Realistic Image Synthesis

- Rendering Equation -

Philipp Slusallek Karol Myszkowski Gurprit Singh

Realistic Image Synthesis SS20 - Rendering Equation

Angle and Solid Angle

- θ the angle subtended by a curve in the plane is the length of the projected arc on the unit circle.
- Ω, ω the solid angle subtended by an object is the surface area of its projection onto the unit sphere
 - Solid angle units: steradians [sr]



Solid Angle for a Small Area

The solid angle subtended by an (infinitely) small surface patch *S* with area d*A* is obtained by dividing the projected area d $A \cos \theta$ by the square of the distance to the origin:

$$\mathrm{d}\omega, d\Omega = \frac{dA\,\cos\theta}{r^2}$$



Solid Angle in Spherical Coordinates

- Infinitesimally small solid angle
 - $du = r d\theta$
 - $dv = r\sin\theta \, d\phi$
 - $dA = du \, dv = r^2 \sin \theta \, d\theta \, d\phi$
 - $\Rightarrow d\omega = \frac{dA}{r^2} = \sin\theta \, d\theta \, d\phi$
- Finite solid angle of an surface S
 - $\omega = \int_S \sin\theta \, d\theta \, d\phi$

Definition:

- We denote the entire Sphere with Ω and the (positive) hemisphere at a point x centered around its normal vector with Ω_+



Radiometry

 Radiometry is the science of measuring radiant energy transfers. Radiometric quantities have physical meaning and can be directly measured using proper equipment such as spectral photometers.

Radiometric Quantities

- Energy [watt second] $n \cdot hv$

 Φ, P

L

E

R

- Radiant power (total flux) [watt]
- Radiance [watt/(m² sr)]
- Irradiance (flux density) [watt/m²]
- Radiosity (flux density) [watt/m²]

Radiometric Quantities: Radiance

- Radiance is used to describe radiant energy transfer.
- Radiance L is defined as the total flux (radiant power) traveling at some point x in a specified direction ω, per unit area perpendicular to the direction of travel, per unit solid angle.
- Thus, the differential flux $d^2\Phi$ radiated through the differential solid angle $d\omega$, from the projected differential area $dA \cos\theta$ is:

$$d^2 \Phi = L(x, \omega) dA \cos \theta \ d\omega$$

or

$$L(x,\omega) = \frac{d^2\Phi}{dA\,\cos\theta\,d\omega}$$



• From here on we distinguish between the direction ω and the (differential) solid angle $d\omega$!!!

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Radiance in Space



Flux leaving surface 1 must be equal to flux arriving on surface 2

 $L_1 \cdot d\Omega_1 \cdot dA_1 = L_2 \cdot d\Omega_2 \cdot dA_2$ From geometry follows $d\Omega_1 = \frac{dA_2}{r^2} \qquad d\Omega_2 = \frac{dA_1}{r^2}$ Def: Ray *Throughput* $T = d\Omega_1 \cdot dA_1 = d\Omega_2 \cdot dA_2 = \frac{dA_1 \cdot dA_2}{r^2} \implies L_1 = L_2$

The radiance in the direction of a light ray remains constant as it propagates along the ray.

Sensors response is proportional to radiance (human eye, camera)

Radiometric Quantities: Irradiance

 Irradiance E is the total radiant power per unit area (flux density) *incident* onto a surface with a fixed orientation. To obtain the total flux incident to *dA*, the incoming radiance L_i is integrated over the upper hemisphere Ω₊ above the surface:

$$E = \frac{d\Phi}{dA}$$
$$d \Phi = \left[\int_{\Omega_{+}} L_{i}(x,\theta,\phi) \cos \theta \, d\omega \right] dA$$
$$E = \int_{\Omega_{+}} L_{i}(x,\theta,\phi) \cos \theta \, d\omega = \int_{0}^{2\pi} \int_{0}^{\pi/2} L_{i}(x,\theta,\phi) \cos \theta \sin \theta \, d\theta \, d\phi$$

Radiometric Quantities: Radiosity

 Radiosity B is defined as the total radiant power per unit area (flux density) *leaving* a surface. To obtain the total flux radiated from *dA*, the outgoing radiance *L_o* is integrated over the upper hemisphere Ω₊ above the surface.

$$B = \frac{d\Phi}{dA}$$
$$d \Phi = \left[\int_{\Omega_{+}} L_{o}(x,\theta,\phi) \cos \theta \, d\omega \right] dA$$
$$B = \int_{\Omega_{+}} L_{o}(x,\theta,\phi) \cos \theta \, d\omega = \int_{0}^{2\pi} \int_{0}^{\pi/2} L_{o}(x,\theta,\phi) \cos \theta \sin \theta \, d\theta \, d\phi$$

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Bidirectional Reflectance Distribution Function

- BRDF f_r describes surface reflection at a point x for light incident from direction $\omega_i = (\theta_i, \varphi_i)$ reflected into direction $\omega_o = (\theta_o, \varphi_o)$
- Bidirectional (six dimensional function)
 - Depends on two directions ω_i and ω_o (2D plus 2D = 4D)
 - Also depends on location x (2D)

Distribution function

- Can be infinite but integrates to finite value
- Strictly positive (physics!)

