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Realistic Image Synthesis (SS 2018)
Assignment 2

Submission deadline for the exercises: **May 22, 2018**

2.1 Area of a disc (20 points)

A common estimator for the integral $I = \int_0^1 f(x)\mathrm{d}x$ is the following:

$$E = \frac{1}{n}\sum_{i=1}^{n} f(x_i) \text{ with } x_1, ..., x_n \sim U(0,1)$$

The notation $U(0,1)$ corresponds to the uniform distribution between 0 and 1.

- Write down the formula if the x_i follow an arbitrary probability density function (*pdf*) p instead of the uniform distribution.
- What can be the benefit of using a different pdf as you did above?
- Using a Monte Carlo integration method, devise an algorithm to compute the number π.
  Hint: Since the area of the unit disk is π, the idea is to compute numerically the integral $\pi = \int_D 1\,\mathrm{d}x\mathrm{d}y$ where the domain D is the unit disk.

2.2 Discrete Probability: Dice (15 points)

You are given a fair six-sided dice. How many times on average are you going to throw it util you get a 6? In general: given a discrete probability space $(\Omega, F, \mathbb{P})$ and an event $e \in F$, that occurs with some probability $\mathbb{P}(e) = p_e$ define the random variable that gives the number of trials until e is "drawn" and compute its expected value.

2.3 Variance (20 points)

Given is the integral:

$$G = \int_0^1 \cos\left(\frac{\pi x}{2}\right) dx.$$

Straightforward Monte Carlo method sampling uniformly on $(0,1)$ will form the estimator:

$$g = \cos\left(\frac{\pi x}{2}\right).$$

Show that the analytical evaluation of the variance results in $var(g) \approx 0.0947$.

2.4 Cosine-weighted Random Hemisphere Sampling (30 points)

It is often necessary to generate directions (ω) or points (ϕ_i, θ_i) on the unit hemisphere, so that the PDF of the distribution would be:

$$p(\omega) = \frac{\cos\omega}{\pi} \tag{1}$$

$$p(\phi, \theta) = \frac{\cos\theta\sin\theta}{\pi}. \tag{2}$$

This distribution is very useful and is used for example when generating direction of photons, reflecting from diffuse (Lambertian) surfaces. Show that the following transformation is correct, given points (x_i, y_i) uniformly distributed on the unit square $[0, 1] \times [0, 1]$:

$$\phi_i = 2\pi x_i \tag{3}$$

$$\theta_i = \arccos\sqrt{y_i}. \tag{4}$$

Hint: One could use a method of inversion to actually arrive at Formulae (3, 4) from Distribution (2). Another way is to show that lifting the points uniformly generated on the unit circle so that they reach the surface of hemisphere also produces a needed distribution. The third way is to take Formulae (3, 4) for granted and to show that the number of samples it produces in the hemispherical segment depends on the solid angle subtended by this segment in respect to the hemisphere center.

Hint: The inversion technique requires to invert the commulative distribution function $F(t) = \int_0^t f(x)\,\mathrm{d}x$ of the probability distribution function f. The variable defined by $Y = F^{-1}(U)$, where U is uniformly distributed over $[0, 1]$, then follows the distribution f.

Hint: To sample from a 2D distribution $f(x, y)$, you can first obtain a value for x by sampling from the marginal $f(x) = \int f(x, y)\,\mathrm{d}y$. Then, to obtain the second coordinate y, you can use the rule of multiplication: $f(y|x) = \dfrac{f(x, y)}{f(x)}$.

2.5 Uniform Sampling of a Hemisphere (20 points)

Following the same idea as in the previous exercise, derive a method to sample a hemisphere uniformly w.r.t solid angle, given two random numbers $(x_i, y_i) \in [0, 1]^2$ generated uniformly on the unit square.
Hint: the target distribution has the form $p(w) = \frac{1}{2\pi}$, or in spherical coordinates $p(\theta, \phi) = \frac{sin(\theta)}{2\pi}$.

Procedure of Submitting

Write your solutions and submit them on May 22, 2018, before the lecture. You can also e-mail the solution as a pdf to your tutor.