

\[ L(x, \omega) = \int_0^z T_r(x, x_t) \sigma_s(x_t) L_s(x_t, \omega) dt \]

Approximate with Riemann summation
Ray Marching

\[ L(x, \bar{\omega}) \approx \sum_{k=0}^{N} T_r(x, x_{t,k}) \sigma_s(x_{t,k}) L_s(x_{t,k}, \bar{\omega}) \Delta t \]
Ray Marching

\[ L(x, \bar{\omega}) \approx \sum_{k=0}^{N} T_r(x, x_{t,k}) \sigma_s(x_{t,k}) L_s(x_{t,k}, \bar{\omega}) \Delta t \]
Ray Marching

\[ L(x, \omega) \approx \sum_{k=0}^{N} T_r(x, x_{t,k}) \sigma_s(x_{t,k}) L_s(x_{t,k}, \omega) \Delta t \]
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\[ L(x, \bar{\omega}) \approx \sum_{k=0}^{N} T_r(x, x_{t,k}) \sigma_s(x_{t,k}) L_s(x_{t,k}, \bar{\omega}) \Delta t \]

Homogeneous volume: \( T_r(x, x_{t,k}) = e^{-\sigma_t ||x, x_{t,k}||} \)
Ray Marching

\[ L(x, \bar{\omega}) \approx \sum_{k=0}^{N} T_r(x, x_{t,k}) \sigma_s(x_{t,k}) L_s(x_{t,k}, \bar{\omega}) \Delta t \]

Heterogeneous volume: \( T_r(x, x_{t,k}) = T_r(x, x_{t,k-1}) e^{-\sigma_t(x_{t,k}) \Delta t} \)

Assume constant extinction along each segment
Ray Marching

\[ L(x, \bar{\omega}) \approx \sum_{k=0}^{N} T_r(x, x_{t,k}) \sigma_s(x_{t,k}) L_s(x_{t,k}, \bar{\omega}) \Delta t \]
Ray Marching

\[ L_s(\mathbf{x}_t, \bar{\omega}) = \int_{S^2} f_p(\mathbf{x}_t, \bar{\omega}, \bar{\omega}') L_i(\mathbf{x}_t, \bar{\omega}') d\bar{\omega}' \]
Ray Marching

\[ L_s(x_t, \bar{\omega}) \approx \frac{1}{M} \sum_{j=1}^{M} \frac{f_p(x_t, \bar{\omega}, \bar{\omega}'_j)L_i(x_t, \bar{\omega}'_j)}{p(\bar{\omega}'_j)} \]
Ray Marching

\[ L_s(x_t, \bar{\omega}) \approx \frac{1}{M} \sum_{j=1}^{M} \frac{f_p(x_t, \bar{\omega}, \bar{\omega}_j') L_i(x_t, \bar{\omega}_j')}{p(\bar{\omega}_j')} \]
Ray Marching

\[ L_s(x_t, \bar{\omega}) \approx \frac{1}{M} \sum_{j=1}^{M} f_p(x_t, \bar{\omega}, \bar{\omega}_j') \frac{L_i(x_t, \bar{\omega}_j')}{p(\bar{\omega}_j')} \]
Ray Marching

\[ L_s(x_t, \bar{\omega}) \approx \frac{1}{M} \sum_{j=1}^{M} \frac{f_p(x_t, \bar{\omega}, \bar{\omega}_j') L_i(x_t, \bar{\omega}_j')}{p(\bar{\omega}_j')} \]

Another ray marching needed to estimate the transmittance along the connection ray (in the heterogeneous media)
Ray Marching in Heterogeneous Media

Marching towards the light source

- Connections are expensive, many, and uniformly distributed along the primary ray
Decoupled Transmittance and in-scattering

1. Ray march and cache transmittance

- Choose step-size w.r.t. frequency content to accurately capture variations
Decoupled Transmittance and in-scattering

1. Ray march and cache transmittance

- Choose step-size w.r.t. frequency content to accurately capture variations
Decoupled Transmittance and in-scattering

