Realistic Image Synthesis

- Rendering Equation -

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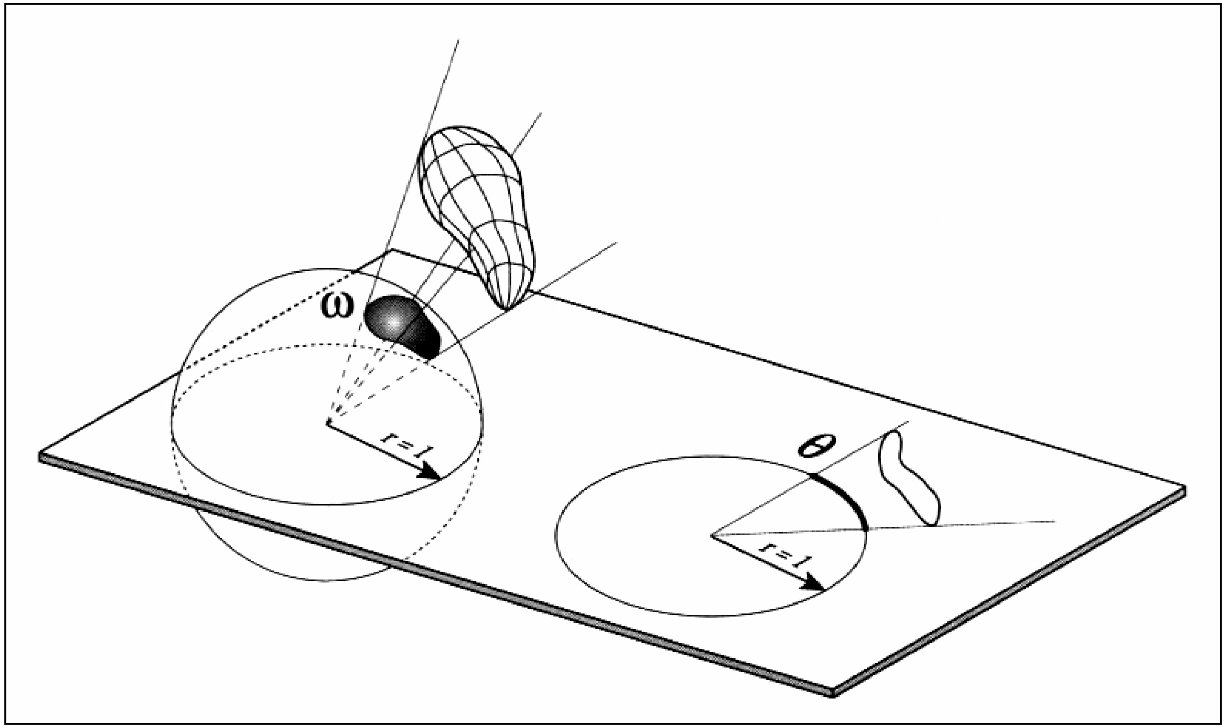
Angle and Solid Angle

θ **the angle subtended by a curve in the plane is the length of the projected arc on the unit circle.**

Ω, ω **the solid angle subtended by an object is the surface area of its projection onto the unit sphere**

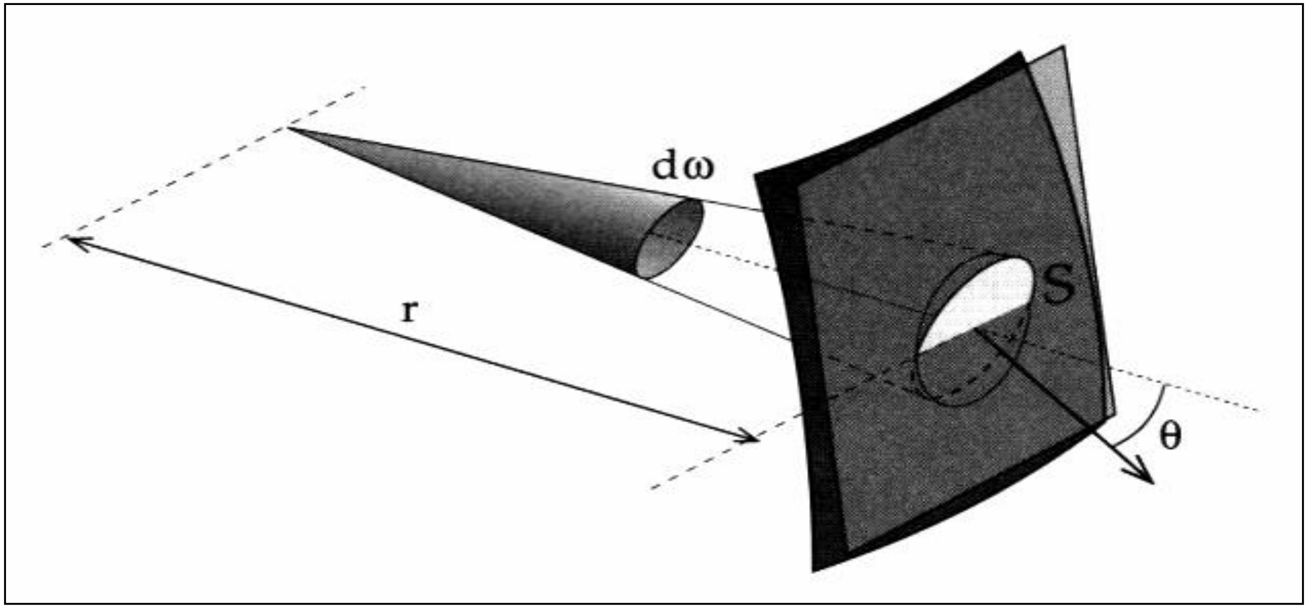
Solid angle units: steradians [sr]



