Computer Graphics

Sampling Theory & Anti-Aliasing

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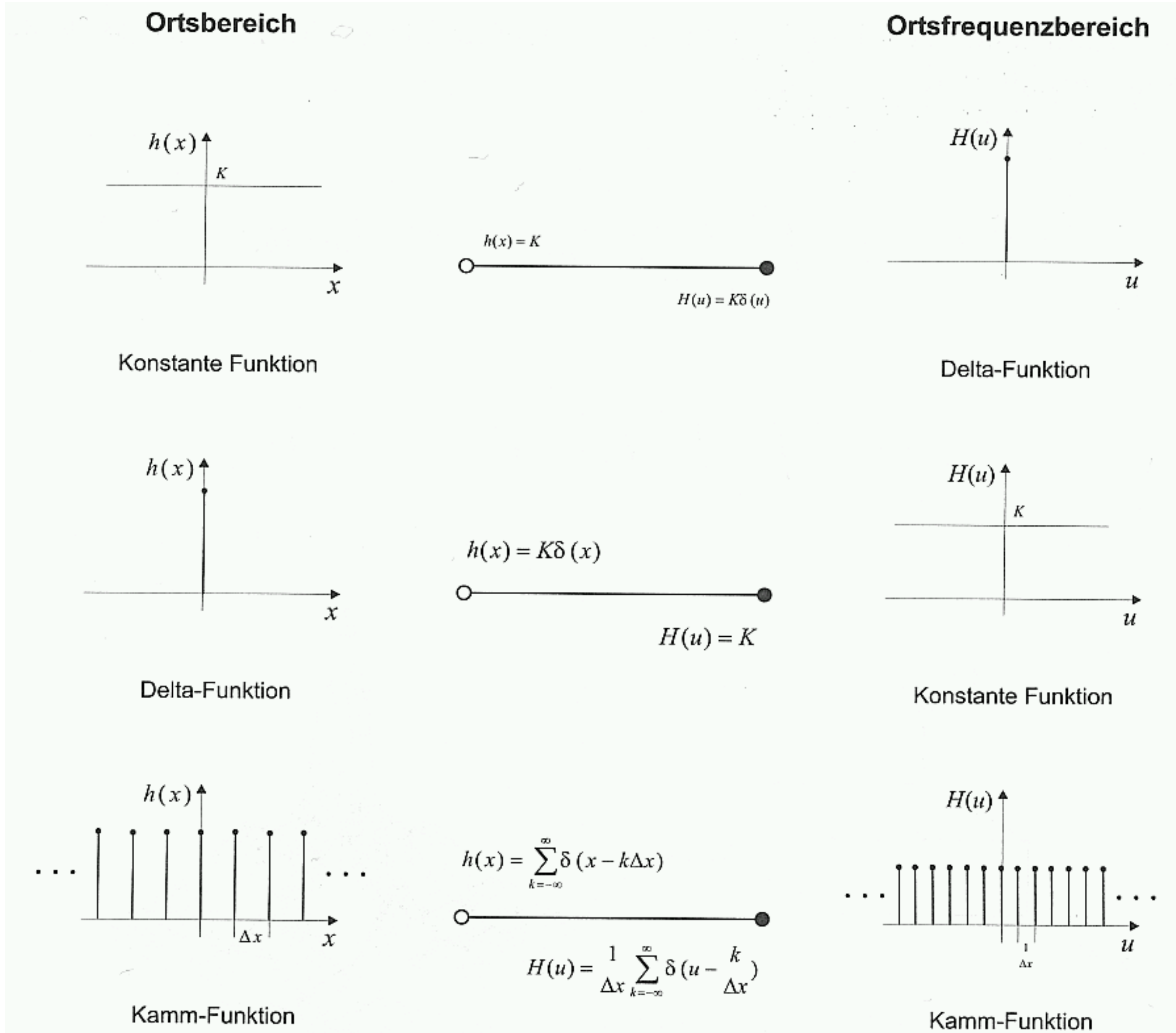
Dirac Comb Function (1)

- **Constant & δ-function**
- **Comb/Shah function**

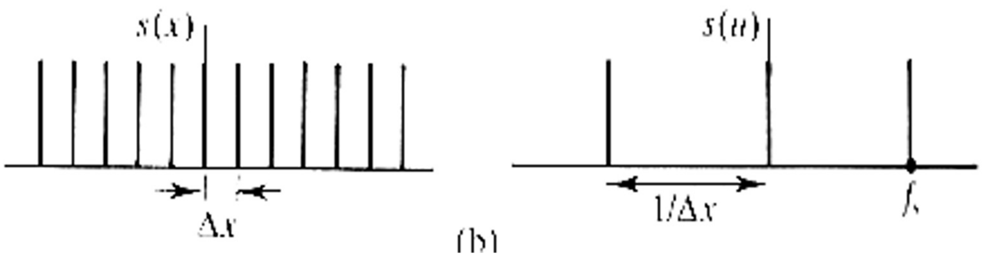


Dirac Comb (2)

- **Constant & δ-Function**
  - Duality
    $f(x) = K$
    $F(\omega) = K\delta(\omega)$
  - And vice versa
- **Comb function**
  - Duality: the dual of a comb function is again a comb function
    - Inverse wavelength/distances
    - Amplitude scales with inverse wavelength

$$f(x) = \sum_{k=-\infty}^{\infty} \delta(x - k\Delta x)$$

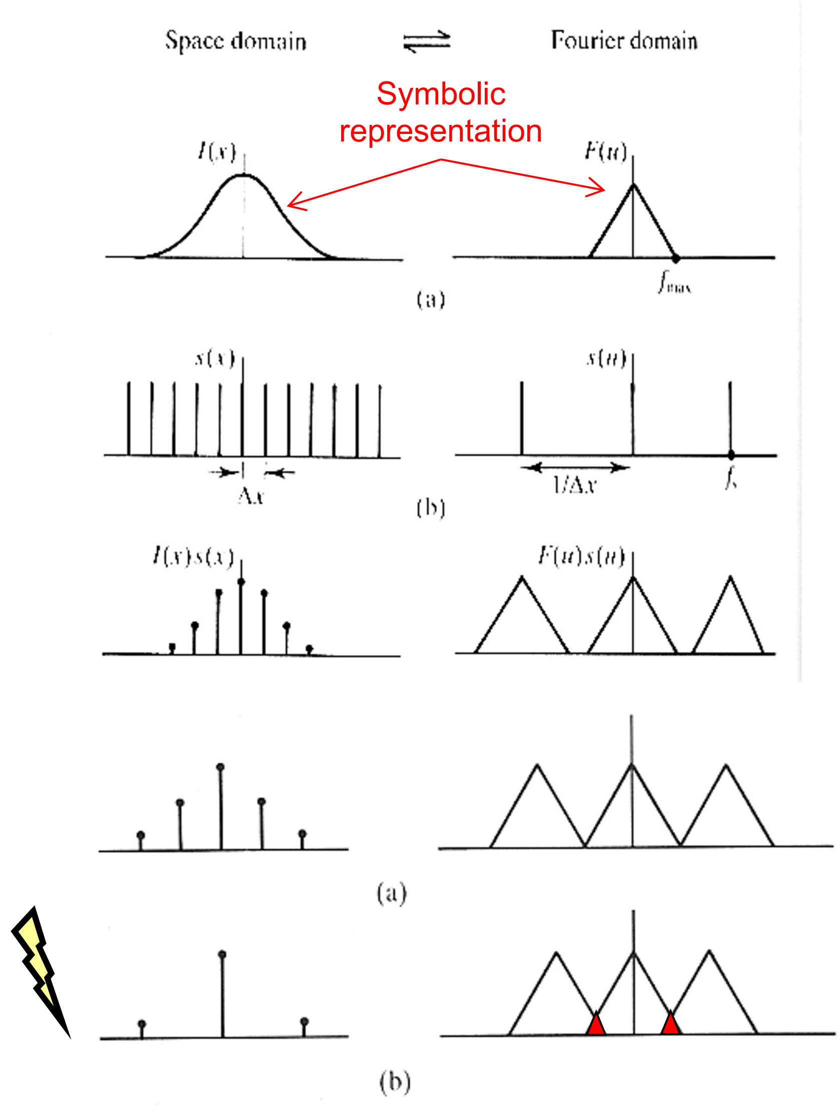
$$F(\omega) = \frac{1}{\Delta x} \sum_{k=-\infty}^{\infty} \delta\left(\omega - k\frac{1}{\Delta x}\right)$$



