

Computer Graphics

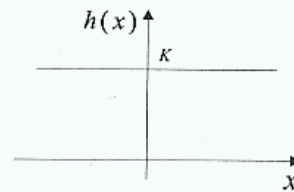
Sampling Theory & Anti-Aliasing

Philipp Slusallek

Dirac Comb Function (1)

- **Constant & δ -function**

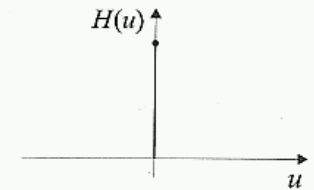
Ortsbereich



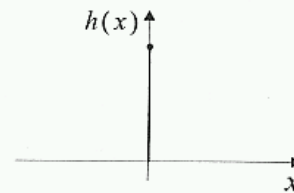
Konstante Funktion



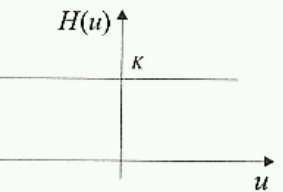
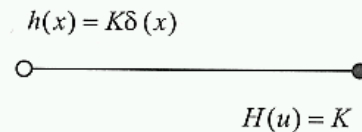
Ortsfrequenzbereich



Delta-Funktion

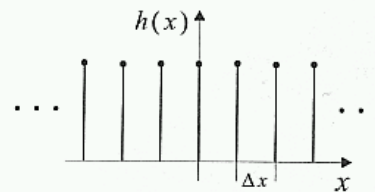


Delta-Funktion

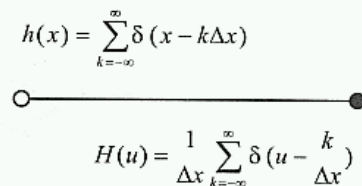


Konstante Funktion

- **Comb/Shah function**



Kamm-Funktion



Kamm-Funktion

Dirac Comb (2)

- **Constant & δ -Function**

- Duality

$$f(x) = K$$

$$F(\omega) = K \delta(\omega)$$

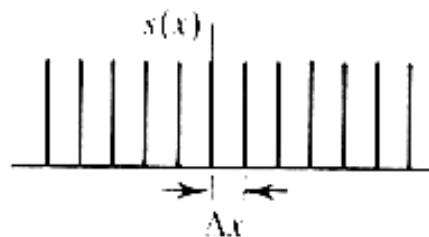
- And vice versa

- **Comb function**

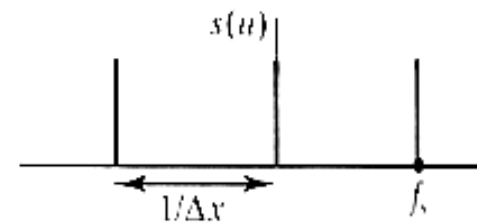
- Duality: the dual of a comb function is again a comb function

- Inverse wavelength
- Amplitude scales with inverse wavelength

$$f(x) = \sum_{k=-\infty}^{\infty} \delta(x - k\Delta x)$$
$$F(\omega) = \frac{1}{\Delta x} \sum_{k=-\infty}^{\infty} \delta\left(\omega - k\frac{1}{\Delta x}\right)$$



(b)



Sampling

- **Continuous function**

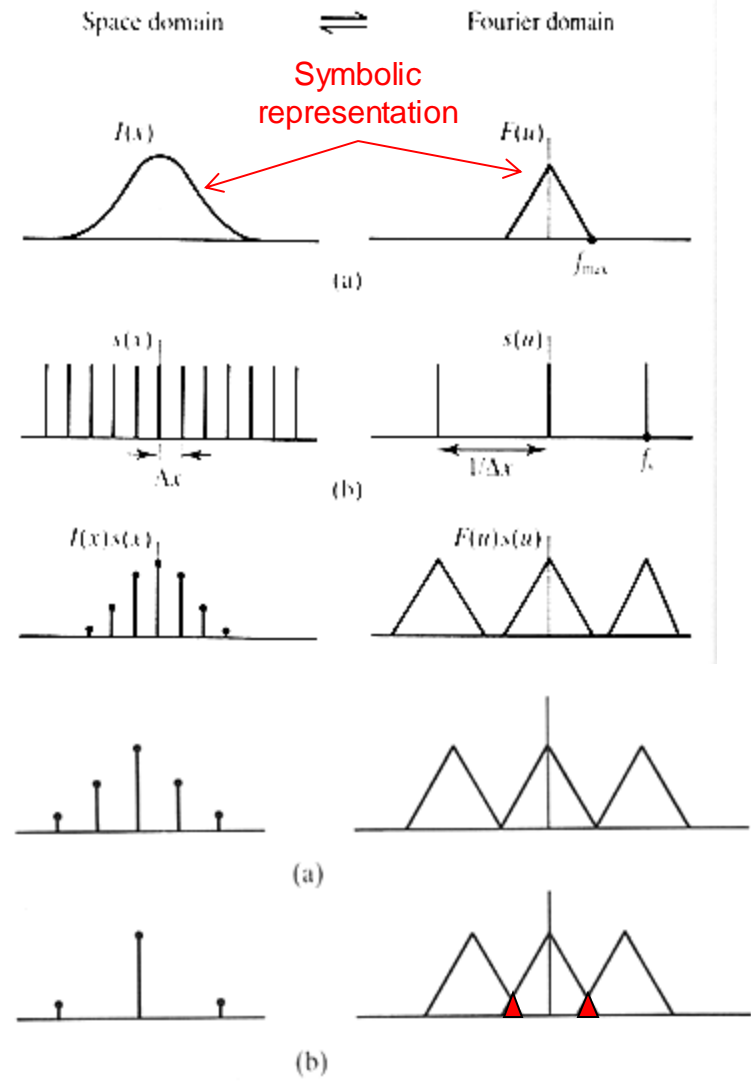
- Assume band-limited
- Finite support of Fourier transform
 - Depicted symbolically here as triangle-shaped finite spectrum (not meant to be a tent function)

- **Sampling at discrete points**

- Multiplication with Comb function in spatial domain
- Corresponds to convolution in Fourier domain
 - ⇒ Multiple copies of the original spectrum (convolution theorem!)