Definition of BRDF:

 Outgoing radiance per incident irradiance

$$- f_r(\omega_i, x, \omega_o) = \frac{L_o(x, \omega_o)}{dE_i}$$

$$- f_r(\omega_i, x, \omega_o) = \frac{L_o(x, \omega_o)}{L_i(x, \omega_i) \cos \theta_i \, d\omega_i}$$



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BRDF Properties

Helmholtz reciprocity principle

- BRDF remains unchanged if incident and reflected directions are interchanged
- Due to physics (linearity)

$$f_r(\omega_i, x, \omega_o) = f_r(\omega_o, x, \omega_i)$$



- Smooth surface: Isotropic BRDF
 - Reflectivity is independent of rotation around surface normal
 - BRDF directional dependence has only 3 instead of 4 degrees of freedom

$$f_r(\omega_i, x, \omega_o) = f_r(x, \theta_i, \theta_o, \varphi_i - \varphi_o)$$



BRDF Properties

Characteristics

- BRDF units [sr ⁻¹]
 - Not very intuitive
- Range of values:
 - From 0 (complete absorption) to
 - ∞ (perfect mirror reflection, δ -function)
 - Because it relates the density *L* to an absolute value
- Energy conservation law
 - Integrating over all outgoing light:
 - No more energy can be reflected than was incoming
 - In other words the directional-hemispherical reflectance must be smaller than 1

-
$$\rho_{dh} = \int_{\Omega_+} f_r(\omega_i, x, \omega_o) \cos\theta \ d\omega_o \le 1$$
, $\forall \omega_i$

- Reflection only at the point of entry $(x_i = x_o)$

• Subsurface scattering (e.g. in skin) is not included in this formulation

Directional Hemispherical Reflectance

- More intuitive measure of reflectance is the directionalhemispherical reflectance:
 - The fraction of the incident radiant flux density incoming from a given direction that is reflected by the surface in all possible directions.
 - Dimensionless number in [0,1]
 - Can change with the angle of incidence

$$\rho_{dh}(\omega_i) = \frac{dB}{dE(\omega_i)} = \frac{\int_{\Omega_+} L_o(x, \omega_o) \cos \theta_o \, d\omega_o}{dE(\omega_i)} = \int_{\Omega_+} f_r(\omega_i, x, \omega_o) \cos \theta_o \, d\omega_o$$
$$\frac{L_o(x, \omega_o)}{dE(\omega_i)} = f_r(\omega_i, x, \omega_o)$$

Lambertian Diffuse Reflection

- Diffuse reflectance is modeled by assuming that light is equally likely to be scattered in any direction, regardless of the incident direction
- Therefore the BRDF and reflected radiance are constant: – $f_r(\omega_i, x, \omega_o) = \rho$ and $L_o = const$
- Also, directional-hemispherical reflectance ρ_d becomes independent of direction. This dimensionless constant, which corresponds to the intuitive meaning of reflectance, is then called the diffuse reflectance ρ_d :

-
$$\rho_d = \int_{\Omega_+} \rho \cos \theta_o d\omega_o = \rho \int_0^{2\pi} \int_0^{\pi/2} \cos \theta_o \sin \theta_o d\theta_o d\varphi_o = \pi \rho$$

 E_i

N $L_o = \text{const}$

• Irradiance *E* and radiosity *B* for the Lambertian surface are related as:

$$-\rho_d = \frac{B}{E} \longrightarrow B = \int_{\Omega_+} L_o(x,\theta,\phi) \cos\theta \, d\omega = L_o \cdot \pi$$

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Reflection Equation

• Putting at all together:

– The light reflected at a point x in direction ω is given as

$$L_r(x,\omega_o) = \int_{\omega_i \in \Omega_+} f_r(\omega_i, x, \omega_o) L_i(x, \omega_i) \cos \theta_i \, d\omega_i$$

Visible surface radiance

- Surface position
- Outgoing direction
- Incoming illumination direction

Reflected light

- Incoming radiance
- Direction-dependent reflectance

$$\begin{array}{c} L_i(x,\omega_i) \\ e f_r(\omega_i,x,\omega_0) \\ \theta_i \\ \theta_i \\ \theta_i \end{array}$$

х

$$L_r(x,\omega_o)$$

 ω_i

X

Reflection Equation: Properties

Reflection operator is linear

- Superposition holds
- Solution could be computed separately for each light source
 - And be accumulated

BRDF is a six-dimensional function

- Difficult to represent and compute accurately
- Measurements are expensive and need much storage
 - But often compresses well

Light Transport in a Scene

Scene

- Lights (emitters)
- Object surfaces (partially absorbing)

Illuminated object surfaces become emitters, too !

