Computer Graphics

Spectral Analysis

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Motivation

Image Processing and Rendering

Digital Signal Processing

[Egan et al. 2009]
Fourier Transformation: Audio Signal Analogy
Fourier Transformation

\[ + \]

\[ + \]

\[ = \]
Spatial Frequency

- Inverse of period length of some structure in an image
- Unit [1/pixel]

Lowest frequency
Image average

... Nyquist frequency

Highest frequency
½ of image resolution
Spatial Frequency
Fourier Transformation

**Analysis:**
Fourier Transformation

\[ F(k) = F_x[f(x)](k) = \int_{-\infty}^{\infty} f(x)e^{-i2\pi k x} \, dx \]

**Synthesis:**
Inverse Fourier Transformation

\[ f(x) = F_x^{-1}[F(k)](x) = \int_{-\infty}^{\infty} F(k)e^{i2\pi k x} \, dk \]

Representation via complex exponential:
- \( e^{ix} = \cos(x) + i \sin(x) \) (see Taylor expansion)
- Use to describe phase information: shifting of the pattern.
Division into odd and even parts

Division into even and odd parts
- Even: \( f(x) = f(-x) \) (symmetry about y-axis): Described by cosine
- Odd: \( f(x) = -f(-x) \) (symmetry about origin): Described by sine.

\[
f(x) = \frac{1}{2} [f(x) + f(-x)] + \frac{1}{2} [f(x) - f(-x)] = E(x) + O(x)
\]

**Analysis**

\[
F(k) = \int_{-\infty}^{\infty} f(x) (\cos(-2\pi k x) + i \sin(-2\pi k x)) dx = b(k) - ia(k)
\]

**Even term**

\[
b(k) = \int_{-\infty}^{\infty} f(x) \cos(2\pi k x) dx = \int_{-\infty}^{\infty} (E(x) + O(x)) \cos(2\pi k x) dx = \int_{-\infty}^{\infty} E(x) \cos(2\pi k x) dx
\]

**Odd term**

\[
a(k) = \int_{-\infty}^{\infty} f(x) \sin(2\pi k x) dx = \int_{-\infty}^{\infty} (E(x) + O(x)) \sin(2\pi k x) dx = \int_{-\infty}^{\infty} O(x) \sin(2\pi k x) dx
\]

**Synthesis**

\[
f(x) = \int_{-\infty}^{\infty} F(k) (\cos(2\pi k x) + i \sin(2\pi k x)) dk = E(x) + O(x)
\]

**Even term**

\[
E(x) = \int_{-\infty}^{\infty} F(k) \cos(2\pi k x) dk = \int_{-\infty}^{\infty} (b(k) - ia(k)) \cos(2\pi k x) dk = \int_{-\infty}^{\infty} b(k) \cos(2\pi k x) dk
\]

**Odd term**

\[
O(x) = \int_{-\infty}^{\infty} F(k) i \sin(2\pi k x) dk = \int_{-\infty}^{\infty} (b(k) - ia(k)) i \sin(2\pi k x) dk = \int_{-\infty}^{\infty} a(k) \sin(2\pi k x) dk
\]
Spatial vs Frequency Domain: Important basis functions

Box ↔ (normalized) sinc
\[ \text{sinc}(x) = \frac{\sin(x\pi)}{x\pi} \]
\[ \text{sinc}(0) = 1 \]
\[ \int \text{sinc}(x) \, dx = 1 \]
- Negative values
- Infinite support!

Gaussian ↔ Gaussian
- Inverse width

Tent ↔ \( \text{sinc}^2 \)
- Tent is convolution of box function with itself
Spatial vs Frequency Domain: Transform behavior

Fourier transform of a box function

\[ \text{rect}(at) \rightarrow \frac{1}{|a|} \sin \left( \frac{\omega}{2a} \right). \]

Wide box
\[ a = 0.5 \]

Narrow box
\[ a = 2 \]

Wide sinc

Narrow sinc

Sinc
Example: Fourier Synthesis (Inverse Fourier Transformation)

- **Square wave: periodic, uneven function**
  \[
  f(x) = \begin{cases} 
  0.5 & \forall 0 < (x \mod 2\pi) < \pi \\ 
  -0.5 & \forall \pi < (x \mod 2\pi) < 2\pi 
  \end{cases}
  \]
  \[
  a_k = \int \sin(2\pi kx)f(x) \, dx 
  \]
  \[
  f(x) = \sum_k a_k \sin(2\pi kx)
  \]
  - \(a_0 = 0\)
  - \(a_1 = 1\)
  - \(a_2 = 0\)
  - \(a_3 = 1/3\)
  - \(a_4 = 0\)
  - \(a_5 = 1/5\)
  - \(a_6 = 0\)
  - \(a_7 = 1/7\)
  - \(a_8 = 0\)
  - \(a_9 = 1/9\)
  - \(
  \begin{array}{c}
  0 \quad 2\pi \\
  4\pi \\
  6\pi \\
  8\pi \\
  10\pi \\
  \end{array}
  \]
  - \(\begin{array}{c}
  0 \quad 2\pi \\
  4\pi \\
  6\pi \\
  8\pi \\
  10\pi \\
  \end{array}
  \)
Discrete Fourier Transformation

Periodic in space ⇔ discrete in frequency (vice ver.)