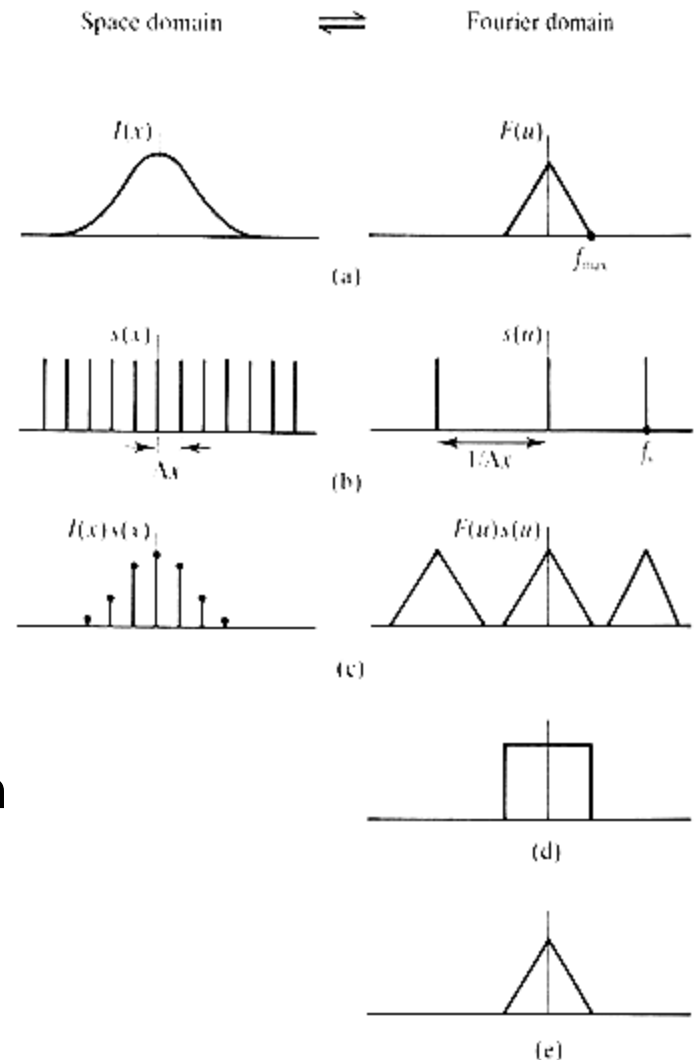
- **Frequency bands overlap ?**

- No : Sampling was high enough
- Yes: **aliasing artifacts**



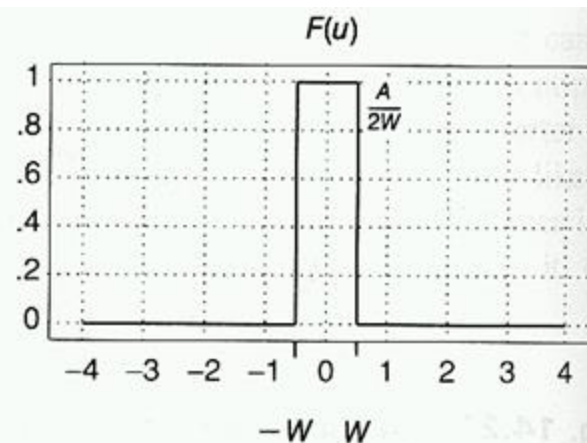
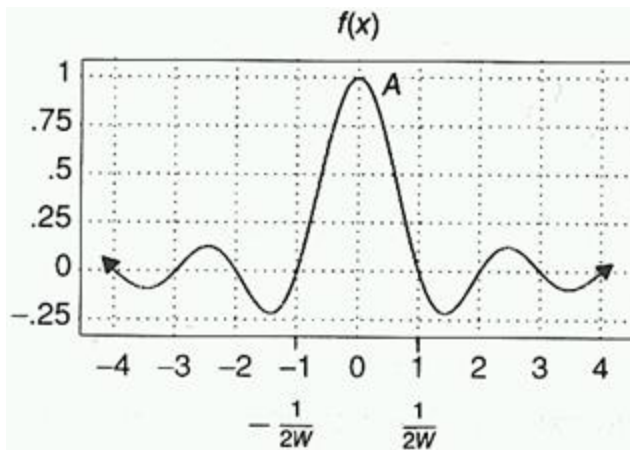
Reconstruction

- **Only original frequency band desired**
- **Filtering**
 - In Fourier domain:
 - Multiplication with windowing function around origin (low-pass filter)
 - In spatial domain
 - Convolution with inverse Fourier transform of windowing function
- **Optimal filtering function**
 - Box function in Fourier domain
 - Corresponds to *sinc* in spatial domain
 - Unlimited region of support
 - Spatial domain only allows approximations due to finite support of practical filters

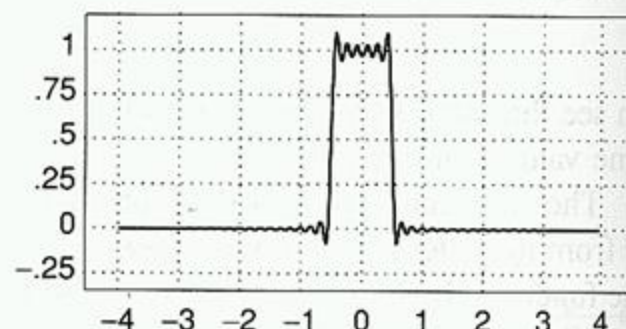
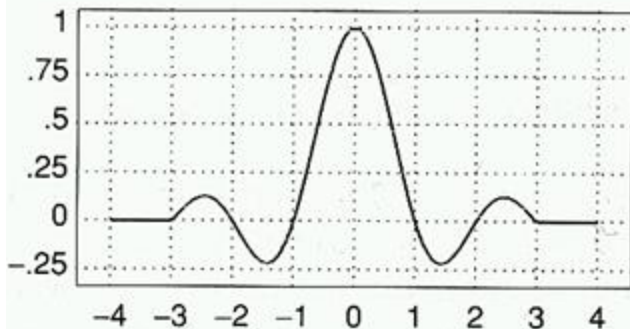


Reconstruction Filter

- Simply cutting off the spatial support of the *sinc* function to limit support is **NOT** a good solution
 - Re-introduces high-frequencies \Rightarrow spatial ringing



(a)



(b)

Sampling and Reconstruction

Original function and its band-limited frequency spectrum

Signal sampling beyond Nyquist:

Mult./conv. with comb

Frequency spectrum is replicated

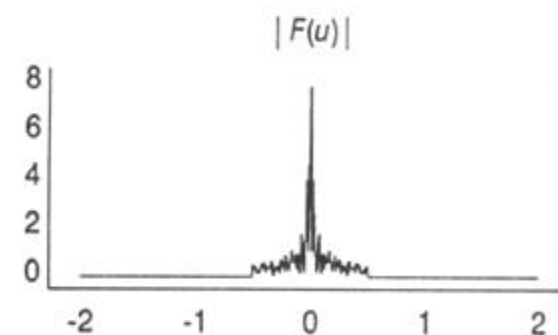
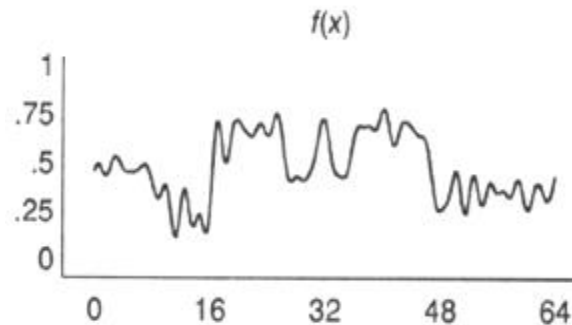
Comb dense enough (sampling rate $> 2 \cdot \text{bandlimit}$)

Bands do not overlap

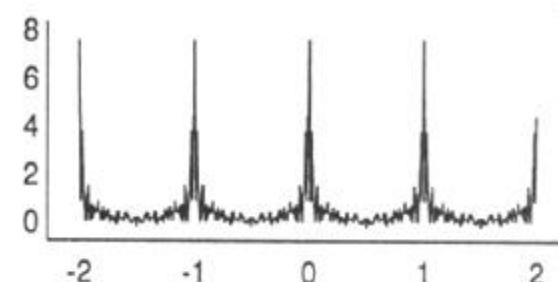
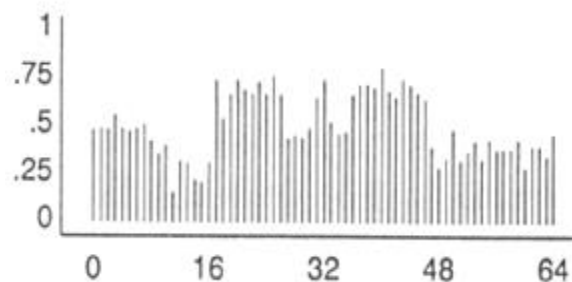
Ideal filtering

Fourier: box (mult.)
Space: *sinc* (conv.)

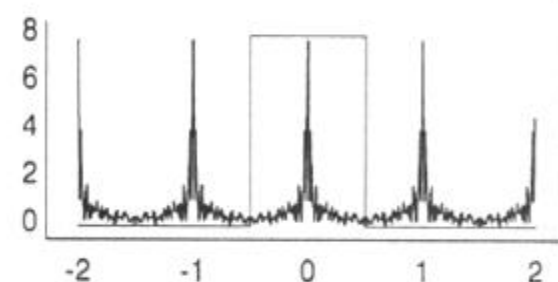
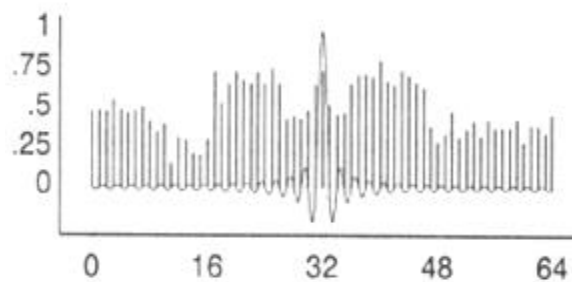
Only one copy



(a)



(b)

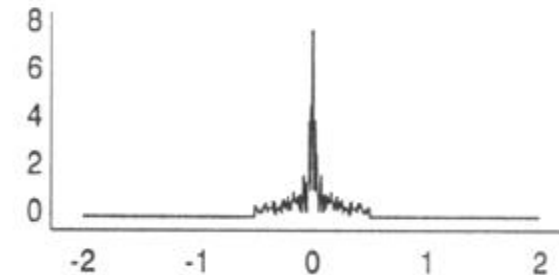
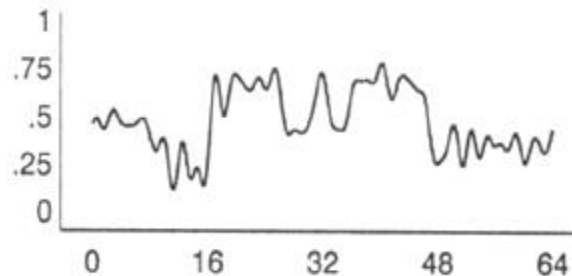


(c)

Sampling and Reconstruction

Reconstruction
with ideal *sinc*

Identical signal

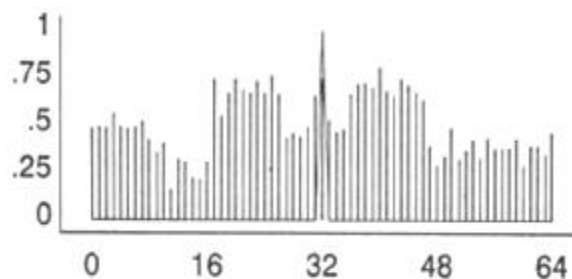


Non-ideal filtering

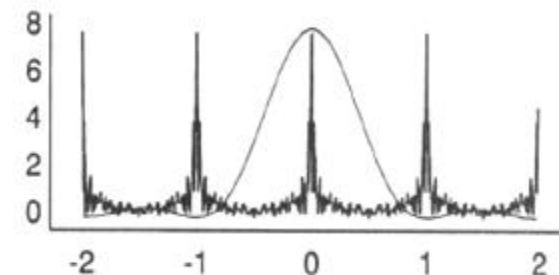
Fourier: sinc^2 (mult.)
Space: tent (conv.)