Sampling

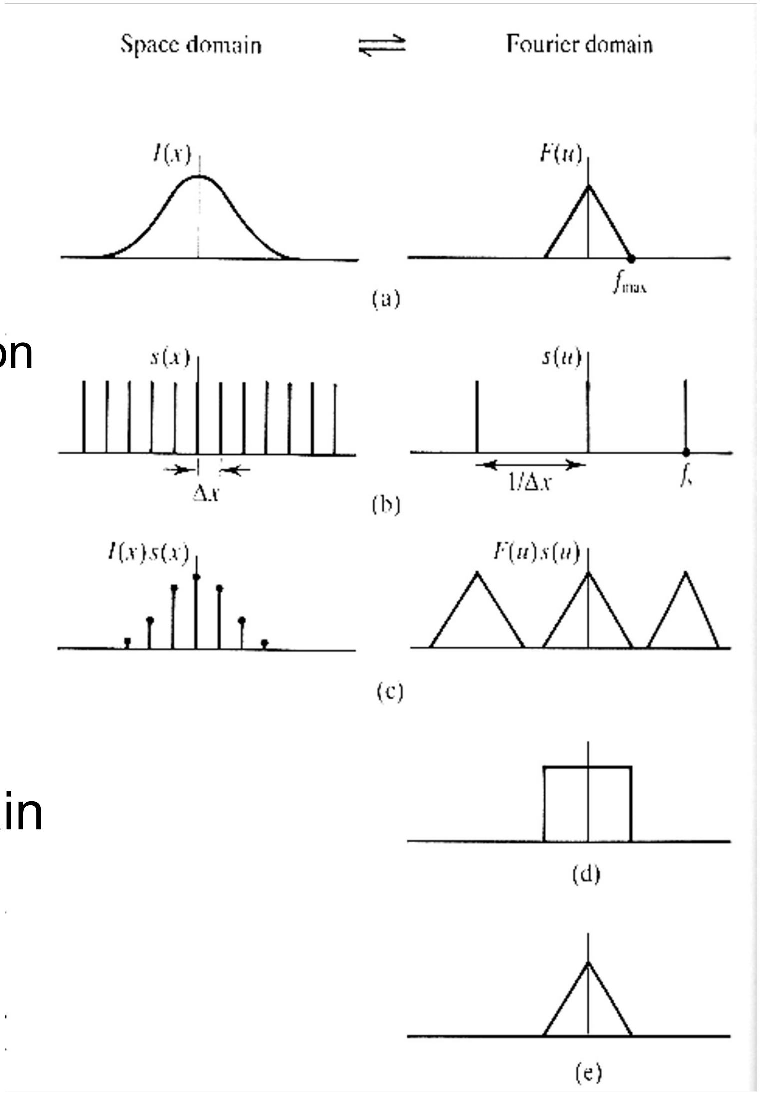
- **Continuous function**
  - Assume band-limited
  - Finite support of Fourier transform
    - Depicted symbolically here as triangle-shaped finite spectrum (not meant to be a tent function!)
- **Sampling at discrete points**
  - Multiplication with Comb function in spatial domain
  - Corresponds to convolution in Fourier domain

    $\Rightarrow$ Multiple copies of the original spectrum (convolution theorem!)
- **Frequency bands overlap ?**
  - No : Sampling was high enough (Nyquist limit!)
  - Yes: aliasing artifacts !!!



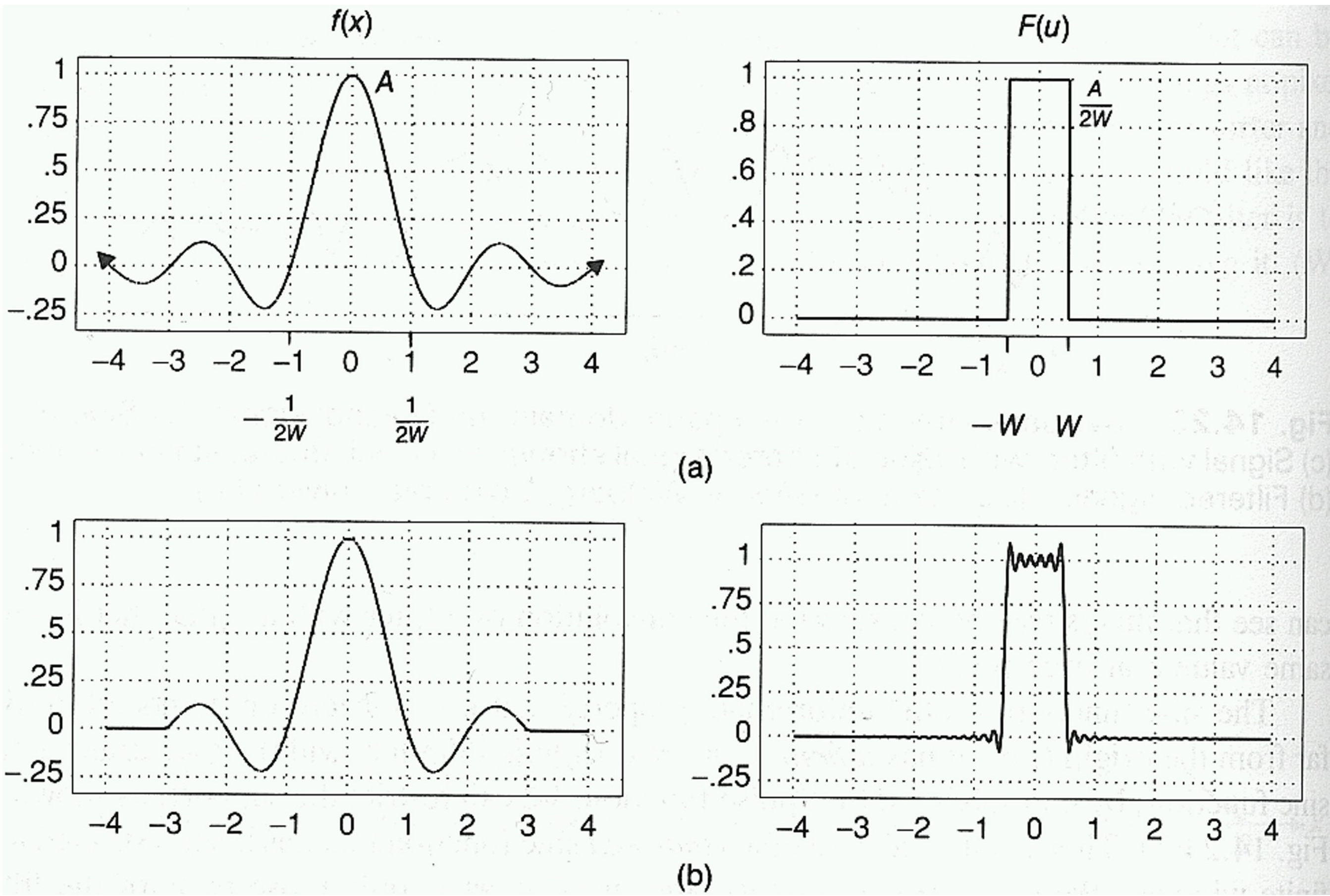
Reconstruction

- **Only original frequency band desired**
- **Filtering**
  - In Fourier domain:
    - Multiplication with windowing function around origin (low-pass filter)
  - In spatial domain
    - Convolution with inverse Fourier transform of windowing function
- **Optimal filtering function**
  - Box function in Fourier domain
  - Corresponds to *sinc* in spatial domain
    - Unlimited region of support
    - Spatial domain only allows approximations due to finite support of practical filters



Reconstruction Filter

- **Simply cutting off the spatial support of the *sinc* function to limit support is NOT a good solution**
  - Re-introduces high-frequencies ⇒ spatial ringing



Sampling and Reconstruction

Original function and its band-limited frequency spectrum

Signal sampling beyond Nyquist:

Mult./conv. with comb

Frequency spectrum is replicated

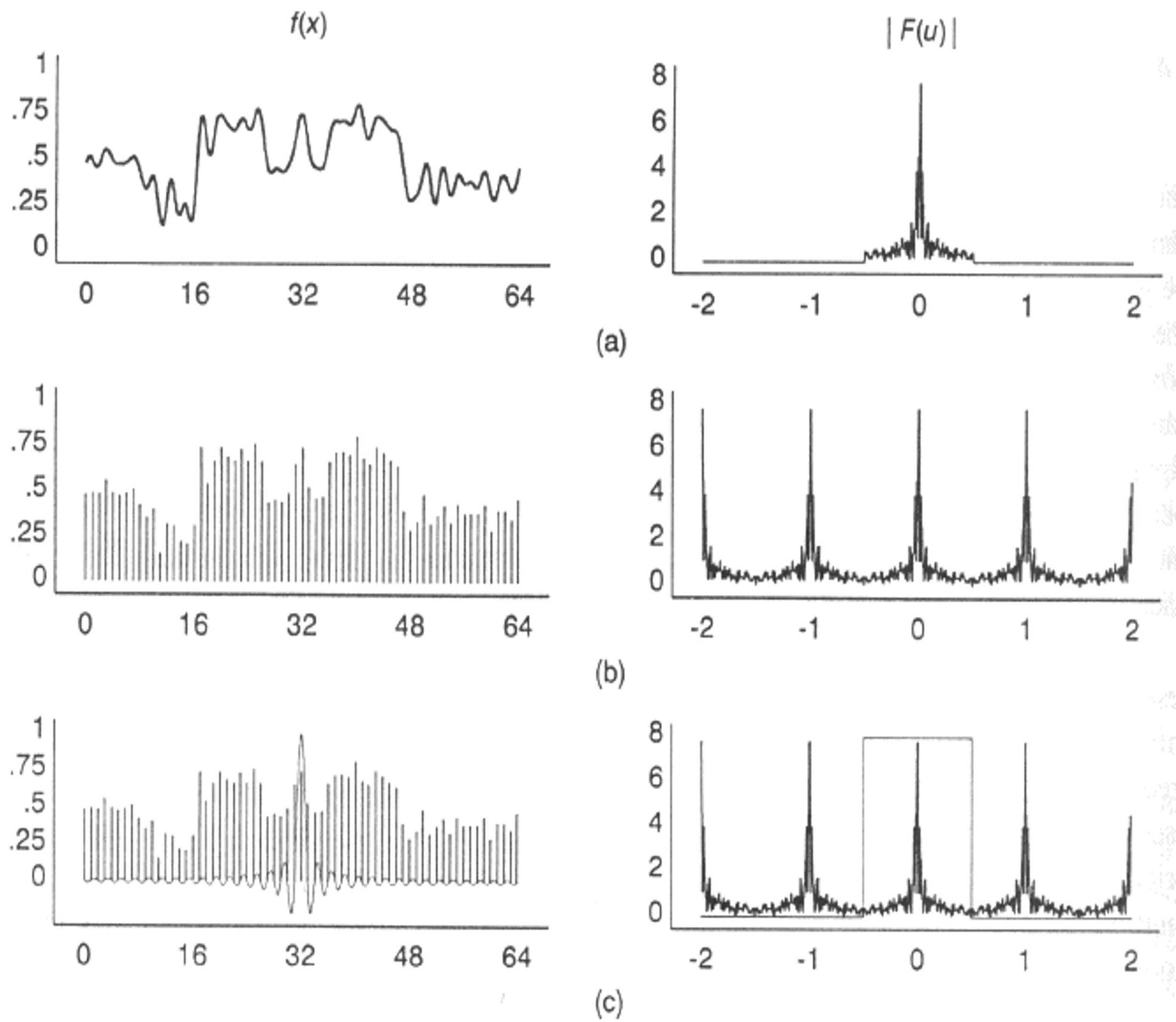
Comb dense enough (sampling rate > 2*bandlimit)

Bands do not overlap

Ideal filtering

Fourier: box (mult.)
Space: *sinc* (conv.)