2. Estimate in-scattering using MC integration
   - Distribute samples proportional to (part of) the integrand
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- Distribute samples proportional to (part of) the integrand

\[ p(x_t) \propto T_r(x, x_t) \]
2. Estimate in-scattering using MC integration

- Distribute samples proportional to (part of) the integrand
Decoupled Transmittance and in-scattering

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\[ p(x_t) \propto T_r(x, x_t) \]
Decoupled Transmittance and in-scattering

2. Estimate in-scattering using MC integration

- Distribute samples proportional to (part of) the integrand

\[ p(x_t) \propto \frac{1}{d^2} \]

\( d \): distance to light
Decoupled Transmittance and in-scattering

2. Estimate in-scattering using MC integration

- Distribute samples proportional to (part of) the integrand

\[ p(x_t) \propto \frac{1}{d^2} \]

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\[ d : \text{distance to light} \]
2. Estimate in-scattering using MC integration

- Distribute samples proportional to (part of) the integrand

\[ p(x_t) \propto \frac{1}{d^2} \]

\(d\) : distance to light

Equiangular sampling
Kulla and Fajardo [2012]
Decoupled Transmittance and in-scattering

Ray Marching

Equi-angular sampling
Realistic Image Synthesis SS2020

Volumetric Path Tracing

Motivation

Same as with path tracing: avoid the exponential growth

Single scattering

Multiple scattering
Volumetric Path Tracing
Volumetric Path Tracing

Motivation

Same as with path tracing: avoid the exponential growth

Paths can:

Reflect / Refract off surfaces

Scatter inside a volume
Volumetric Rendering Equation

\[ L(x, \bar{\omega}) = \int_{0}^{z} T_r(x, x_t) \sigma_a(x_t) L_e(x_t, \bar{\omega}) dt \]

\[ + \int_{0}^{z} T_r(x, x_t) \sigma_s(x_t) L_s(x_t, \omega) dt \]

\[ + T_r(x, x_z) L(x_z, \bar{\omega}) \]
Volumetric Rendering Equation

\[ L(x, \omega) = \int_0^Z T_r(x, x_t) \sigma_a(x_t) L_e(x_t, \omega) \, dt \]

\[ + \int_0^Z T_r(x, x_t) \sigma_s(x_t) L_s(x_t, \omega) \, dt \]

\[ + T_r(x, x_z) L(x_z, \omega) \]

Accumulated emitted radiance
Volumetric Rendering Equation

\[ L(x, \omega) = \int_{0}^{z} T_r(x, x_t) \sigma_a(x_t) L_e(x_t, \omega) dt \]

Accumulated emitted radiance

\[ + \int_{0}^{z} T_r(x, x_t) \sigma_s(x_t) L_s(x_t, \omega) dt \]

Accumulated in-scattered radiance

\[ + T_r(x, x_z) L(x_z, \omega) \]
Volumetric Rendering Equation

\[ L(x, \omega) = \int_0^z T_r(x, x_t) \sigma_a(x_t) L_e(x_t, \omega) \, dt \]
\[ + \int_0^z T_r(x, x_t) \sigma_s(x_t) L_s(x_t, \omega) \, dt \]
\[ + T_r(x, x_z) L(x_z, \omega) \]

Accumulated emitted radiance

Accumulated in-scattered radiance

Attenuated background radiance
Volumetric Rendering Equation

\[ L(x, \omega) = \int_0^z T_r(x, x_t) \left[ \sigma_a(x_t) L_e(x_t, \omega) + \sigma_s(x_t) L_s(x_t, \omega) \right] dt + T_r(x, x_z) L(x_z, \omega) \]