Solid Angle for a Small Area

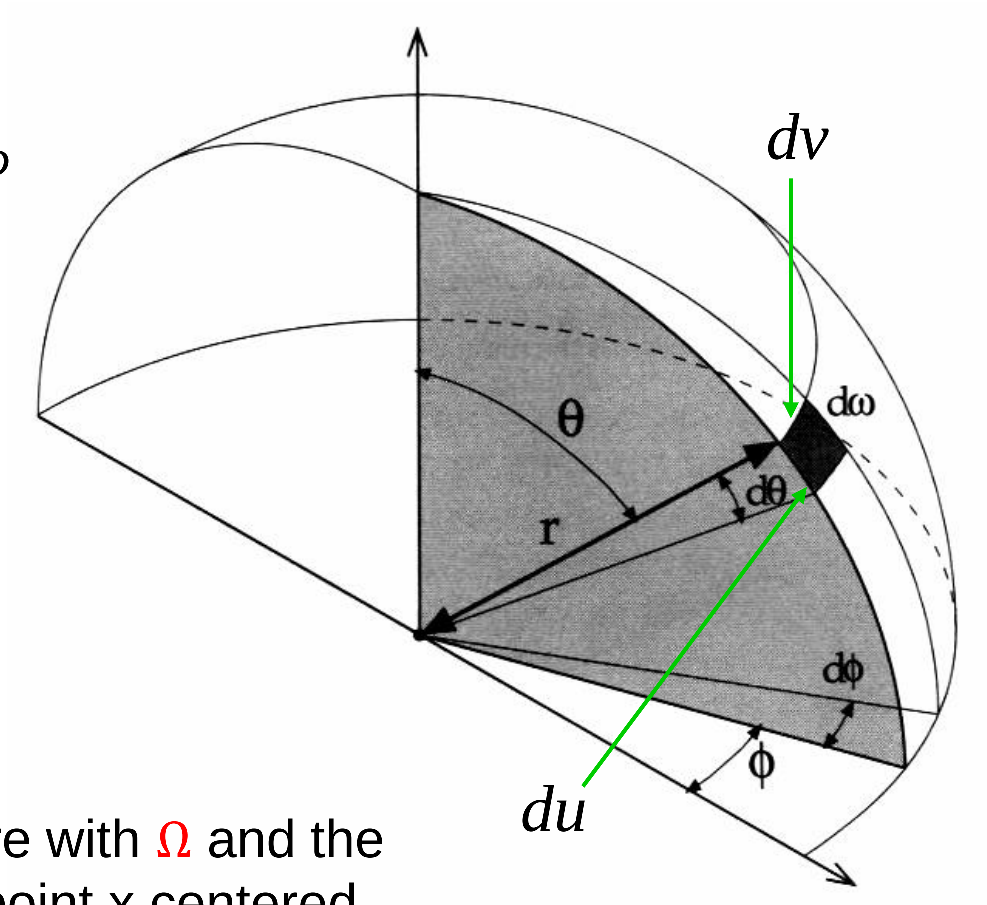
The solid angle subtended by an (infinitely) small surface patch *S* with area d*A* is obtained by dividing the projected area $\mathrm{d}A\ \cos\theta$ by the square of the distance to the origin:

$$\mathrm{d}\omega, d\Omega = \frac{dA\ \cos\theta}{r^2}$$



Solid Angle in Spherical Coordinates

- **Infinitesimally small solid angle**
  - $du = r\,d\theta$
  - $dv = r\sin\theta\,d\phi$
  - $dA = du\,dv = r^2\sin\theta\,d\theta\,d\phi$
  - $\Rightarrow d\omega = \frac{dA}{r^2} = \sin\theta\,d\theta\,d\phi$



- **Finite solid angle of an surface S**
  - $\omega = \int_S \sin\theta\,d\theta\,d\phi$
- **Definition:**
  - We denote the entire Sphere with Ω and the (positive) hemisphere at a point x centered around its normal vector with Ω_+

Radiometry

- Radiometry is the science of measuring radiant energy transfers. Radiometric quantities have physical meaning and can be directly measured using proper equipment such as spectral photometers.
- **Radiometric Quantities**
  - Energy [watt second] $n \cdot h\nu$
  - Radiant power (total flux) [watt] Φ, P
  - **Radiance** [watt/(m^2 sr)] L
  - Irradiance (flux density) [watt/m^2] E
  - Radiosity (flux density) [watt/m^2] B