Only one copy



Sampling and Reconstruction

Reconstruction with ideal *sinc*

Identical signal

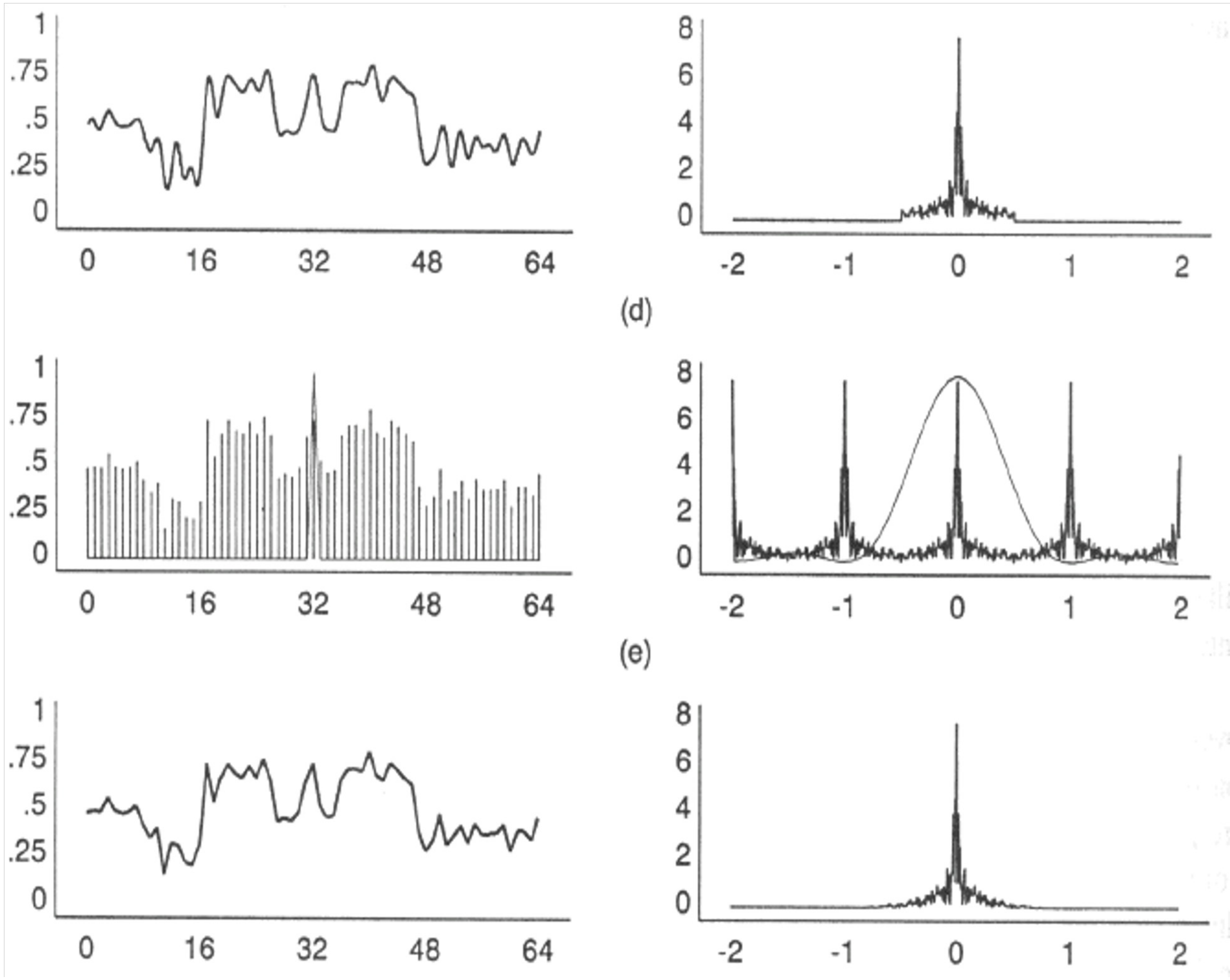
Non-ideal filtering

Fourier: *sinc*^{2} (mult.)
Space: tent (conv.)

Artificial high frequen. are not cut off

⇒ Aliasing artifacts

Reconstruction with tent function (= piecewise linear interpolation)



Sampling at Too Low Frequency

Original function and its band-limited frequency spectrum

Signal sampling below Nyquist:

Mult./conv. with comb

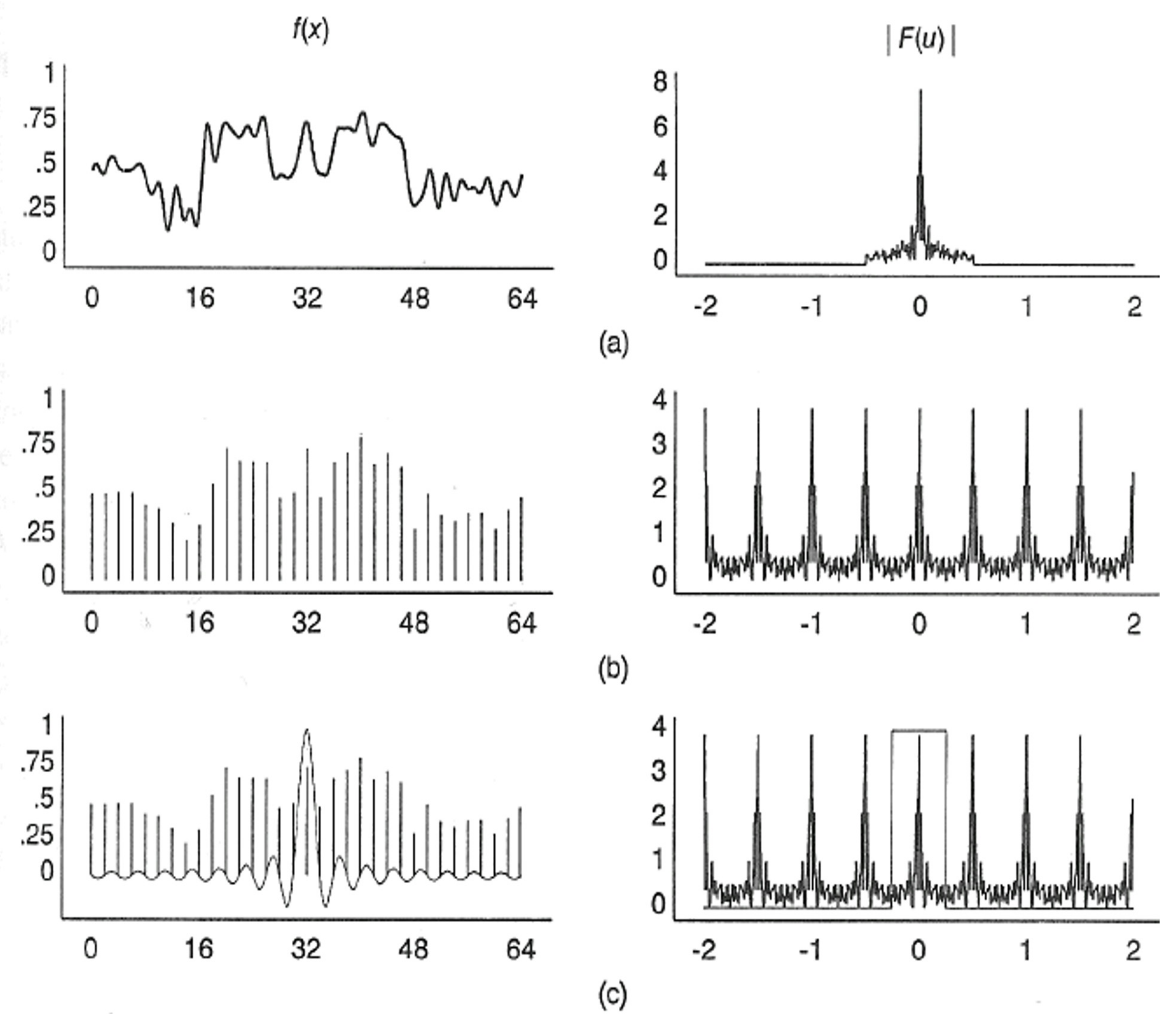
Comb spaced too far (sampling rate ≤ 2*bandlimit)

Spectral band overlap: artificial low frequenci.