Accumulated emitted + in-scattered radiance
Attenuated background radiance
Volumetric Rendering Equation

\[
L(x, \vec{\omega}) = \int_0^z \left[ T_r(x, x_t) \left[ \sigma_a(x_t)L_e(x_t, \vec{\omega}) + \sigma_s(x_t)L_s(x_t, \vec{\omega}) \right] \right] dt \\
+ T_r(x, x_z)L(x_z, \vec{\omega})
\]
1-Sample Monte Carlo Estimator

\[
\langle L(x, \bar{w}) \rangle = \frac{T_r(x, x_t)}{p(t)} \left[ \sigma_a(x_t) L_e(x_t, \bar{w}) + \sigma_s(x_t) L_s(x_t, \bar{w}) \right] \\
+ \frac{T_r(x, x_z)}{P(z)} L(x_z, \bar{w})
\]
1-Sample Monte Carlo Estimator

\[
\langle L(x, \omega) \rangle = \frac{T_r(x, x_t)}{p(t)} \left[ \sigma_a(x_t)L_e(x_t, \omega) + \sigma_s(x_t)L_s(x_t, \omega) \right] \\
+ \frac{T_r(x, x_z)}{P(z)} L(x_z, \omega)
\]

- \( p(t) \) Probability density of distance \( t \)
- \( P(z) \) Probability of exceeding distance \( z \)
1-Sample Monte Carlo Estimator

\[
\langle L(x, \bar{\omega}) \rangle = \frac{T_r(x, x_t)}{p(t)} \left[ \sigma_a(x_t) L_e(x_t, \bar{\omega}) + \sigma_s(x_t) \frac{f_p(x, \bar{\omega}, \bar{\omega}_i) L(x_t, \bar{\omega})}{p(\bar{\omega}_i)} \right] \\
+ \frac{T_r(x, x_z)}{P(\bar{z})} L(x_z, \bar{\omega})
\]

\( p(t) \) Probability density of distance \( t \)

\( P(\bar{z}) \) Probability of exceeding distance \( \bar{z} \)

\( p(\bar{\omega}_i) \) Probability density of direction \( \bar{\omega}_i \)
Volumetric Path Tracing

1. Sample distance to next interaction

2. Scatter in the volume or bounce off a surface
Volumetric Path Tracing

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1. Sample distance to next interaction
2. Scatter in the volume or bounce off a surface
Volumetric Path Tracing with NEE
Sampling the Phase Function

Isotropic:

Henyey-Greenstein:
Sampling the Phase Function

**Isotropic:** Uniform sphere sampling

**Henyey-Greenstein:**
Sampling the Phase Function

**Isotropic:** Uniform sphere sampling

**Henyey-Greenstein:** Using the inversion method we can derive

\[
\cos \theta = \frac{1}{2g} \left( 1 + g^2 - \left( \frac{1 - g^2}{1 - g + 2g \xi_1} \right)^2 \right)
\]

\[
\phi = 2\pi \xi_2
\]

PDF is the value of the HG phase function
Free-path Sampling

Free-path or free-flight distance:

- Distance to the next interaction in the medium

- Dense media (e.g. milk): short mean-free path

- Thin media (e.g. atmosphere): long mean-free path
Free-path Sampling

Free-path or free-flight distance:

- Distance to the next interaction in the medium

- Dense media (e.g. milk): short mean-free path

- Thin media (e.g. atmosphere): long mean-free path

Ideally, we want to sample according to (part of) of the integrand:

\[
p(x_t|(x, \omega)) \propto T_r(x, x_t)
\]
Free-path Sampling

Free-path or free-flight distance:

- Distance to the next interaction in the medium

- Dense media (e.g. milk): short mean-free path

- Thin media (e.g. atmosphere): long mean-free path

Ideally, we want to sample according to (part of) of the integrand:

\[ p(x_t | (x, \vec{\omega})) \propto T_r(x, x_t) \]

simplified notation

\[ p(t) \propto T_r(t) \]
Free-path Sampling

Homogeneous media: \[ T_r(t) = e^{-\sigma t} \]
Free-path Sampling

Homogeneous media:

\[ T_r(t) = e^{-\sigma_i t} \]

PDF:

\[ p(t) \propto e^{-\sigma_i t} \]
Free-path Sampling

Homogeneous media:

\[ T_r(t) = e^{-\sigma_i t} \]

PDF:

\[ p(t) \propto e^{-\sigma_i t} \]

\[ p(t) = \frac{e^{-\sigma_i t}}{\int_0^\infty e^{-\sigma_i s} ds} = \sigma_i e^{-\sigma_i t} \]
Free-path Sampling