- Radiosity = Irradiance minus absorbed photon flux
 - Radiosity: photons per second per m^2 leaving surface
 - Irradiance: photons per second per m^2 incident on surface
- Light bounces between all mutually visible surfaces

Invariance of radiance in free space (vacuum)

- No absorption in-between objects
- Hold also in clean air (approximately!)

• Dynamic Energy Equilibrium

Emitted photons = absorbed photons (+ photons escaping scene)

Global Illumination Problem

Definition: Rendering Equation

Light exiting at some point

Given by emitted light plus reflected incoming light at x

•
$$L_o(x, \omega_o) = L_e(x, \omega_o) + L_r(x, \omega_o)$$

= $L_e(x, \omega_o) + \int_{\omega_i \in \Omega_+} f_r(\omega_i, x, \omega_o) L_i(x, \omega_i) \cos \theta_i d\omega_i$

- Coupling output back to input
 - Light incident at x is the light exiting at some other point y

•
$$L_i(x, \omega_i) = L_o(y, -\omega_i) = L_o(RT(x, w_i), -\omega_i)$$

- With the visibility or ray-tracing operator RT

•
$$y = RT(x, \omega_i)$$



Definition: Rendering Equation

Rendering Equation

Parameterized by direction

$$L_o(x,\omega_o) = L_e(x,\omega_o) + \int_{\omega_i \in \Omega_+} f_r(\omega_i, x, \omega_o) L_i(x,\omega_i) \cos \theta_i \, d\omega_i$$

- Parameterized by position over all surfaces *S* $L_o(x, \omega_o)$ $= L_e + \int_{y \in S} f_r(\omega_i, x, \omega_o) L_o\left(y, \frac{x - y}{\|x - y\|}\right) V(x, y) G(x, y) dA_y$
 - with V(x, y) giving visibility between x und y,
 - and the Geometric Term G given by

$$- d\omega_i = dA_y \frac{\cos \theta_y}{\|x - y\|^2}$$
$$- G(x, y) = \frac{\cos \theta_x \cos \theta_y}{\|x - y\|^2}$$

Rendering Equation

$$L_o(x,\omega_o) = L_e(x,\omega_o) + \int_{y \in S} f_r(\omega_i, x, \omega_o) L_o(y(x,\omega_i), -\omega_i) V(x, y) G(x, y) dA_y$$

Properties

- Mathematical: Fredholm equation of the 2-nd kind
- Global coupling of illumination
 - Each point potentially influences each other point
 - Often still a sparse operator due to occlusion
- Linear transport operator T
 - Solution can be computed separately for each light source
 - And accumulated
 - Dimmed lights result in dimmed solutions
- Volume effects are not considered !!

Lighting Simulation == Solving the Rendering Equation

RE: In Operator Form

- Transport operator T
 - Built from reflection operator S and propagation operator H

•
$$L = L_o = L_e + TL = L_e + (S \circ H)L$$

• Propagation operator *H*

- Computes light incident at points $L_i(x_i, w_i)$ from excitant light at other locations $L(y_i, w_i)$
- Evaluation of ray tracing operator
- Global operator: needs essentially entire scene
- Reflection (scattering) operator S
 - Computes reflected light field $L(y_i, w_i)$ from incident light $L_i(x_i, w_i)$ evaluating reflection equation
 - Evaluates BRDF for entire incident light field
 - Local operator: operates at one point only





Rendering Equation

Solution Approaches

- Monte Carlo technique (and extensions)
 - Point-wise evaluation of multi-dimensional integral equation
 - Efficient solution for the general case
 - Can cause noise through variance of random evaluation
 - No bias and correlation (in approach)
- Finite Element technique
 - Projection of infinite dimensional equation into function space with finite dimensions
 - Solution is represented as combination of basis functions
 - Constant basis functions in the simplest case
 - · Leads to solution of a linear system of equations
 - Efficient for smooth, slowly varying illumination and reflection
 - Causes bias through correlation between solution of neighboring points