Ideal filtering

Fourier: box (mult.)
Space: *sinc* (conv.)

Band overlap in frequency domain cannot be corrected
⇒ Aliasing



Sampling at Too Low Frequency

Reconstruction with ideal *sinc*

Reconstruction fails (frequency components wrong due to aliasing !)

Non-ideal filtering

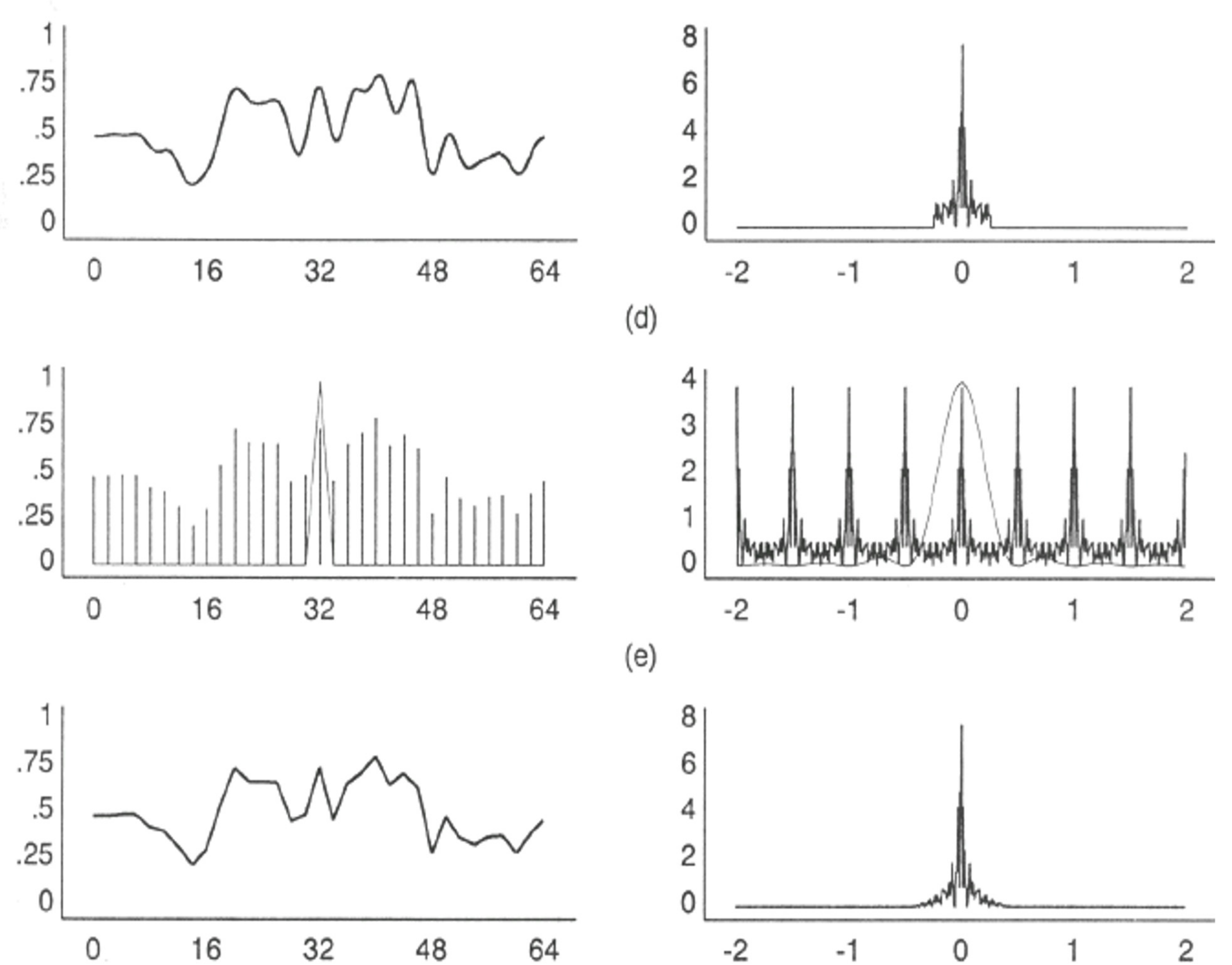
Fourier: *sinc*^{2} (mult.)
Space: tent (conv.)

Artificial high frequen. are not cut off

⇒ Reconstr. artifacts

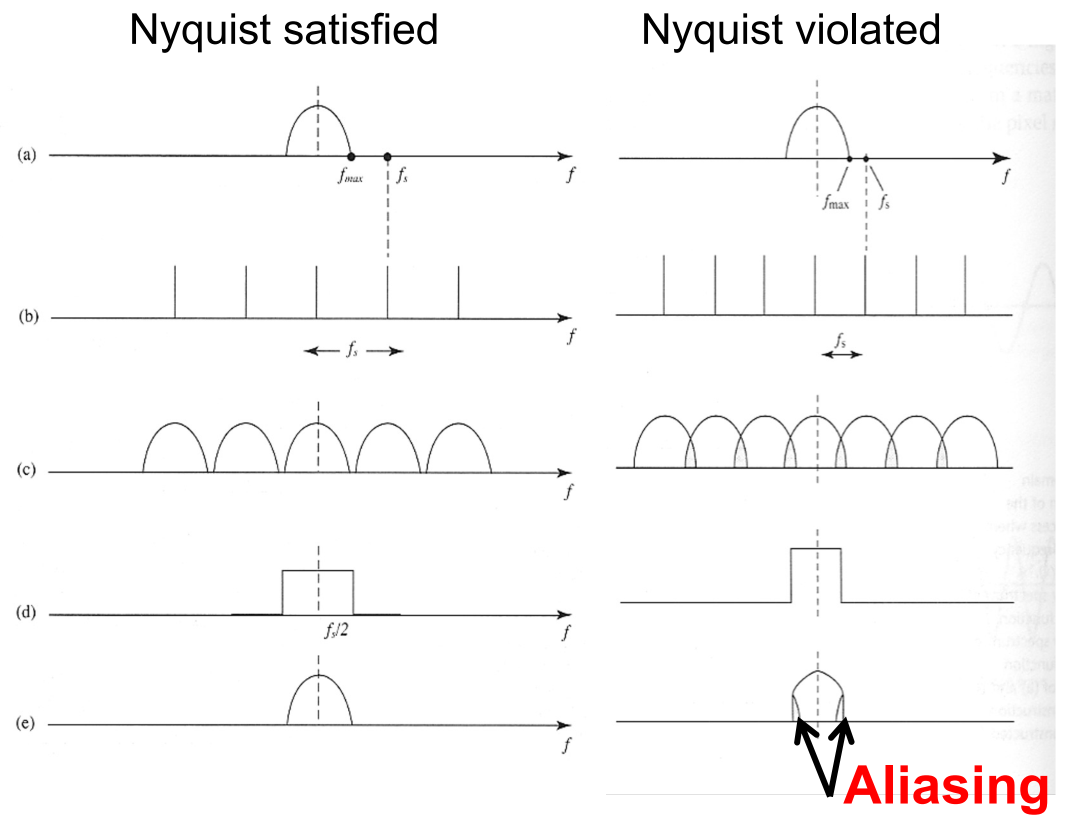
Reconstruction with tent function (= piecewise linear interpolation)

Even worse reconstruction

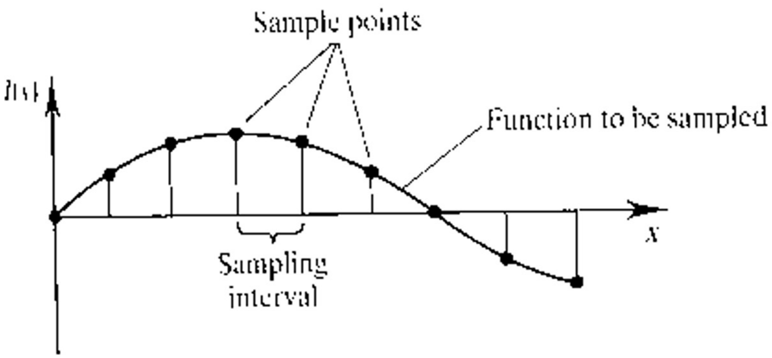


Aliasing

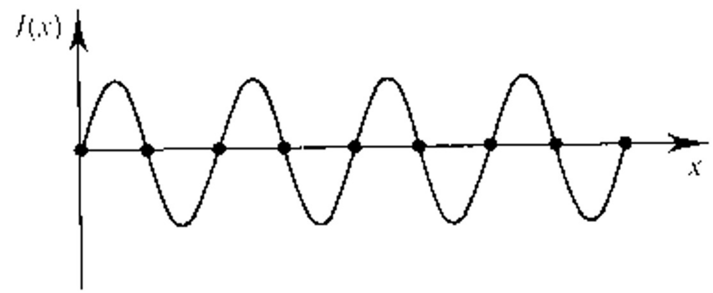
- **High frequency components from the copies appear as low frequencies for the reconstruction process**
- **In Fourier space:**
  - Original spectrum
  - Sampling comb
  - Resulting spectrum
  - Reconstruction filter
  - Reconstructed spectrum