Homogeneous media:

\[ T_r(t) = e^{-\sigma t t} \]

PDF:

\[ p(t) \propto e^{-\sigma t t} \]

\[ p(t) = \int_0^\infty \frac{e^{-\sigma t t}}{e^{-\sigma t s} ds} = \sigma t e^{-\sigma t t} \]

CDF:

\[ P(t) = \int_0^t e^{-\sigma t s} ds = 1 - e^{-\sigma t t} \]
Free-path Sampling

Homogeneous media:

\[ T_r(t) = e^{-\sigma t t} \]

PDF:

\[ p(t) \propto e^{-\sigma t t} \]

\[ p(t) = \frac{e^{-\sigma t t}}{\int_0^\infty e^{-\sigma t s} ds} = \sigma t e^{-\sigma t t} \]

CDF:

\[ P(t) = \int_0^t e^{-\sigma t s} ds = 1 - e^{-\sigma t t} \]

Inverted CDF:

\[ P^{-1}(\xi) = -\frac{\log e(1 - \xi)}{\sigma t} \]
Free-path Sampling

Homogeneous media: \[ T_r(t) = e^{-\sigma t t} \]

Recipe:
Free-path Sampling

Homogeneous media: $T_r(t) = e^{-\sigma_t t}$

Recipe:

Generate a random number $\xi$
Free-path Sampling

Homogeneous media: $T_r(t) = e^{-\sigma_t t}$

Recipe:

Generate a random number $\xi$

Sample distance
Free-path Sampling

Homogeneous media: \[ T_r(t) = e^{-\sigma_t t} \]

Recipe:

Generate a random number \( \xi \)

Sample distance \[ t = -\frac{\log_e(1 - \xi)}{\sigma_t} \]
Free-path Sampling

Homogeneous media: \( T_r(t) = e^{-\sigma_t t} \)

Recipe:

Generate a random number \( \xi \)

Sample distance \( t = -\frac{\log_e(1 - \xi)}{\sigma_t} \)

Compute PDF
Free-path Sampling

Homogeneous media: \( T_r(t) = e^{-\sigma t} \)

Recipe:

Generate a random number \( \xi \)

Sample distance \( t = -\frac{\log_e(1 - \xi)}{\sigma_t} \)

Compute PDF \( p(t) = \sigma_t e^{-\sigma t} \)
Free-path Sampling

Homogeneous media: \( T_r(t) = e^{-\sigma t} \)

Recipe:

- Generate a random number \( \xi \)

\[
t = -\frac{\log_e(1 - \xi)}{\sigma_t}
\]

- Sample distance

- Compute PDF

\[
p(t) = \sigma_t e^{-\sigma t}
\]

Surface hit before reaching \( t \)
Free-path Sampling

Homogeneous media: \( T_r(t) = e^{-\sigma_t t} \)

Recipe:

Generate a random number \( \xi \)

Sample distance \( t = -\frac{\log_e(1 - \xi)}{\sigma_t} \)

Compute PDF \( p(t) = \sigma_t e^{-\sigma_t t} \)
Free-path Sampling

Homogeneous media: \[ T_r(t) = e^{-\sigma_t t} \]

Recipe:

Generate a random number \( \xi \)

Sample distance \( t = -\frac{\log_e(1 - \xi)}{\sigma_t} \)

Compute PDF \[ p(t) = \sigma_t e^{-\sigma_t t} \]

Surface hit before reaching \( t \)
Free-path Sampling

Homogeneous media: \( T_r(t) = e^{-\sigma t} \)

Recipe:

Generate a random number \( \xi \)

Sample distance \( t = -\frac{\log_e(1-\xi)}{\sigma_t} = s \)

Compute PDF \( p(t) = \sigma_t e^{-\sigma_t t} \)

Surface hit before reaching \( t \)
Free-path Sampling

Homogeneous media: \( T_r(t) = e^{-\sigma t} \)

Recipe:

- Generate a random number \( \xi \)
- Sample distance \( t = -\frac{\log_e(1 - \xi)}{\sigma t} = s \)
- Compute PDF \( p(t) = \sigma t e^{-\sigma t} \)