**Artificial high frequen.
are not cut off**

⇒ **Aliasing artifacts**

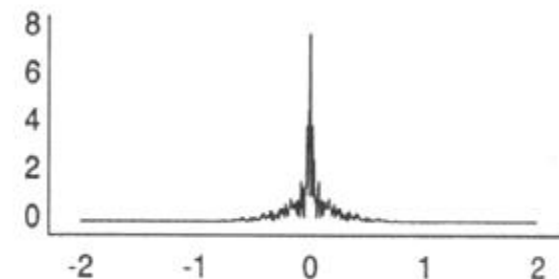
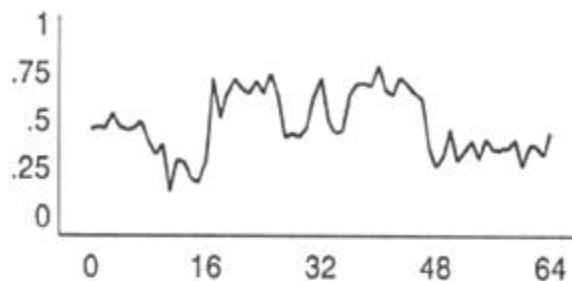


(d)



(e)

Reconstruction with
tent function
(= piecewise linear
interpolation)



Sampling at Too Low Frequency

Original function and its band-limited frequency spectrum

Signal sampling below Nyquist:

Mult./conv. with comb

Comb spaced too far (sampling rate $\leq 2 \cdot \text{bandlimit}$)

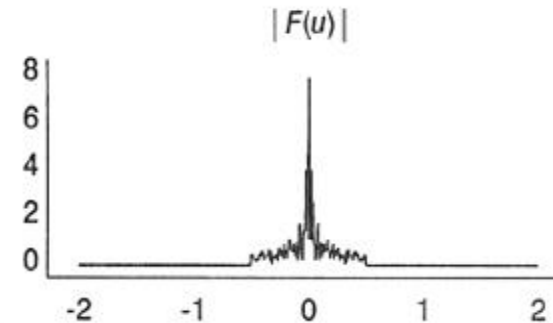
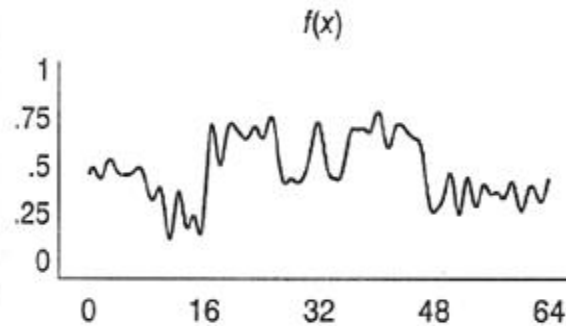
Spectral band overlap: artificial low frequenci.

Ideal filtering

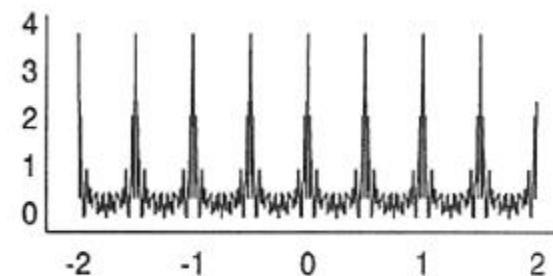
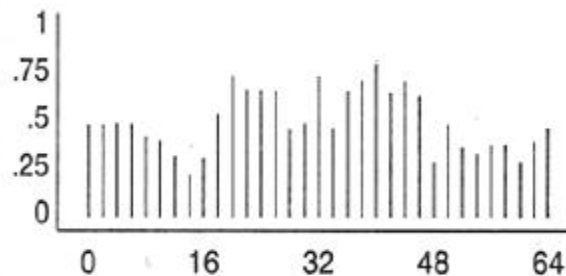
Fourier: box (mult.)
Space: *sinc* (conv.)

Band overlap in frequency domain cannot be corrected

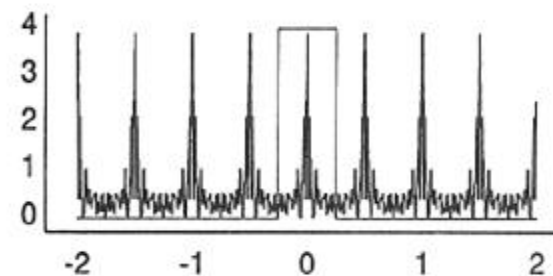
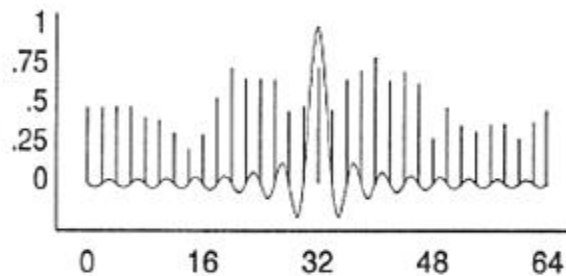
⇒ Aliasing



(a)



(b)



(c)

Sampling at Too Low Frequency

Reconstruction
with ideal *sinc*

Reconstruction fails
(frequency
components wrong
due to aliasing !)

Non-ideal filtering

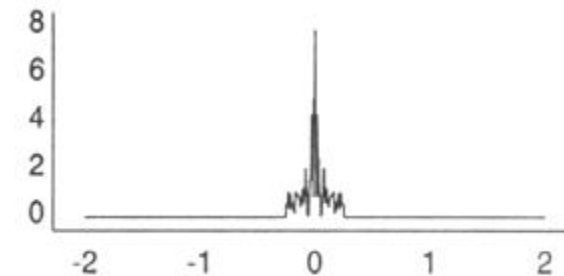
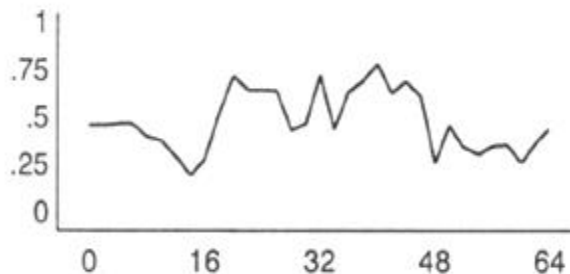
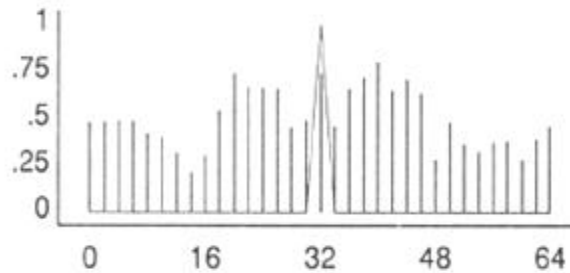
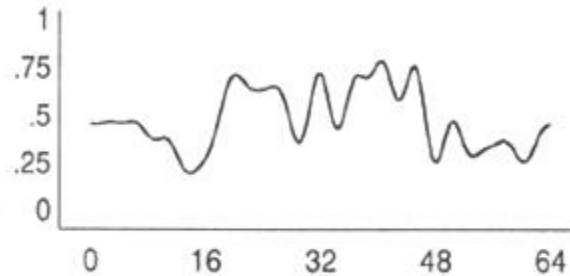
Fourier: sinc^2 (mult.)
Space: tent (conv.)