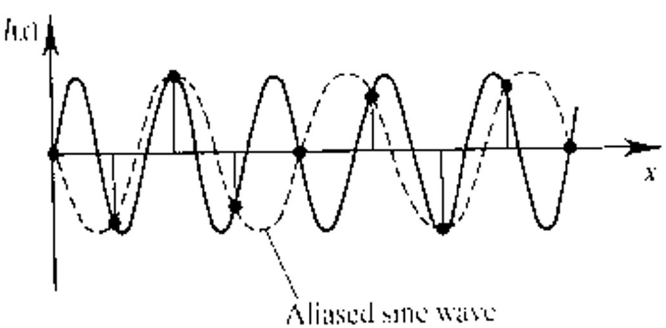
Aliasing in 1D



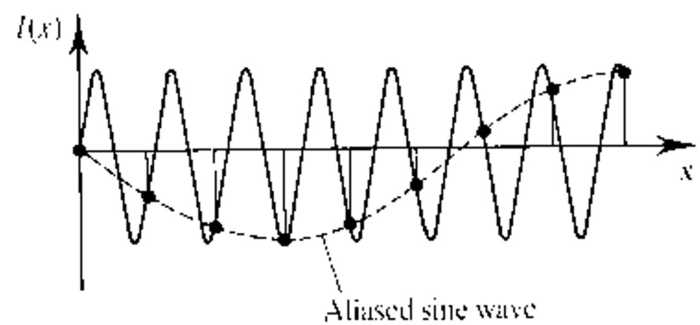
Spatial frequency < Nyquist



Spatial frequency = Nyquist
2 samples / period



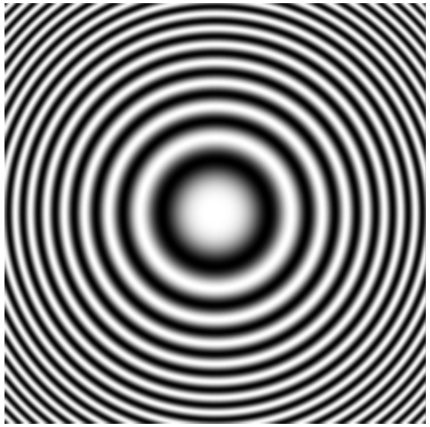
Spatial frequency > Nyquist



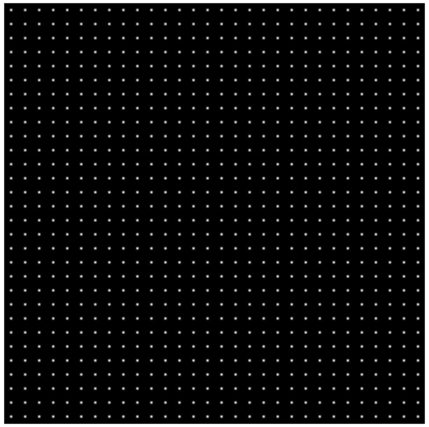
Spatial frequency >> Nyquist

Aliasing in 2D

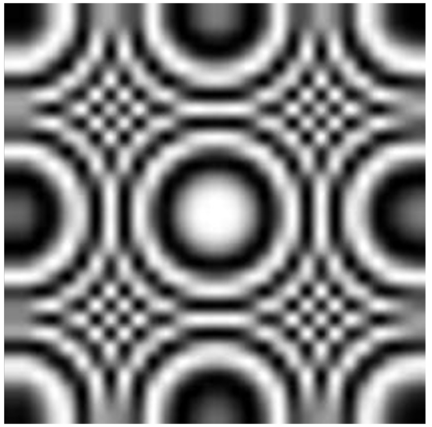
[wikipedia]



This original image



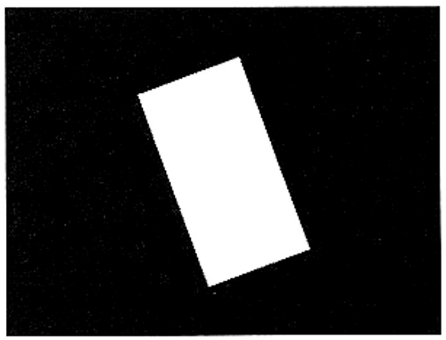
sampled at these locations



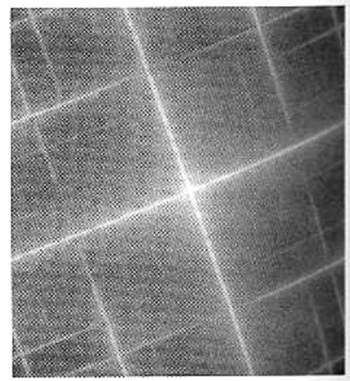
yields this reconstruction.

Aliasing in 2D

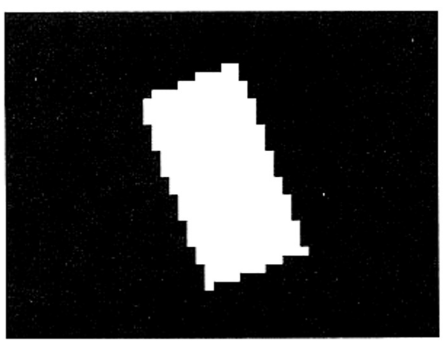
- **Spatial sampling ⇒ repeated frequency spectrum**
- **Spatial conv. with box filter ⇒ spectral mult. with *sinc***



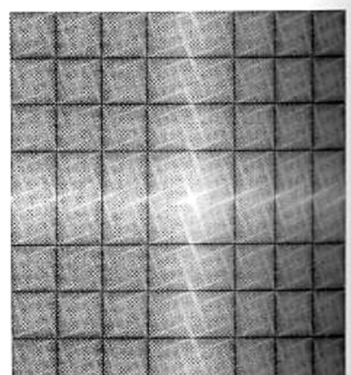
(a) Simulation of a perfect line



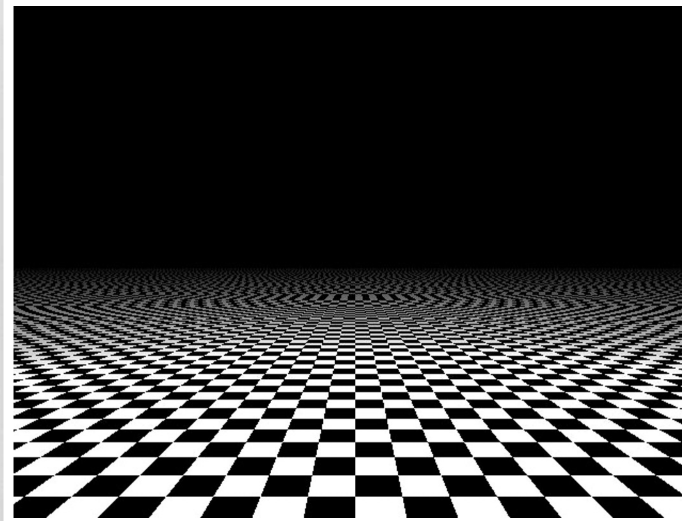
(b) Fourier transform of (a)

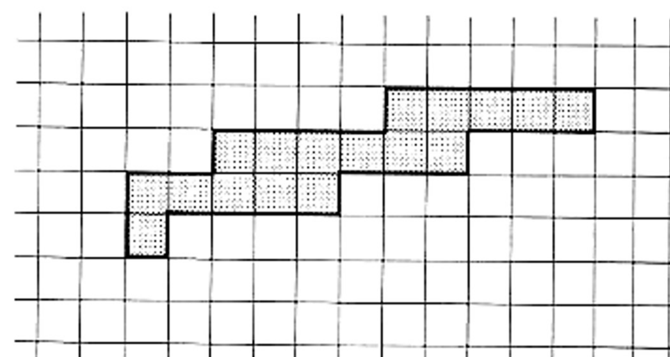


(c) Simulation of a jagged line



(d) Fourier transform of (c)





Causes for Aliasing

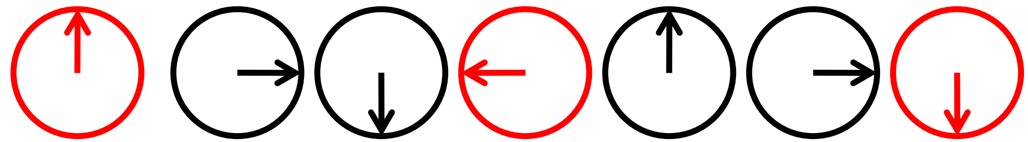
- **It all comes from sampling at discrete points**
  - Multiplication with comb function
  - Comb function: replicates the frequency spectrum
- **Issue when using non-band-limited primitives**
  - E.g., hard edges → infinitely high frequencies
- **In reality, integration over finite region necessary**
  - E.g., finite pixel size in sensor, integrates in the *analog domain*
- **Computer: analytic integration often not possible**
  - No analytic description of radiance or visible geometry available
- **Only way: numerical integration**
  - Estimate integral by taking multiple point samples, average
    - Leads to aliasing
  - Computationally expensive & approximate
- **Important:**
  - Distinction between sampling errors and reconstruction errors

Sampling Artifacts

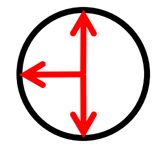
- **Spatial aliasing**
  - Staircases, Moiré patterns (interference), etc…
- **Solutions**
  - Increasing the sampling rate
    - OK, but we have infinite frequencies at sharp edges
  - Post-filtering (after reconstruction)
    - Too late, does not work - only leads to blurred artifacts!
  - Pre-filtering (blurring) of sharp features in *analog* domain (edges)
    - Slowly make geometry “fade out” at the edges?
    - Correct solution in principle, but blurred images might not be useful
    - Analytic low-pass filtering hard to implement
  - Super-sampling (see later)
    - On the fly re-sampling: densely sample, filter, down sample

Sampling Artifacts in Time

- **Temporal aliasing**
  - Video of cartwheel, ...