Surface hit before reaching \( t \)
Free-path Sampling

Homogeneous media: \( T_r(t) = e^{-\sigma_t t} \)

Recipe:

- Generate a random number \( \xi \)
- Sample distance \( t = -\frac{\log_e(1 - \xi)}{\sigma_t} = s \)
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Surface hit before reaching \( t \)
Free-path Sampling

Homogeneous media: \( T_r(t) = e^{-\sigma_t t} \)

Recipe:

Generate a random number \( \xi \)

Sample distance \( t = -\frac{\log_e(1 - \xi)}{\sigma_t} = s \)

Compute PDF \( p(t) = \sigma_t e^{-\sigma_t t} = e^{-\sigma_t s} \)

Note: This is now a probability, not a probability density

Surface hit before reaching \( t \)
What about heterogeneous media?
Free-path Sampling

Heterogeneous medium:

\[ T_r(t) = e^{\int_0^t -\sigma_t(s)ds} \]
Free-path Sampling

Heterogeneous medium: \[ T_r(t) = e^{\int_0^t -\sigma(s)ds} \]

- Closed form solutions exist but for only simple media
  e.g., linearly or exponentially varying extinction
Free-path Sampling

Heterogeneous medium: \[ T_r(t) = e^{\int_0^t -\sigma_t(s) \, ds} \]

- Closed form solutions exist but for only simple media
e.g., linearly or exponentially varying extinction

- Other solutions:
  - Regular tracking (3D DDA)
  - Ray marching
  - Delta tracking
Free-path Sampling

How to sample the flight distance to the next interaction?
Free-path Sampling

How to sample the flight distance to the next interaction?

\[ T(t) = e^{-\int_0^t \mu_t(s) \, ds} = P(X > t) \]
Free-path Sampling

How to sample the flight distance to the next interaction?

\[ T(t) = e^{-\int_0^t \mu_t(s)ds} = P(X > t) \]

Random variable representing flight distance

CDF

\[ P(X \leq t) = F(t) \]
Free-path Sampling

How to sample the flight distance to the next interaction?

\[ T(t) = e^{-\int_0^t \mu(s)ds} = \begin{cases} P(X > t) \\ P(X \leq t) = F(t) \end{cases} \]

Random variable representing flight distance

CDF

Partition of unity
How to sample the flight distance to the next interaction?

\[ T(t) = e^{-\int_0^t \mu_t(s) \, ds} = \begin{cases} P(X > t) \\ P(X \leq t) = F(t) \end{cases} \]

\[ F(t) = 1 - T(t) \]

Random variable representing flight distance

CDF

Partition of unity

Recipe for generating samples
Free-path Sampling

Cumulative distribution function (CDF)

\[ F(t) = 1 - T(t) = 1 - e^{-\tau(t)} \]
Free-path Sampling

Cumulative distribution function (CDF)

\[ F(t) = 1 - T(t) = 1 - e^{-\tau(t)} \]

Probability density function (PDF)

\[ p(t) = \frac{dF(t)}{dt} = \frac{d}{dt} \left( 1 - e^{-\tau(t)} \right) = \mu(t)e^{-\tau(t)} \]
Free-path Sampling

Cumulative distribution function (**CDF**)  
\[ F(t) = 1 - T(t) = 1 - e^{-\tau(t)} \]

Probability density function (**PDF**)  
\[ p(t) = \frac{dF(t)}{dt} = \frac{d}{dt} \left( 1 - e^{-\tau(t)} \right) = \mu_t(t)e^{-\tau(t)} \]

Inverted cumulative distrib. function (**CDF⁻¹**)  
\[ \xi = 1 - e^{-\tau(t)} \]  
\[ \text{Solve for } t \]
Free-path Sampling

Cumulative distribution function (CDF)

\[ F(t) = 1 - T(t) = 1 - e^{-\tau(t)} \]

Probability density function (PDF)

\[ p(t) = \frac{dF(t)}{dt} = \frac{d}{dt} \left( 1 - e^{-\tau(t)} \right) = \mu_t(t)e^{-\tau(t)} \]