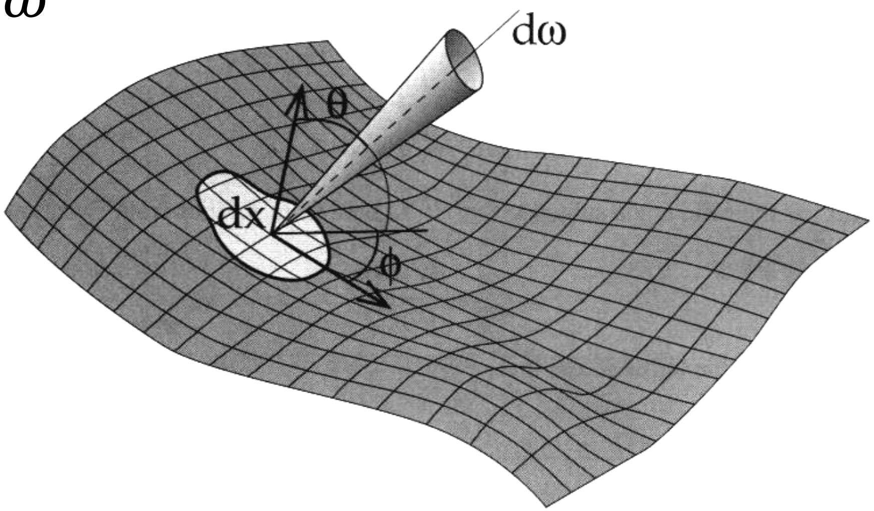
Radiometric Quantities: Radiance

- Radiance is used to describe radiant energy transfer.
- Radiance L is defined as the total flux (radiant power) traveling at some point x in a specified direction ω, per unit area perpendicular to the direction of travel, per unit solid angle.
- Thus, the differential flux $d^2\Phi$ radiated through the differential solid angle $d\omega$, from the projected differential area $dA\ \cos\theta$ is:

$$d^2\Phi = L(x,\omega)dA\ \cos\theta\ \ d\omega$$

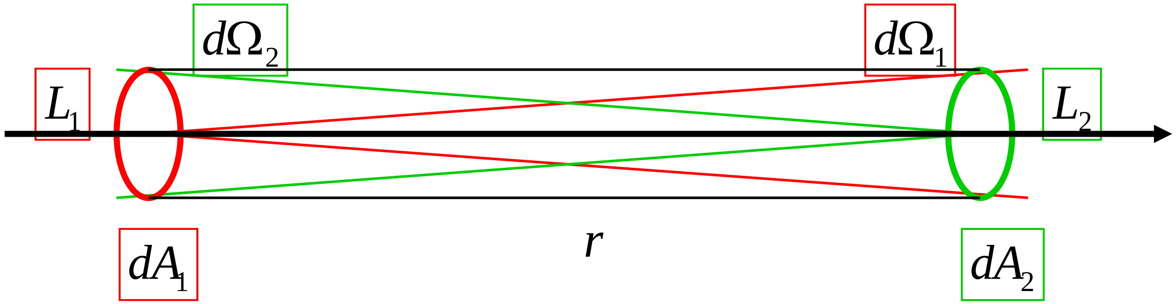
or

$$L(x,\omega) = \frac{d^2\Phi}{dA\ \cos\theta\ d\omega}$$



- **From here on we distinguish between the direction $\boldsymbol{\omega}$ and the (differential) solid angle $\boldsymbol{d\omega}$!!!**

Radiance in Space



Flux leaving surface 1 must be equal to flux arriving on surface 2

$$L_1 \cdot d\Omega_1 \cdot dA_1 = L_2 \cdot d\Omega_2 \cdot dA_2$$

From geometry follows $$d\Omega_1 = \frac{dA_2}{r^2} \qquad d\Omega_2 = \frac{dA_1}{r^2}$$

Def: Ray ***Throughput*** $$T = d\Omega_1 \cdot dA_1 = d\Omega_2 \cdot dA_2 = \frac{dA_1 \cdot dA_2}{r^2} \implies L_1 = L_2$$

The radiance in the direction of a light ray remains constant as it propagates along the ray.

Sensors response is proportional to radiance (human eye, camera)

Radiometric Quantities: Irradiance

- **Irradiance E is the total radiant power per unit area (flux density)** ***incident*** **onto a surface with a fixed orientation. To obtain the total flux incident to dA, the incoming radiance L_i is integrated over the upper hemisphere Ω_+ above the surface:**

$$E \equiv \frac{d\Phi}{dA}$$

$$d\,\Phi = \left[\int_{\Omega_+} L_i(x,\theta,\phi)\cos\theta\, d\omega\right] dA$$

$$E = \int_{\Omega_+} L_i(x,\theta,\phi)\cos\theta\, d\omega = \int_0^{2\pi}\int_0^{\pi/2} L_i(x,\theta,\phi)\cos\theta\sin\theta\, d\theta\, d\phi$$

Radiometric Quantities: Radiosity

- **Radiosity B is defined as the total radiant power per unit area (flux density) *leaving* a surface. To obtain the total flux radiated from dA, the outgoing radiance L_o is integrated over the upper hemisphere Ω_+ above the surface.**

