Computer Graphics

- Basics of Ray Tracing -

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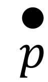
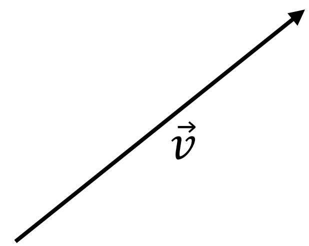
VECTOR ALGEBRA

Points and Vectors

- **Point**
  - $p = \begin{bmatrix} p_x \\ p_y \\ p_z \end{bmatrix}$ $\qquad p \in \mathbb{R}^3$



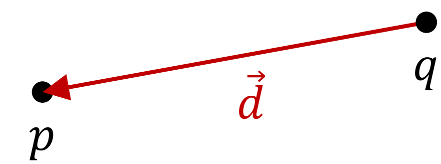
- **Vector**
  - $\vec{v} = \begin{bmatrix} v_x \\ v_y \\ v_z \end{bmatrix}$ $\qquad \vec{v} \in \mathbb{R}^3$



Points and Vectors

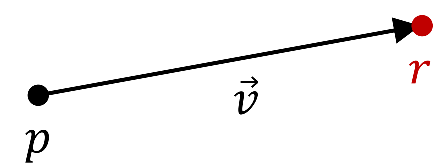
- **Point operators**
  - Point difference:

$$\textcolor{red}{\vec{d}} = p - q := \begin{bmatrix} p_x - q_x \\ p_y - q_y \\ p_z - q_z \end{bmatrix}$$



  - Add vector to a point:

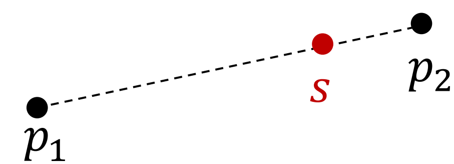
$$\textcolor{red}{r} = p + \vec{v} := \begin{bmatrix} p_x + v_x \\ p_y + v_y \\ p_z + v_z \end{bmatrix}$$



  - Affine combination of n points:

$$\textcolor{red}{s} = \lambda_1 p_1 + \lambda_2 p_2 + \cdots + \lambda_n p_n$$

where $\lambda_1 + \lambda_2 + \cdots + \lambda_n = 1$

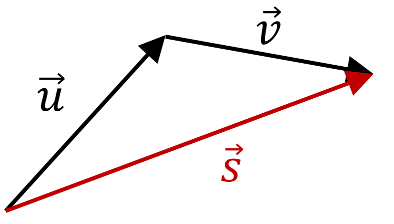


Points and Vectors

- **Vector operators**
  - Addition, subtraction:

    $\vec{s} = \vec{u} + \vec{v}$

    $\vec{u} = \vec{s} - \vec{v}$



  - Vector length

    $$|\vec{v}| = \sqrt{v_x^2 + v_y^2 + v_z^2}$$

  - Dot product:

    $\vec{u} \cdot \vec{v} = u_x v_x + u_y v_y + u_z v_z$

    $\vec{u} \cdot \vec{v} = |\vec{u}||\vec{v}| \cos(\vec{u}, \vec{v})$

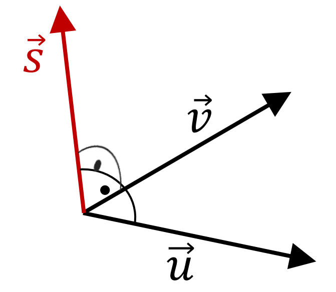
    - 1 parallel
    - > 0 acute angle
    - $= 0$ right angle
    - < 0 obtuse angle

$\vec{u}^2 := \vec{u} \cdot \vec{u}$

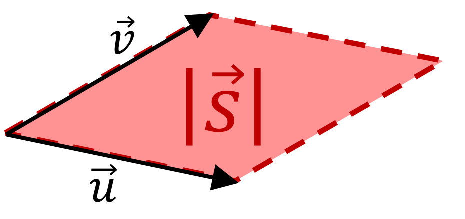
Points and Vectors

- **Vector operators**
  - Cross product:

$$\vec{s} = \vec{u} \times \vec{v} := \begin{bmatrix} u_y v_z - u_z v_y \\ u_z v_x - u_x v_z \\ u_x v_y - u_y v_x \end{bmatrix}$$

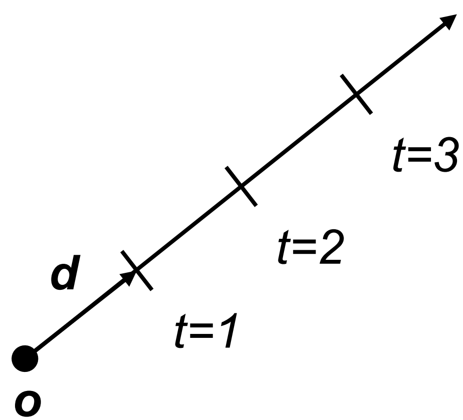


$$|\vec{s}| = |\vec{u}||\vec{v}| \sin(\vec{u}, \vec{v})$$



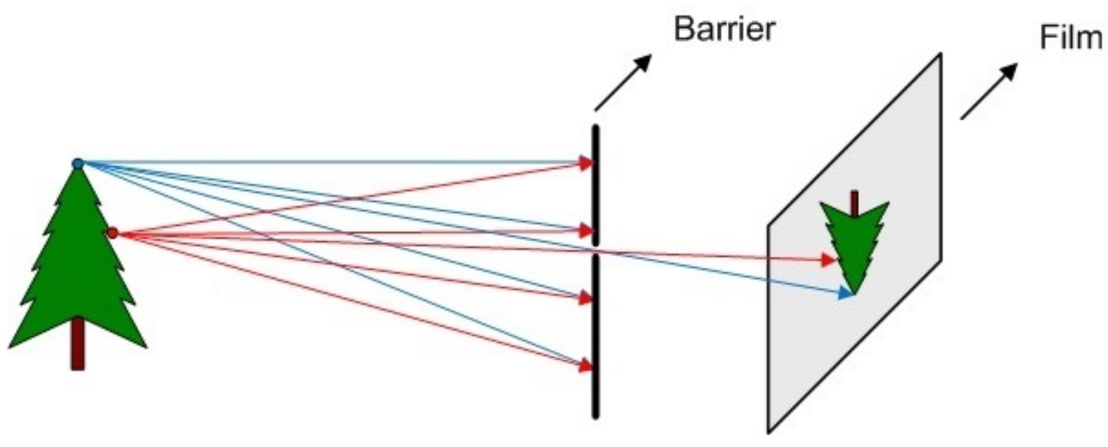
Ray

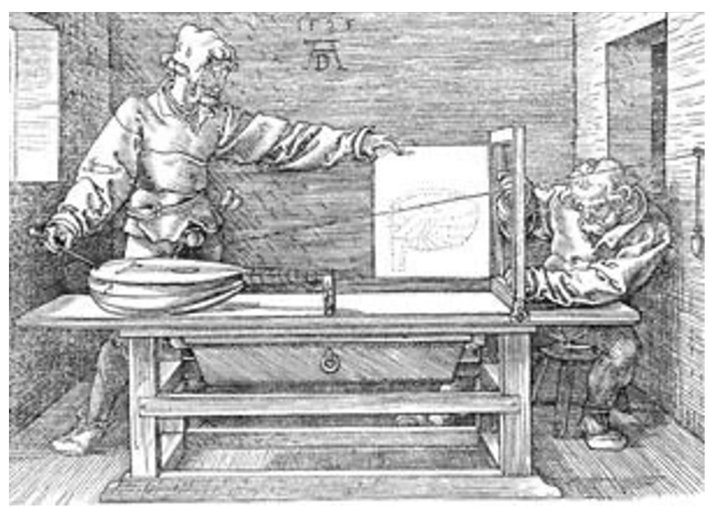
- **Ray parameterization**
  - $r(t) = o + t\vec{d}$, $t \in \mathbb{R}; o, \vec{d} \in \mathbb{R}^3$: origin and direction
- **Ray**
  - All points on the graph of $r(t)$, with t $\in \mathbb{R}_{0+}$