Artificial high frequen.
are not cut off

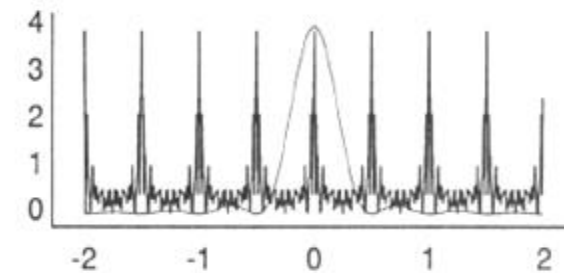
⇒ Aliasing artifacts

Reconstruction with
tent function
(= piecewise linear
interpolation)

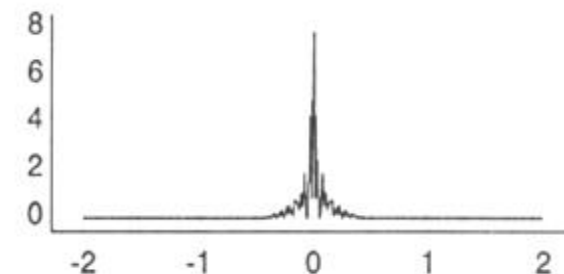
Even worse
reconstruction



(d)



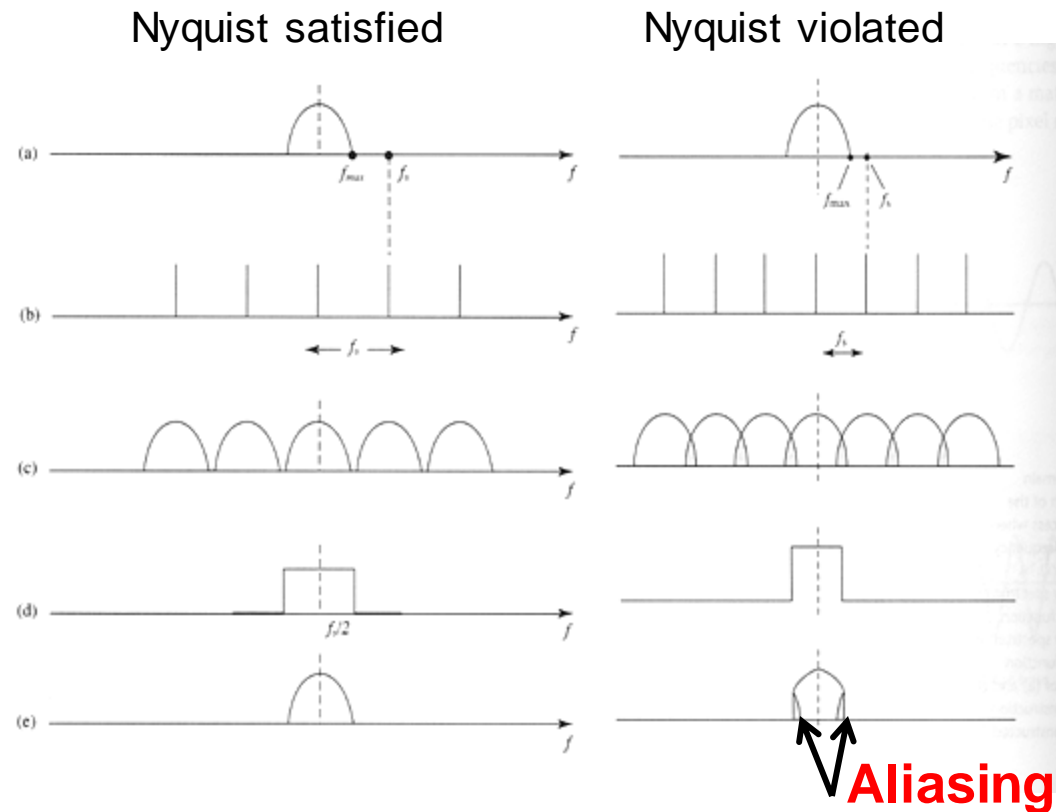
(e)



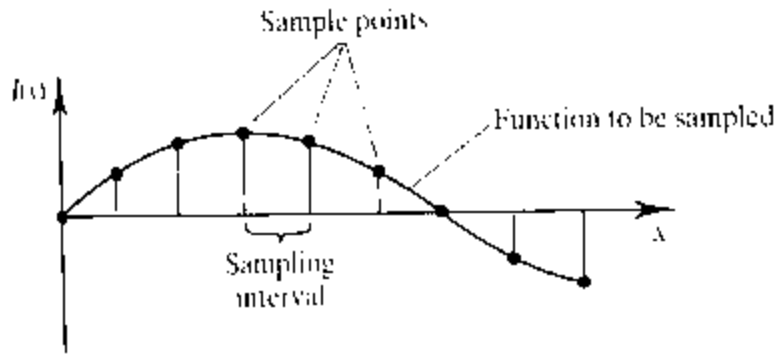
Aliasing

- High frequency components from the copies appear as low frequencies for the reconstruction process
- In Fourier space:

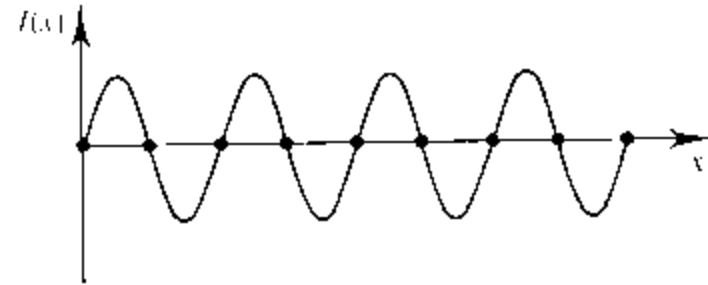
- Original spectrum
- Sampling comb
- Resulting spectrum
- Reconstruction filter
- Reconstructed spectrum



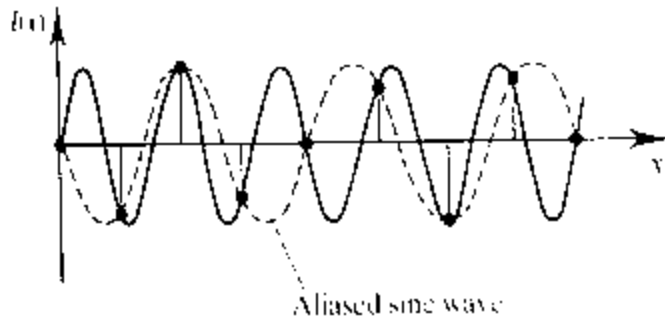
Aliasing in 1D



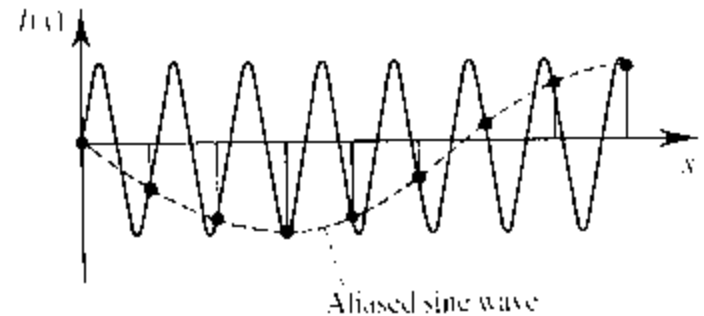
Spatial frequency < Nyquist



**Spatial frequency = Nyquist
2 samples / period**

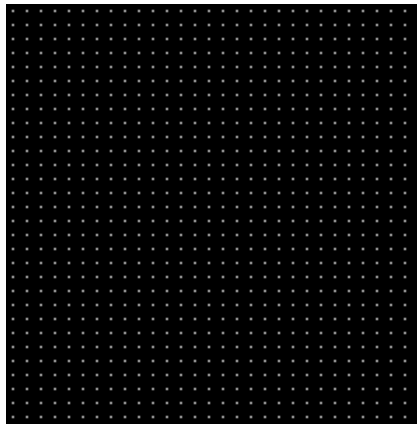
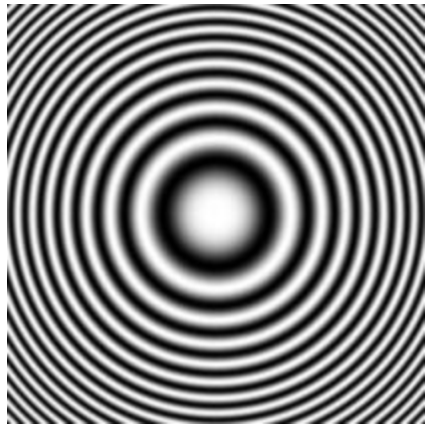