Inverted cumulative distr. function (CDF⁻¹)

\[ \xi = 1 - e^{-\tau(t)} \quad \text{Solve for } t \]

\[ \tau(t) = -\ln(1 - \xi) \]
Free-path Sampling

Cumulative distribution function (CDF)

\[ F(t) = 1 - T(t) = 1 - e^{-\tau(t)} \]

Probability density function (PDF)

\[ p(t) = \frac{dF(t)}{dt} = \frac{d}{dt} \left( 1 - e^{-\tau(t)} \right) = \mu_t(t)e^{-\tau(t)} \]

Inverted cumulative distr. function (CDF⁻¹)

\[ \xi = 1 - e^{-\tau(t)} \quad \text{Solve for } t \]

\[ \int_0^t \mu_t(s)ds = -\ln(1 - \xi) \]
Free-path Sampling

Cumulative distribution function (CDF)

\[ F(t) = 1 - T(t) = 1 - e^{-\tau(t)} \]

Probability density function (PDF)

\[ p(t) = \frac{dF(t)}{dt} = \frac{d}{dt} \left( 1 - e^{-\tau(t)} \right) = \mu_t(t) e^{-\tau(t)} \]

Inverted cumulative distr. function (CDF⁻¹)

\[ \xi = 1 - e^{-\tau(t)} \quad \text{Solve for } t \]

\[ \int_0^t \mu_t(s) ds = -\ln(1 - \xi) \]

Approaches for finding t:
1) ANALYTIC (closed-form CDF⁻¹)
2) SEMI-ANALYTIC (regular tracking)
3) APPROXIMATE (ray marching)
Free-path Sampling

Inverted cumulative distr. function (CDF⁻¹)

\[
\int_0^t \mu_t(s) \, ds = -\ln(1 - \xi)
\]
Free-path Sampling

Inverted cumulative distr. function (CDF\(^{-1}\))

\[
\int_{0}^{t} \mu_t(s) ds = -\ln(1 - \xi)
\]

Some simple volumes permit closed-form solutions

Example: \textbf{homogeneous} medium (\(\mu_t(x) = \mu_t\))
Free-path Sampling

Inverted cumulative distr. function (\(CDF^{-1}\))

\[
\int_0^t \mu_t(s) \, ds = -\ln(1 - \xi)
\]

Some simple volumes permit closed-form solutions

Example: **homogeneous** medium \((\mu_t(x) = \mu_t)\)

**Opt. thickness**

\[
\int_0^t \mu_t(s) \, ds = t \mu_t
\]
Free-path Sampling

Inverted cumulative distr. function ($\text{CDF}^{-1}$)

$$\int_0^t \mu_t(s) \, ds = - \ln(1 - \xi)$$

Some simple volumes permit closed-form solutions

Example: **homogeneous** medium ($\mu_t(x) = \mu_t$)

**Opt. thickness**

$$\int_0^t \mu_t(s) \, ds = t \mu_t \quad \Rightarrow$$

**Expression for $t$**

$$t = -\frac{\ln(1 - \xi)}{\mu_t}$$
Free-path Sampling

Inverted cumulative distr. function ($\text{CDF}^{-1}$)

$$\int_0^t \mu_t(s)\,ds = -\ln(1 - \xi)$$

Some simple volumes permit closed-form solutions

Example: **homogeneous** medium ($\mu_t(x) = \mu_t$)

Opt. thickness

$$\int_0^t \mu_t(s)\,ds = t\mu_t \quad \Rightarrow \quad F^{-1}(\xi) = -\frac{\ln(1 - \xi)}{\mu_t}$$
Analytic Approach

Inverted cumulative distr. function (CDF^{-1})

\[ \int_{0}^{t} \mu_t(s) \, ds = -\ln(1 - \xi) \]

Homogeneous volume
Analytic Approach

Inverted cumulative distr. function (CDF⁻¹)

\[ \int_0^t \mu_t(s) \, ds = -\ln(1 - \xi) \]

Sampling in homogeneous vol:

1) Draw a random number \( \xi \)
2) Set \( t = -\frac{\ln(1 - \xi)}{\mu_t} \)
3) Set \( p(t) = \mu_t e^{-t \mu_t} \)