CAMERA MODELS

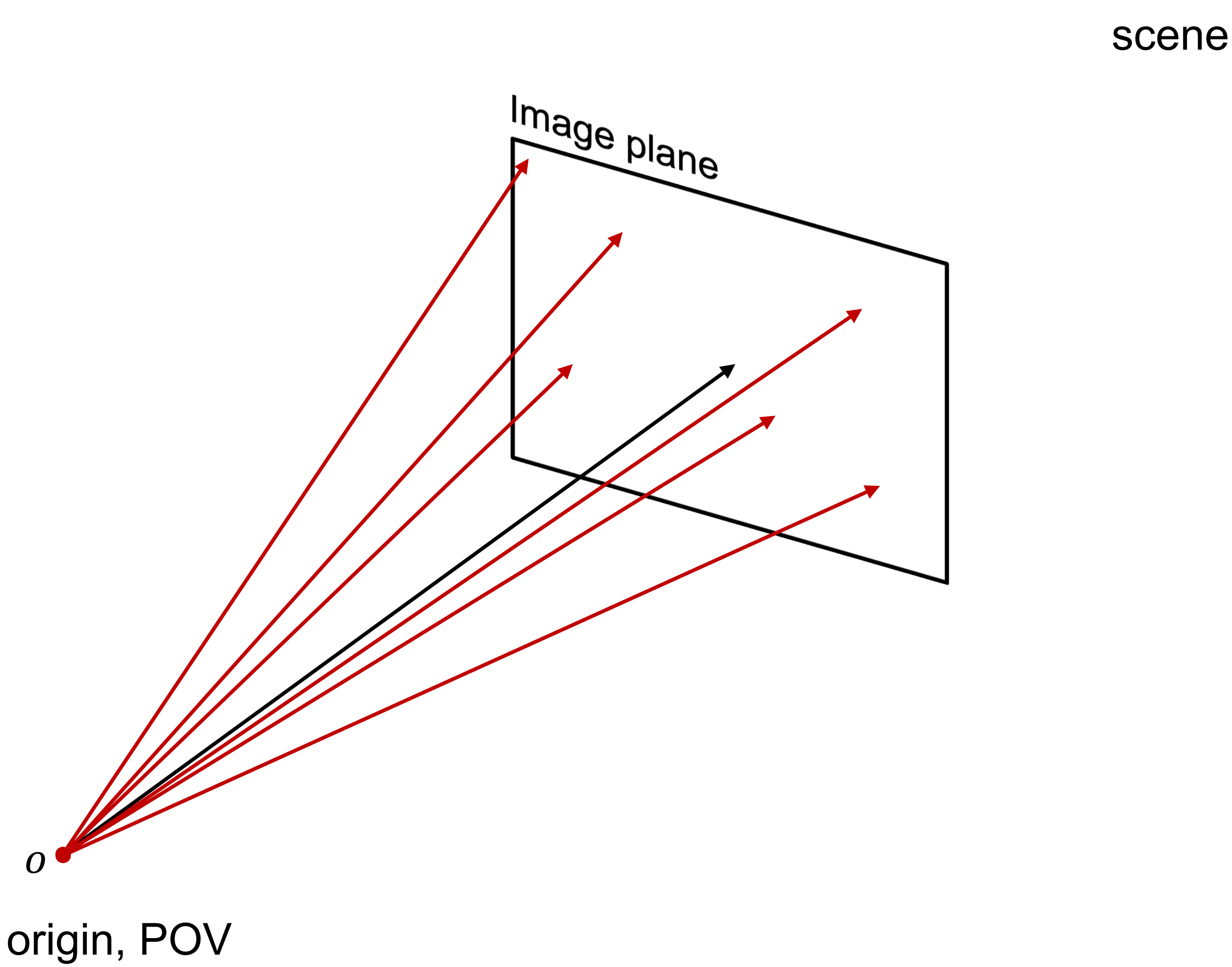
Pinhole Camera Model



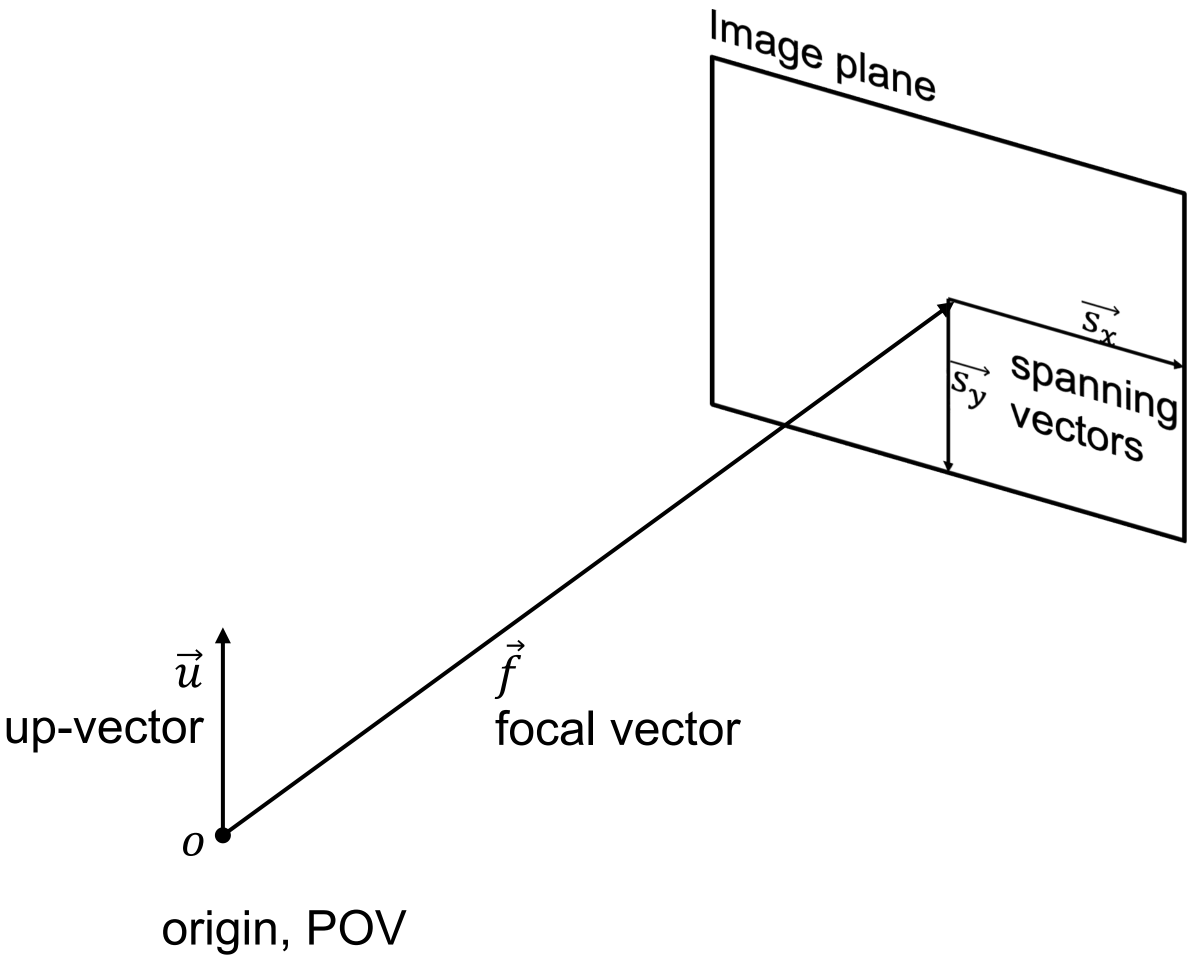


Dürer, 1525

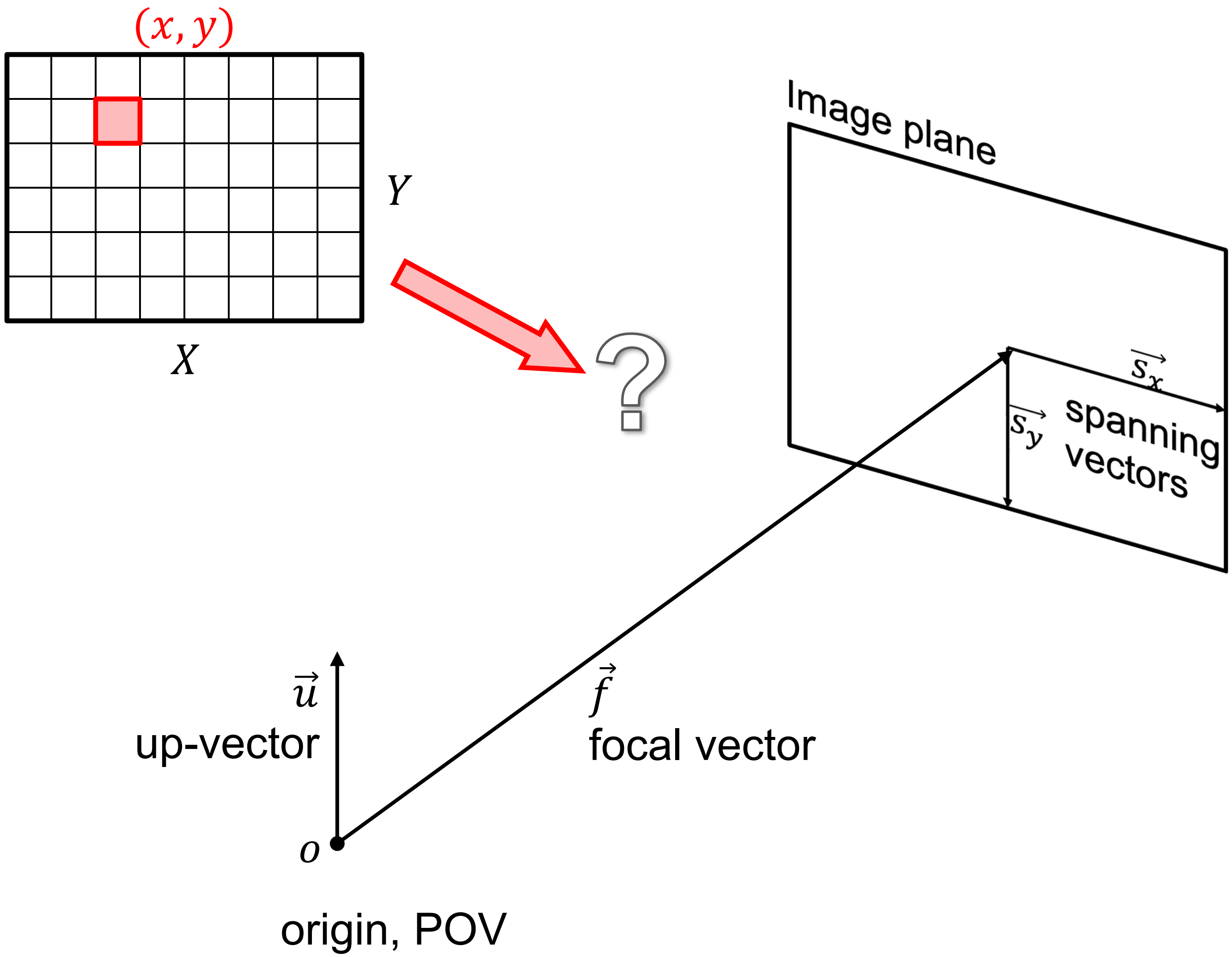
Pinhole Camera Model



Pinhole Camera Model



Pinhole Camera Model



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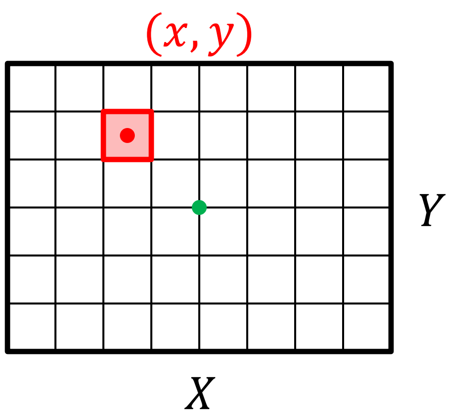
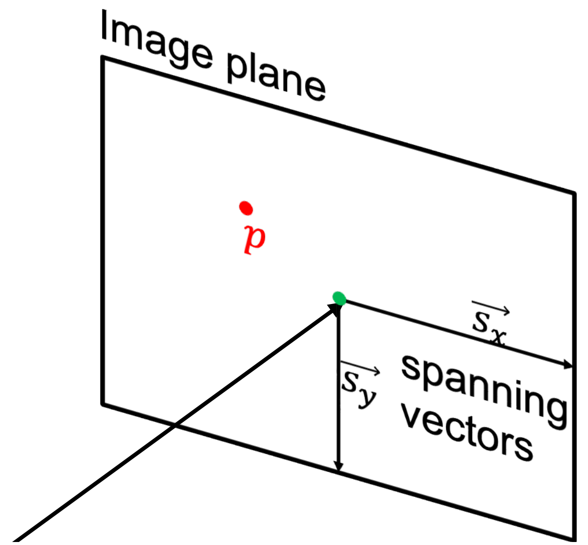