Spatial frequency > Nyquist

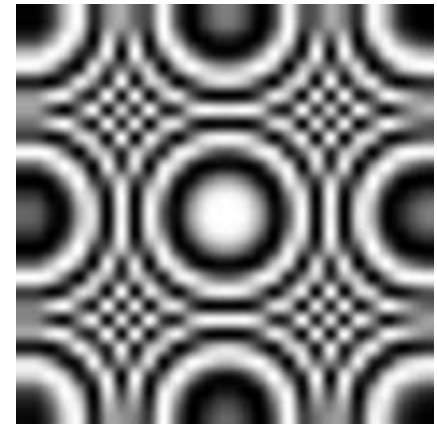


Spatial frequency >> Nyquist

Aliasing in 2D



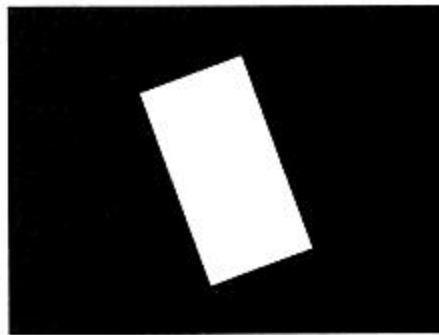
[wikipedia]



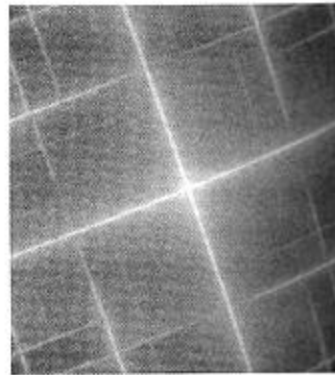
This original image sampled at these locations yields this reconstruction.

Aliasing in 2D

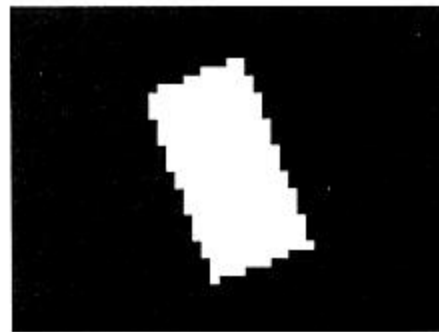
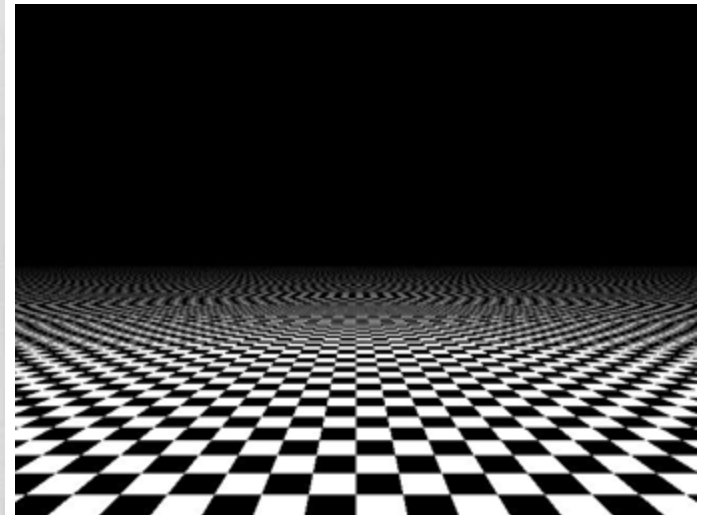
- Spatial sampling \Rightarrow repeated frequency spectrum
- Spatial conv. with box filter \Rightarrow spectral mult. with *sinc*



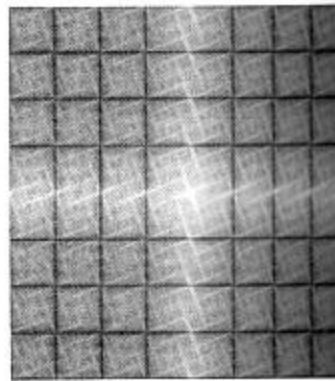
(a) Simulation of a perfect line



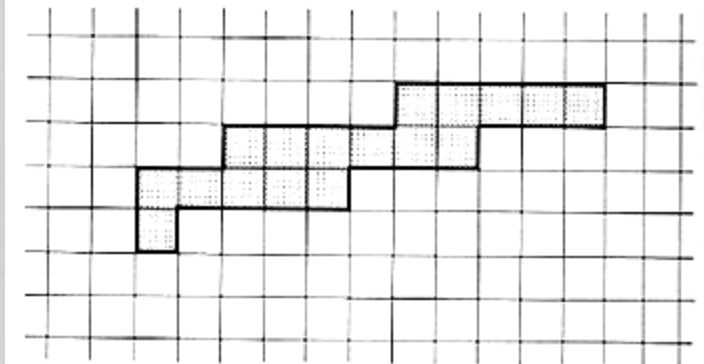
(b) Fourier transform of (a)



(c) Simulation of a jagged line



(d) Fourier transform of (c)



Causes for Aliasing

- **It all comes from sampling at discrete points**
 - Multiplication with comb function
 - Comb function: repeats the frequency spectrum
- **Issue when using non-band-limited primitives**
 - E.g., hard edges → infinitely high frequencies
- **In reality, integration over finite region necessary**
 - E.g., finite pixel size in sensor, integrates in the *analog domain*
- **Computer: analytic integration often not possible**
 - No analytic description of radiance or visible geometry available
- **Only way: numerical integration**
 - Estimate integral by taking multiple point samples, average
 - Leads to aliasing
 - Computationally expensive & approximate
- **Important:**
 - Distinction between **sampling errors** and **reconstruction errors**

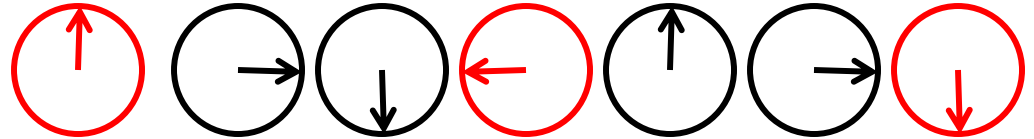
Sampling Artifacts

- **Spatial aliasing**
 - Stair cases, Moiré patterns (interference), etc...
- **Solutions**
 - Increasing the sampling rate
 - OK, but we have infinite frequencies at sharp edges
 - Post-filtering (after reconstruction)
 - Too late, does not work - only leads to blurred stair cases
 - Pre-filtering (blurring) of sharp features in analog domain (edges)
 - Slowly make geometry “fade out” at the edges?
 - Correct solution in principle, but blurred images might not be useful
 - Analytic low-pass filtering hard to implement
 - Super-sampling (see later)
 - On the fly re-sampling: densely sample, filter, down sample

Sampling Artifacts in Time

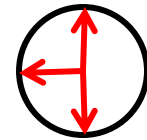
- **Temporal aliasing**

- Video of cart wheel, ...



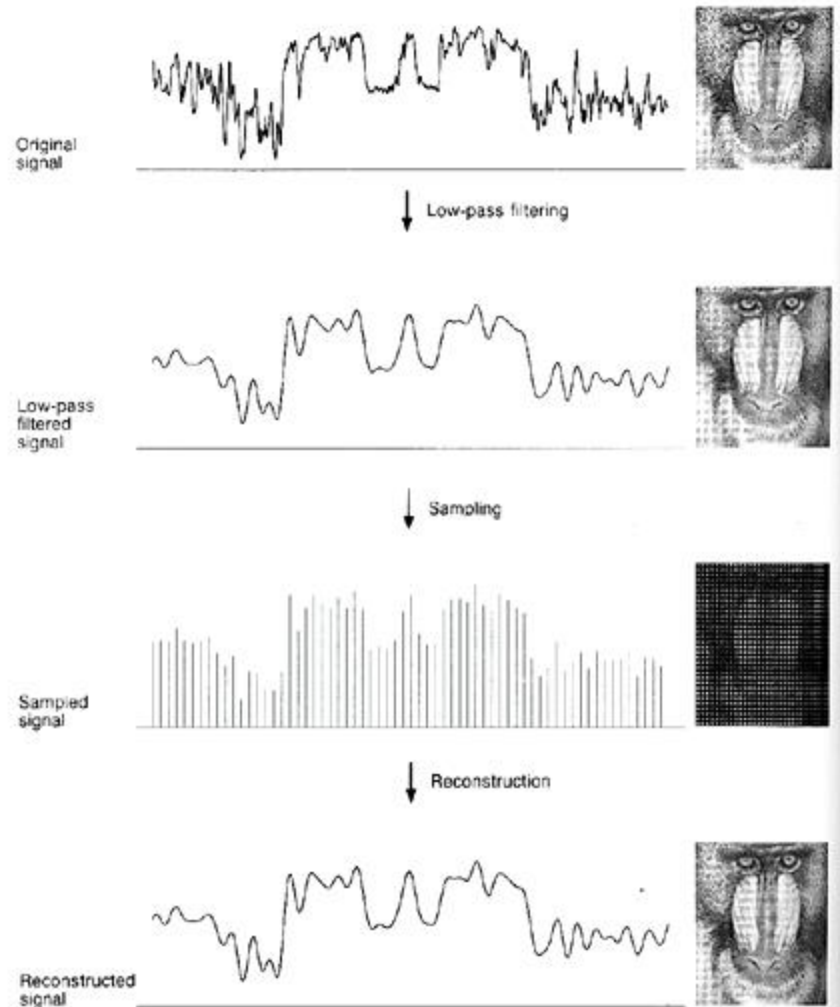
- **Solutions**

- Increasing the frame rate
 - OK
- Post-filtering (averaging several frames)
 - Does not work – creates replicas of details
- Pre-filtering (motion blur)
 - Should be done on the original analog signal
 - Possible for simple geometry (e.g., cartoons)
 - Problems with texture, etc...
- Super-sampling (see later)