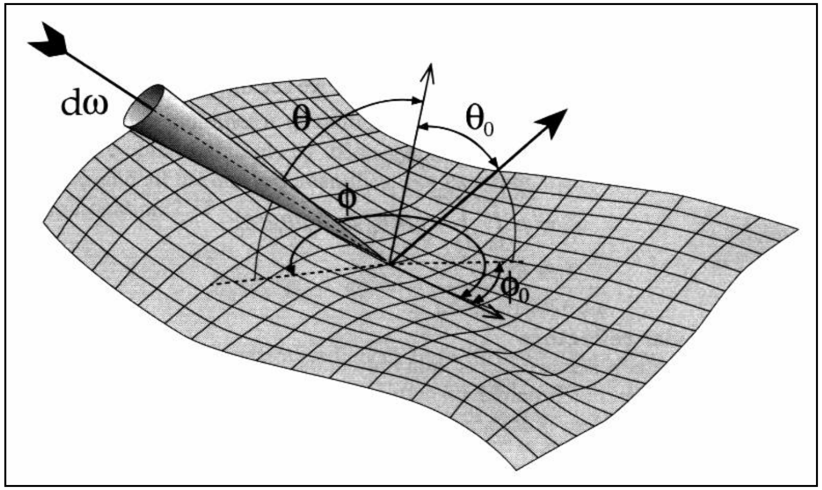
$$B \equiv \frac{d\Phi}{dA}$$

$$d\Phi = \left[\int_{\Omega_+} L_o(x,\theta,\phi)\cos\theta\, d\omega \right] dA$$

$$B = \int_{\Omega_+} L_o(x,\theta,\phi)\cos\theta\, d\omega = \int_0^{2\pi} \int_0^{\pi/2} L_o(x,\theta,\phi)\cos\theta \sin\theta\, d\theta\, d\phi$$

Bidirectional Reflectance Distribution Function

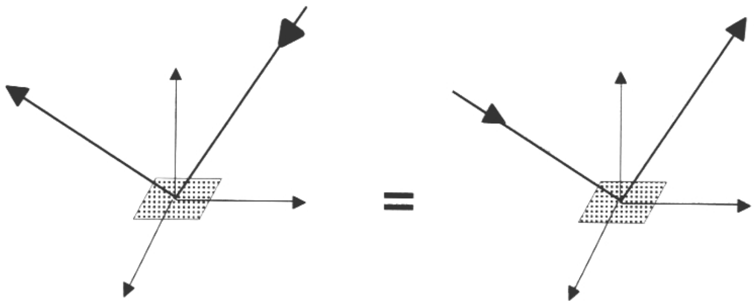
- **BRDF f_r describes surface reflection at a point x for light incident from direction $\omega_i = (\theta_i, \varphi_i)$ reflected into direction $\omega_o = (\theta_o, \varphi_o)$**
- **Bidirectional (six dimensional function)**
  - Depends on two directions ω_i and ω_o (2D plus 2D = 4D)
  - Also depends on location x (2D)
- **Distribution function**
  - Can be infinite but integrates to finite value
  - Strictly positive (physics!)
- **Definition of BRDF:**
  - Outgoing radiance per incident irradiance
  - $f_r(\omega_i, x, \omega_o) = \frac{L_o(x,\omega_o)}{dE_i}$
  - $f_r(\omega_i, x, \omega_o) = \frac{L_o(x,\omega_o)}{L_i(x,\omega_i)\cos\theta_i\, d\omega_i}$



BRDF Properties

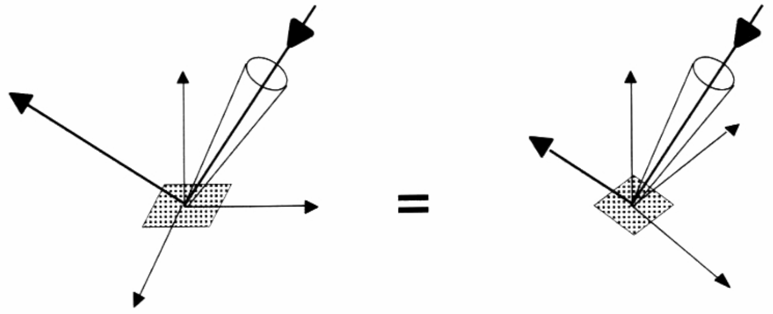
- **Helmholtz reciprocity principle**
  - BRDF remains unchanged if incident and reflected directions are interchanged
  - Due to physics (linearity)

$$f_r(\omega_i, x, \omega_o) = f_r(\omega_o, x, \omega_i)$$



- **Smooth surface: Isotropic BRDF**
  - Reflectivity is independent of rotation around surface normal
  - BRDF directional dependence has only 3 instead of 4 degrees of freedom

$$f_r(\omega_i, x, \omega_o) = f_r(x, \theta_i, \theta_o, \varphi_i - \varphi_o)$$



BRDF Properties

- **Characteristics**
  - BRDF units [sr^{-1}]
    - Not very intuitive
  - Range of values:
    - From 0 (complete absorption) to
    - ∞ (perfect mirror reflection, δ-function)
      - Because it relates the density L to an absolute value
  - Energy conservation law
    - Integrating over all outgoing light:
      - No more energy can be reflected than was incoming
    - In other words the directional-hemispherical reflectance must be smaller than 1
      - $\rho_{dh} = \int_{\Omega_+} f_r(\omega_i, x, \omega_o) cos\theta \, d\omega_o \leq 1, \;\; \forall\omega_i$
  - Reflection only at the point of entry ($x_i = x_o$)
    - Subsurface scattering (e.g. in skin) is not included in this formulation