Pixel coordinates to normalized Image plane coordinates

$$p_x = 2\frac{x+0.5}{X} - 1$$
$$p_y = 2\frac{y+0.5}{Y} - 1$$

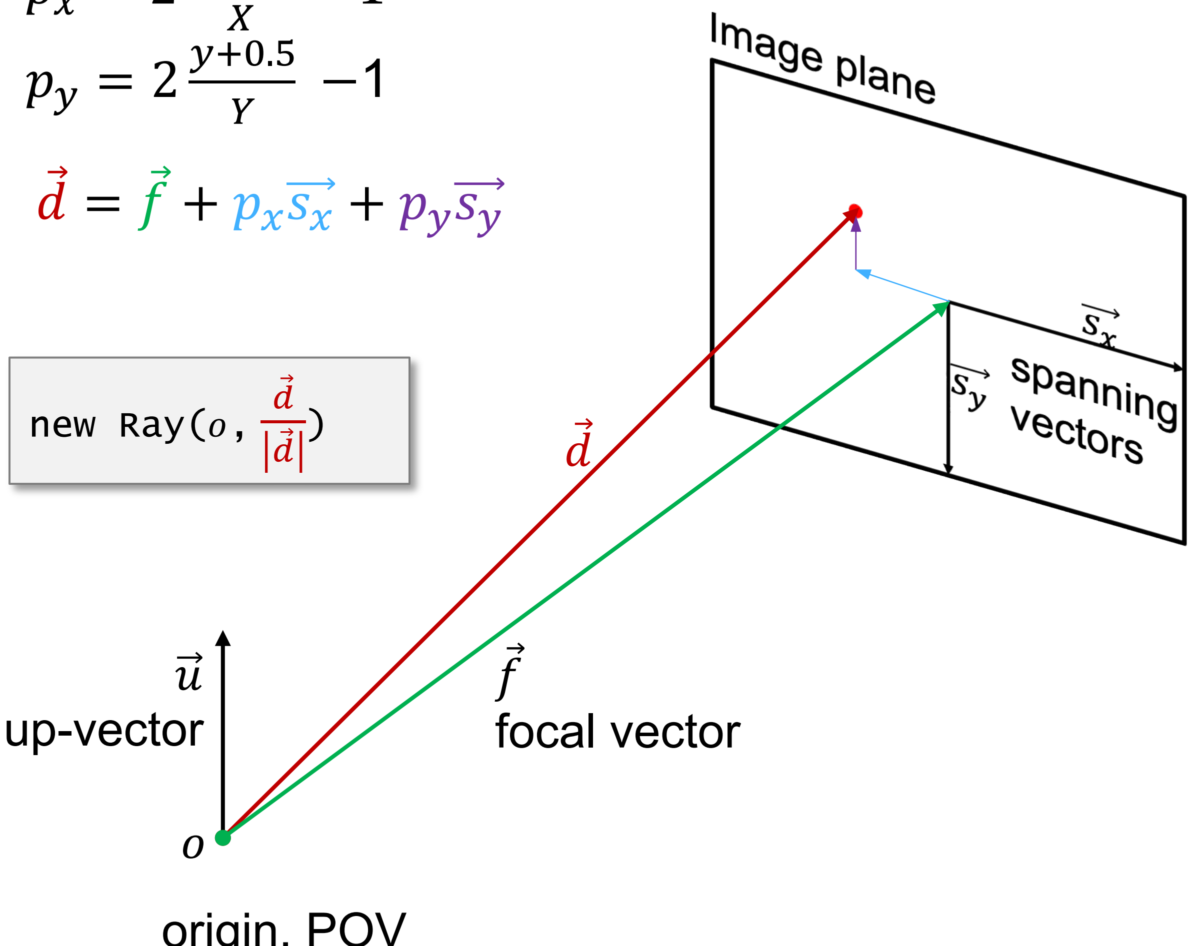
Pinhole Camera Model

$$p_x = 2\,\frac{x+0.5}{X} - 1$$

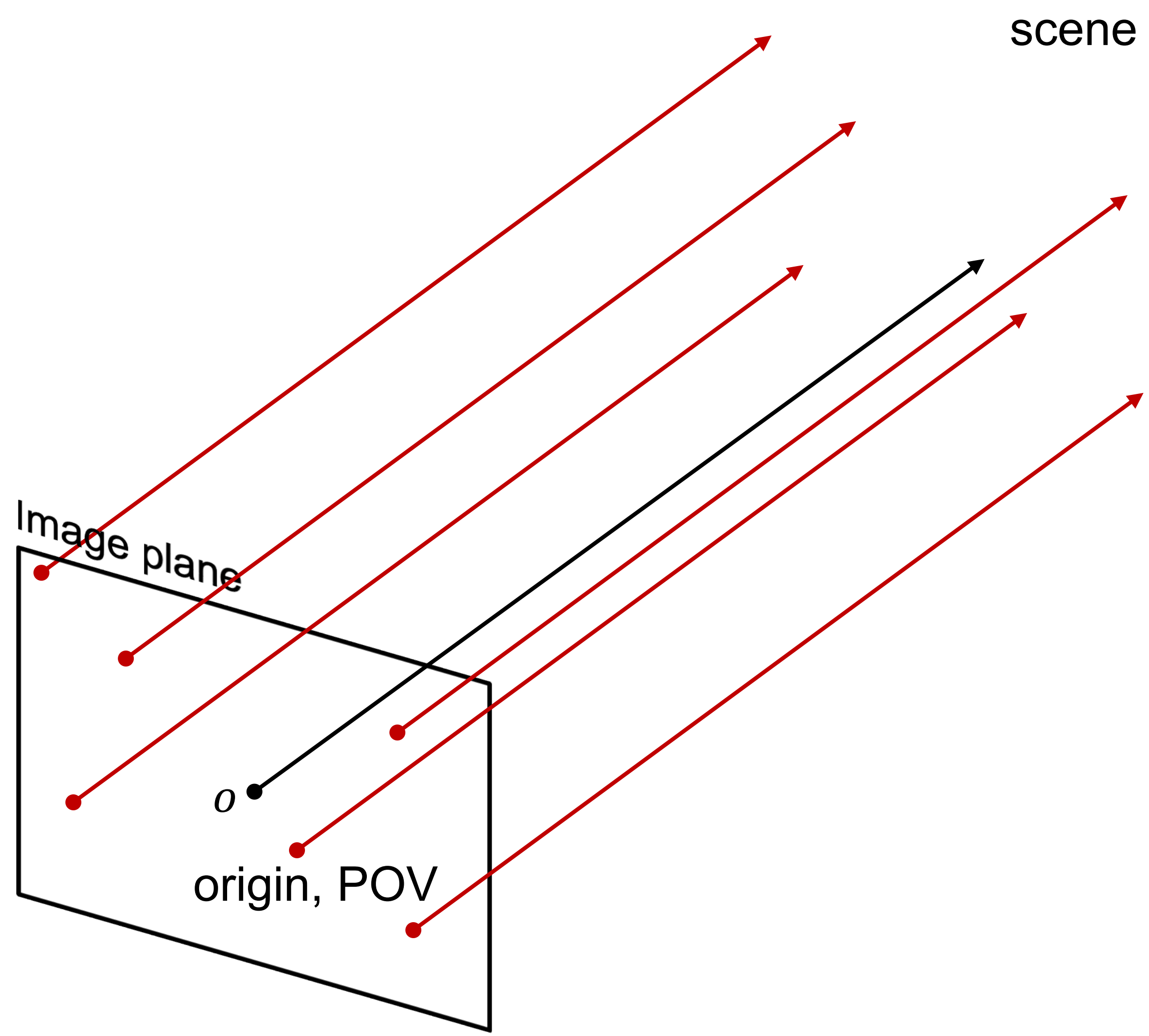
$$p_y = 2\,\frac{y+0.5}{Y} - 1$$

$$\vec{d} = \vec{f} + p_x\overrightarrow{s_x} + p_y\overrightarrow{s_y}$$

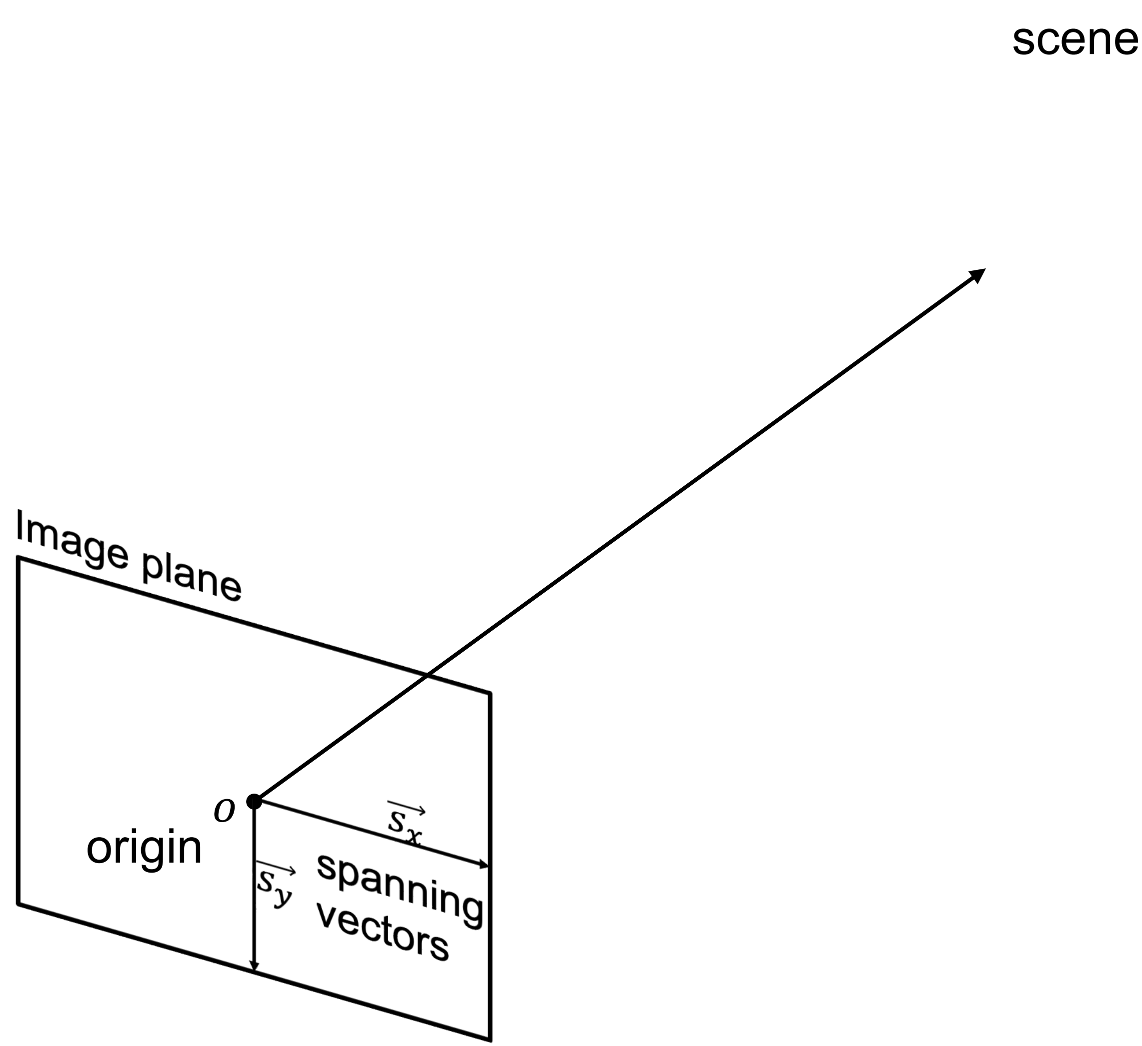
`new Ray(`o`,` $\frac{\vec{d}}{|\vec{d}|}$`)`