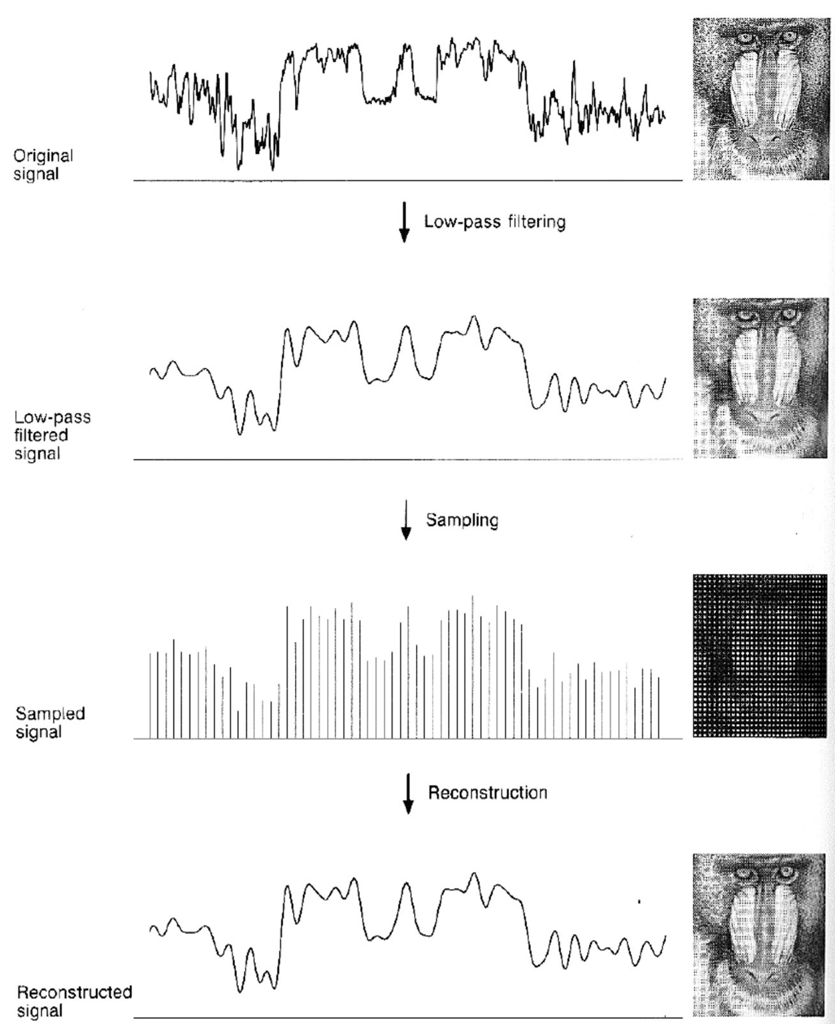
- **Solutions**
  - Increasing the frame rate
    - OK
  - Post-filtering (averaging several frames)
    - Does not work – creates replicas of details



  - Pre-filtering (motion blur)
    - Should be done on the original analog signal
    - Possible for simple geometry (e.g., cartoons)
    - Problems with texture, etc…
  - Super-sampling (see later)

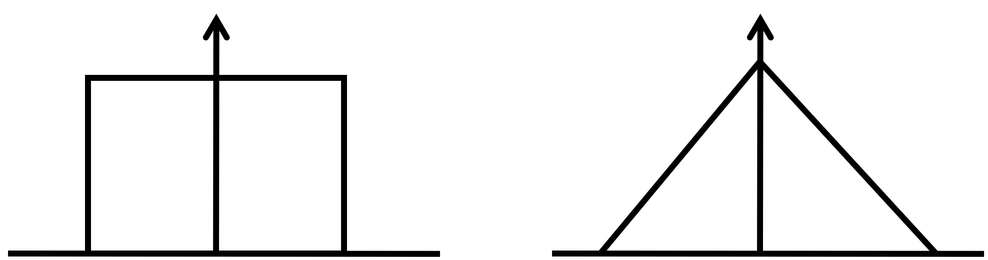
Antialiasing by Pre-Filtering

- **Filtering before sampling**
  - Analog/analytic original signal
  - Band-limiting the signal
  - Reduces Nyquist frequency for chosen sampling-rate
- **Ideal reconstruction**
  - Convolution with *sinc*
- **Practical reconstruction**
  - Convolution with
    - Box filter, Bartlett (tent)
    - → Reconstruction error



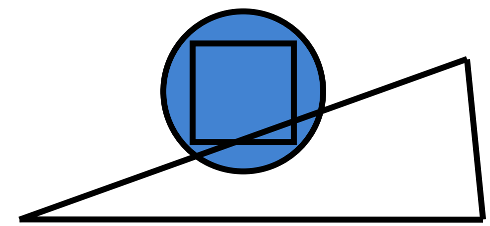
Sources of High Frequencies

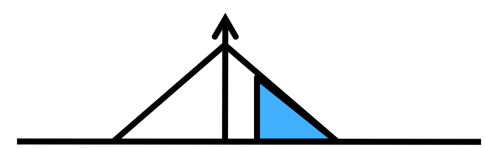
- **Geometry**
  - Edges, vertices, sharp boundaries
  - Silhouettes (view dependent)
  - …
- **Texture**
  - E.g., checkerboard pattern, other discontinuities, …
- **Illumination**
  - Shadows, lighting effects, projections, …
- **Analytic filtering almost impossible**
  - Even with the simplest filters



Comparison

- **Analytic low-pass filtering (pixel/triangle overlap)**
  - Ideally eliminates aliasing completely
  - Complex to implement
    - Compute distance from pixel to a line
    - Weighted or unweighted area evaluation
    - Filter values can be stored in look-up tables
    - Fails at corners
    - Possibly taking into account slope

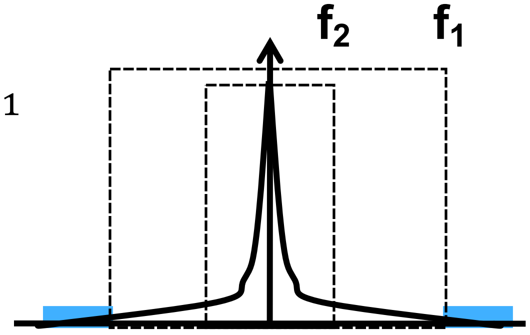




- **Over-/Super-sampling**
  - Very easy to implement
  - Does not eliminate aliasing completely
    - Sharp edges contain **infinitely** high frequencies
  - But it helps: …

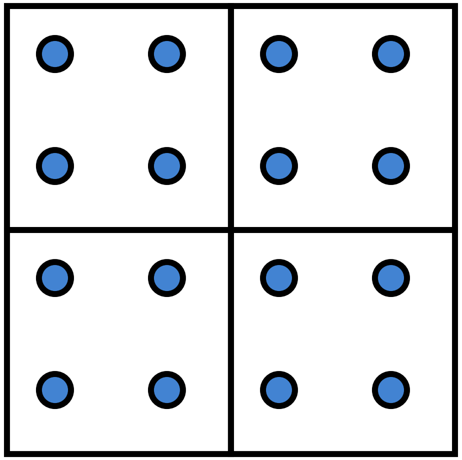
Re-Sampling Pipeline

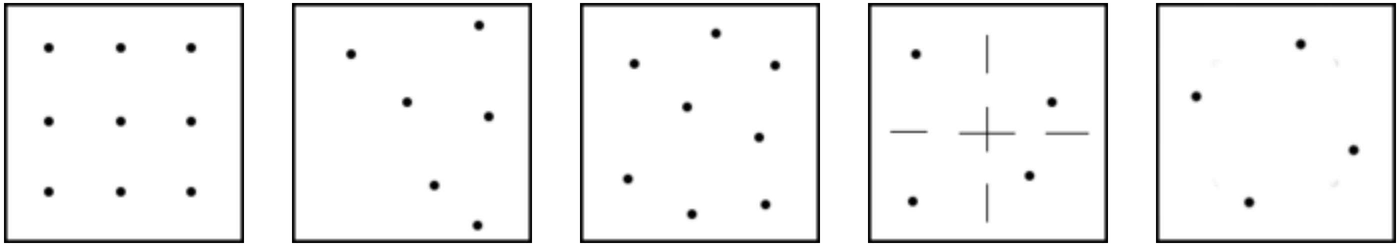
- **Assumption**
  - Energy in higher frequencies typically decreases quickly
  - Idea: Reduced aliasing by sampling at higher frequency
- **Algorithm**
  - Super-sampling
    - Sample continuous signal with high frequency f_1
    - Aliasing (only here!) with energy beyond f_1 (assumed to be small)
  - Reconstruction of signal
    - Filtering with $g_1(x)$: e.g., convolution with $sinc_{f1}$
    - Exact representation with sampled values !!
  - Analytic low-pass filtering of signal
    - Filtering with filter $g_2(x)$ where $f_2 << f_1$
    - Signal is now band-limited w.r.t. f_2
  - Re-sampling with a sampling frequency that is compatible with f_2
    - No additional aliasing
  - Filters $g_1(x)$ & $g_2(x)$ can be combined!