Directional Hemispherical Reflectance

- More intuitive measure of reflectance is the **directional-hemispherical reflectance**:
  - The fraction of the incident radiant flux density incoming from a given direction that is reflected by the surface in all possible directions.
  - Dimensionless number in [0,1]
  - Can change with the angle of incidence

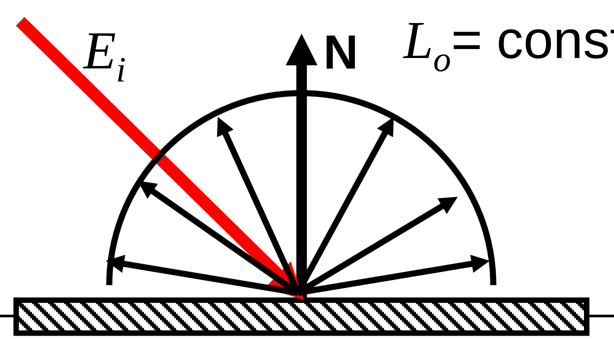
$$\rho_{dh}(\omega_i) = \frac{dB}{dE(\omega_i)} = \frac{\int_{\Omega_+} L_o(x, \omega_o) \cos\theta_o \, d\omega_o}{dE(\omega_i)} = \int_{\Omega_+} f_r(\omega_i, x, \omega_o) \cos\theta_o \, d\omega_o$$

$$\frac{L_o(x, \omega_o)}{dE(\omega_i)} = f_r(\omega_i, x, \omega_o)$$

Lambertian Diffuse Reflection

- Diffuse reflectance is modeled by assuming that light is equally likely to be scattered in any direction, regardless of the incident direction
- Therefore the BRDF and reflected radiance are constant:
  - $f_r(\omega_i, x, \omega_o) = \rho$ and $\mathrm{L_o} = \mathrm{const}$
- Also, directional-hemispherical reflectance ρ_d becomes independent of direction. This dimensionless constant, which corresponds to the intuitive meaning of reflectance, is then called the diffuse reflectance ρ_d:
  - $\rho_d = \int_{\Omega_+} \rho \cos\theta_o d\omega_o = \rho \int_0^{2\pi} \int_0^{\pi/2} \cos\theta_o \sin\theta_o \, d\theta_o d\varphi_o = \pi\rho$
- Irradiance E and radiosity B for the Lambertian surface are related as:
  - $\rho_d = \frac{B}{E}$

$$B = \int_{\Omega_+} L_o(x, \theta, \phi) \cos\theta \, d\omega = L_o \cdot \pi$$

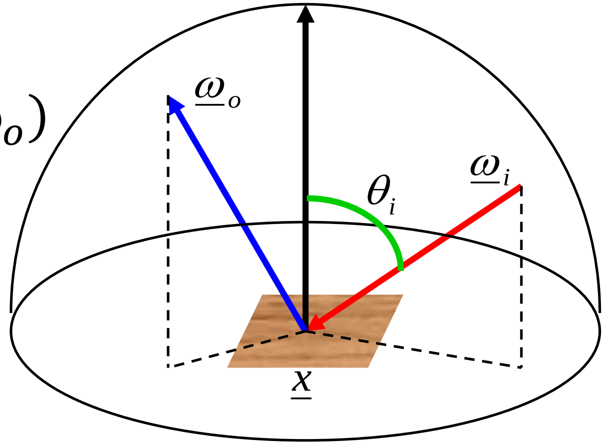


Reflection Equation

- **Putting at all together:**
  - The light reflected at a point x in direction ω is given as

$$L_r(x,\omega_o) = \int_{\omega_i \in \Omega_+} f_r(\omega_i, x, \omega_o)\, L_i(x,\omega_i) \cos\theta_i \, d\omega_i$$

- **Visible surface radiance** $L_r(x,\omega_o)$
  - Surface position x
  - Outgoing direction ω_o
  - Incoming illumination direction ω_i
- **Reflected light**
  - Incoming radiance $L_i(x,\omega_i)$
  - Direction-dependent reflectance $f_r(\omega_i, x, \omega_o)$



Reflection Equation: Properties

- **Reflection operator is linear**
  - Superposition holds
  - Solution could be computed separately for each light source
    - And be accumulated
- **BRDF is a six-dimensional function**
  - Difficult to represent and compute accurately
  - Measurements are expensive and need much storage
    - But often compresses well