Orthographic Camera Model



Orthographic Camera Model

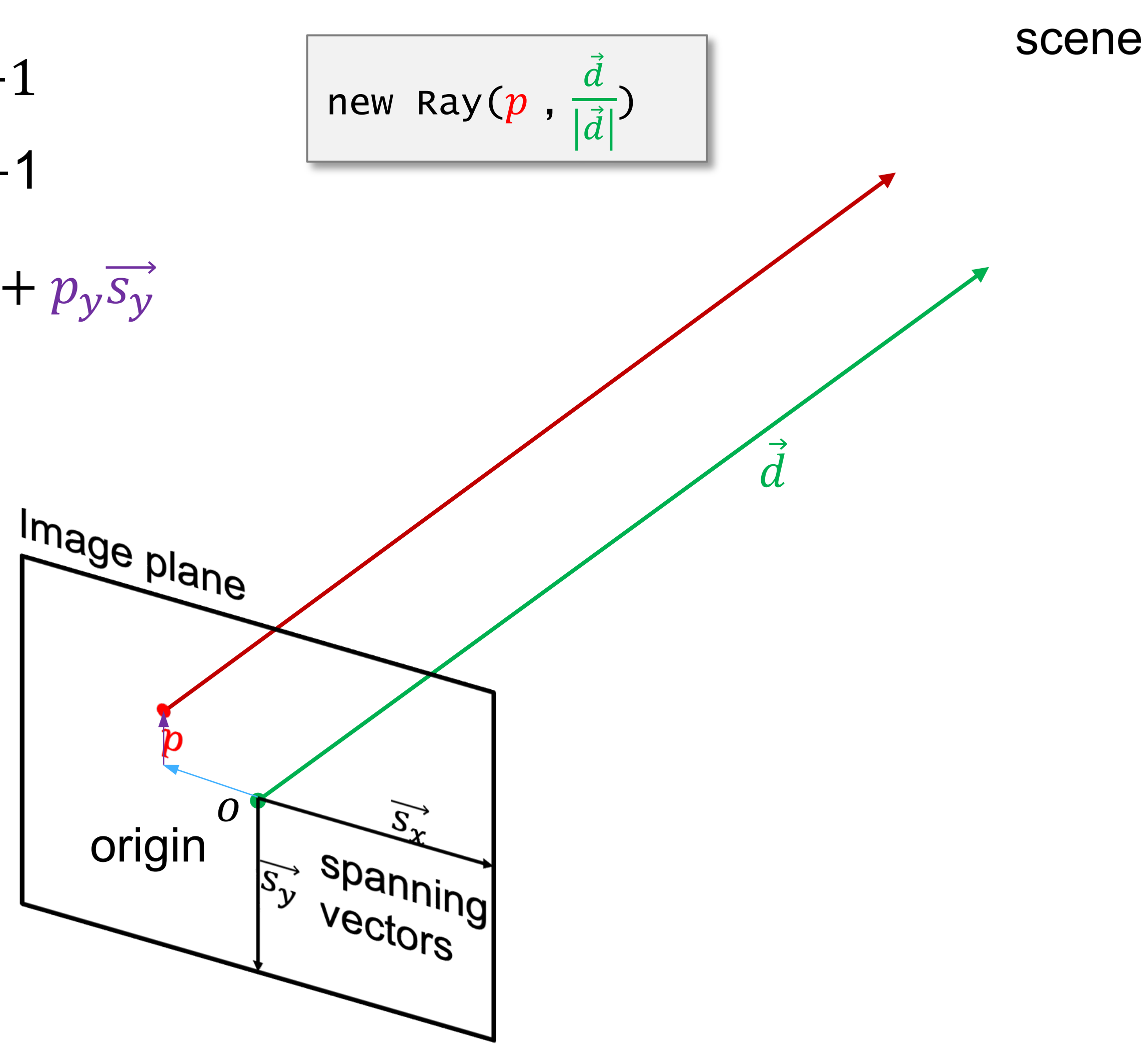


Orthographic Camera Model

$$p_x = 2\,\frac{x+0.5}{X} - 1$$
$$p_y = 2\,\frac{y+0.5}{Y} - 1$$

$$p = o + p_x\,\overrightarrow{s_x} + p_y\overrightarrow{s_y}$$
$$\vec{d} = \overrightarrow{s_x} \times \overrightarrow{s_y}$$

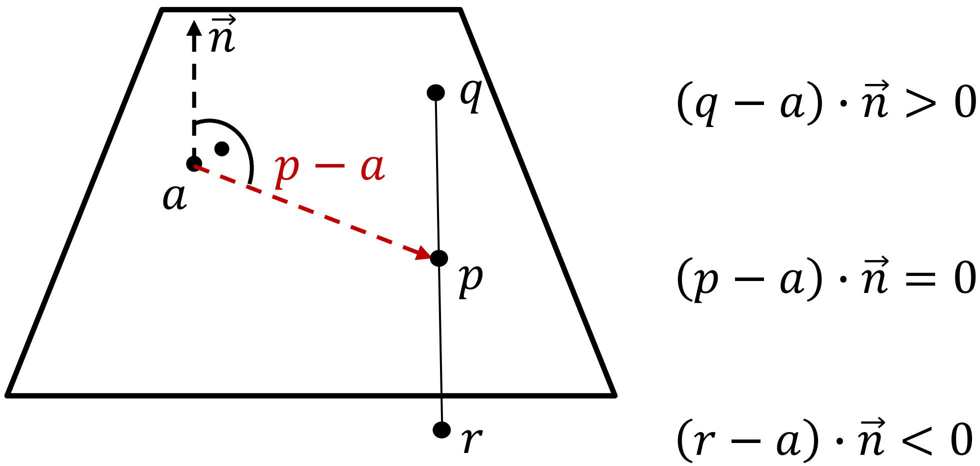
```
new Ray(p , d⃗/|d⃗|)
```



RAY-PRIMITIVE INTERSECTIONS

Plane

- **Plane** P
  - $\vec{n}, a \in \mathbb{R}^3$: normal and point in P
  - $P \coloneqq \{p \in \mathbb{R}^3 : (p - a) \cdot \vec{n} = 0\}$



Ray-Plane Intersection

- **Given**
  - Ray: $r(t) = o + t\vec{d}$, $t \in \mathbb{R}_{0+}$; $o, \vec{d} \in \mathbb{R}^3$
  - Plane: $\vec{n}, a \in \mathbb{R}^3$: normal and point in P
- **Compute intersection point**
  - p – the intersection point

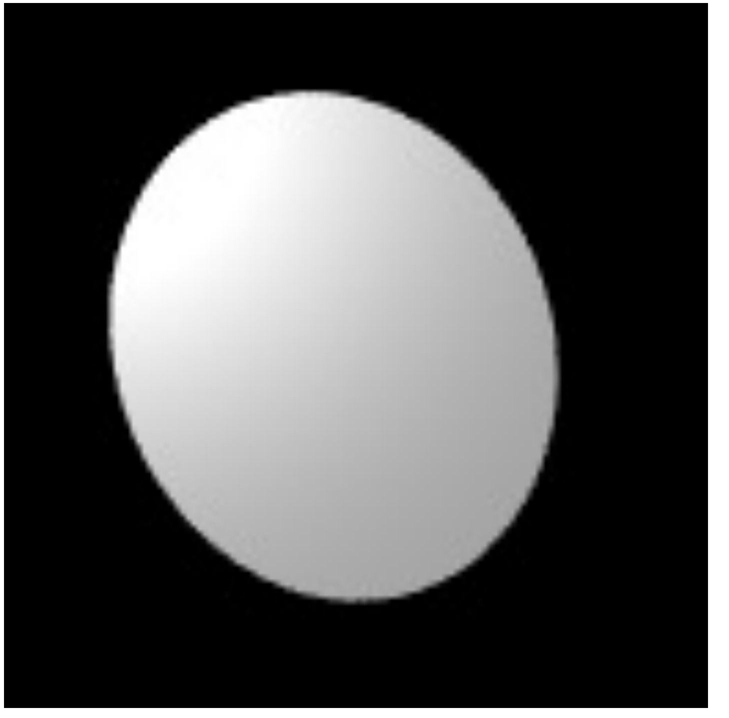
$$\begin{cases} (p - a) \cdot \vec{n} = 0 \\ o + t\vec{d} = p \end{cases}$$

$$\left(o + t\vec{d} - a\right) \cdot \vec{n} = 0$$

Solve for $t \in \mathbb{R}_{0+}$

Ray-Disc Intersection

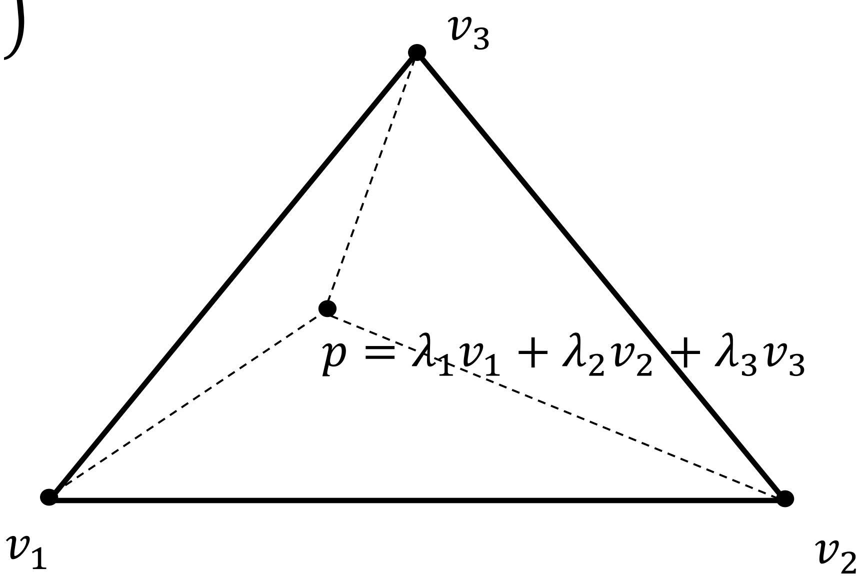
- **Disc** D
  - $\vec{n}, a \in \mathbb{R}^3, r \in \mathbb{R}_+$:
    - normal of disc plane
    - center point of the disc
    - radius of the disc
  - $D := \left\{ \begin{matrix} p \in \mathbb{R}^3: \\ (p-a)\cdot\vec{n} = 0 \\ \wedge\, |p-a| \le r \end{matrix} \right\}$