Homogeneous volume
Analytic Approach

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Regular Tracking (Semi-Analytic)

\[
\int_0^t \mu_t(s) \, ds = -\ln(1 - \xi)
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Regular Tracking (Semi-Analytic)

For piecewise-simple (e.g. piecewise-constant), summation replaces integration

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(Hierarchical) voxel grid
Regular Tracking (Semi-Analytic)

For piecewise-simple (e.g. piecewise-constant), summation replaces integration

\[ \int_0^t \mu_t(s) ds = -\ln(1 - \xi) \]

\[ \sum_{i=1}^{k} \mu_{t,i} \Delta_i = -\ln(1 - \xi) \]

(Hierarchical) voxel grid
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Regular tracking:
1) Draw a random number \( \xi \)
2) While LHS < RHS
   move to the next intersection
3) Find the exact location
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(Hierarchical) voxel grid

Start

LHS > RHS
Regular Tracking (Semi-Analytic)

For piecewise-simple (e.g. piecewise-constant), summation replaces integration

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(Hierarchical) voxel grid

Start

Sampled collision LHS=RHS

LHS > RHS
Ray Marching

Find the collision distance approximately

\[ \int_0^t \mu_t(s) ds = -\ln(1 - \xi) \]

\[ \sum_{i=1}^{k} \mu_{t,i} \Delta = -\ln(1 - \xi) \]

(Hierarchical) voxel grid

Constant step
Ray Marching

Find the collision distance approximately

\[
\int_{0}^{t} \mu_t(s) \, ds = -\ln(1 - \xi)
\]

\[
\sum_{i=1}^{k} \mu_{t,i} \Delta = -\ln(1 - \xi)
\]

Ray marching:
1) Draw a random number \( \xi \)
2) While LHS < RHS
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3) Find the exact location in the last segment analytically
Ray Marching

Find the collision distance approximately

\[
\int_0^t \mu_t(s) \, ds = -\ln(1 - \xi)
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(Hierarchical) voxel grid

LHS > RHS
Ray Marching

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Ray marching:
1) Draw a random number \( \xi \)
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   make a (fixed-size) step
3) Find the exact location in the last segment analytically

General volume

Ignored thin features = bias

LHS > RHS

Sampled collision LHS=RHS
Free-path Sampling

| ANALYTIC CDF<sup>-1</sup> | REGULAR TRACKING | RAY MARCHING |
Free-path Sampling

<table>
<thead>
<tr>
<th>ANALYTIC CDF⁻¹</th>
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Free-path Sampling

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Common approach: sample optical thickness, find corresponding distance
Delta Tracking

a.k.a. Woodcock tracking, pseudo scattering, hole tracking, null-collision method,…
Physical Interpretation

Add **FICTITIOUS MATTER** to homogenize heterogeneous extinction
Physical Interpretation

Add **FICTITIOUS MATTER** to homogenize heterogeneous extinction

- albedo $\alpha(x) = 1$
Physical Interpretation

Add **FICTITIOUS MATTER** to homogenize heterogeneous extinction

- albedo \( \alpha(x) = 1 \)
- phase function \( f_P(\omega, \bar{\omega}) = \delta(\omega - \bar{\omega}) \)
Add **FICTITIOUS MATTER** to homogenize heterogeneous extinction

- albedo  \(\alpha(x) = 1\)
- phase function  \(f_p(\omega, \bar{\omega}) = \delta(\omega - \bar{\omega})\)

**Presence of fictitious matter does not impact light transport**
Physical Interpretation

HOMOGENIZATION
Physical Interpretation

HOMOGENIZATION

Real particle
Physical Interpretation

HOMOGENIZATION

Volume bounds

Real particle
Physical Interpretation

HOMOGENIZATION

Volume bounds

Fictitious particle

Real particle
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Physical Interpretation

HOMOGENIZATION

- Volume bounds
- Fictitious particle
- Real particle
Physical Interpretation

HOMOGENIZATION

[Diagram showing a representation of a material's microstructure with labels for volume bounds, fictitious particle, and real particle.]
Physical Interpretation