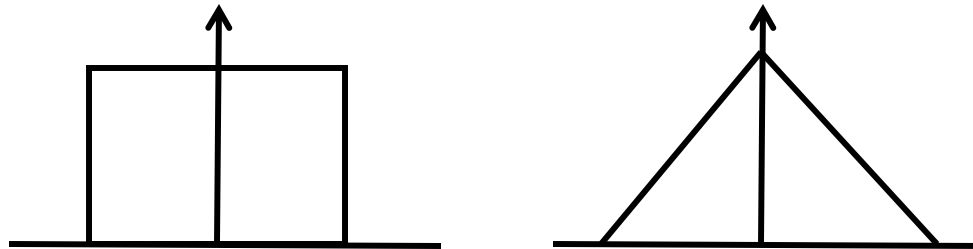
Antialiasing by Pre-Filtering

- **Filtering before sampling**
 - Analog/analytic original signal
 - Band-limiting the signal
 - Reduces Nyquist frequency for chosen sampling-rate
- **Ideal reconstruction**
 - Convolution with *sinc*
- **Practical reconstruction**
 - Convolution with
 - Box filter, Bartlett (tent)
 - Reconstruction error



Sources of High Frequencies

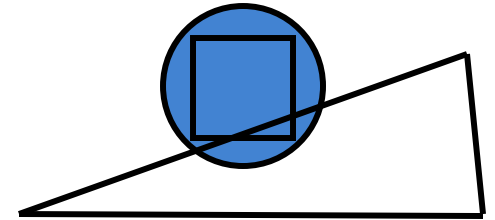
- **Geometry**
 - Edges, vertices, sharp boundaries
 - Silhouettes (view dependent)
 - ...
- **Texture**
 - E.g., checkerboard pattern, other discontinuities, ...
- **Illumination**
 - Shadows, lighting effects, projections, ...
- **Analytic filtering almost impossible**
 - Even with the most simple filters



Comparison

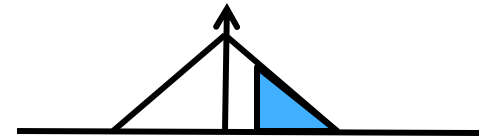
- **Analytic low-pass filtering (pixel/triangle overlap)**

- Ideally eliminates aliasing completely
- Complex to implement
 - Compute distance from pixel to a line
 - Weighted or unweighted area evaluation
 - Filter values can be stored in look-up tables
 - Fails at corners
 - Possibly taking into account slope



- **Over-/Super-sampling**

- Very easy to implement
- Does not eliminate aliasing completely
 - Sharp edges contain **infinitely** high frequencies
- But it helps: ...



Re-Sampling Pipeline

- **Assumption**

- Energy in higher frequencies typically decreases quickly
- Reduced aliasing by intermediate sampling at higher frequency

- **Algorithm**

- Super-sampling

- Sample continuous signal with high frequency f_1
- Aliasing (**only here!**) with energy beyond f_1 (**assumed to be small**)

- Reconstruction of signal

- Filtering with $g_1(x)$: e.g. convolution with sinc_{f_1}
- Exact representation with sampled values !!

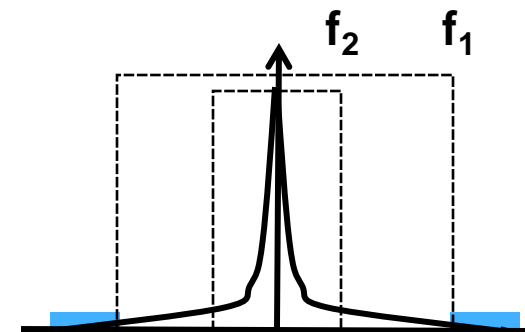
- Analytic low-pass filtering of signal

- Filtering with filter $g_2(x)$ where $f_2 \ll f_1$
- Signal is now band-limited w.r.t. f_2

- Re-sampling with a sampling frequency that is compatible with f_2

- No additional aliasing

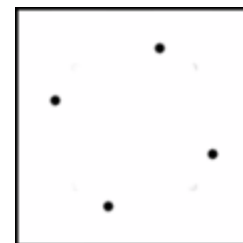
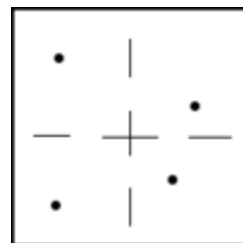
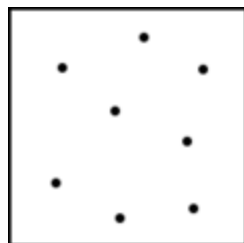
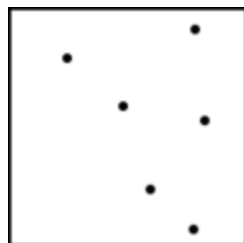
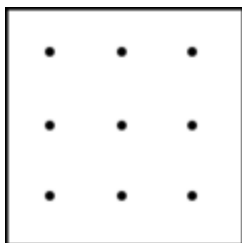
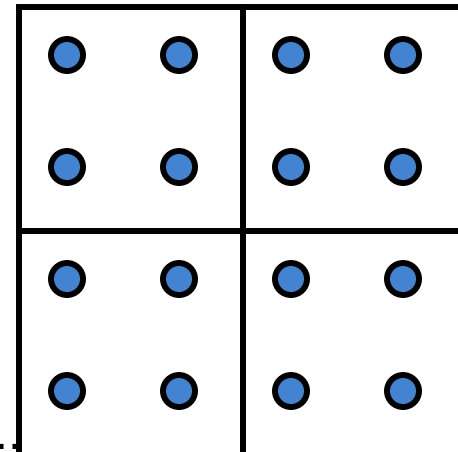
- Filters $g_1(x)$ & $g_2(x)$ can be combined – **with no additional aliasing**



Super-Sampling in Practice

- **Regular super-sampling**

- Averaging of N samples per pixel
- N : 4 (quite good), 16 (often sufficient)
- Samples: rays, z-buffer, motion, reflection, ...
- Filter weights
 - Box filter
 - Others: B-spline, pyramid (Bartlett), hexagonal, ...
- Sampling Patterns (left to right)
 - Regular: aliasing likely
 - Random: often clumps, incomplete coverage
 - Poisson Disc: close to perfect, but can be costly
 - Jittered: randomized regular sampling
 - Most often: rotated grid pattern



Super-Sampling Caveats

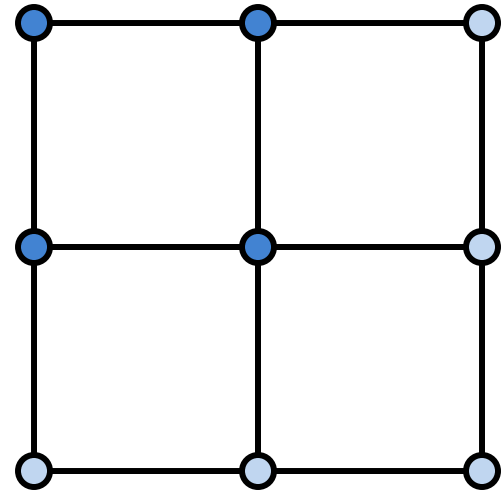
- **Popular mistake**

- Sampling at the corners of every pixel
- Pixel color by averaging from corners
- Free super-sampling ???