Super-Sampling in Practice

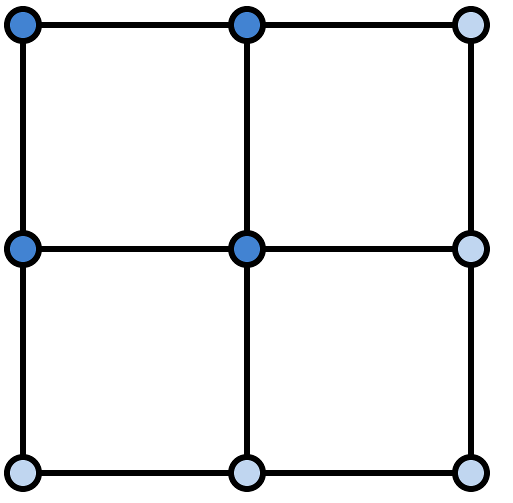
- **Regular super-sampling**
  - Averaging of *N* samples per pixel
  - *N*: 4 (quite good already), 16 (often sufficient)
  - Samples: rays, z-buffer, motion, reflection, ...
  - Filter weights
    - Box filter
    - Others: B-spline, pyramid (Bartlett), hexagonal, ...
  - Sampling Patterns (left to right)
    - Regular: aliasing likely
    - Random: often clumps, incomplete coverage
    - Poisson Disc: close to perfect (“blue noise”), but can be costly
    - Jittered: randomized regular sampling, avoid biggest issues
    - Most often (in HW): rotated grid pattern





Super-Sampling Caveats

- **Popular mistake**
  - Sampling at the corners of every pixel
  - Pixel color by averaging from corners
  - Free super-sampling ???



- **Problem**
  - Wrong reconstruction filter !!!
  - Same sampling frequency, but post-filtering with a tent function
  - Blurring: loss of information
- **Post-reconstruction blur**

1x1 Sampling, 3x3 Blur

1x1 Sampling, 7x7 Blur

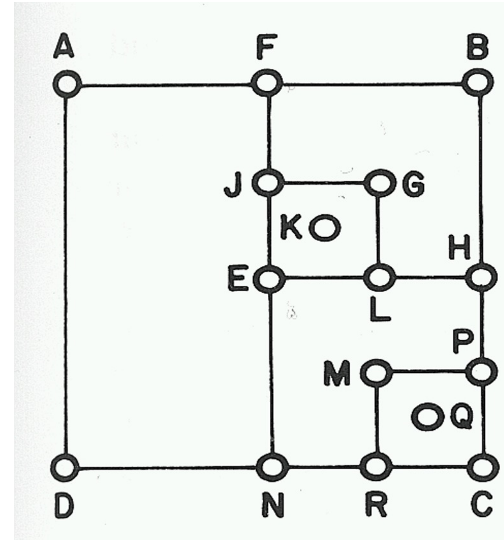
- **There is no "free" super-sampling/lunch**

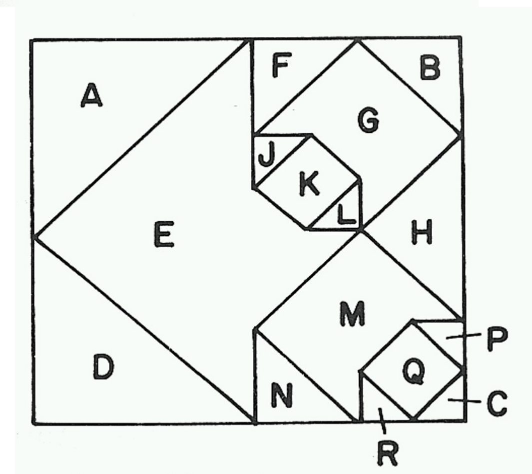
Adaptive Super-Sampling

- **Idea: locally adapt sampling density**
  - Slowly varying signal (mostly low frequencies): low sampling rate
  - Strong changes (mostly high frequencies): high sampling rate
- **Decide sampling density locally**
- **Decision criterion:**
  - Differences of pixel values
  - Contrast (relative difference)
    - |A-B| / (|A|+|B|)
  - Others

Adaptive Super-Sampling

- **Recursive algorithm**
  - Sampling at pixel corners and center
  - Decision criterion for corner-center pairs
    - Differences, contrast, object/shader-IDs, ...
  - Subdivide quadrant by adding 3 diag. points
  - Filtering with weighted averaging
    - Tile: ¼ from each quadrant
    - Leaf quadrant: ½ (center + corner)
  - Box filter with final weight proport. to area →



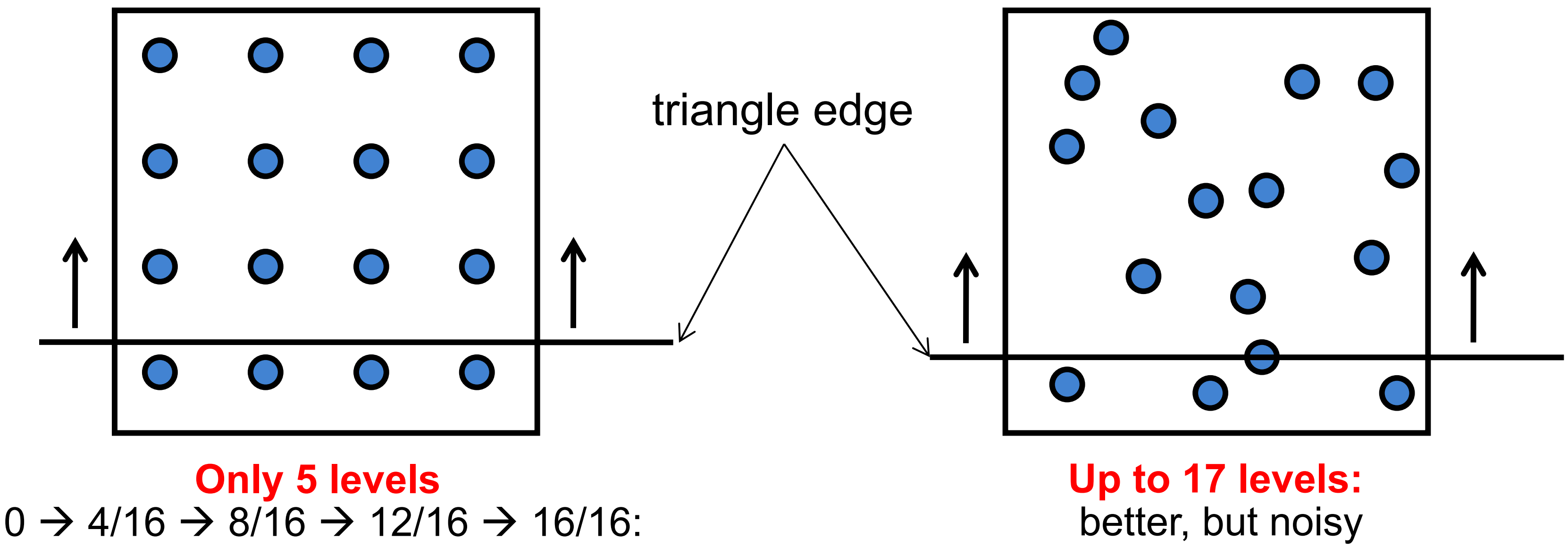


$$\frac{1}{4}\left(\frac{A+E}{2}+\frac{D+E}{2}+\frac{1}{4}\left[\frac{F+G}{2}+\frac{B+G}{2}+\frac{H+G}{2}+\frac{1}{4}\left\{\frac{J+K}{2}+\frac{G+K}{2}+\frac{L+K}{2}+\frac{E+K}{2}\right\}\right]+\frac{1}{4}\left[\frac{E+M}{2}+\frac{H+M}{2}+\frac{N+M}{2}+\frac{1}{4}\left\{\frac{M+Q}{2}+\frac{P+Q}{2}+\frac{C+Q}{2}+\frac{R+Q}{2}\right\}\right]\right)$$

- **Extension**
  - Jittering of sample points

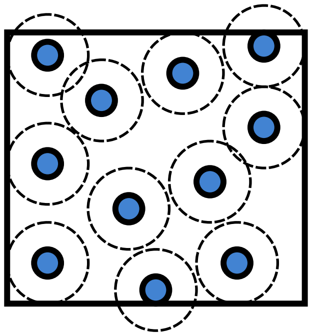
Stochastic Super-Sampling

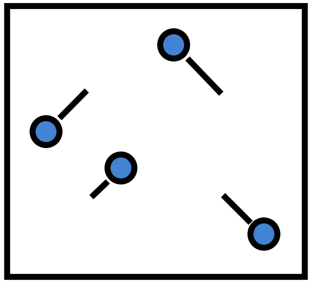
- **Problems with regular super-sampling**
  - Nyquist frequency for aliasing only shifted
  - Expensive: e.g., 4-fold or 16-fold effort
  - Non-adaptive: same effort everywhere
  - Too regular: reduction of effective number of axis-aligned levels
- **Introduce irregular sampling pattern**

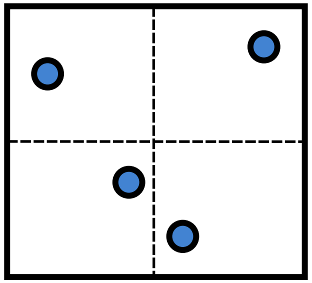


Stochastic Sampling

- **Requirements**
  - Even sample distribution: no clustering
  - Little correlation between positions: no alignment
  - Blue noise property: Shift error to higher frequencies
  - Incremental generation: on demand as needed
- **Generation of samples**
  - Poisson-disk sampling
    - Random generation of samples
    - Rejection if closer than min distance to other samples
  - Jittered sampling
    - Random perturbation from regular positions
  - Stratified sampling
    - Subdivision into areas with one random sample in each
    - Improves even distribution
  - Quasi-random numbers (Quasi-Monte Carlo)
    - E.g., Halton sequence
    - Advanced feature: see RIS course for more details