HOMOGENIZATION

Volume bounds

Fictitious particle

Real particle
Physical Interpretation

HOMOGENIZATION

Volume bounds
Fictitious particle
Real particle
Physical Interpretation

HOMOGENIZATION

- Volume bounds
- Fictitious particle
- Real particle
Physical Interpretation

HOMOGENIZATION

Volume bounds

Real particle
Stochastic Sampling

Volume bounds

Real medium
Stochastic Sampling

Volume bounds

Fictitious medium

\[ \mu_t(x) + \mu_n(x) \]

Real medium
Stochastic Sampling

Volume bounds

Fictitious medium

Majorant $\bar{\mu} = \mu_t(x) + \mu_n(x)$

Real medium
Stochastic Sampling

Majorant $\bar{\mu} = \mu_t(x) + \mu_n(x)$
Stochastic Sampling

Tentative collision

Majorant $\bar{\mu} = \mu_t(x) + \mu_n(x)$

$\ln(1 - \xi) = \frac{\bar{\mu}}{\mu}$
Stochastic Sampling

\[ P_r(x) = \frac{\mu_t(x)}{\mu} \]

Majorant \( \mu = \mu_t(x) + \mu_n(x) \)

\[ \ln(1 - \xi) \]

\[ \frac{1}{\mu} \]
Stochastic Sampling

\[ P_r(x) = \frac{\mu_t(x)}{\bar{\mu}} \quad P_n(x) = \frac{\mu_n(x)}{\bar{\mu}} \]

**Majorant**

\[ \bar{\mu} = \mu_t(x) + \mu_n(x) \]

\[ \ln(1 - \xi) \]

\[ \frac{1}{\bar{\mu}} \]
Stochastic Sampling

\[ P_r(x) = \frac{\mu_t(x)}{\bar{\mu}} \quad \text{and} \quad P_n(x) = \frac{\mu_n(x)}{\bar{\mu}} \]

Majorant \( \bar{\mu} = \mu_t(x) + \mu_n(x) \)

\[ \ln(1 - \xi) \]

\[ \frac{\mu}{\bar{\mu}} \]
Stochastic Sampling

\[ P_r(x) = \frac{\mu_t(x)}{\bar{\mu}} \quad P_n(x) = \frac{\mu_n(x)}{\bar{\mu}} \]

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Majorant \( \bar{\mu} = \mu_t(x) + \mu_n(x) \)

Tentative collision

Distance

Extinction
Stochastic Sampling

\[ P_r(x) = \frac{\mu_t(x)}{\bar{\mu}} \quad P_n(x) = \frac{\mu_n(x)}{\bar{\mu}} \]

Majorant \( \bar{\mu} = \mu_t(x) + \mu_n(x) \)
Stochastic Sampling

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Majorant \( \bar{\mu} = \mu_t(x) + \mu_n(x) \)
Stochastic Sampling

\[ P_r(x) = \frac{\mu_t(x)}{\bar{\mu}} \quad P_n(x) = \frac{\mu_n(x)}{\bar{\mu}} \]

Majorant \[ \bar{\mu} = \mu_t(x) + \mu_n(x) \]
Impact of Majorant

$$\bar{\mu} = \mu_t(x) + \mu_n(x)$$
Impact of Majorant

\[ \bar{\mu} = \mu_t(x) + \mu_n(x) \]
Impact of Majorant

Tight majorant = GOOD
(few rejected collisions)
Impact of Majorant

Majorant $\bar{\mu} = \mu_t(x) + \mu_n(x)$
Impact of Majorant

Loose majorant = BAD (many expensive rejected)

Majorant $\bar{\mu} = \mu_t(x) + \mu_n(x)$

Extinction

Distance

Sampled free path

$\mu_t$
Next Lecture: Many-Light Methods
Next Lecture: Many-Light Methods

Path Tracing
21 mins

Path Tracing
13 hours

Yuksel & Yuksel 2017
Next Lecture: Many-Light Methods

Path Tracing
21 mins

Path Tracing
13 hours

Many-Light Method
(21 mins)

Yuksel & Yuksel 2017
Acknowledgements

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