- **Intersection**
  - Intersect ray with plane
  - Discard intersection if $|p-a|$ >r

Triangle

- **Triangle** T
  - $v_1, v_2, v_3 \in \mathbb{R}^3$: vertices
  - Affine: $\lambda_1 v_1 + \lambda_2 v_2 + \lambda_3 v_3 \rightarrow$ points in the plane
  - $\lambda_1, \lambda_2, \lambda_3 \in \mathbb{R}_{0+} \rightarrow$ points in the triangle
  - $T = \left\{ \begin{matrix} p \in \mathbb{R}^3 : \exists \lambda_{1,2,3} \in \mathbb{R}_{0+} : \\ \lambda_1 + \lambda_2 + \lambda_3 = 1 \wedge \\ p = \lambda_1 v_1 + \lambda_2 v_2 + \lambda_3 v_3 \end{matrix} \right\}$



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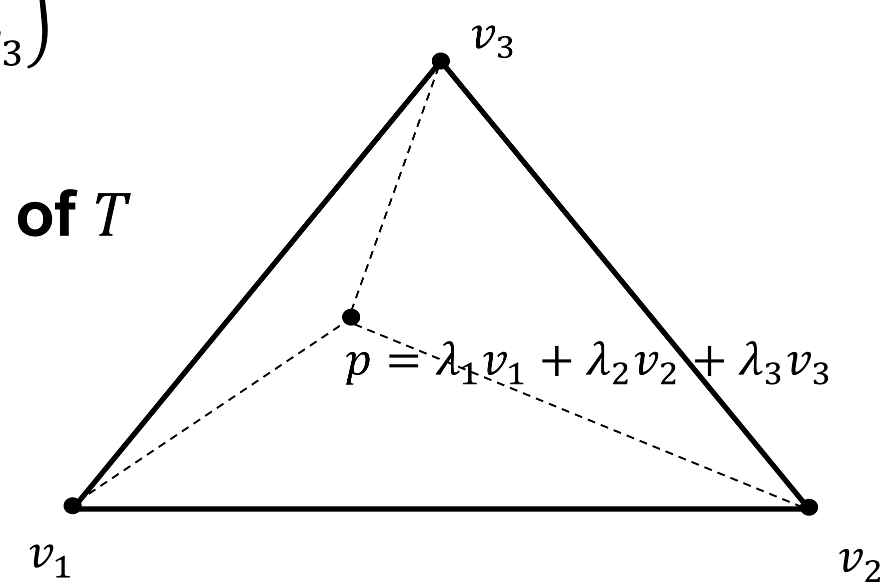
- **Barycentric coordinates of** T
  - $\lambda_{1,2,3}$
  - unique



Triangle

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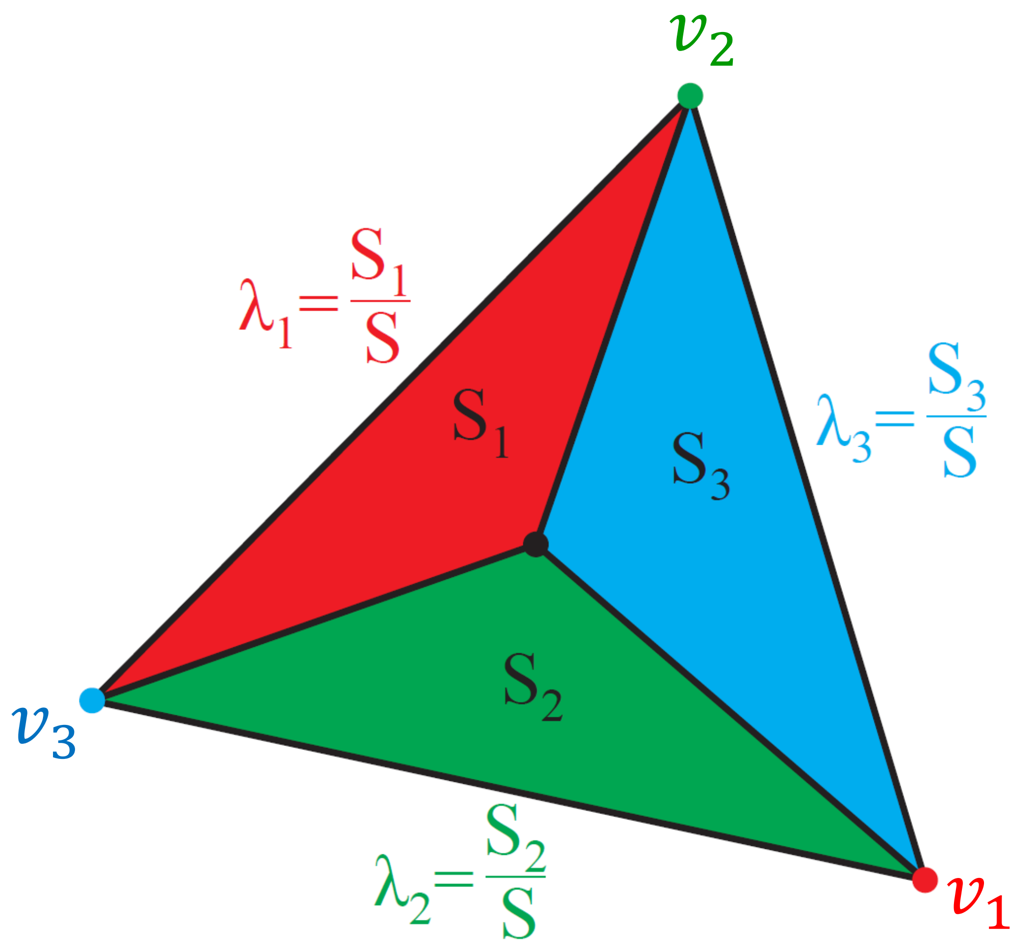

Triangle

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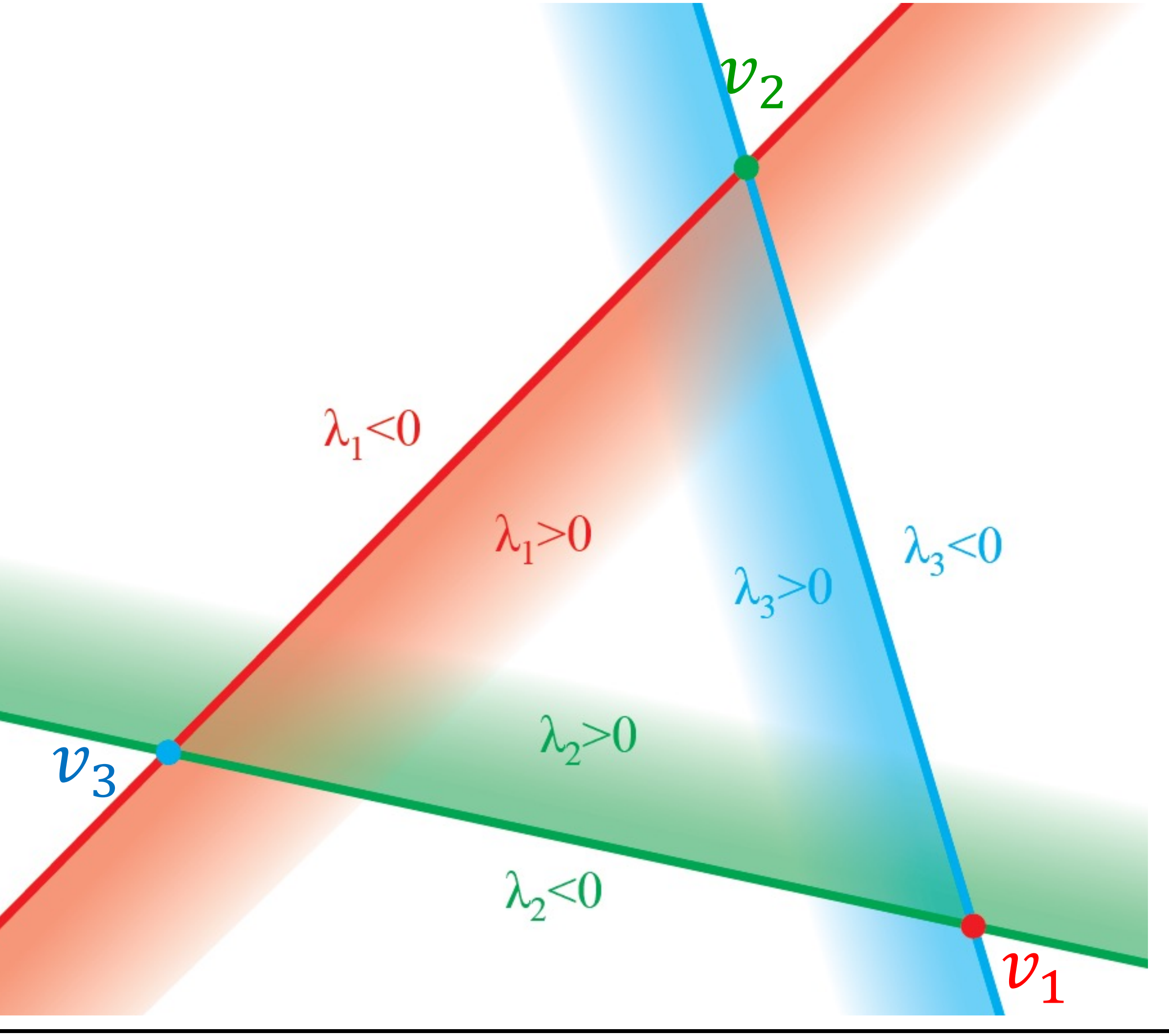
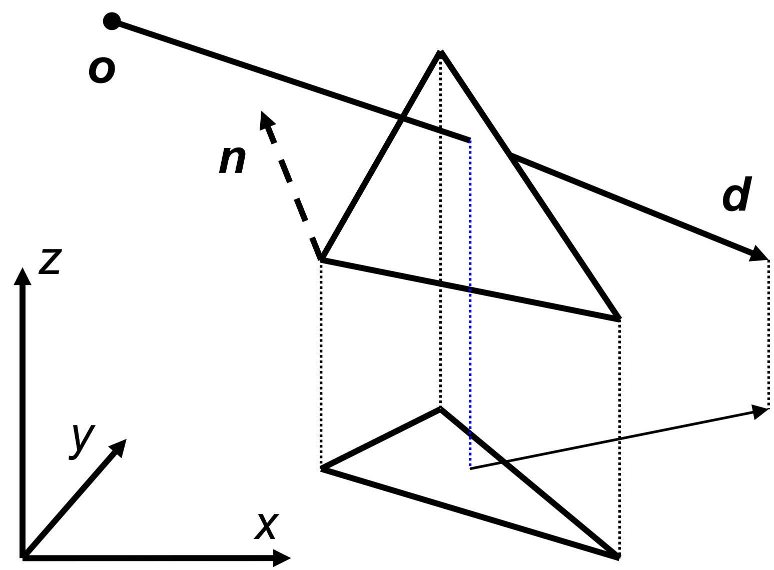