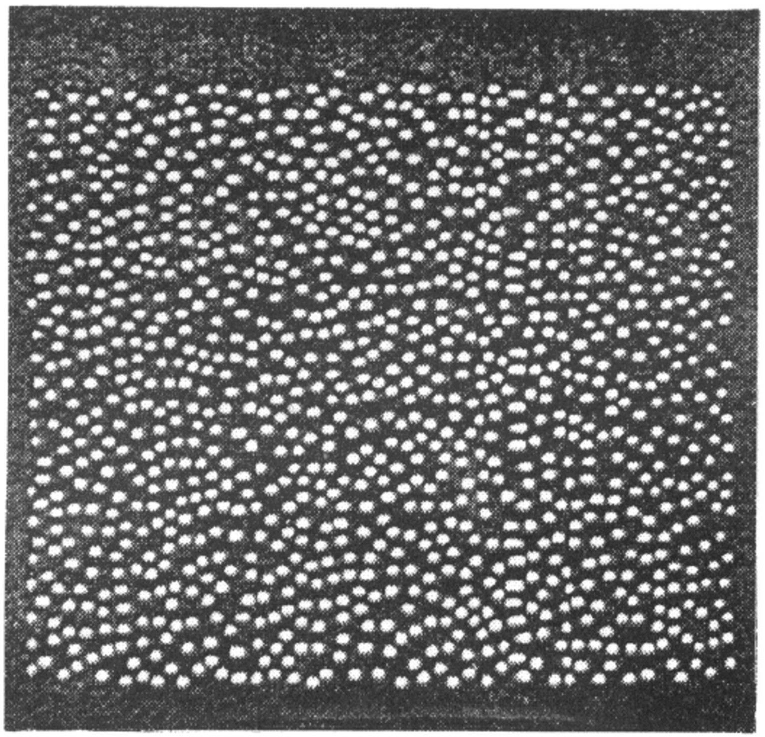




Poisson-Disk Sample Distribut.

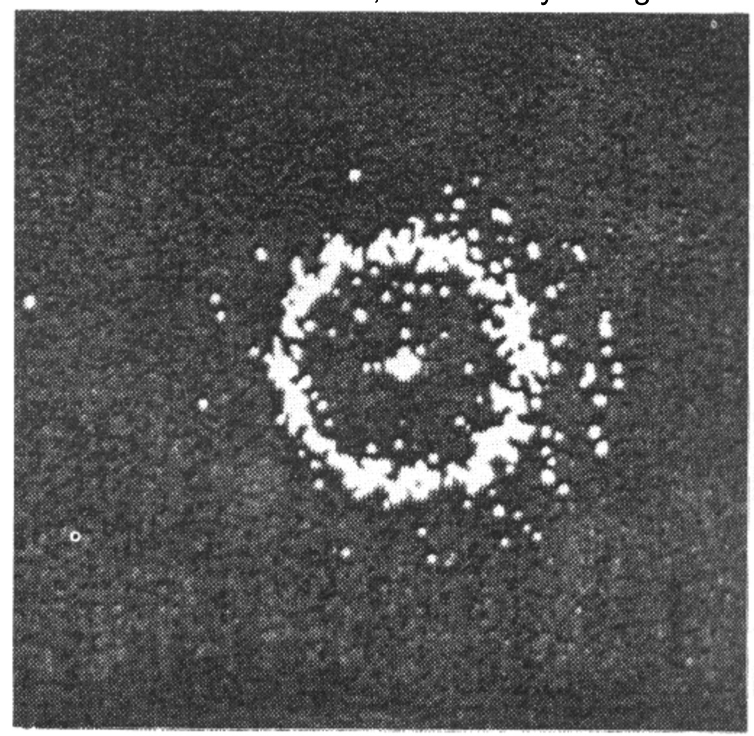
- **Motivation**
  - Distribution of the optical receptors on the retina (here: ape)

(a)



Distribution of the photo-receptors

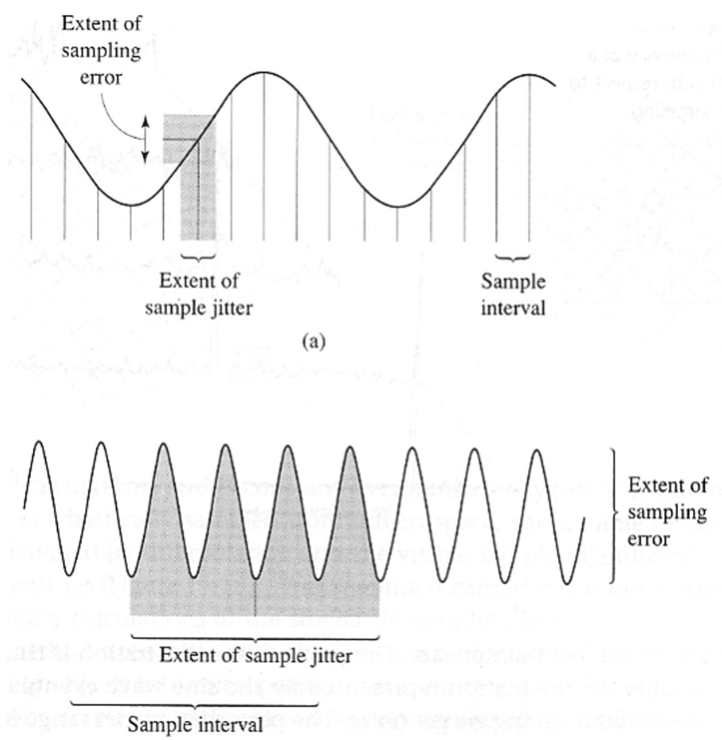
(b)



Fourier analysis

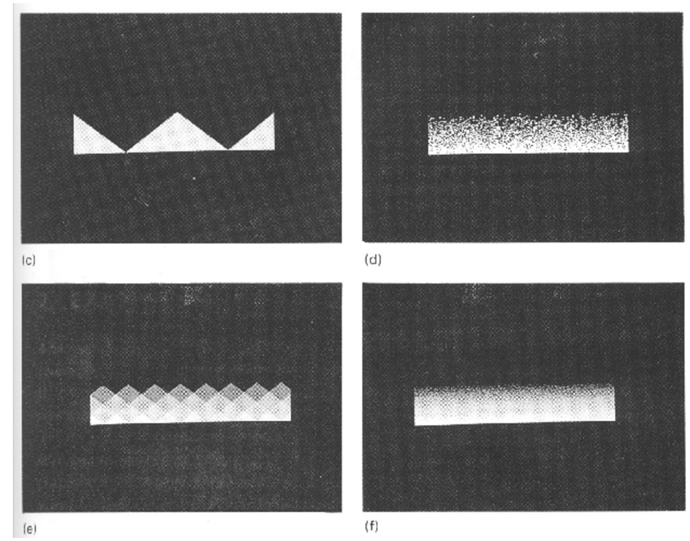
Stochastic Sampling

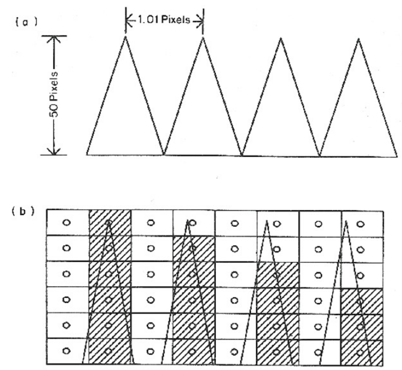
- **Slowly varying function in sample domain**
  - Closely reconstructs target value with few samples
- **Quickly varying function in sample domain**
  - Transforms energy in high-frequency bands into noise
  - Reconstructs average value as sample count increases



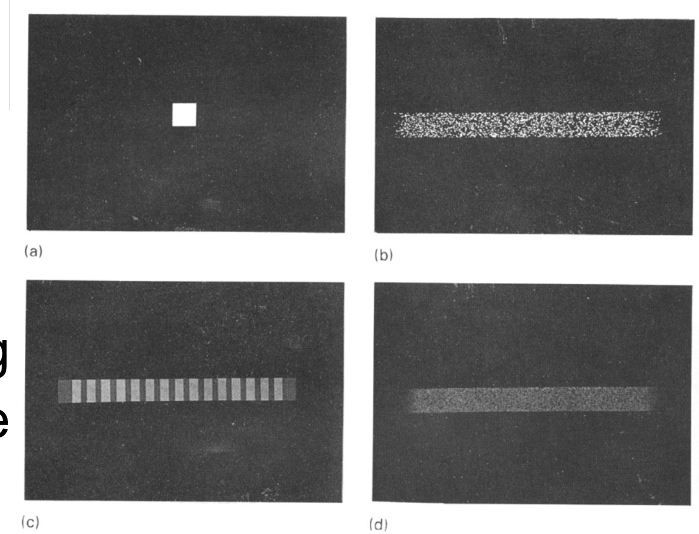
Examples

- **Spatial sampling: triangle comb**
  - (c) 1 sample/pixel, no jittering: aliasing
  - (d) 1 spp, jittering: noise
  - (e) 16 spp, no jittering: less aliasing
  - (f) 16 spp, jittering: less noise





- **Temporal sampling: motion blur**
  - (a) 1 time sample, no jittering: aliasing
  - (b) 1 time sample, jittering/pixel: noise
  - (c) 16 samples, no jittering: less aliasing
  - (d) 16 samples, jittering/pixel: less noise



Comparison

- **Regular, 1x1**
- **Regular, 3x3**
- **Regular, 7x7**
- **Jittered, 3x3**
- **Jittered, 7x7**