Light Transport in a Scene

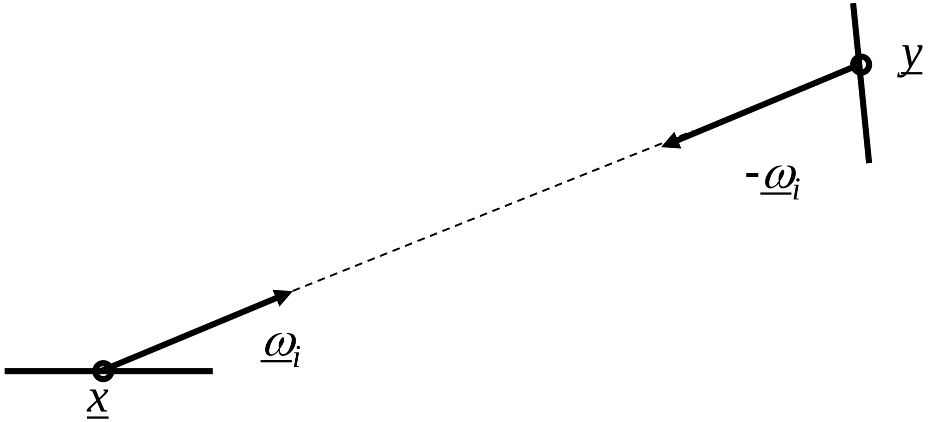
- **Scene**
  - Lights (emitters)
  - Object surfaces (partially absorbing)
- **Illuminated object surfaces become emitters, too !**
  - Radiosity = Irradiance minus absorbed photon flux
    - Radiosity: photons per second per m^2 leaving surface
    - Irradiance: photons per second per m^2 incident on surface
- **Light bounces between all mutually visible surfaces**
- **Invariance of radiance in free space (vacuum)**
  - No absorption in-between objects
  - Hold also in clean air (approximately!)
- **Dynamic Energy Equilibrium**
  - Emitted photons = absorbed photons (+ photons escaping scene)

Global Illumination Problem

Definition: Rendering Equation

- **Light exiting at some point**
  - Given by emitted light plus reflected incoming light at x
    - $L_o(x, \omega_o) = L_e(x, \omega_o) + L_r(x, \omega_o)$

      $= L_e(x, \omega_o) + \int_{\omega_i \in \Omega_+} f_r(\omega_i, x, \omega_o)\, L_i(x, \omega_i) \cos\theta_i \, d\omega_i$
- **Coupling output back to input**
  - Light incident at x is the light exiting at some other point y
    - $L_i(x, \omega_i) = L_o(y, -\omega_i) = L_o(RT(x, w_i), -\omega_i)$
  - With the visibility or ray-tracing operator RT
    - $y = RT(x, \omega_i)$



Definition: Rendering Equation

- **Rendering Equation**
  - Parameterized by direction

$$L_o(x,\omega_o) = L_e(x,\omega_o) + \int_{\omega_i \in \Omega_+} f_r(\omega_i, x, \omega_o)\, L_i(x,\omega_i) \cos\theta_i \, d\omega_i$$

  - Parameterized by position over all surfaces S

$$L_o(x,\omega_o) = L_e + \int_{y \in S} f_r(\omega_i, x, \omega_o)\, L_o\left(y, \frac{x-y}{\|x-y\|}\right) V(x,y) G(x,y) dA_y$$

    - with $V(x,y)$ giving visibility between x und y,
    - and the Geometric Term G given by
      - $d\omega_i = dA_y \frac{\cos\theta_y}{\|x-y\|^2}$
      - $\mathrm{G}(x,y) = \frac{\cos\theta_x \cos\theta_y}{\|x-y\|^2}$

Rendering Equation

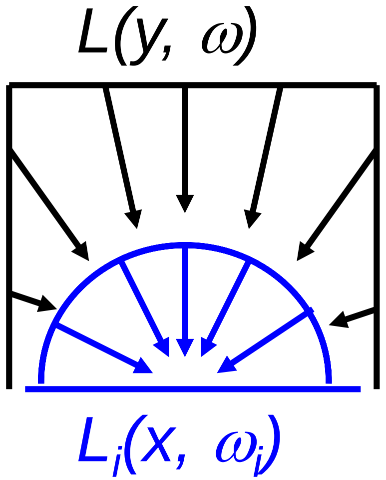
$$L_o(x,\omega_o) = L_e(x,\omega_o) + \int_{y \in S} f_r(\omega_i, x, \omega_o)\, L_o(y(x,\omega_i), -\omega_i) V(x,y) G(x,y)\, dA_y$$

- **Properties**
  - Mathematical: Fredholm equation of the 2-nd kind
  - Global coupling of illumination
    - Each point potentially influences each other point
    - Often still a sparse operator due to occlusion
  - Linear transport operator **T**
    - Solution can be computed separately for each light source
      - And accumulated
      - Dimmed lights result in dimmed solutions
  - Volume effects are not considered !!

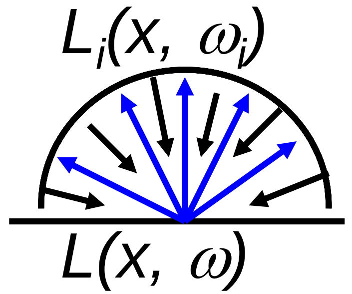
► Lighting Simulation == Solving the Rendering Equation

RE: In Operator Form

- **Transport operator *T***
  - Built from reflection operator ***S*** and propagation operator ***H***
    - $L = L_o = L_e + \boldsymbol{T}L = L_e + (\boldsymbol{S} \circ \boldsymbol{H})L$



- **Propagation operator *H***
  - Computes light incident at points $L_i(x_i, w_i)$ from excitant light at other locations $L(y_i, w_i)$
  - Evaluation of ray tracing operator
  - Global operator: needs essentially entire scene



- **Reflection (scattering) operator *S***
  - Computes reflected light field $L(y_i, w_i)$ from incident light $L_i(x_i, w_i)$ evaluating reflection equation
  - Evaluates BRDF for entire incident light field
  - Local operator: operates at one point only