- **Problem**

- Wrong reconstruction filter !!!
- Same sampling frequency, but post-filtering with a tent function
- Blurring: loss of information

- **Post-reconstruction blur**



1x1 Sampling, 3x3 Blur



1x1 Sampling, 7x7 Blur

- **There is no “free” Super-sampling**

Adaptive Super-Sampling

- **Idea: locally adapt sampling density**
 - Slowly varying signal (mostly low frequencies): low sampling rate
 - Strong changes (mostly high frequencies): high sampling rate
- **Decide sampling density locally**
- **Decision criterion:**
 - Differences of pixel values
 - Contrast (relative difference)
 - $|A-B| / (|A|+|B|)$
 - Others

Adaptive Super-Sampling

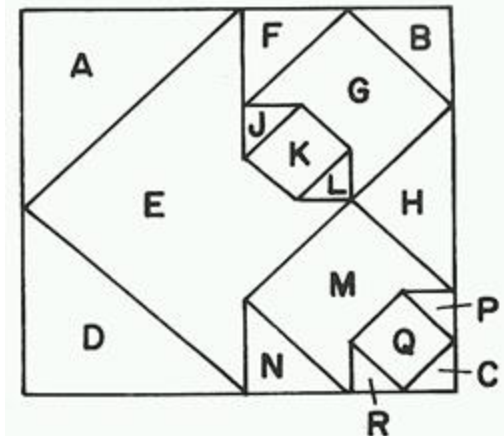
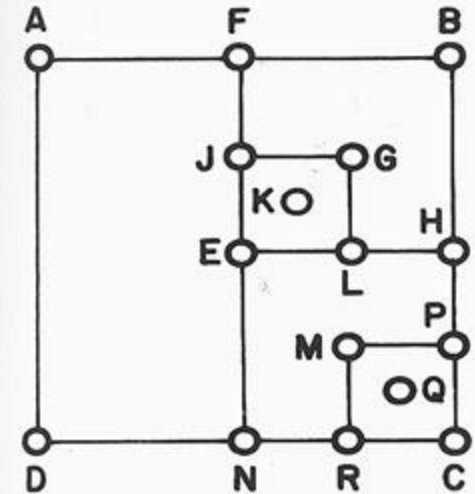
- **Recursive algorithm**

- Sampling at pixel corners and center
- Decision criterion for corner-center pairs
 - Differences, contrast, object/shader-IDs, ...
- Subdivide quadrant by adding 3 diag. points
- Filtering with weighted averaging
 - Tile: $\frac{1}{4}$ from each quadrant
 - Leaf quadrant: $\frac{1}{2}$ (center + corner)
- Box filter with final weight proport. to area \rightarrow

$$\frac{1}{4} \left(\begin{aligned} & \frac{A+E}{2} + \frac{D+E}{2} + \frac{1}{4} \left[\frac{F+G}{2} + \frac{B+G}{2} + \frac{H+G}{2} + \frac{1}{4} \left\{ \frac{J+K}{2} + \frac{G+K}{2} + \frac{L+K}{2} + \frac{E+K}{2} \right\} \right] \\ & + \frac{1}{4} \left[\frac{E+M}{2} + \frac{H+M}{2} + \frac{N+M}{2} + \frac{1}{4} \left\{ \frac{M+Q}{2} + \frac{P+Q}{2} + \frac{C+Q}{2} + \frac{R+Q}{2} \right\} \right] \end{aligned} \right)$$

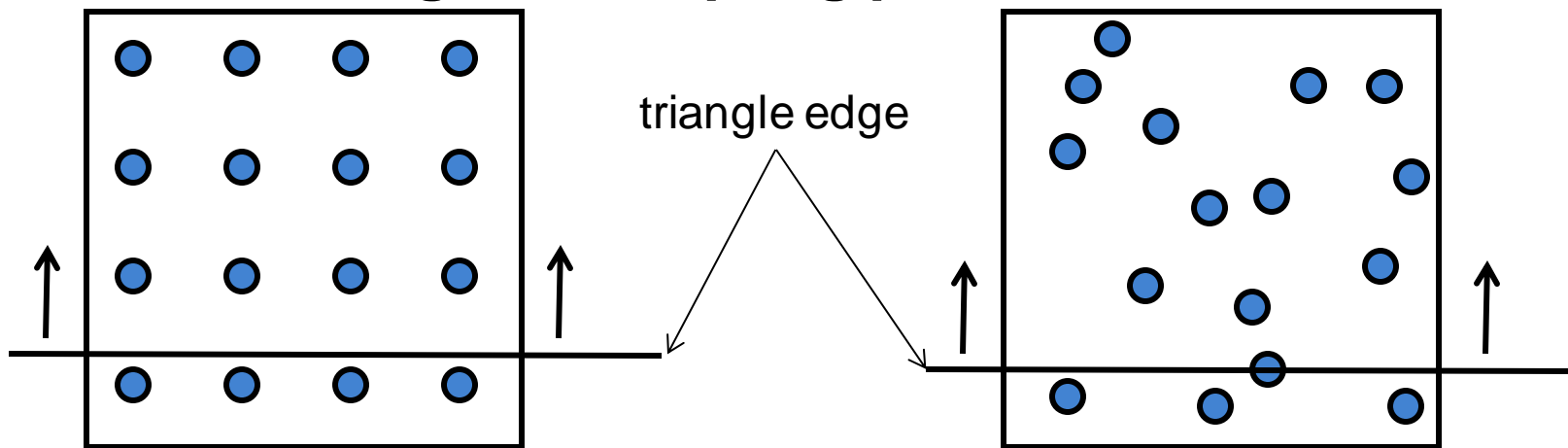
- **Extension**

- Jittering of sample points



Stochastic Super-Sampling

- **Problems with regular super-sampling**
 - Nyquist frequency for aliasing only shifted
 - Expensive: 4-fold to 16-fold effort
 - Non-adaptive: same effort everywhere
 - Too regular: reduction of effective number of axis-aligned levels
- **Introduce irregular sampling pattern**



Only 5 levels

$0 \rightarrow 4/16 \rightarrow 8/16 \rightarrow 12/16 \rightarrow 16/16$:

Up to 17 levels:

better, but noisy

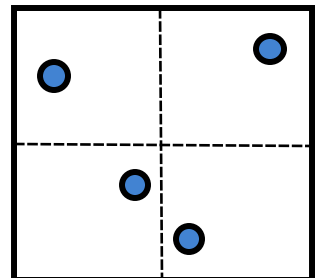
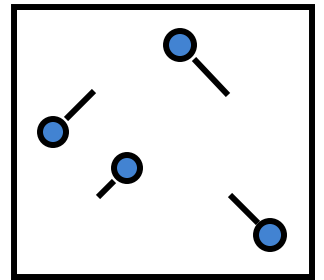
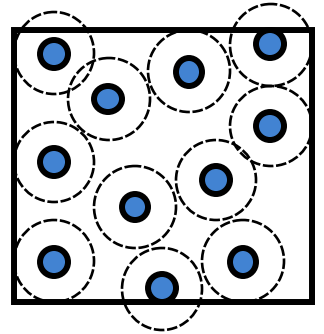
Stochastic Sampling

- **Requirements**

- Even sample distribution: no clustering
- Little correlation between positions: no alignment
- Incremental generation: on demand as needed

- **Generation of samples**

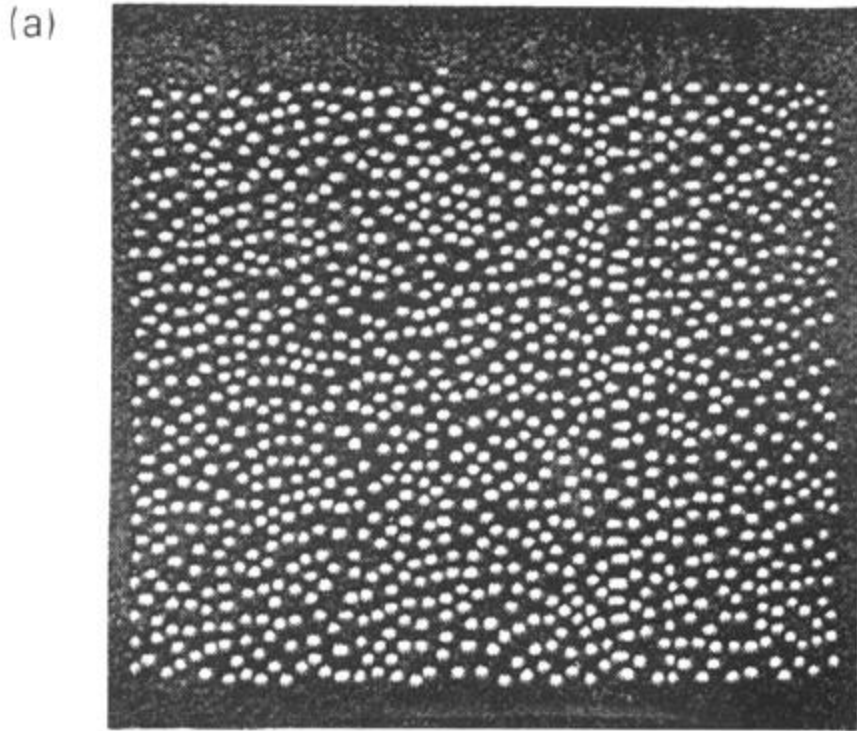
- Poisson-disk sampling
 - Random generation of samples
 - Rejection if closer than min distance to other samples
- Jittered sampling
 - Random perturbation from regular positions
- Stratified sampling
 - Subdivision into areas with one random sample in each
 - Improves even distribution
- Quasi-random numbers (Quasi-Monte Carlo)
 - E.g. Halton sequence
 - Advanced feature: see RIS course for more details



Poisson-Disk Sample Distribut.