- Any periodic, continuous function can be expressed as the sum of an (infinite) number of sine or cosine waves:

\[ f(x) = \sum_k a_k \sin(2\pi k x) + b_k \cos(2\pi k x) \]

Decomposition of signal into different frequency bands: spectral analysis

- Frequency band: \( k \) (must be an integer)
  - \( k = 0 \) : mean value
  - \( k = 1 \) : function period, lowest possible frequency
  - \( k_{\text{max}} ? \) : band limit, no higher frequency present in signal

- Fourier coefficients: \( a_k, b_k \) (real-valued, as before)
  - Even function \( f(x) = f(-x) \) : \( a_k = 0 \)
  - Odd function \( f(x) = -f(-x) \) : \( b_k = 0 \)
Discrete Fourier Transformation

Equally-spaced function samples ($N$ samples)
- Function values known only at discrete points, e.g.
  - Idealized Physical measurements
  - Pixel positions in an image!
    - Represented via sum of Delta distribution (Fourier integrals $\rightarrow$ sums)

Fourier analysis

$$a_k = \sum_{i} \sin \left( \frac{2\pi ki}{N} \right) f_i$$

$$b_k = \sum_{i} \cos \left( \frac{2\pi ki}{N} \right) f_i$$

- Sum over all $N$ measurement points
- $k = 0, 1, 2, \ldots$? Highest possible frequency?
  - Nyquist frequency: highest frequency that can be represented
  - Defined as 1/2 the sampling frequency
- Sampling rate $N$: determined by image resolution (pixel size)
- 2 samples / period $\leftrightarrow$ 0.5 cycles per pixel $\Rightarrow k_{\text{max}} \leq N / 2$
Spatial vs. Frequency Domain: Examples

Sine wave with positive offset

Square wave

Scanline of an image
2D Fourier Transformation

- 2D Fourier Transformation can be separated into two 1D Fourier transformations along $x$ and $y$ directions.
- Discontinuities: orthogonal direction in Fourier domain!

**Rendered Image**

**Fourier Transform**

- (a) Bush
- (b) Arcos da Lapa (Rio de Janeiro)

**Fourier transform** $|F(u, v)|$
Power Spectra
With Sneak Peak into Realistic Image Synthesis

Power spectrum describes the distribution of power into frequency components.

White Noise

Blue Noise

Monte Carlo Renderer

Result

Monte Carlo Renderer

Result

Zero Frequency
Convolution: Motivation

Describes many natural processes:
- Room Impulse Measurement in Acoustics
- Image Processing: Filtering

Jochen Schulz
Convolution

\[(f \otimes g)(x) = \int_{-\infty}^{\infty} f(\tau)g(x - \tau) d\tau\]

Expensive operation in image space
- For each \(x\) integrate over non-zero domain
Convolution: Fourier vs Image Space

<table>
<thead>
<tr>
<th>Image Domain</th>
<th>Fourier domain</th>
</tr>
</thead>
<tbody>
<tr>
<td>Convolution</td>
<td>→ Multiplication</td>
</tr>
<tr>
<td>Multiplication</td>
<td>→ Convolution</td>
</tr>
</tbody>
</table>

Multiplication in transformed Fourier domain is cheaper than direct convolution in image domain!

\[
\text{rect}(t) \ast \text{rect}(t) = \omega(t)
\]

\[
si\left(\frac{\omega}{2}\right) \cdot si\left(\frac{\omega}{2}\right) = X(j\omega) = si\left(\frac{\omega}{2}\right).
\]
Convolution and Filtering

Technical realization
- In image domain
- Pixel mask with weights

Problems (e.g. sinc)
- Large filter support
  - Large mask (resolution of the image)
  - A lot of computation
- Negative weights might introduce problems if not handled properly
Filtering

Low-pass filtering
- Multiplication with box in frequency domain
- Convolution with $sinc$ in spatial domain

High-pass filtering
- Multiplication with $(1 - \text{box})$ in frequency domain
- Only high frequencies

Band-pass filtering
- Only intermediate

(a)

(b)

(c)

(d)
Low Pass Filtering: Blurring
High-Pass Filtering

Enhances discontinuities in image
- Useful for edge detection
Anything Clear?