Rendering Equation

- **Solution Approaches**
  - Monte Carlo technique (and extensions)
    - Point-wise evaluation of multi-dimensional integral equation
    - Efficient solution for the general case
    - Can cause noise through variance of random evaluation
    - No bias and correlation (in approach)
  - Finite Element technique
    - Projection of infinite dimensional equation into function space with finite dimensions
      - Solution is represented as combination of basis functions
      - Constant basis functions in the simplest case
    - Leads to solution of a linear system of equations
    - Efficient for smooth, slowly varying illumination and reflection
    - Causes bias through correlation between solution of neighboring points

Discretization of Rendering Equation

- **Simplification of the rendering equation**

$$L_o(x,\theta_o,\varphi_o) = L_e(x,\theta_o,\varphi_o) + \int_{\Omega} \rho_{bd}(x,\theta_o,\varphi_o,\theta,\phi) L_i(x,\theta,\varphi)\cos\theta d\omega$$

$$\Downarrow$$

$$B_i = E_i + \rho_i \sum_{j=1}^{N} B_j F_{ij}$$

  - All surfaces in the scene are Lambertian
  - Equation expressed in terms of the radiosity quantities
  - Integration domain split into N pieces corresponding to discrete patches in the scene
  - Constant radiosity and reflectance assumptions for each patch
- **We are going to discuss all these steps in detail**

Lambertian Diffuse Reflection

- **Diffuse reflectance is modeled by assuming that light is equally likely to be scattered in any direction, regardless of the incident direction:**

$$\rho_{bd}(x,\theta_o,\varphi_o,\theta,\varphi)=\rho(x)$$

- **Directional-hemispherical reflectance** ρ_d **becomes independent of direction:**

$$\rho_d(x)=\int_\Omega \rho(x)\cos\theta_o d\omega_o=\rho(x)\int_0^{2\pi}\int_0^{\pi/2}\cos\theta_o\sin\theta_o d\theta_o d\varphi_o=\pi\rho(x)$$

$$\rho(x)=\frac{\rho_d(x)}{\pi}$$

- **Then the rendering equation simplifies to:**

$$L_o(x,\theta_o,\varphi_o)=L_e(x,\theta_o,\varphi_o)+\frac{\rho_d(x)}{\pi}\int_\Omega L_i(x,\theta,\varphi)\cos\theta d\omega$$

Further Simplifications

- **For diffuse surfaces**
  - **the radiance** $L_o(x,\theta_o,\varphi_o) \equiv L_o(x)$ **does not depend on the outgoing direction,**
  - **the incoming radiance** L_i **still depends on the incoming direction**

$$L_o(x) = L_e(x) + \frac{\rho_d(x)}{\pi}\int_\Omega L_i(x,\theta,\varphi)\cos\theta d\omega$$

- **Now let us replace radiances by radiosities:**

$$B(x) = \int_\Omega L(x)\cos\theta\, d\omega = \int_0^{2\pi}\int_0^{\pi/2} L(x)\cos\theta\sin\theta\, d\theta\, d\phi = \pi L(x)$$

$$\pi L_o(x) = \pi L_e(x) + \pi\frac{\rho_d(x)}{\pi}\int_\Omega L_i(x,\theta,\varphi)\cos\theta d\omega$$

$$B(x) = E(x) + \rho_d(x)\int_\Omega L_i(x,\theta,\varphi)\cos\theta d\omega$$

Transforming the Hemispherical Integral into a Surface Integral

- **The invariance of radiance along a line of sight states that:**

$$L_i(x,\theta,\varphi) = L\ (y,\theta',\varphi')$$

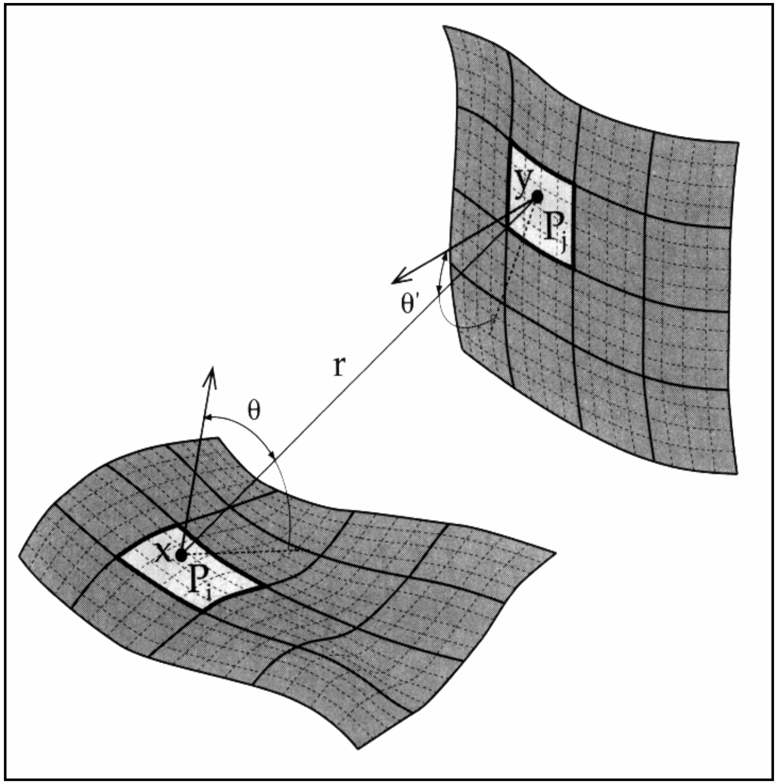
$$L(y,\theta',\varphi') = \frac{B(y)}{\pi}$$