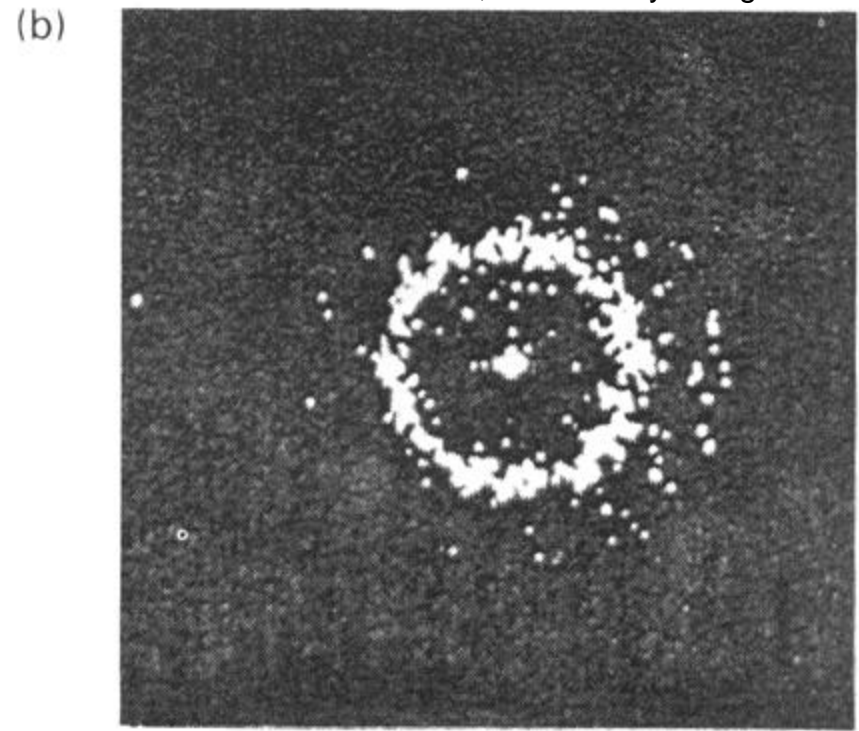
- **Motivation**

- Distribution of the optical receptors on the retina (here: ape)



Distribution of the photo-receptors

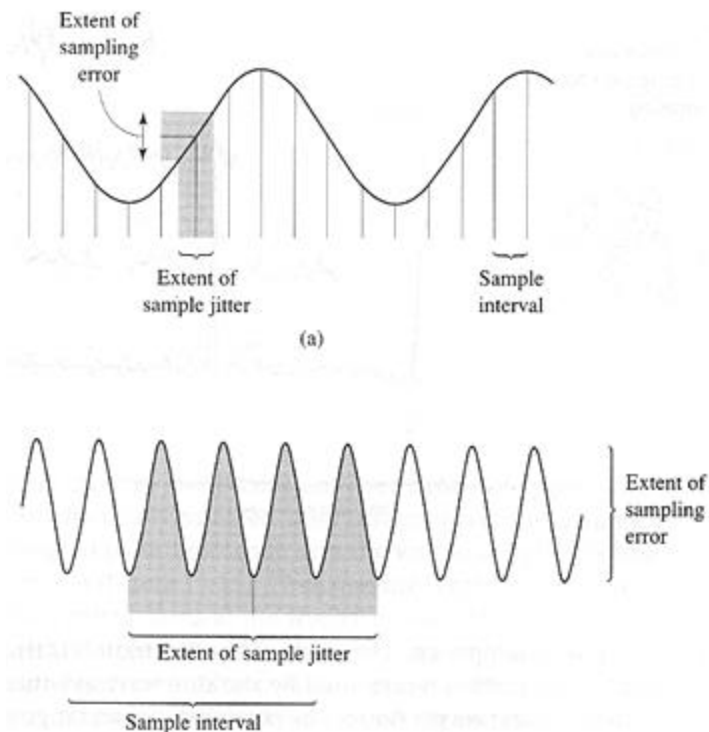
© Andrew Glassner, Intro to Raytracing



Fourier analysis

Stochastic Sampling

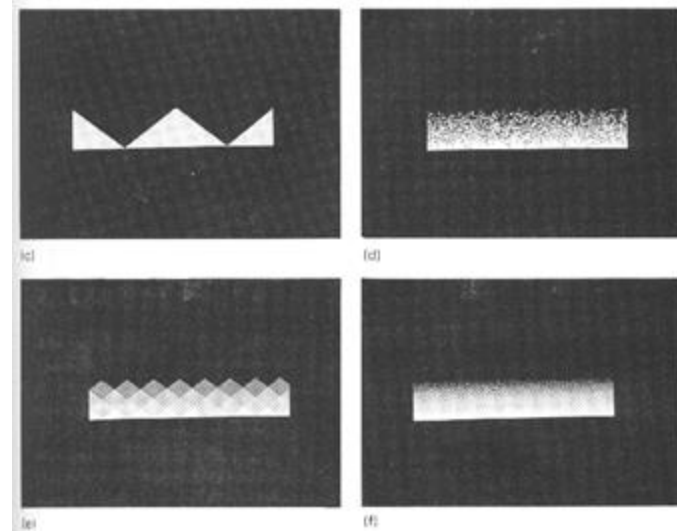
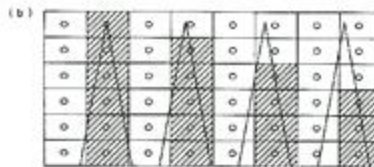
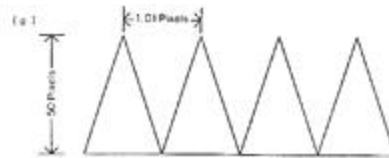
- **Slowly varying function in sample domain**
 - Closely reconstructs target value with few samples
- **Quickly varying function in sample domain**
 - Transforms energy in high-frequency bands into noise
 - Reconstructs average value as sample count increases



Examples

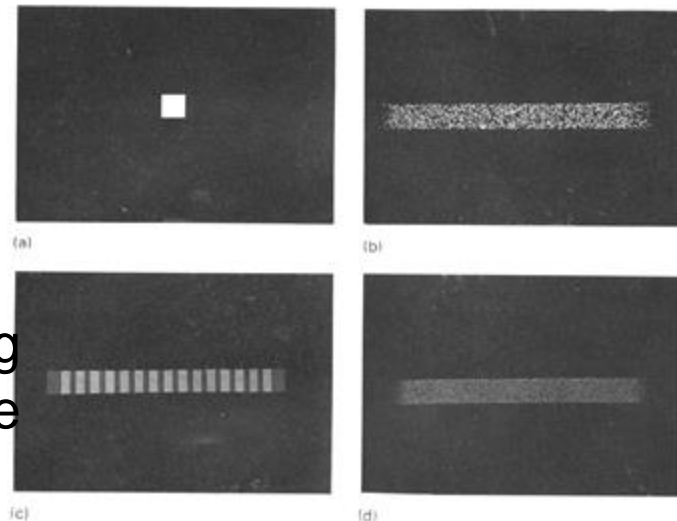
- **Spatial sampling: triangle comb**

- (c) 1 sample/pixel, no jittering: aliasing
- (d) 1 spp, jittering: noise
- (e) 16 spp, no jittering: less aliasing
- (f) 16 spp, jittering: less noise



- **Temporal sampling: motion blur**

- (a) 1 time sample, no jittering: aliasing
- (b) 1 time sample, jittering/pixel: noise
- (c) 16 samples, no jittering: less aliasing
- (d) 16 samples, jittering/pixel: less noise



Comparison

- **Regular, 1x1**
- **Regular, 3x3**
- **Regular, 7x7**
- **Jittered, 3x3**
- **Jittered, 7x7**