Discretization of Rendering Equation

Simplification of the rendering equation

- All surfaces in the scene are Lambertian
- Equation expressed in terms of the radiosity quantities
- Integration domain split into N pieces corresponding to discrete patches in the scene
- Constant radiosity and reflectance assumptions for each patch

• We are going to discuss all these steps in detail

Lambertian Diffuse Reflection

 Diffuse reflectance is modeled by assuming that light is equally likely to be scattered in any direction, regardless of the incident direction:

$$\rho_{bd}(x,\theta_o,\varphi_o,\theta,\varphi) = \rho(x)$$

• Directional-hemispherical reflectance ρ_d becomes independent of direction:

$$\rho_d(x) = \int_{\Omega} \rho(x) \cos \theta_o d\omega_o = \rho(x) \int_{0}^{2\pi} \int_{0}^{\pi/2} \cos \theta_o \sin \theta_o d\theta_o d\varphi_o = \pi \rho(x)$$
$$\rho(x) = \frac{\rho_d(x)}{\pi}$$

• Then the rendering equation simplifies to:

$$L_o(x,\theta_o,\varphi_o) = L_e(x,\theta_o,\varphi_o) + \frac{\rho_d(x)}{\pi} \int_{\Omega} L_i(x,\theta,\varphi) \cos\theta d\omega$$

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Further Simplifications

- For diffuse surfaces
 - the radiance $L_o(x, \theta_o, \varphi_o) \equiv L_o(x)$ does not depend on the outgoing direction,
 - the incoming radiance L_i still depends on the incoming direction

$$L_o(x) = L_e(x) + \frac{\rho_d(x)}{\pi} \int_{\Omega} L_i(x,\theta,\varphi) \cos\theta d\omega$$

• Now let us replace radiances by radiosities:

$$B(x) = \int_{\Omega} L(x) \cos \theta \, d\omega = \int_{0}^{2\pi} \int_{0}^{\pi/2} L(x) \cos \theta \sin \theta \, d\theta \, d\phi = \pi L(x)$$
$$\pi L_{o}(x) = \pi L_{e}(x) + \pi \frac{\rho_{d}(x)}{\pi} \int_{\Omega} L_{i}(x,\theta,\phi) \cos \theta d\omega \longleftarrow$$
$$B(x) = E(x) + \rho_{d}(x) \int_{\Omega} L_{i}(x,\theta,\phi) \cos \theta d\omega$$

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Transforming the Hemispherical Integral into a Surface Integral

• The invariance of radiance along a line of sight states that:

$$L_{i}(x,\theta,\varphi) = L(y,\theta',\varphi')$$

$$L(y, \theta', \varphi') = \frac{B(y)}{\pi}$$

• Now let us replace integration over the hemisphere by integration over all surfaces *y* taking into account their visibility from *x*:

$$d\omega = \frac{\cos\theta' dy}{r^2}$$

$$V(x, y) = \begin{cases} 1 & \text{if } x \text{ and } y \text{ are mutually visible} \\ 0 & \text{if } x \text{ and } y \text{ are mutually visible} \end{cases}$$

$$B(x) = E(x) + \rho_d(x) \int_{y \in S} B(y) \frac{\cos \theta \cos \theta}{\pi r^2} V(x, y) dy$$



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Discrete Formulation

- The integral over all surfaces in the scene in the previous slide is broken into *N* pieces, each corresponding to a discrete patch.
- It is assumed that each patch has a uniform radiosity at each point y in patch P_i.

$$B(x) = E(x) + \rho_d(x) \sum_{j=1}^N B_j \int_{y \in P_j} \frac{\cos \theta \cos \theta}{\pi r^2} V(x, y) dy$$



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Radiosity Equation and Form Factors

 The constant radiosity value for each patch is computed as an area-weighted average of radiosity:

$$B_i = \frac{1}{A_i} \int_{x \in P_i} B(x) dx \qquad E_i = \frac{1}{A_i} \int_{x \in P_i} E(x) dx$$

• Then assuming also that reflectance is constant across each patch $\rho_d(x) = \rho_i$, the radiosity equation can be formulated as:

$$B_{i} = E_{i} + \rho_{i} \sum_{j=1}^{N} B_{j} \frac{1}{A_{i}} \int_{x \in P_{i}} \int_{y \in P_{j}} \frac{\cos \theta \cos \theta'}{\pi r^{2}} V(x, y) dx dy$$

$$B_i = E_i + \rho_i \sum_{j=1}^N B_j F_{ij}$$

• where F_{ij} is the form factor:

$$F_{ij} = \frac{1}{A_i} \int_{x \in P_i} \int_{y \in P_j} \frac{\cos \theta \cos \theta'}{\pi r^2} V(x, y) dx dy$$

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