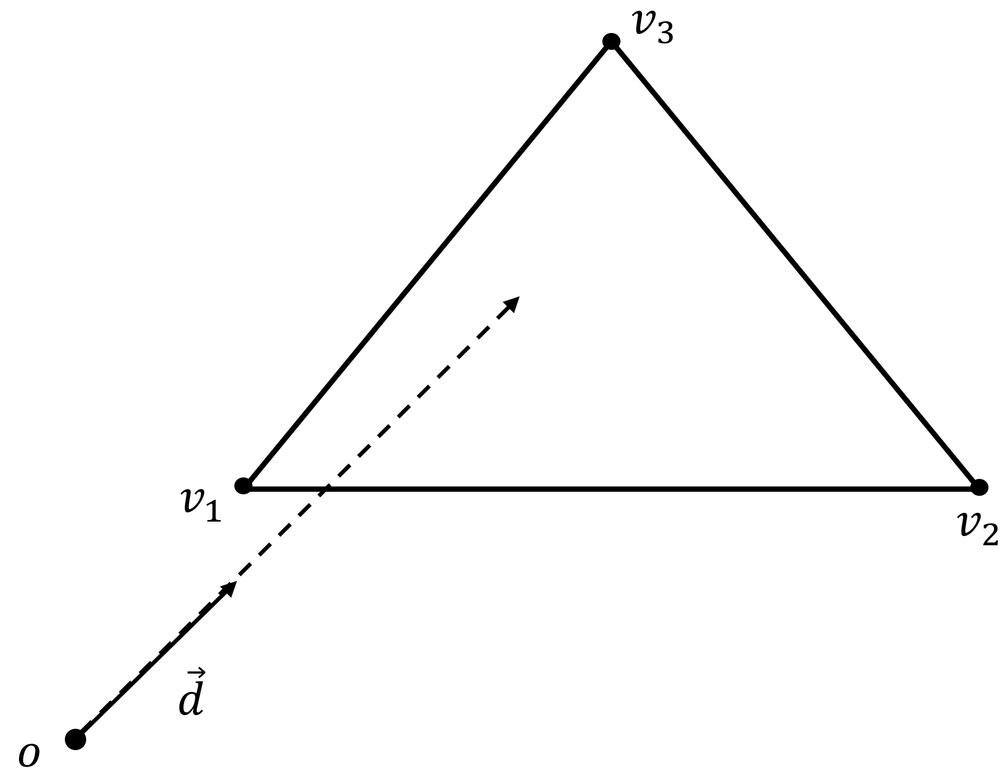
Triangle Intersection: Plane-Based

- **Compute intersection with triangle plane**
- **Project onto a coordinate plane**
  - Use the most aligned coordinate plane
    - xy: if n_z is maximal, etc.
  - Coordinate plane and 2D vertices can be pre-computed
- **Compute barycentric coordinates**
  - Signed areas of subtriangles
- **Test for positive BCs**



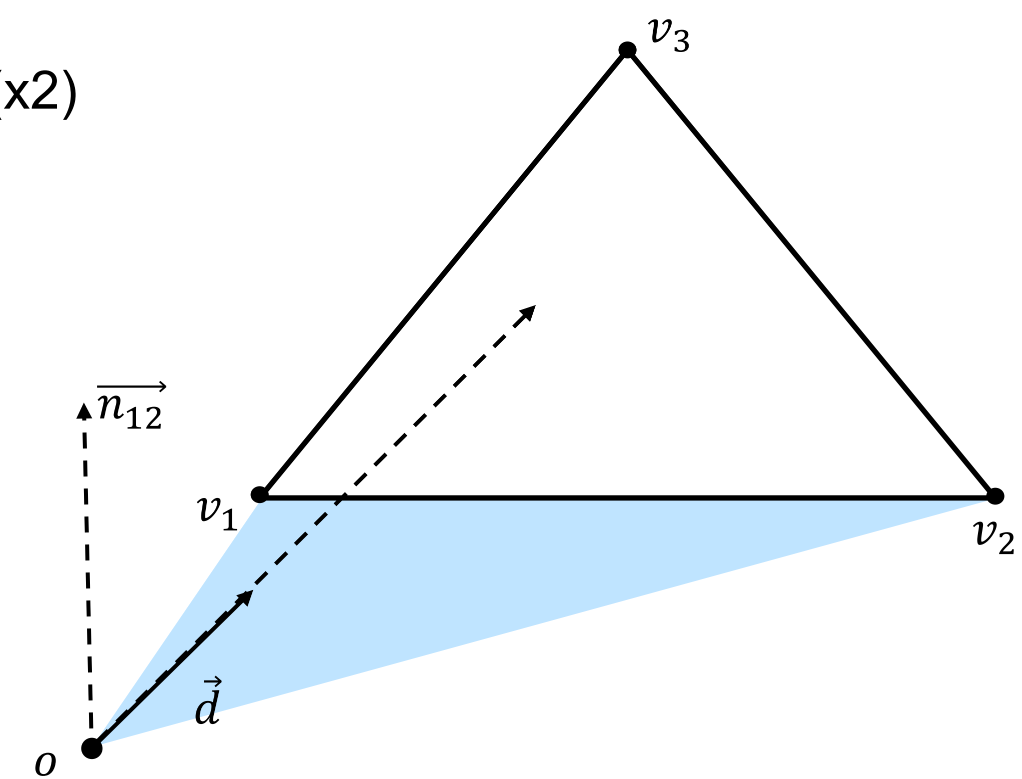
Triangle Intersection Edge-Based (1)

- **3D Linear Function across triangle**
  - Ray: $o + t\vec{d}$, t $\in \mathbb{R}$; $o, \vec{d} \in \mathbb{R}^3$
  - Triangle: $v_1, v_2, v_3 \in \mathbb{R}^3$



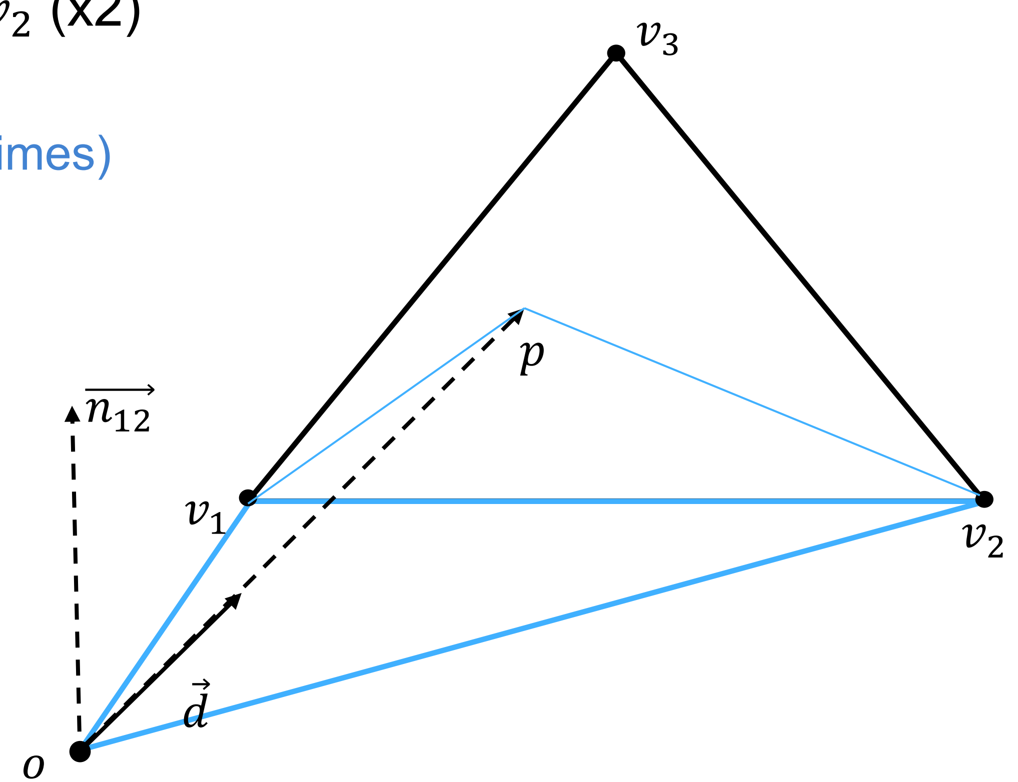
Triangle Intersection Edge-Based (2)

- **3D Linear Function across triangle**
  - Ray: $o + t\vec{d}$, $t \in \mathbb{R}$; $o, \vec{d} \in \mathbb{R}^3$
  - Triangle: $v_1, v_2, v_3 \in \mathbb{R}^3$
  - $\overrightarrow{n_{12}} = (v_2 - o)\times(v_1 - o)$
  - $|\overrightarrow{n_{12}}|$ is the area of ov_1v_2 (x2)



Triangle Intersection Edge-Based (3)

- **3D Linear Function across triangle**
  - Ray: $o + t\vec{d}$, t ∈ $\mathbb{R}$; $o, \vec{d} \in \mathbb{R}^3$
  - Triangle: $v_1, v_2, v_3 \in \mathbb{R}^3$
  - $\overrightarrow{n_{12}} = (v_2 - o) \times (v_1 - o)$
  - $|\overrightarrow{n_{12}}|$ is the area of ov_1v_2 (x2)
  - $\lambda_3^*(t) = \overrightarrow{n_{12}} \cdot t\vec{d}$
    - Volume of ov_1v_2p (6 times)
    - For $t = t_{hit}$