- **Now let us replace integration over the hemisphere by integration over all surfaces** y **taking into account their visibility from** x**:**

$$d\omega = \frac{\cos\theta' dy}{r^2}$$

$$V(x,y) = \begin{cases} 1 & if\ \ x\ and\ \ y\ are\ mutually\ visible \\ 0 & otherwise \end{cases}$$

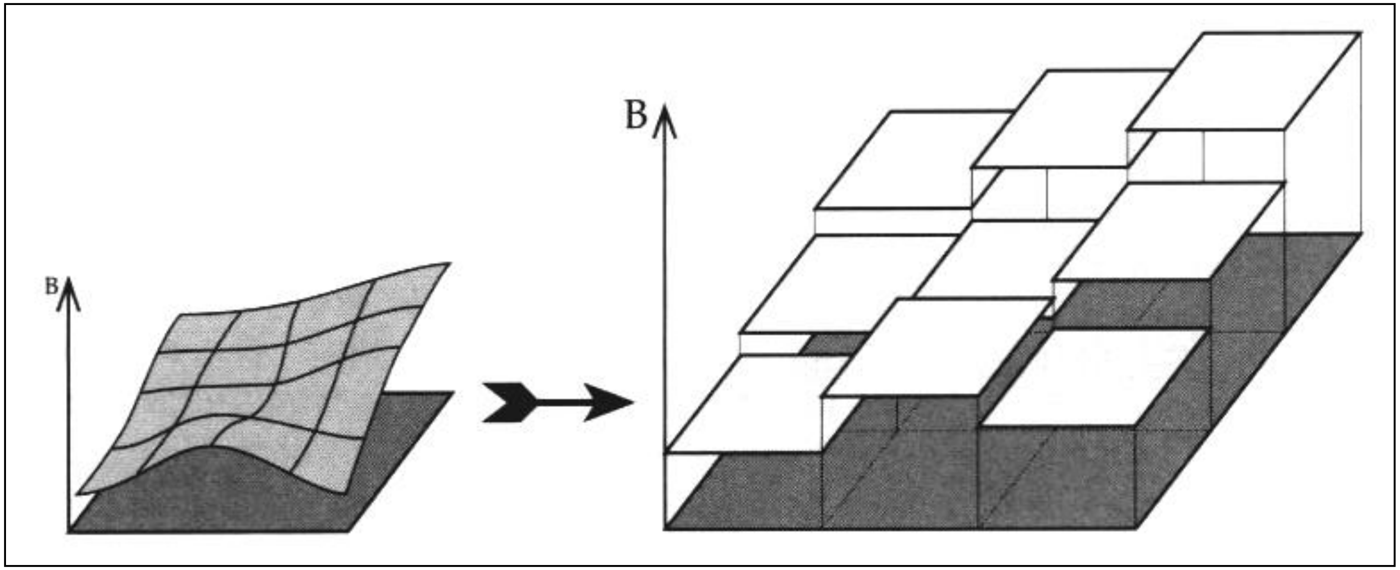
$$B(x) = E(x) + \rho_d(x) \int_{y \varepsilon S} B(y) \frac{\cos\theta \cos\theta'}{\pi r^2} V(x,y) dy$$



Discrete Formulation

- **The integral over all surfaces in the scene in the previous slide is broken into N pieces, each corresponding to a discrete patch.**
- **It is assumed that each patch has a uniform radiosity at each point y in patch P_j.**

$$B(x) = E(x) + \rho_d(x) \sum_{j=1}^{N} B_j \int_{y \varepsilon P_j} \frac{\cos\theta \cos\theta'}{\pi r^2} V(x,y)\,dy$$



Radiosity Equation and Form Factors

- **The constant radiosity value for each patch is computed as an area-weighted average of radiosity:**

$$B_i = \frac{1}{A_i} \int_{x \varepsilon P_i} B(x) dx \qquad E_i = \frac{1}{A_i} \int_{x \varepsilon P_i} E(x) dx$$

- **Then assuming also that reflectance is constant across each patch $\rho_d(x) = \rho_i$, the radiosity equation can be formulated as:**

$$B_i = E_i + \rho_i \sum_{j=1}^{N} B_j \frac{1}{A_i} \int_{x \varepsilon P_i} \int_{y \varepsilon P_j} \frac{\cos\theta \cos\theta'}{\pi r^2} V(x,y) dx dy$$

$$B_i = E_i + \rho_i \sum_{j=1}^{N} B_j F_{ij}$$

- **where F_{ij} is the form factor:**

$$F_{ij} = \frac{1}{A_i} \int_{x \varepsilon P_i} \int_{y \varepsilon P_j} \frac{\cos\theta \cos\theta'}{\pi r^2} V(x,y) dx dy$$