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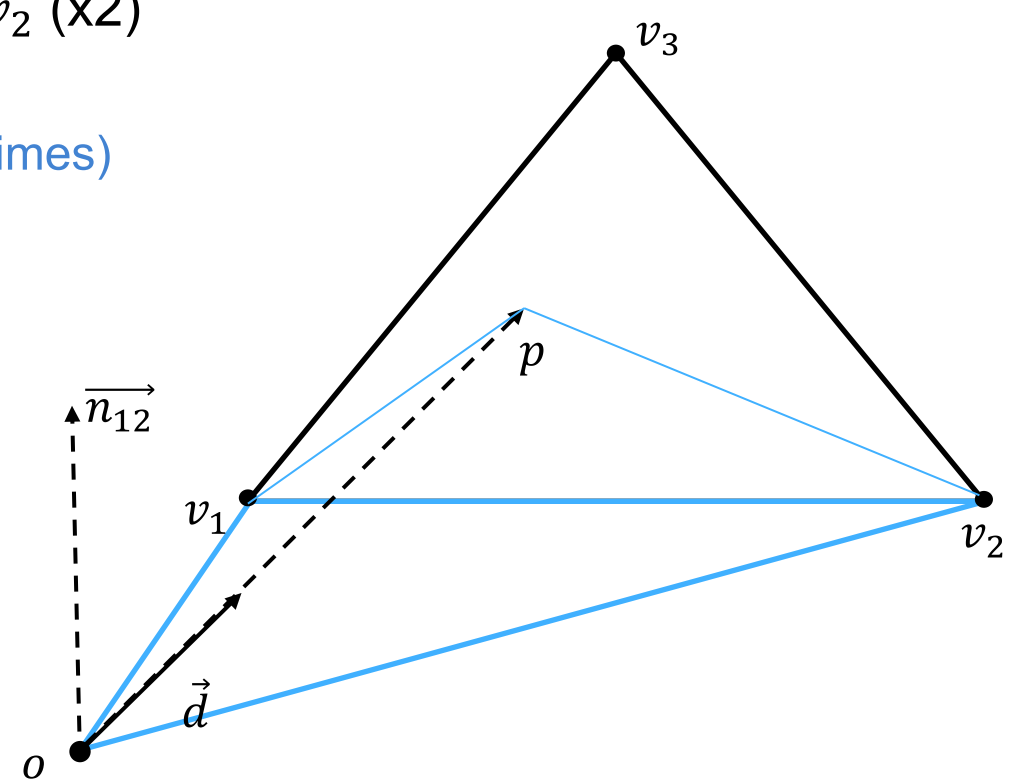
    - Volume of ov_1v_2p (6 times)
    - For $t = t_{hit}$
  - $\lambda_1^*(t) = \overrightarrow{n_{23}} \cdot t\vec{d}$
  - $\lambda_2^*(t) = \overrightarrow{n_{31}} \cdot t\vec{d}$



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  - $\lambda_1^*(t) = \overrightarrow{n_{23}} \cdot t\vec{d}$
  - $\lambda_2^*(t) = \overrightarrow{n_{31}} \cdot t\vec{d}$
  - Normalize:
    - $\lambda_i = \frac{\lambda_i^*(t)}{\lambda_1^*(t)+\lambda_2^*(t)+\lambda_3^*(t)}$, t cancles out

Triangle Intersection Edge-Based (3)

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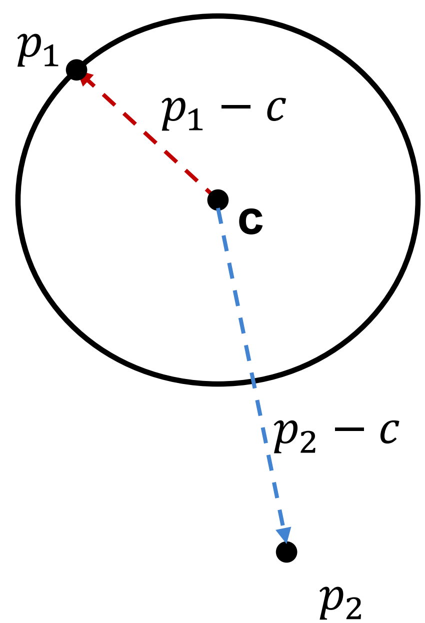
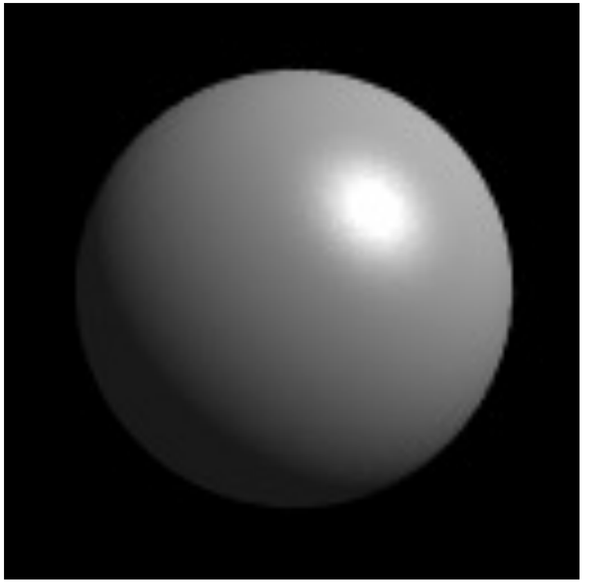
  - $\lambda_1^*(t) = \overrightarrow{n_{23}} \cdot t\vec{d}$
  - $\lambda_2^*(t) = \overrightarrow{n_{31}} \cdot t\vec{d}$
  - Normalize:
    - $\lambda_i = \frac{\lambda_i^*(t)}{\lambda_1^*(t)+\lambda_2^*(t)+\lambda_3^*(t)}$, t cancles out
  - Check if positive
    - just check sign of $\overrightarrow{n_{12}} \cdot \vec{d}$, $\overrightarrow{n_{23}} \cdot \vec{d}$, $\overrightarrow{n_{31}} \cdot \vec{d}$
    - no division

Basic Math - Sphere

- **Sphere** S
  - $c \in \mathbb{R}^3, r \in \mathbb{R}_{0+}$: center and radius
  - $S = \{p \in \mathbb{R}^3 : (p - c)^2 - r^2 = 0\}$
  - The distance between the points on the sphere and its center equals the radius





Ray-Sphere Intersection

- **Given**
  - Ray: $r(t) = o + t\vec{d}$, $t \in \mathbb{R}_{0+};\ o, \vec{d} \in \mathbb{R}^3$
  - Sphere: $c \in \mathbb{R}^3, r \in \mathbb{R}_{0+}$: center and radius
- **Compute intersection point**
  - p – the intersection point
  - Algebraic approach: substitute ray equation:

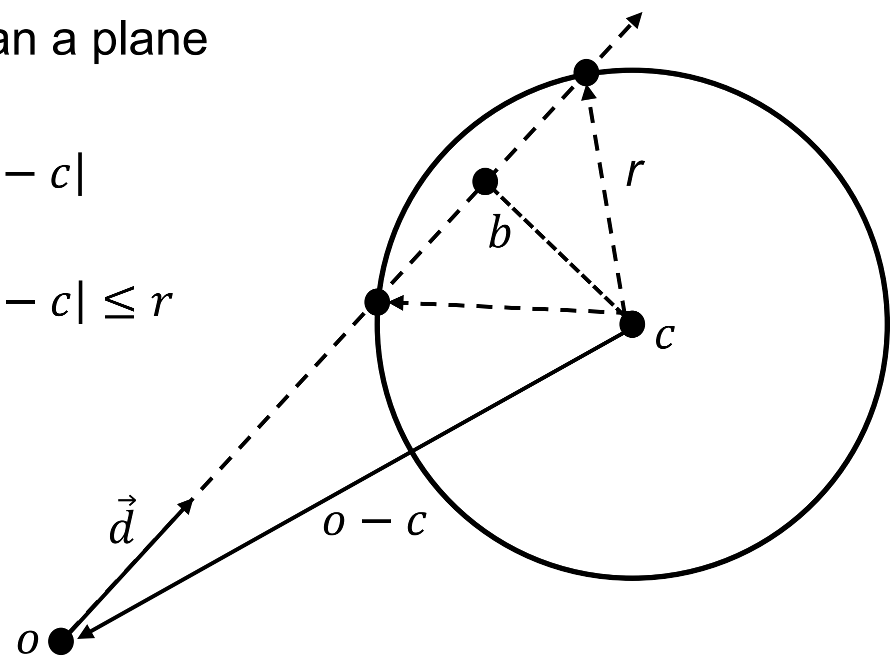
$$\begin{cases} (p - c)^2 - r^2 = 0 \\ o + t\vec{d} = p \end{cases}$$

$$t^2\vec{d}^2 + 2t\vec{d}(o - c)^2 - r^2 = 0$$

Solve for $t \in \mathbb{R}_{0+}$

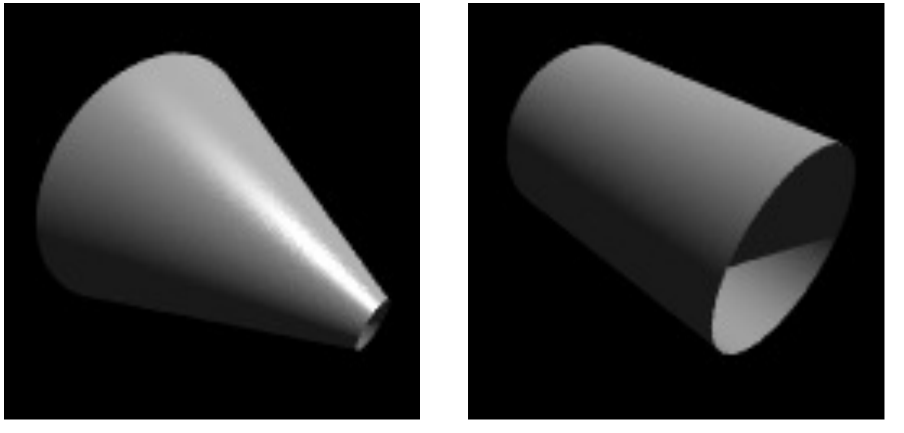
Ray-Sphere Intersection (2)

- **Given**
  - Ray: $r(t) = o + t\vec{d}$, $t \in \mathbb{R}_{0+};\ o, \vec{d} \in \mathbb{R}^3$
  - Sphere: $c \in \mathbb{R}^3, r \in \mathbb{R}_{0+}$: center and radius
- **Compute intersection point**
  - p – the intersection point
  - Geometric approach
    - Ray and center span a plane
    - Solve in 2D
    - Compute $|b - o|, |b - c|$
      - $\measuredangle obc = 90°$
    - Intersection(s) if $|b - c| \leq r$

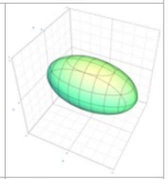
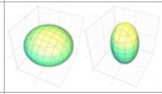
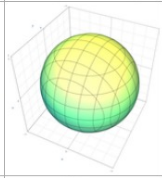
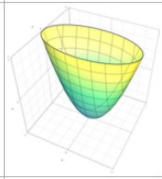
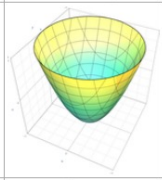
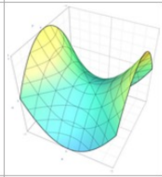
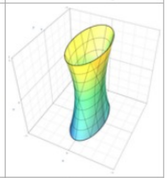
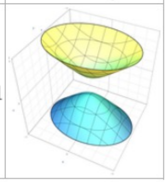


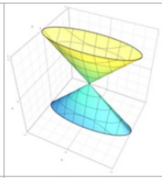
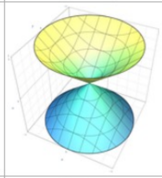
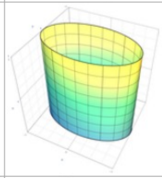
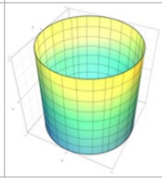
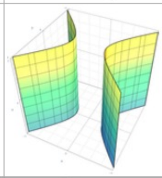
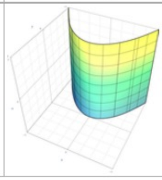
Quadrics

- **Implicit**
  - $f(x, y, z)$
  - $Q(f, v) \coloneqq \{p \in \mathbb{R}^3 : f(p_x, p_y, p_z) = v\}$
- **Intersection**
  - Substitute the ray equation:
    - $f(o_x + td_x, o_y + td_y, o_z + td_z) = v$
  - Solve for $t \in \mathbb{R}_{0+}$



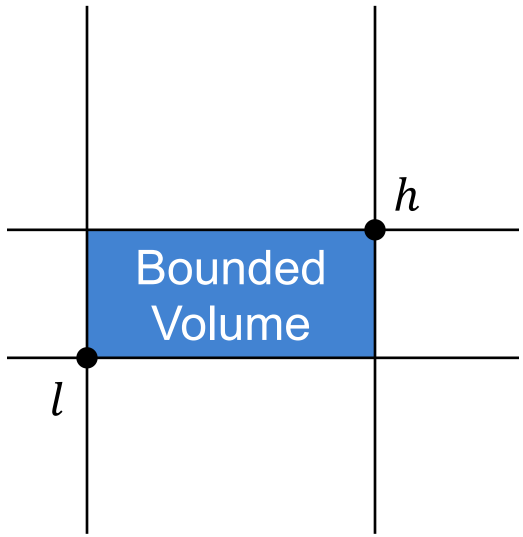
Quadrics

Non-degenerate real quadric surfaces		
Ellipsoid	$\frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c^2}=1$	
Spheroid (special case of ellipsoid)	$\frac{x^2}{a^2}+\frac{y^2}{a^2}+\frac{z^2}{b^2}=1$	
Sphere (special case of spheroid)	$\frac{x^2}{a^2}+\frac{y^2}{a^2}+\frac{z^2}{a^2}=1$	
Elliptic paraboloid	$\frac{x^2}{a^2}+\frac{y^2}{b^2}-z=0$	
Circular paraboloid(special case of elliptic paraboloid)	$\frac{x^2}{a^2}+\frac{y^2}{a^2}-z=0$	
Hyperbolic paraboloid	$\frac{x^2}{a^2}-\frac{y^2}{b^2}-z=0$	
Hyperboloid of one sheet	$\frac{x^2}{a^2}+\frac{y^2}{b^2}-\frac{z^2}{c^2}=1$	
Hyperboloid of two sheets	$\frac{x^2}{a^2}+\frac{y^2}{b^2}-\frac{z^2}{c^2}=-1$	

Degenerate quadric surfaces		
Cone	$\frac{x^2}{a^2}+\frac{y^2}{b^2}-\frac{z^2}{c^2}=0$	
Circular Cone (special case of cone)	$\frac{x^2}{a^2}+\frac{y^2}{a^2}-\frac{z^2}{b^2}=0$	
Elliptic cylinder	$\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$	
Circular cylinder (special case of elliptic cylinder)	$\frac{x^2}{a^2}+\frac{y^2}{a^2}=1$	
Hyperbolic cylinder	$\frac{x^2}{a^2}-\frac{y^2}{b^2}=1$	
Parabolic cylinder	$x^2+2ay=0$	

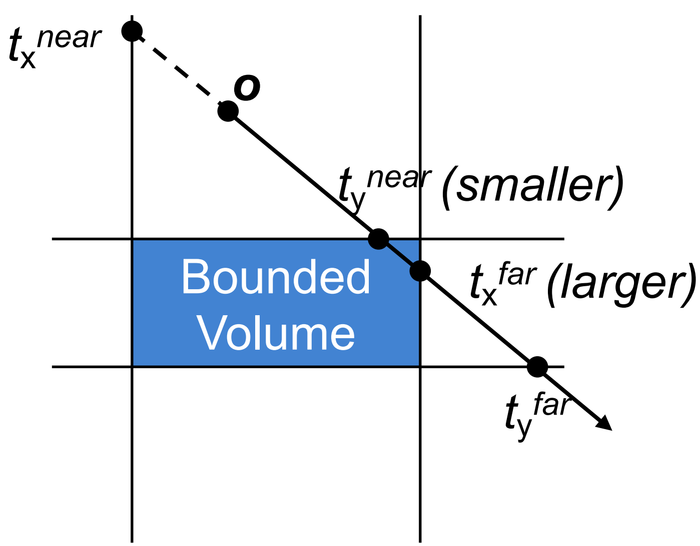
Axis Aligned Bounding Box

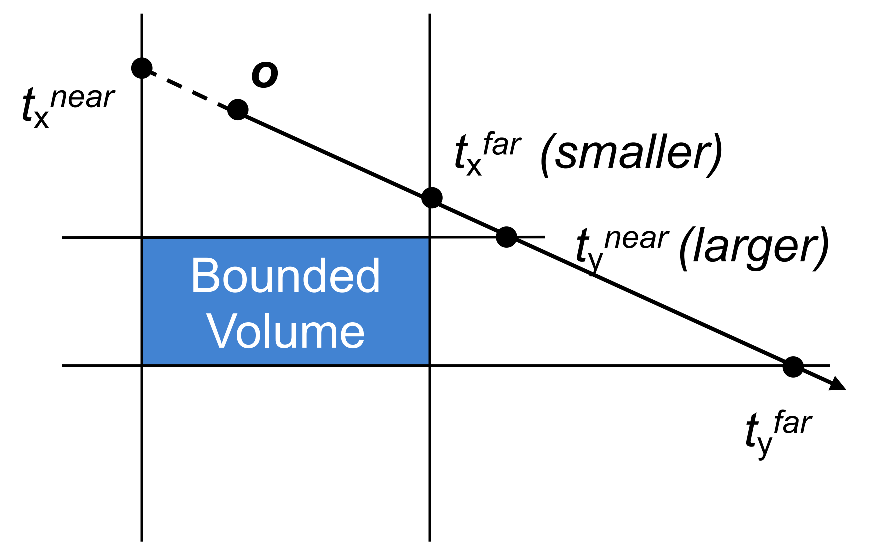
- **Given**
  - Ray: $r(t) = o + t\vec{d}$, $t \in \mathbb{R}_{0+}$; $o, \vec{d} \in \mathbb{R}^3$
  - Axis aligned bounding box (AABB): $l, h \in \mathbb{R}^3$



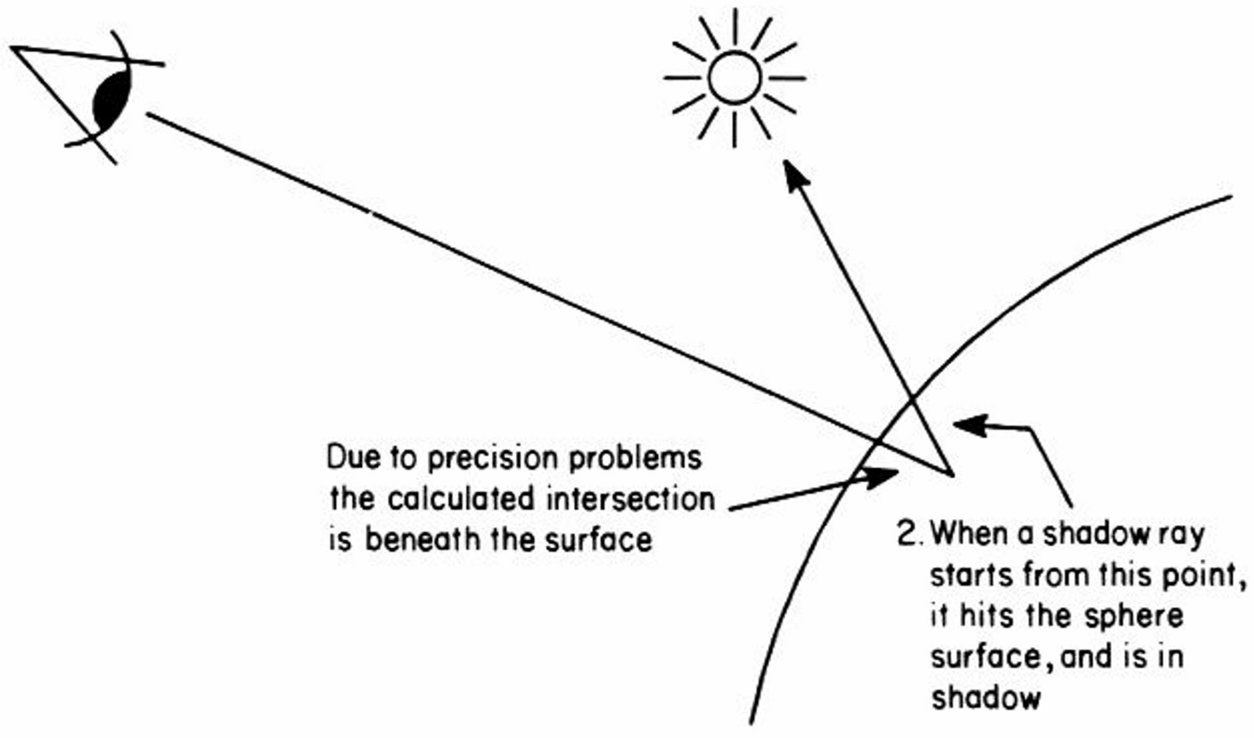
Ray-Box Intersection

- **Given**
  - Ray: $r(t) = o + t\vec{d}$, $t \in \mathbb{R}_{0+};\ o, \vec{d} \in \mathbb{R}^3$
  - Axis aligned bounding box (AABB): $l, h \in \mathbb{R}^3$
- **"Slabs test" for ray-box intersection**
  - Ray enters the box in all dimensions before exiting in any
  - $\max(\{t_i^{near} | i = x, y, z\}) < \min(\{t_i^{far} | i = x, y, z\})$





Precision Problems



Problem in surface intersection.

History of Intersection Algorithms

- **Ray-geometry intersection algorithms**
  - Polygons: [Appel ’68]
  - Quadrics, CSG: [Goldstein & Nagel ’71]
  - Recursive Ray Tracing: [Whitted ’79]
  - Tori: [Roth ’82]
  - Bicubic patches: [Whitted ’80, Kajiya ’82]
  - Algebraic surfaces: [Hanrahan ’82]
  - Swept surfaces: [Kajiya ’83, van Wijk ’84]
  - Fractals: [Kajiya ’83]
  - Deformations: [Barr ’86]
  - NURBS: [Stürzlinger ’98]
  - Subdivision surfaces: [Kobbelt et al ’98]